

**UNCLASSIFIED**

---

---

**AD 401 744**

*Reproduced  
by the*

**DEFENSE DOCUMENTATION CENTER**

FOR

**SCIENTIFIC AND TECHNICAL INFORMATION**

**CAMERON STATION, ALEXANDRIA, VIRGINIA**



---

---

**UNCLASSIFIED**

**NOTICE:** When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

6332

FTD-TT-63-164

401744

CATALOGED BY ASTIA  
AS AD NO.

401744

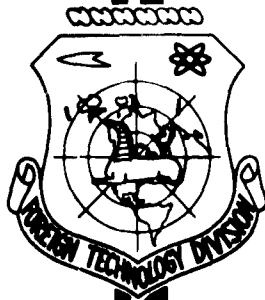
# TRANSLATION

APPLICATION OF THE ASYMPTOTIC METHOD OF CERTAIN  
PROBLEMS OF THE DYNAMICS OF VEHICLES

By

V. A. Yaroshevskiy

## FOREIGN TECHNOLOGY DIVISION



AIR FORCE SYSTEMS COMMAND

WRIGHT-PATTERSON AIR FORCE BASE

OHIO

## UNEDITED ROUGH DRAFT TRANSLATION

APPLICATION OF THE ASYMPTOTIC METHOD OF CERTAIN  
PROBLEMS OF THE DYNAMICS OF VEHICLES

By: V. A. Yaroshevskiy

English Pages: 19

Source: Russian Periodical, Inzhermerhyy  
Zhurnal, Vol. 2, Nr. 2, 1962, pp. 211-221.

T-21  
SOV/258-62-2-2-1/18

THIS TRANSLATION IS A RENDITION OF THE ORIGINAL FOREIGN TEXT WITHOUT ANY ANALYTICAL OR EDITORIAL COMMENT. STATEMENTS OR THEORIES ADVOCATED OR IMPLIED ARE THOSE OF THE SOURCE AND DO NOT NECESSARILY REFLECT THE POSITION OR OPINION OF THE FOREIGN TECHNOLOGY DIVISION.

PREPARED BY:

TRANSLATION SERVICES BRANCH  
FOREIGN TECHNOLOGY DIVISION  
WP-APB, OHIO.

APPLICATION OF THE ASYMPTOTIC METHOD OF CERTAIN  
PROBLEMS OF THE DYNAMICS OF VEHICLES

V. A. Yaroshevskiy

The present article generalizes and reduces to a compact form the rule of the conservation of the "adiabatic invariant."

As an example of applying the formulas obtained, we will examine problems concerning orbital decay of a satellite, the skip-glide motion of a glider, and the oscillations of a vehicle about the center of mass on a flight path with a variable velocity.

1. When investigating a number of questions concerning the dynamics of vehicles, the problem reduces to finding a solution of a system of equations in which the principal second-order equation describes a rapid quasiperiodic motion, and the parameters of the system are variable, slowly changing magnitudes whose change is described by a system of second-order equations.

Let us examine the system of equations

$$\frac{d^2y}{dt^2} + F\left(\tau, r_j, y, \frac{dy}{dt}\right) = 0, \quad (1)$$

$$\frac{dr_j}{dt} - \epsilon S_i\left(\tau, r_j, y, \frac{dy}{dt}\right) = 0 \quad i = 1, 2, \dots, n. \quad (2)$$

Here  $\varepsilon$  is the parameter of smallness,  $\tau = \varepsilon t$  is "slow" time,  $F$  and  $s_1$  are differentiable functions of both arguments.

At constant values of  $\tau$  and  $r_j$  and when  $\varepsilon = 0$ , the first equation describes a rapid periodic motion.

To obtain the relationships characterizing the change of "amplitude" values of the variable  $y$  and the "mean" values of the variables  $r_1$ , we will use in this work a method analogous to the method developed by Kuzmak [1, 2].

We will represent the unknown functions as functions of two variables, "slow" time  $\tau$  and the phase of the periodic motion  $\varphi$ :

$$\begin{aligned} y &= y_0(\tau, \varphi) + \varepsilon y_1(\tau, \varphi) + O(\varepsilon^2), & r_i &= r_{i0}(\tau) + \varepsilon r_{i1}(\tau, \varphi) + O(\varepsilon^2), \\ \frac{d\varphi}{dt} &= \omega_0(\tau) + \varepsilon \omega_1(\tau) + O(\varepsilon^2). \end{aligned}$$

To avoid the appearance of secular members, we will require that the functions be periodic with respect to  $\varphi$ . Then the order of smallness of the terms of the asymptotic expansion will be retained in the interval  $\tau \sim 1$ , i.e.,  $t \sim 1/\varepsilon$  [1].

