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⑥ OPTIMUM SYNTHESIS OF SAMPLED-DATA SYSTEMS
WITH NON-STATIONARY RANDOM INPUTS

by

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ABSTRACT

This thesis presents a criterion for use in obtaining a "best" approximation to a certain desired sampled-data system under conditions such that the input to the system is a non-stationary random signal corrupted by additive noise. The criterion is applied to both a time-invariant system and time-varying system. Physically realizable solutions for the optimum system are obtained in each case.

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CHAPTER I

INTRODUCTION

Although the sampling theorem was expressed in a general form by Cauchy^{3*} more than a century ago, sampled-data systems received little attention until the early 1940's when radar fire-control problems became of interest.

Sampled-data systems have come of age within the past decade, primarily because of the increased use of digital computers in control systems, and the greatly expanded use of telemetry with the associated importance given to efficient use of available communication channel capacity.

The techniques for the synthesis of sampled-data systems have lagged considerably behind the analogous state-of-the-art of continuous-data systems. However, design methods developed for continuous-data systems have been almost universally extended and applied to sampled-data systems.

One of the first design philosophies for sampled-data systems was proposed by Linvill,⁸ who viewed sampling as a form of

*Numerical superscripts refer to entries in the bibliography.

amplitude modulation. Probably the most important philosophy was proposed by Ragazzini and Zadeh,¹¹ who introduced the theory of the z transform. Other basic design philosophies have been proposed by Brown,² who used a difference equation approach, and Smith, Lawden, and Bailey,¹³ who interpreted sampled-data system design as a prediction problem.

Statistical methods have played an increasingly important role in the design of control and communications systems in the past fifteen years. The application of statistical methods to system design as originally presented by Wiener¹⁵ has been refined and extended in many ways and by many authors.¹⁶

This thesis presents a synthesis procedure which may be used to obtain optimum compensation for sampled-data systems with non-stationary random inputs, the criterion for the optimization being a mean amplitude-set-time weighted error criterion. Results are obtained for both time-invariant systems and time-varying systems. Solutions are presented in the form of equations which may be synthesized by standard methods. The conditions for the physical realization of the solutions are discussed in each case.

CHAPTER II

THE OPTIMIZATION CRITERION

2.1 The Criterion for Continuous Systems

Murphy and Sahara¹⁰ have proposed, and have defined for continuous systems, an optimization criterion of rather general form which is based upon the minimization of a function of the system error which may be weighted with respect to error amplitude, the relative time of the error occurrence, and the subset of the sample space from which the error is taken. In its most general form this criterion is

$$C = \int_{-\infty}^{\infty} m(x) \int_{-\infty}^{\infty} p(x,y) \lim_{T_0 \rightarrow \infty} \frac{1}{2T_0} \int_{-T_0}^{T_0} w(x,y;T_0;t) F[e(x,y;t)] dt dy dx, \quad (2.1)$$

where t is the real time variable, T_0 is the length of the system memory, x and y are the coordinates of the error sample space, $e(x,y;t)$ is the system error, $F[e(x,y;t)]$ is an amplitude weighting function, $w(x,y;T_0;t)$ is a time weighting function associated with the relative occurrence time of the error, $p(x,y)$ is a probability density function associated with the error sample space, and $m(x)$ is the subset weighting parameter.

Such a criterion is ideally suited for use in designing control systems which must operate over a wide range of ambient conditions. The parameter $m(x)$ can be varied according to the relative importance of errors occurring under the various conditions and then weighted by the probability that the system will be operating under these particular conditions.

2.2 Extension of the Criterion to Sampled-Data Systems

If the system under consideration is completely sampled, that is, if all parts of the system operate with sampled data, the integrals in equation (2.1) may be replaced by discrete summations. Similarly, the limiting process may be replaced with its discrete equivalent. The criterion for sampled-data systems thus becomes

$$C = \sum_{i=-I}^I m_i \sum_{j=-J}^J P_{ij} \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N w(i,j;NT;nT) F[e(i,j;nT)]. \quad (2.2)$$

In equation (2.2) T is the system sampling interval, $F[e(i,j;nT)]$ is a function of the system error amplitude, $w(i,j;NT;nT)$ is a time weighting function, NT is the length of the system memory, P_{ij} is the probability associated with the j th sample point in the i th subset of the error sample space, m_i is the weighting parameter of the i th subset, J is the number of sample points in the largest subset, and I is the number of subsets.