Separating in all equations the terms of the zero and first order of smallness, we derive

$$\begin{aligned} & \left\{ \omega_0^2 \frac{\partial^2 y_0}{\partial \varphi^2} + F(\tau, r_{i0}, y_0, 0) + \varepsilon \left[ \frac{d\omega_0}{d\tau} \frac{\partial y_0}{\partial \varphi} + 2\omega_0 \frac{\partial^2 y_0}{\partial \varphi \partial \tau} + \frac{\partial F}{\partial y}(\tau, r_{i0}, y_0, 0) \omega_0 \frac{\partial y_0}{\partial \varphi} + \right. \right. \\ & \left. \left. + \omega_0^2 \frac{\partial^2 y_1}{\partial \varphi^2} + \frac{\partial F}{\partial y}(\tau, r_{i0}, y_0, 0) y_1 + 2\omega_0 \omega_1 \frac{\partial^2 y_0}{\partial \varphi^2} + \right. \right. \\ & \left. \left. + \sum_{i=1}^n \frac{\partial F}{\partial r_i}(\tau, r_{i0}, y_0, 0) r_{i1} \right\} + O(\varepsilon^2) = 0. \end{aligned} \quad (3)$$

Here  $\partial F / \partial y'$  designates a partial derivative of the function  $F$  with respect to the argument  $\varepsilon dy/dt$ .

$$\varepsilon \left\{ \omega_0 \frac{\partial r_{11}}{\partial \varphi} + \frac{dr_{10}}{d\tau} - s_1 \left( \tau, r_{10}, y_0, \omega_0 \frac{\partial y_0}{\partial \varphi} \right) \right\} + O(\varepsilon^2) = 0 \quad (4)$$

Hereafter such partial derivatives of the function  $F$  will be designated as  $\partial F_0 / \partial y$ ,  $\partial F_0 / \partial y'$  and  $\partial F_0 / \partial r_j$ , and the functions themselves  $F(\tau, r_{j0}, y_0, 0)$  and  $s_1(\tau, r_{j0}, y_0, \omega_0 \partial y_0 / \partial \varphi)$  as  $F_0$  and  $S_{10}$ .

Integrating the expression in  $\varepsilon$  in Eq. (4) with respect to  $\varphi$  from  $\varphi$  to  $\varphi + 2\pi$  and taking into account that  $r_{11}$  is a function periodic with respect to  $\varphi$ , we obtain

$$\frac{dr_{10}}{d\tau} - \bar{s}_{10} = 0, \quad (5)$$

where

$$\bar{s}_{10} = \frac{1}{2\pi} \int_{\varphi}^{\varphi+2\pi} s_1 \left( \tau, r_{10}, y_0, \omega_0 \frac{\partial y_0}{\partial \varphi} \right) d\varphi$$

(naturally, when integrating here with respect to  $\varphi$  the quantities  $\tau$  and  $r_j$  are considered parameters).

The term  $r_{j1}(\tau, \varphi)$  is determined by the relation:

$$r_{11}(\tau, \varphi) = \frac{1}{\omega_0} \int_{\varphi_0(\tau)}^{\varphi} [s_{10} - \bar{s}_{10}] d\varphi, \quad (6)$$

where  $\varphi_0(\tau)$  is some function which is nonessential when calculating the zero terms of the asymptotic expansion.

The term  $y_0$  is a solution of the "standard" equation

$$\omega_0^2(\tau) \frac{\partial^2 y_0}{\partial \varphi^2} + F[\tau, r_{10}, y_0, 0] = 0. \quad (7)$$

The value of  $\omega_0$  is conveniently determined from the hypothesis that  $y_0(\varphi) = y_0(\varphi + 2\pi)$ .

The change in time of the amplitude values  $y_0$ , which is of particular interest to us, cannot be determined from this equation, therefore we must attract the "condition of periodicity" of the function  $y_1$  [1].

The equation (linear) for determining the function  $y_1$  has the following form (see Eq. (3)):

$$\begin{aligned} & \omega_0^2(\tau) \frac{\partial^2 y_1}{\partial \varphi^2} + \frac{\partial F_0}{\partial y} y_1 = \\ & - \left[ \frac{d\omega_0}{d\tau} \frac{dy_0}{d\varphi} + 2\omega_0 \frac{\partial^2 y_0}{\partial \varphi \partial \tau} + \frac{\partial F_0}{\partial y'} \omega_0 \frac{\partial y_0}{\partial \varphi} + 2\omega_0 \omega_1 \frac{\partial^2 y_0}{\partial \varphi^2} + \sum_{l=1}^n \frac{\partial F_0}{\partial r_l} r_{l1} \right] \end{aligned} \quad (8)$$

Differentiating Eq. (7) in terms of  $\varphi$ , we are easily convinced that the term  $\partial y_0 / \partial \varphi$  satisfies Eq. (8) without the right part. Then, multiplying Eq. (7) by  $y_1$  and (12) by  $\partial y_0 / \partial \varphi$ , subtracting the obtained relations from one another, integrating in terms of  $\varphi$  from  $\varphi$  to  $\varphi + 2\pi$  and taking into account that  $\int_{\varphi}^{\varphi+2\pi} \frac{\partial^2 y_0}{\partial \varphi^2} \frac{\partial y_0}{\partial \varphi} d\varphi = 0$ , we easily find the first conditions needed for the periodicity of the function  $y_1$  in terms of  $\varphi$ :

$$\begin{aligned} & \int_{\varphi}^{\varphi+2\pi} \left\{ \frac{d\omega_0}{d\tau} \frac{\partial y_0}{\partial \varphi} + 2\omega_0 \frac{\partial^2 y_0}{\partial \varphi \partial \tau} + \frac{\partial F_0}{\partial y'} \omega_0 \frac{\partial y_0}{\partial \varphi} + \right. \\ & \left. + \sum_{l=1}^n \frac{\partial F_0}{\partial r_l} r_{l1} \right\} \frac{\partial y_0}{\partial \varphi} d\varphi = 0 \end{aligned} \quad (9)$$

We will not examine the second condition of periodicity which enables us to determine the correction  $\omega_1$ .

Taking into account that  $y_0$  is a solution of Eq. (7), we find that

$$\frac{\partial y_0}{\partial \varphi} = \pm \frac{1}{\omega_0} \left( -2 \int_{y_m}^{y_0} F_0 dy_0 \right)^{1/2} \quad (10)$$

where  $y_m$  is the amplitude value of the function  $y_0$  indifferently, more or less, corresponding to the vanishing of the derivative  $\partial y_0 / \partial \varphi$  (the plus sign is ascribed to the increasing portion of  $y_0$  and the minus sign to the decreasing portion of  $y_0$ ).

It is evident that relation (10) is equivalent to the determination of the "instantaneous amplitude  $y_m$ " for values  $y$  and  $dy/dt$  by means of "frozen" Eq. (1) ( $\tau = \text{const}$ ,  $r_j = \text{const}$ ,  $\epsilon = 0$ ).

We will transform Eq. (9) to the following form:

$$\frac{d}{d\tau} \left[ \int_{\varphi}^{\varphi+2\pi} \omega_0 \left( \frac{\partial y_0}{\partial \varphi} \right)^2 d\varphi \right] + \int_{\varphi}^{\varphi+2\pi} \omega_0 \frac{\partial F_0}{\partial y'} \left( \frac{\partial y_0}{\partial \varphi} \right)^2 d\varphi + \\ + \sum_{l=1}^n \int_{\varphi}^{\varphi+2\pi} \frac{\partial F_0}{\partial r_l} \frac{\partial y_0}{\partial \varphi} r_{l1} d\varphi = 0.$$

When integrating with respect to  $\varphi$ , the magnitudes  $\tau$  and  $r_{j0}$  must be considered as parameters, therefore  $\partial y_0 / \partial \varphi d\varphi = dy_0$ . Taking this into account as well as relation (10), we will reduce Eq. (9) to the following form:

$$\frac{d}{d\tau} \oint \left( -2 \int_{y_m}^{y_0} F_0 dy_0 \right)^{1/2} dy_0 + \\ + \oint \frac{\partial F_0}{\partial y'} \left( -2 \int_{y_m}^{y_0} F_0 dy_0 \right)^{1/2} dy_0 + \sum_{l=1}^n \oint \frac{\partial F_0}{\partial r_l} r_{l1} dy_0 = 0. \quad (11)$$

(The sign  $\oint$  means that integration is performed for the complete period of the change  $y_0$ ).

Integrating by parts and taking into account that  $r_{j1}$  is a periodic function determined by formula (6), and

$$\oint \frac{\partial F_0}{\partial r_l} dy_0 = 0.$$

we obtain

$$\oint \frac{\partial F_0}{\partial r_l} r_{l1} dy_0 = \oint \left( - \int_{y_m}^{y_0} \frac{\partial F_0}{\partial r_l} dy_0 \right) \frac{y_0 - \bar{y}_0}{\left( -2 \int_{y_m}^{y_0} F_0 dy_0 \right)^{1/2}} dy_0. \quad (12)$$

Having differentiated the first term in Eq. (11), we derive

$$\begin{aligned}
\frac{d}{d\tau} \oint \left( -2 \int_{y_m}^{y_0} F_0 dy_0 \right)^{1/2} dy_0 &= \oint \frac{\sum_{l=1}^n \left( - \int_{y_m}^{y_0} \frac{\partial F_0}{\partial r_l} dy_0 \right) s_{l_0}}{\left( -2 \int_{y_m}^{y_0} F_0 dy_0 \right)^{3/2}} dy_0 + \\
&+ \frac{\partial}{\partial \tau} \Big|_{r_{j_0} = \text{const}} \left[ \oint \left( -2 \int_{y_m}^{y_0} F_0 dy_0 \right)^{1/2} dy_0 \right].
\end{aligned} \tag{13}$$

Substituting (12) and (13) into (11), we obtain

$$\begin{aligned}
\frac{\partial}{\partial \tau} \Big|_{r_{j_0} = \text{const}} \oint \frac{dy_0}{dt}(y_0) dy_0 + \oint \frac{\partial F_0}{\partial y_l} \frac{dy_0}{dt}(y_0) dy_0 + \\
\oint \frac{\sum_{l=1}^n \left( - \int_{y_m}^{y_0} \frac{\partial F_0}{\partial r_l} dy_0 \right) s_{l_0} dy_0}{\frac{dy_0}{dt}(y_0)} = 0.
\end{aligned} \tag{14}$$

Here  $dy_0/dt(y_0)$  is the solution of Eq. (1) with frozen values of  $r_{j_0}$  and  $\tau$  when  $\varepsilon = 0$ .

Since

$$\int_{y_m}^{y_0} \frac{\partial F_0}{\partial r_l} dy_0 = - \frac{1}{2} \frac{\partial}{\partial r_l} \left[ \frac{dy_0}{dt}(y_0) \right]^2 = - \frac{dy_0}{dt} \frac{\partial}{\partial r_l} \left[ \frac{dy_0}{dt} \right],$$

then the third integral in (14) can be presented in the following form:

$$\oint \frac{\sum_{l=1}^n \left( - \int_{y_m}^{y_0} \frac{\partial F_0}{\partial r_l} dy_0 \right) s_{l_0} dy_0}{\frac{dy_0}{dt}(y_0)} = \oint \sum_{l=1}^n \frac{\partial}{\partial r_l} \left[ \frac{dy_0}{dt}(y_0) \right] s_{l_0} dy_0.$$