Although the summation indices n , j , and i have no physical interpretation for negative integers for a finite process starting at time $nT = 0$, the criterion is written in a symmetric form to

allow application of statistical design methods. The weighting functions, the error, and thus the criterion are assumed to be zero for terms arising from negative summation integers.

2.3 A Restriction

Although the amplitude weighting function may, in general, be an arbitrary function of the error, the function

$$F[e(i,j;nT)] = e^2(i,j;nT) \quad (2.3)$$

is chosen for use in the sequel. Certainly, this choice of amplitude weighting will not always be the ideal in any system considered, however, there are two compelling reasons for its selection. First, a specific function is necessary in order that the minimization process may be carried out, and second, it is of relatively simple form, which leads to a workable design. With this restriction the criterion, equation (2.2) becomes

$$C = \sum_{i=-I}^I m_i \sum_{j=-J}^J P_{ij} \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N w(i,j;NT;nT) e^2(i,j;nT). \quad (2.4)$$

The criterion defined by equation (2.4) may be considered to be a generalization of the familiar mean-square-error criterion which has found much use in the design of optimum control and communications systems.

The following more compact forms of equation (2.4) will often be used for brevity in writing equations in the sequel.

$$C = \langle\langle\langle w(i,j;NT;nT)e^{2(i,j;nT)} \rangle_n \rangle_j \rangle_i \quad (2.5)$$

or simply

$$C = \langle\langle\langle w e^2 \rangle_n \rangle_j \rangle_i \quad (2.6)$$

The symbols $\langle\langle\langle \rangle_n \rangle_j \rangle_i$ represent the statistical average or expected value of the argument taken with respect to the three indicated indices, the argument being the quantities between the innermost brackets.

CHAPTER III

SOME PRELIMINARY CONSIDERATIONS

The basic problem considered in this thesis is the problem of obtaining a "best" approximation to a given desired system. The basic definitions and relationships necessary to attain this end will be presented in this chapter.

3.1 The Model

A functional diagram of the model which will be used in this thesis to define and clarify the mathematical conventions and notation for the systems or processes under consideration is shown in Figure 1, page 8.

As indicated by Figure 1, the system sampled error, $e(nT)$, is defined as the difference between a certain desired output, $c_d(nT)$, and the actual system sampled output, $c_a(nT)$.

The input to the actual system, the reference input, $r(t)$, is assumed to consist of a signal, $s(t)$, containing the desired input information, and a noise, $n(t)$, which is an undesirable corruption of the input information.

The weighting sequence of the desired system is defined to be $g_d(kT)$, while that of the actual system is defined as $g_a(kT)$. Both the desired system and the actual system are assumed to be

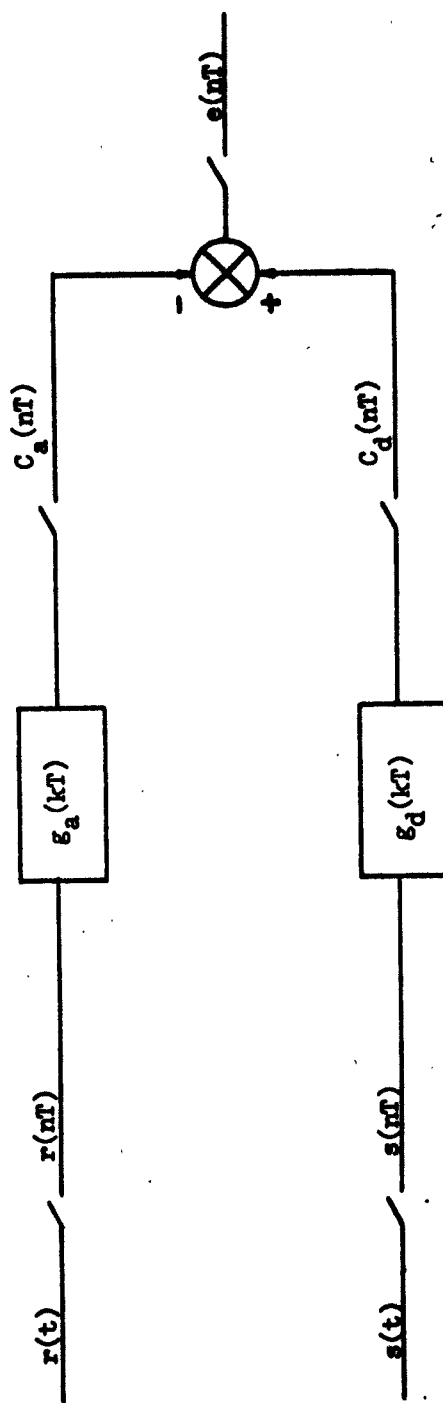


Figure 1
The System Model

linear and finite. The weighting sequence of the desired system is a set of linear operators which, when applied to the sampled signal input, yields the desired output sequence. Similarly, the weighting sequence of the actual system is the set of linear operators which, when applied to the sampled reference input, will yield the actual system output sequence.