We will introduce the following designations for averageable values:

$$\begin{aligned} \overline{\frac{\partial F_0}{\partial y'}} &= \frac{\oint \frac{\partial F_0}{\partial y'} \frac{dy_0}{dt} dy_0}{\oint \frac{dy_0}{dt} dy_0} = \overline{\frac{\partial F_0}{\partial y'}}(\tau, r_{j_0}, y_m), \\ &= \frac{\oint \frac{\partial}{\partial r_j} \left[ \frac{dy_0}{dt} \right] s_{j_0} dy_0}{\oint \frac{\partial}{\partial r_j} \left[ \frac{dy_0}{dt} \right] dy_0} = \overline{s_{j_0}}(\tau, r_{j_0}, y_m), \\ &\quad \oint \frac{dy_0}{dt} dy_0 = D(\tau, r_{j_0}, y_m) \end{aligned}$$

is the "adiabatic invariant" [3].

Then the concluding result can be presented in the following form:

$$\frac{\partial}{\partial \tau} \Big|_{r_{j_0} = \text{const}} D + \sum_{j=1}^n \frac{\partial D}{\partial r_{j_0}} \overline{s_{j_0}} + D \overline{\frac{\partial F_0}{\partial y'}} = 0. \quad (15)$$

The values of  $r_{j_0}$  are determined from Eq. (5)

$$\frac{dr_{j_0}}{d\tau} = \overline{s_{j_0}}(\tau, r_{j_0}, y_m).$$

Using these equations we can determine the change in time of the amplitude of the zero term of the asymptotic expansion for  $\underline{y}$  (i.e.,  $y_m$ ) and also the change in time of the zero terms of the asymptotic expansion (mean values) for  $r_1$  (i.e.,  $r_{1_0}$ ). The difference between the exact solution and the zero terms of the asymptotic expansion in the interval  $t \sim 1/\varepsilon$  retains an order of smallness  $\varepsilon$ .

The first two terms of Eq. (15) can be combined into a "unique derivative" of the integral  $D$  in time, which is distinguished from the real, complete derivative in that the values  $dr_{j_0}/dt = \overline{s_{j_0}}$  are replaced by the values  $\overline{s_{j_0}}$ . If the right-hand parts of Eq. (2) were to depend on  $\underline{y}$  and  $dy/dt$ , then (when  $\partial F/\partial y' = 0$ ) we would obtain the known rule of the conservation of the adiabatic invariant  $D = \text{const}$  [3]. Equation (15) generalizes this rule.

The magnitude  $\overline{\frac{\partial F_0}{\partial y'}}$  corresponds to the term  $\partial F_0/\partial y'$  averaged

over a period with a weight of  $(dy_0/dt)^2$ , the magnitudes  $\overline{s_{j_0}}$  correspond to the terms  $s_{j_0}$  averaged over a period with a weight

$$\frac{\partial}{\partial r_j} \left[ \left( \frac{dy_0}{dt} \right)^2 \right].$$

It is easy to see that in the important special case where

$$F \left( \tau, r_j, y, \varepsilon \frac{dy}{dt} \right) = F_1 \left( \tau, y, \varepsilon \frac{dy}{dt} \right) F_2(\tau, r_j),$$

the magnitude  $\overline{s_{j_0}}$ , like  $\overline{\partial F_0 / \partial y'}$ , is a magnitude averaged over a period with a weight of  $(dy_0/dt)^2$ .

We note in conclusion that these results can easily be generalized for the case where the term  $\varepsilon dy/dt$  is replaced by the term  $\varepsilon f(dy/dt)$ .

2. Let us examine, as the first example, the problem of the decay of an elliptical orbit of a satellite owing to a small aerodynamic drag.

The initial system of equations is written in the following form [4]:

$$\frac{d^2 \xi}{d\varphi^2} + \xi = \eta, \quad (16)$$

$$\frac{d\eta}{d\varphi} = \eta \frac{C_x S R_0}{m} \frac{\rho(\xi)}{\xi} \sqrt{1 + \left( \frac{1}{\xi} \frac{d\xi}{d\varphi} \right)^2}, \quad (17)$$

where

$$\xi = \frac{R_0}{R}, \quad \eta = \frac{g_0 R_0^2}{\left( R^2 \frac{d\varphi}{dt} \right)^2}$$

is the magnitude which is the invariant on a Keplerian trajectory and in this case slowly changes,  $R$  is the distance from the vehicle to the center of the planet,  $R_0$  is the planet radius,  $g_0$  is the acceleration of gravity at the planet surface,  $\varphi$  is the central angle,  $\rho$  is the density of the atmosphere,  $C_x$  is the drag coefficient,  $S$  is the

characteristic area,  $\underline{m}$  is the satellite mass. The meaning of the designations can be cleared up from Fig. 1. The perigean point corresponds to the value  $\xi_p = \frac{R_0}{R_p}$ .

Let us apply the method cited in part 1 to this system (16), (17). Taking into account that

$$\frac{d\xi}{d\varphi}(\xi_p, \eta, \xi) = \sqrt{(\xi_p - \eta)^2 - (\xi - \eta)^2}, \quad (18)$$

we will find the value of the adiabatic invariant

$$D = \oint \frac{d\xi}{d\varphi} d\xi = \pi (\xi_p - \eta)^2$$

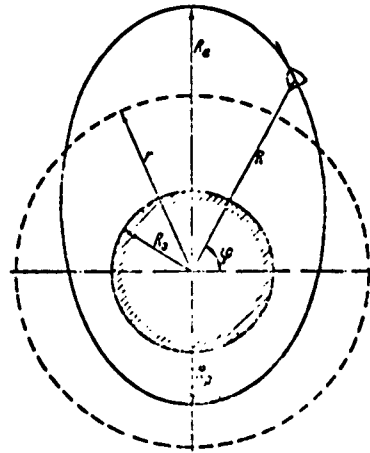


Fig. 1.

Equation (15) has the following form here:

$$\frac{\partial D}{\partial \xi_p} \frac{d\xi_p}{d\varphi} + \frac{\partial D}{\partial \eta} \frac{d\eta}{d\varphi} = 0.$$

Hence

$$\frac{d\xi_p}{d\varphi} = - \frac{d\eta}{d\varphi} (\eta_0, \xi_p)$$

( $\eta_0$  is the zero term of the asymptotic expansion for  $\eta$  or the average value for  $\eta$  during the circling time).

Equation (5) is written as

$$\frac{d\eta_0}{d\varphi} = \overline{\frac{d\eta}{d\varphi}} (\eta_0, \xi_0).$$

Here

$$\overline{\frac{d\eta}{d\varphi}} = \frac{C_x S R_0}{m} \eta_0 \frac{\int_{\eta_0 - \xi_p}^{\xi_p} \frac{\rho(\xi)}{\xi} \sqrt{1 + \left(\frac{1}{\xi} \frac{d\xi}{d\varphi}\right)^2} \frac{d\xi}{d\xi|d\varphi}}{\int_{\eta_0 - \xi_p}^{\xi_p} \frac{d\xi}{d\xi|d\varphi}} \quad (19)$$

$$\overline{\frac{d\eta}{d\varphi}} = \frac{C_x S R_0}{m} \eta_0 \frac{\int_{\eta_0 - \xi_p}^{\xi_p} \frac{\rho(\xi)}{\xi} \sqrt{1 + \left(\frac{1}{\xi} \frac{d\xi}{d\varphi}\right)^2} \frac{\partial}{\partial \eta} \left(\frac{d\xi}{d\varphi}\right) d\xi}{\int_{\eta_0 - \xi_p}^{\xi_p} \frac{\partial}{\partial \eta} \left(\frac{d\xi}{d\varphi}\right) d\xi} \quad (20)$$

where  $d\xi|d\varphi$  are determined by formula (18).

Let us extract the explicit expressions for these derivatives in two extreme cases.

a) Considering the eccentricity of the orbit as small ( $\xi_p - \xi \leq \xi_p$ ), approximating the dependence of density on height  $H = R - R_0$  by the function  $\rho = \rho_0 \exp(-\lambda H)$ , and introducing the concept of an "average" radius of the orbit  $r = \frac{R_0}{\eta_0}$ , we will transform Eqs. (19) and (20) to the following (dimensional) form:

$$\begin{aligned} \frac{dR_p}{d\varphi} &= -\frac{\rho_0 C_x S r^2}{m} e^{-\lambda(r-R_0)} \{I_0[\lambda(r-R_p)] - I_1[\lambda(r-R_p)]\} \times \\ &\quad \times \left\{ 1 + 0 \left[ \lambda \frac{(r-R_p)^2}{r} \right] + 0 \left[ \frac{r-R_p}{r} \right] \right\} \\ \frac{dr}{d\varphi} &= -\frac{\rho_0 C_x S r^2}{m} e^{-\lambda(r-R_0)} I_0[\lambda(r-R_p)] \left\{ 1 + 0 \left[ \lambda \frac{(r-R_p)^2}{r} \right] + 0 \left[ \frac{r-R_p}{r} \right] \right\}. \end{aligned}$$

Designating the parameter  $\lambda(r - R_p)$ , which in the indicated assumption is proportional to the eccentricity, in terms of  $x$ , we derive the known result (see, for example, [5])

$$\frac{dr}{dx} = \frac{1}{\lambda} \frac{I_0(x)}{I_1(x)},$$

where  $I_0$  and  $I_1$  are Bessel's functions of the imaginary argument.

At large values of  $x$ ,  $I_0/I_1 \rightarrow 1$ . Consequently, if the eccentricity is comparatively large, the height of the perigee  $R_p$  decreases slowly in comparison with the average radius of the orbit  $\bar{r}$  (and the height of the apogee  $R_a$ ), and the eccentricity rapidly decreases.

At small values of  $x$ ,  $I_0/I_1 \approx 2|x$ ,  $x \sim \exp \lambda r/2$ , the orbit changes to a circular orbit, the average radius of the orbit and the height of the perigee virtually coincide and they decrease with the same velocity.