The summing device is assumed to be linear and ideal, and all samplers are assumed to be ideal and to operate in synchronism. The sampling frequency is assumed to be at least twice the highest signal frequency in the system.

Although Figure 1 may be interpreted equally well to represent either a control or a communications system, it seems desirable to amplify the diagram in the case of a general closed-loop control system. Figure 2, page 10, presents a functional diagram of such a closed-loop or feedback control system which may be used to represent the actual system (labeled $g_a(kT)$ in Figure 1) in greater detail when such a system is considered.

In Figure 2, $p(kT)$ is defined as the weighting sequence of the system plant, and $g_c(kT)$ is defined as the weighting sequence of a compensating device placed in the system in order to realize the optimum weighting sequence for the actual system with a contaminated input signal. The symbols $\sigma(nT)$ and $\gamma(nT)$ denote the output sequences of the compensating device and the system plant respectively. All assumptions made with reference to Figure 1 may also be applied to Figure 2.

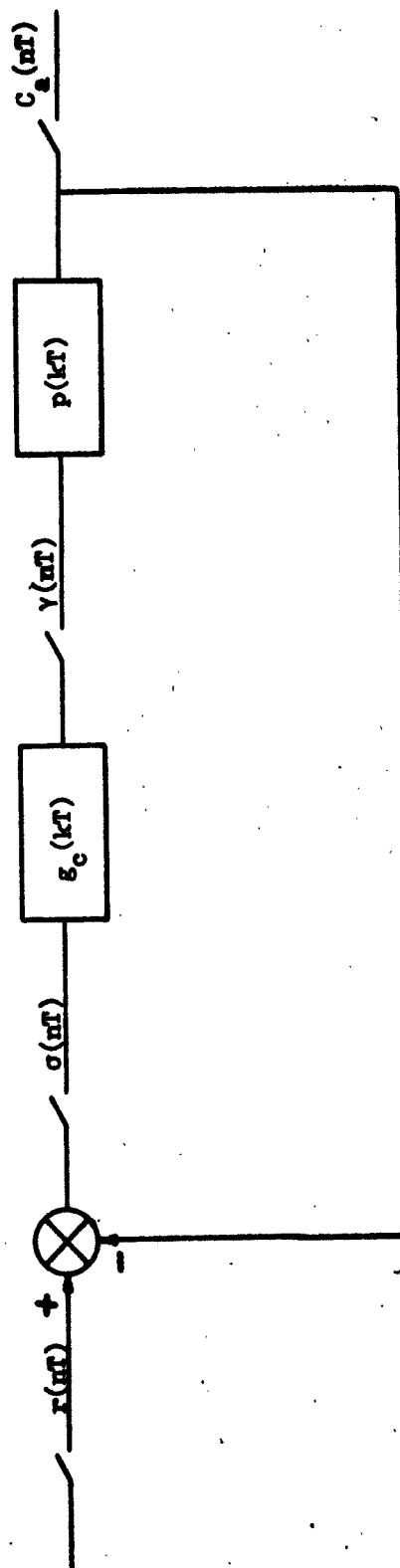


Figure 2
The Closed-Loop Control System

3.2 The Convolution Summation

In analogy with the convolution integral which expresses the relationship between the input and output signals of continuous-data systems, a convolution summation may be defined which relates the input and output sequences of linear, constant parameter sampled-data systems. Using symbols defined with reference to Figure 1, page 8, the following convolution summations can be written.

$$c_d(nT) = \sum_{k=-\infty}^{\infty} g_d(nT-kT)s(kT) \quad (3.1a)$$

or

$$c_d(nT) = \sum_{k=-\infty}^{\infty} s(nT-kT)g_d(kT) \quad (3.1b)$$

Although it is assumed that most systems or processes to be considered are of the so-called terminal classification and therefore have a finite memory and operate with sets of signals which are all members of finite sample spaces, the limits of summation in equation (3.1) may be extended to infinity as indicated since the additional terms created are all zero and therefore contribute nothing to the sums.