b) If the eccentricity of the original orbit is large, then the main deceleration of the satellite occurs in the portion of the path adjacent to the perigean point, wherein the role of this portion of the path grows with an increase of  $\lambda$ . The dependence of density on height for a large range of altitudes, characteristic for an orbit with a large eccentricity, can be approximated by the function

$$\rho = \rho_0 e^{-\int_0^H \lambda(\epsilon H) dH}$$

where  $\lambda$  is a slowly changing function  $\lambda(\epsilon H)$ . In the section  $R - R_p \ll \ll R_p$ , which is of interest to us, we can consider that

$$\rho = \rho_p e^{-\lambda_p (R - R_p)}$$

$$(\rho_p = \rho(R_p), \lambda_p = \lambda(R_p)).$$

Simplifying the expressions for derivatives (19) and (20) in accord with the assumptions made, we obtain (again in a dimensional form)

$$\frac{dr}{d\varphi} = -\frac{C_x S r p_p}{m \sqrt{2\pi \lambda_p \left( \frac{1}{R_p} - \frac{1}{r} \right)}} \times$$

$$\times \left\{ 1 + 0 \left[ \frac{1}{\lambda_p^2} \frac{d\lambda_p}{dH} \right] + 0 \left[ \frac{1}{\lambda_p (R_a - R_p)} \right] + 0 \left[ \frac{R_a - R_p}{\lambda_p R_a R_p} \right] \right\},$$

$$\frac{dR_p}{d\varphi} = -\frac{C_x S p_p}{m 2 \sqrt{2\pi} \lambda_p \left( \frac{1}{R_p} - \frac{1}{r} \right)^{3/2}} \times$$

$$\times \left\{ 1 + 0 \left[ \frac{1}{\lambda_p^2} \frac{d\lambda_p}{dH} \right] + 0 \left[ \frac{1}{\lambda_p (R_a - R_p)} \right] + 0 \left[ \frac{R_a - R_p}{\lambda_p R_a R_p} \right] \right\}.$$

A similar result is obtained by a different method in El'yasberg's study [6]. It follows from the reduced relations that

$$\frac{d(1/r)}{dR_p} = -2\lambda(R_p) \left( \frac{1}{R_p} - \frac{1}{r} \right),$$

i.e., we obtain an ordinary linear equation which is easily integrated.

For convenience we can combine the results obtained for small and large eccentricities and write them in the following form:

$$\frac{dr}{d\varphi} = -\frac{C_x S r^2}{m} \rho(R_p) I_0(x) e^{-x}, \quad \frac{dR_p}{d\varphi} = -\frac{C_x S r^2}{m} \rho(R_p) [I_0(x) - I_1(x)] e^{-x},$$

where  $x = \lambda(R_p) \frac{r}{R_p} (r - R_p)$  (at large values of  $x$   $I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$ ,  $I_0(x) - I_1(x) \approx \frac{e^x}{\sqrt{\pi}(2x)^{3/2}}$ ). Since  $\lambda r$  has a order of 100-900, we

can expect that the ranges of variability of the formulas overlap.

3. As the second example we will examine the motion of a flying vehicle with a large positive quality (the so-called skip trajectory) [7].

We will introduce the following assumptions:

$$H \ll R_0, \quad |\theta| \ll 1, \quad \rho = \rho_0 e^{-\lambda H}, \quad C_x = \text{const.}$$

Then the equation of motion can be rewritten in the following form:

$$\begin{aligned} \frac{d^2H}{ds^2} &= \frac{C_y \rho S V^2}{2m} - g_0 + \frac{V^2}{R_0}, \\ \frac{d(V^2)}{ds} &= \frac{C_x \rho S V^2}{m} - 2g_0 \frac{dH}{ds}. \end{aligned}$$

Here  $H$  - height,  $V$  - velocity,  $\theta$  - the local angle of inclination of the trajectory,  $s$  - a variable of the path,  $C_x$  and  $C_y$  - coefficients of drag and lift (see, for example, [7]).

Considering that the velocity changes negligibly during the oscillation period with respect to  $H$  and that  $C_y$  changes slowly, we will apply formulas (5) and (15) to this system of equations.

We are easily satisfied that when calculating the averaged derivatives  $\overline{dV^2/ds}$  and  $\overline{d^2H/ds^2}$ , the term  $2gdH/ds$  does not have any effect; it vanished on averaging. Therefore, this term is not taken into account when calculating the amplitude and mean value of the zero terms of the asymptotic expansion for height and velocity respectively.

Then, if we eliminate the variable  $ds$ , we derive in plane  $(H, V)$  the following equation of motion:

$$\frac{d^2y}{dx^2} = -\sqrt{R_0 \lambda} \frac{C_y}{C_x} + \frac{e^{2x} - 1}{y},$$

$$\text{where } x = \ln \frac{\sqrt{R_0 g_0}}{V}, \quad y = \frac{C_x S}{2m} \sqrt{\frac{R_0}{\lambda}} \rho.$$

We can apply to this equation the method cited in Kuzmak's study [1], considering  $\exp(2x)-1$  a slowly changing coefficient, which is valid for large values of the quality and at velocities smaller than circular velocity.

Without going into detailed calculations, we can extract the final formula which determines the dependence of the amplitude values of

height on the mean velocity:

$$\left( \frac{\frac{R_0 g_0}{V_{\text{initial}}^2} - 1}{\frac{R_0 g_0}{V^2} - 1} \right)^{3/2} \frac{C_y}{C_{y, \text{initial}}} h \left( \frac{C_y S R_0 \rho_{\text{max}}}{2m \left( \frac{R_0 g_0}{V^2} - 1 \right)} \right) = \text{const.}$$

Here the argument of the function  $h$  is the ratio of the maximal amplitude of density  $\rho_{\text{max}}$  to the density corresponding to a trajectory of quasistationary gliding

$$\rho_* = \frac{2m(R_0 g_0 - V^2)}{C_y S R_0 V^2}.$$

The function  $h$  is determined by the formula

$$h(u) = \int_{z_1(u)}^{z_2(u)} \sqrt{u - z + \ln \frac{z}{u}} dz.$$

where  $z_1(u)$  and  $z_2(u) = u$  are the roots of the equation  $u - z + \ln \frac{z}{u} = 0$ .