3.3 The Weighted Second-Order Correlation Sequence

Murphy and Bold⁹ have defined weighted two-space correlation functions for continuous stationary signals, and Murphy and Sahara¹⁰ have defined weighted second-order correlation functions for continuous

non-stationary signals. In analogy with these definitions, weighted second-order autocorrelation sequences can be defined. Using symbols which were defined with reference to Figure 1, page 8, the weighted second-order autocorrelation sequence of the reference input with respect to the weighting function $w(i, j; NT; nT)$ is

$$\phi_{wrr}(kT, lT) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} \left\langle \left\langle w(i, j; NT; nT) r(nT+kT) r(nT+lT) \right\rangle_j \right\rangle_1. \quad (3.2)$$

In a similar manner the weighted second-order cross-correlation sequence between the reference input and the signal input with respect to $w(i, j; NT; nT)$ is defined as

$$\phi_{wrs}(kT, lT) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} \left\langle \left\langle w(i, j; NT; nT) r(nT+kT) s(nT+lT) \right\rangle_j \right\rangle_1. \quad (3.3)$$

In equations (3.2) and (3.3) ϕ_w represents the weighted second-order correlation sequence of the signal or signals identified by the subscripts appurtenant to the symbol.

3.4 The Pulse-Spectral Density

The spectral density of a continuous signal is defined as the Fourier or Laplace transform of the autocorrelation function of the signal. The discrete equivalent of this statement defines the pulse-spectral density of a sampled signal as the two-dimensional z transform of the autocorrelation sequence of the signal.

The relationships between the weighted second-order autocorrelation sequence and the weighted pulse-spectral density of

the sampled reference input are

$$\phi_{wrr}(z_1, z_2) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{wrr}(kT, lT) z_1^{-k} z_2^{-l} \quad (3.4a)$$

and

$$\phi_{wrr}(kT, lT) = \left(\frac{1}{2\pi j}\right)^2 \oint_{\Lambda_2} \oint_{\Lambda_1} \mathbb{I}_{wrr}(z_1, z_2) z_1^{k-1} z_2^{l-1} dz_1 dz_2 \quad (3.4b)$$

The weighted pulse-cross-spectral density is defined in a similar manner:

$$\mathbb{I}_{wrs}(z_1, z_2) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{wrs}(kT, lT) z_1^{-k} z_2^{-l}, \quad (3.5a)$$

$$\phi_{wrs}(kT, lT) = \left(\frac{1}{2\pi j}\right)^2 \oint_{\Lambda_2} \oint_{\Lambda_1} \mathbb{I}_{wrs}(z_1, z_2) z_1^{k-1} z_2^{l-1} dz_1 dz_2 \quad (3.5b)$$

In equations (3.4) and (3.5) $\mathbb{I}_w$ represents the weighted pulse-spectral density of the signal or signals identified by the subscripts appurtenant to the symbol. The symbols z_1 and z_2 are the complex variables of the four-dimensional z transform space. The integration contours Λ_1 and Λ_2 are the unit circles in the z_1 and z_2 planes respectively.

3.5 Some Basic Relationships

This section will present in mathematical form the basic relationships between the various signals and components of the

system represented by the model diagrammed in Figures 1 and 2, pages 8 and 10.

With reference to Figure 1 and the associated definitions of Section 3.1, the following relations can be written.

$$e(nT) = c_d(nT) - c_a(nT) \quad (3.6)$$

$$r(t) = s(t) + n(t) \quad (3.7)$$

Since sampling is a linear operation, the sampled input to the actual system is

$$r(nT) = s(nT) + n(nT). \quad (3.8)$$

Following the definition of convolution summations of Section 3.2, the actual system output may be written

$$c_a(nT) = \sum_{k=-\infty}^{\infty} g_a(nT-kT)r(kT) \quad (3.9a)$$

or

$$c_a(nT) = \sum_{k=-\infty}^{\infty} r(nT-kT)g_a(nT). \quad (3.9b)$$

Substitution of equations (3.1b) and (3.9b) into equation (3.6) yields the relationship between the system sampled error and the sampled reference input and the sampled signal input:

$$e(nT) = \sum_{k=-\infty}^{\infty} [s(nT-kT)g_d(kT) - r(nT-kT)g_a(kT)]. \quad (3.10)$$

This result also indicates the effect of the system weighting sequences on the system error.

Referring now to Figure 2 and the associated definitions of Sections 3.1 and 3.2, these relationships may be noted.

$$\sigma(nT) = r(nT) - c_a(nT) \quad (3.11)$$

$$\gamma(nT) = \sum_{k=-\infty}^{\infty} g_c(nT-kT)\sigma(kT) \quad (3.12a)$$

or

$$\gamma(nT) = \sum_{k=-\infty}^{\infty} \sigma(nT-kT)g_c(kT) \quad (3.12b)$$

$$c_a(nT) = \sum_{k=-\infty}^{\infty} p(nT-kT)\gamma(kT) \quad (3.13a)$$

or

$$c_a(nT) = \sum_{k=-\infty}^{\infty} \gamma(nT-kT)p(kT) \quad (3.13b)$$

Combining equations (3.13a) and (3.12b) yields

$$c_a(nT) = \sum_{k=-\infty}^{\infty} p(nT-kT) \sum_{l=-\infty}^{\infty} \sigma(kT-lT) g_c(lT). \quad (3.14)$$

Substituting equation (3.11) into equation (3.14) yields

$$c_a(nT) = \sum_{k=-\infty}^{\infty} p(nT-kT) \sum_{l=-\infty}^{\infty} g_c(lT) [r(kT-lT) - c_a(kT-lT)]. \quad (3.15)$$

Equation (3.15) indicates the implicit relationship between the input and output of the closed-loop system and the weighting sequences of the system components. It should be noted that this equation cannot, in general, be solved explicitly for the output sequence due to the different arguments of the output sequence as it appears in the equation.

CHAPTER IV

THE OPTIMUM SYSTEM

The criterion developed in Sections 2.2 and 2.3 will be applied in this chapter to the model described in Section 3.1 with the aim of finding an optimum system. The optimum system is defined as the physically realizable system which has for its weighting sequence, or equivalently, its pulse-transfer function that sequence or function which minimizes the criterion, equation (2.4). Solutions for this function will be obtained and the conditions for their physical realization will be described. If the system under consideration is a closed-loop control system as shown in Figure 2, page 10, knowledge of the optimum system weighting sequence or transfer function and the conditions on its realizability allows the synthesis of a compensating device to correct the system plant in such a way as to achieve the optimum system.

4.1 Time-Invariant Systems

In this section systems with plants which are invariant with respect to time will be considered.

Again with reference to Figure 1, page 8, substitution of equation (3.9b) into equation (3.6) and the result into equation

(2.4) yields

$$C = \sum_{i=-I}^I m_i \sum_{j=-J}^J P_{ij} \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N [c_d^2(nT) - 2c_d(nT) \sum_{k=-\infty}^{\infty} g_a(kT)r(nT-kT) \\ + \sum_{k=-\infty}^{\infty} g_a(kT)r(nT-kT) \sum_{l=-\infty}^{\infty} g_a(lT)r(nT-lT)]. \quad (4.1)$$

Comparison of this result with the definitions of Section 3.3 suggests that equation (4.1) can be written as

$$C = [\phi_{wc_d} c_d(kT, lT) - \sum_{h=-\infty}^{\infty} g(hT) \phi_{wc_d} r(kT, lT-hT) \\ - \sum_{h=-\infty}^{\infty} g(hT) \phi_{wc_d} r(lT, kT-hT) + \sum_{h=-\infty}^{\infty} g(hT) \\ \cdot \sum_{f=-\infty}^{\infty} g(fT) \phi_{wrr}(kT-hT, lT-fT)] \Big|_{kT=lT=0} \quad (4.2)$$

The interchange of the operations of averaging and summation is usually justifiable in equations describing physical systems. This is discussed in greater detail in Chapter V. Equation (4.2) shows that the optimization criterion is completely determined when the unit impulse response of the system and the weighted second-order correlation sequences of the reference input and the desired output are known.

The pulse-transfer function which minimizes the right side of equation (4.2) may now be found through application of the methods of variational calculus.

Let $\lambda(kT)$ be defined as an arbitrary, physically realizable weighting sequence satisfying the equation

$$\lambda(kT) = 0 \quad (4.3)$$

for $kT \leq 0$ and $kT \geq NT$, where NT is the length of the system memory, and which has no discontinuities at either $kT = 0$ or $kT = NT$.