The graph of the function  $h(u)$  is shown in Fig. 2. When  $u \approx 1$   $h(u) \approx \frac{\pi}{2\sqrt{2}} (u - 1)^2$ , when  $u \gg 1$   $h(u) \approx \frac{2u\sqrt{u}}{3}$ .

Hence it is clear that in "deep" skips ( $u \gg 1$ ) when  $C_y = \text{const}$ , the value of  $\rho_{\text{max}}$  is practically constant (analogous to the case where the height of the perigee is at first almost constant upon decay of an elliptic orbit with a large eccentricity), which coincides with the results obtained earlier [7]. As the velocity decreases  $h \rightarrow 0$ , i.e.,  $\rho_{\text{max}} \rightarrow \rho_*$ , the trajectory becomes a trajectory of quasistationary gliding. Here the minimal (amplitudinal) density  $\rho_{\text{min}}$  is associated with the maximal relation

$$\frac{\rho_{\text{max}}}{\rho_*} \exp\left(-\frac{\rho_{\text{max}}}{\rho_*}\right) = \frac{\rho_{\text{min}}}{\rho_*} \exp\left(-\frac{\rho_{\text{min}}}{\rho_*}\right).$$

The instantaneous oscillation "period" with respect to the velocity is determined by the formula

$$T_V = \frac{2\pi C_x}{\sqrt{R_0 \lambda C_y}} \sqrt{R_0 \beta_0 - V^2} T \left( \frac{P_{max}}{P_0} \right).$$

The function  $T(u) = \frac{\sqrt{2}}{\pi} \frac{u}{u-1} h'(u)$  is shown in Fig. 2. Since  $T \geq 1$ , the number of complete oscillations of the vehicle when  $C_y =$  a const does not exceed

$$n = \int_{V_{end}}^{V_{initial}} \frac{dV}{T_V} < \frac{\sqrt{R_0 \lambda} C_y}{2\pi C_x} \times \arcsin \frac{V}{\sqrt{R_0 \beta_0}} \Big|_{V_{end}}^{V_{initial}}$$

or

$$n < \frac{\sqrt{R_0 \lambda} C_y}{4 C_x} \approx 7,5 \frac{C_y}{C_x}.$$

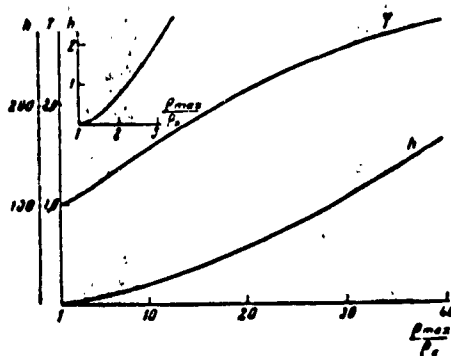


Fig. 2.

4. In the last example we will examine a rapid plane oscillatory motion of a ballistic vehicle in an idealized medium with a constant density in the absence of a gravitational field. For simplicity we will exclude from the examination the damping terms in the equation of the oscillations of the vehicle about the center of mass (although this cannot be done in practical calculations).

The equations are written in the following form:

$$\frac{d^2 \alpha}{dt^2} = k_1 m_x^a \alpha V^2,$$

$$\frac{dV}{dt} = -k_2 C_x \alpha V^2.$$

Here  $k_1 = \frac{\rho S l}{2J}$ ,  $k_2 = \frac{\rho S}{2m}$ ,  $\alpha$  is the angle of attack,  $m \frac{d\alpha}{dt}$  is a derivative of the aerodynamic moment coefficient,  $C_x$  is the coefficient of drag,  $S$  and  $l$  are the characteristic surface and length,  $J$  is the moment of inertia,  $m$  is mass of the vehicle.

If the coefficient  $C_x$  did not depend on  $\alpha$ , the change in amplitude of oscillations  $\alpha_m$  would be determined by the asymptotic formula

$$\alpha_m^2 \sim \frac{1}{\sqrt{-k_1 m^2 V^2}}$$

or

$$\alpha_m^2 V = \text{const.} \quad (21)$$

We will apply formulas (5) and (15) to the extracted equations. Taking into account

$$D = \sqrt{-k_1 m^2 V \alpha_m^2},$$

we obtain equations describing the changes of the zero terms of the asymptotic expansion for velocity  $V_0$  and for the amplitude  $\alpha_m$

$$\frac{dV_0}{dt} = -k_2 V_0^2 \frac{\int_{-a_m}^{a_m} \frac{C_x(\alpha) d\alpha}{\sqrt{\alpha_m^2 - \alpha^2}}}{\int_{-a_m}^{a_m} \frac{d\alpha}{\sqrt{\alpha_m^2 - \alpha^2}}} = -k_2 V_0^2 \overline{C_x(\alpha_m)},$$

$$V_0 \frac{d}{dt} (\alpha_m^2) + \alpha_m^2 \frac{dV_0}{dt} = 0,$$

$$\frac{d\overline{V}}{dt} = -k_2 V_0^2 \frac{\int_{-a_m}^{a_m} C_x(\alpha) \sqrt{\alpha_m^2 - \alpha^2} d\alpha}{\int_{-a_m}^{a_m} \sqrt{\alpha_m^2 - \alpha^2} d\alpha} = -k_2 V_0^2 \overline{\overline{C_x(\alpha_m)}}.$$

Then the equation relating the amplitude to the mean velocity takes the following form:

$$2d \ln \alpha_m + \frac{\overline{C_x(\alpha_m)}}{C_x(\alpha_m)} d \ln V_0 = 0. \quad (22)$$

If  $\overline{C_x(\alpha_m)} \approx \overline{C_x(\alpha_m)}$ , we derive formula (21). We will approximate the dependence  $C_x(\alpha)$  by a function of the form:

$$C_x = a + b|\alpha|^m \quad (a > 0, b > 0).$$

Then

$$\overline{C_x(\alpha_m)} = a + \frac{\Gamma\left(\frac{m}{2} + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\pi \Gamma\left(\frac{m}{2} + 1\right)} b \alpha_m^m$$

$$\overline{\overline{C_x(\alpha_m)}} = a + \frac{2\Gamma\left(\frac{m}{2} + \frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\pi \Gamma\left(\frac{m}{2} + 2\right)} b \alpha_m^m$$

Using these formulas, we can easily integrate Eq. (22) and obtain the relation between  $V_0$  and  $\alpha_m$  in an explicit form.

We will limit ourselves to a qualitative investigation of this equation.

When  $m > 0$

$$1 > \frac{\overline{C_x(\alpha_m)}}{C_x(\alpha_m)} > \frac{2}{2+m}.$$

In the limiting case when  $a = 0$  and  $m \rightarrow \infty$  we derive that this relation approaches zero, i.e.,  $\alpha_m = \text{const}$ , in spite of the fact that the velocity decreases monotonically and the relative change of velocity during the oscillation period with respect to  $\alpha$  can be as small as desired. The physical sense of the last result is that the fall-off of velocity  $V$  mainly occurs only in the neighborhood of amplitude  $\alpha = \alpha_m$ , and outside this neighborhood the velocity can be considered as a virtually constant value. Therefore the motion "inside" each

half-period is conservative, and the frequency change jumplike from half-period to half-period.

An opposite "degenerate" case can be obtained if we assume that  $0 > m > -1$  so that  $C_x(\alpha = 0) = \infty$ .

Then the relation under consideration is greater than unity, i.e., in this case when  $m \rightarrow -1$ , the amplitude can increase proportionally  $1/V$  (see [22]). Both degenerate cases are illustrated in Fig. 3.

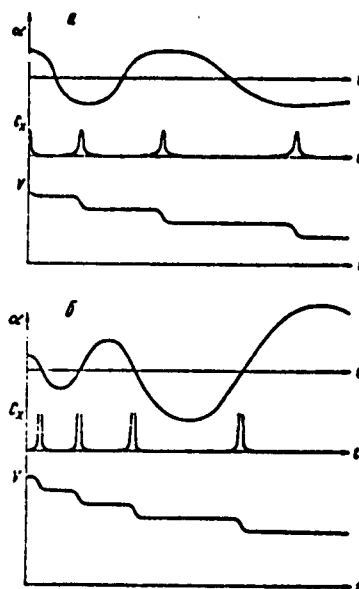


Fig. 3.

#### REFERENCES

1. G. Ye. Kuzmak. Asimptoticheskiye resheniya odnogo nelineynogo differentsial'nogo uravneniya vtorogo poryadka. Prikl. matem. i mekhan. Vol. 21, No. 2, 1957.
2. G. Ye. Kuzmak. Dvizheniye osesimmetrichnogo tverdogo tela okolo nepodvizhnoy toчки pod vozdeystviyem momentov, medlenno izmenyayushchikhsya vo vremeni. Izv. AN SSSR. Otd. tekhn. n., Mekhanika i mashinostroyeniye, No. 4, 1961.
3. L. D. Landau, Ye. M. Lifshits. Mekhanika. Fizmatgiz, 1958.
4. R. E. Roberson. Effect of air drag on elliptic satellite orbits. J. Propulsion, No. 2, 1958 (see also Voprosy raketnoy tekhniki, No. 5, 1958).

5. I. M. Yatsunskiy. O vliyanii geofizicheskikh faktorov na dvizheniye sputnika. Usp. fiz. nauk, Vol. 63, No. 1a, 1957.
6. P. Ye. El'yasberg. Zavisimost' vekovykh izmeneniy elementov orbit ot soprotivleniya vozdukh. V. s. "Iskusstvennyye sputniki Zemli," No. 3. 1959.
7. A. J. Eggers, H. J. Allen, S. E. Neice. A comparative analysis of the performance of long-range hypervelocity vehicles. Nat. Advisory Comm. for Aeronautics Techn. Not., 4046, 1957.

DISTRIBUTION LIST

DEPARTMENT OF DEFENSE	Nr. Copies	MAJOR AIR COMMANDS	Nr. Copies
		AFSC	
		SCFDD	1
		ASTIA	25
HEADQUARTERS USAF		TDBTL	5
		TDBDP	5
AFCIN-3D2	1	ASD (ASYIM)	1
ARL (ARB)	1	AEDC (AEY)	1
		SSD (SSF)	2
		APGC (PCF)	1
OTHER AGENCIES		ESD (ESY)	1
		AFMTC (MTW)	1
CIA	1	RADC (RAY)	1
NSA	6	AFMDC (MDF)	1
DIA	9	AFSWC (SWF)	1
AID	2	TDEPA (Baldwin)	1
OTS	2	AFFTC (FTY)	1
AEC	2		
PWS	1		
NASA	1		
ARMY	3		
NAVY	3		
RAND	1		
NAFEC	1		
PGE	12		