Now let

$$g(kT) = g(kT) + \delta\lambda(kT), \quad (4.4)$$

where δ is an arbitrary small real number. Equation (4.2) now becomes

$$\begin{aligned} C + \Delta C = & \left\{ \phi_{wc_d c_d}(kT, 1T) - \sum_{h=-\infty}^{\infty} [g(hT) + \delta\lambda(hT)] \phi_{wc_d r}(kT, 1T-hT) \right. \\ & - \sum_{h=-\infty}^{\infty} [g(hT) + \delta\lambda(hT)] \phi_{wc_d r}(1T, kT-hT) + \sum_{h=-\infty}^{\infty} [g(hT) + \delta\lambda(hT)] \\ & \left. \cdot \sum_{f=-\infty}^{\infty} [g(fT) + \delta\lambda(fT)] \phi_{wrr}(kT-hT, 1T-fT) \right\} \Big|_{kT=1T=0} \quad (4.5) \end{aligned}$$

Subtracting equation (4.2) from equation (4.5) yields the first variation of C as

$$\begin{aligned}
\Delta C = & \left\{ - \sum_{h=-\infty}^{\infty} \delta\lambda(hT) \phi_{wc_d r}(kT, lT-hT) - \sum_{h=-\infty}^{\infty} \delta\lambda(hT) \phi_{wc_d r}(lT, kT-hT) \right. \\
& + \sum_{h=-\infty}^{\infty} \sum_{f=-\infty}^{\infty} [g(hT) \delta\lambda(fT) + g(fT) \delta\lambda(hT) \\
& \left. + \delta^2 \lambda(hT) \lambda(fT)] \phi_{wrr}(kT-hT, lT-fT) \right\} \Big|_{kT=lT=0}
\end{aligned} \tag{4.6}$$

The necessary condition for an extremum of C is that

$$\frac{\partial \Delta C}{\partial \delta} \Big|_{\delta=0} = 0. \tag{4.7}$$

Applying this to equation (4.6) yields

$$\begin{aligned}
\frac{\partial \Delta C}{\partial \delta} \Big|_{\delta=0} = & \left\{ - \sum_{h=-\infty}^{\infty} \lambda(hT) \phi_{wc_d r}(kT, lT-hT) \right. \\
& - \sum_{h=-\infty}^{\infty} \lambda(hT) \phi_{wc_d r}(lT, kT-hT) + \sum_{h=-\infty}^{\infty} \sum_{f=-\infty}^{\infty} [g(hT) \lambda(fT) \\
& \left. + g(fT) \lambda(fT)] \phi_{wrr}(kT-hT, lT-fT) \right\} \Big|_{kT=lT=0} = 0.
\end{aligned} \tag{4.8}$$

Again, the interchange of operations is usually justifiable in physically based equations. Equation (4.8) may be written in a more meaningful form as

$$\begin{aligned}
& \left\{ \sum_{h=-\infty}^{\infty} \lambda(hT) [\phi_{wc_d r}(kT, lT-hT) + \phi_{wc_d r}(lT, kT-hT)] \right. \\
& - \sum_{h=-\infty}^{\infty} g(hT) \sum_{f=-\infty}^{\infty} \lambda(fT) \phi_{wrr}(kT-hT, lT-fT) \\
& \left. - \sum_{h=-\infty}^{\infty} \lambda(hT) \sum_{f=-\infty}^{\infty} g(fT) \phi_{wrr}(kT-hT, lT-fT) \right\} \Big|_{kT=lT=0} = 0. \quad (4.9)
\end{aligned}$$

Interchanging the order of summation in the first double summation term yields

$$\begin{aligned}
& \left\{ \sum_{h=-\infty}^{\infty} \lambda(hT) [\phi_{wc_d r}(kT, lT-hT) + \phi_{wc_d r}(lT, kT-hT)] \right. \\
& - \sum_{h=-\infty}^{\infty} \lambda(hT) \sum_{f=-\infty}^{\infty} g(fT) \phi_{wrr}(lT-hT, kT-fT) \\
& \left. - \sum_{h=-\infty}^{\infty} \lambda(hT) \sum_{f=-\infty}^{\infty} g(fT) \phi_{wrr}(kT-hT, lT-fT) \right\} \Big|_{kT=lT=0} = 0, \quad (4.10)
\end{aligned}$$

which may be simplified to

$$\begin{aligned}
& \sum_{h=-\infty}^{\infty} \lambda(hT) \left\{ \sum_{f=-\infty}^{\infty} g(fT) [\phi_{wrr}(lT-hT, kT-fT) + \phi_{wrr}(kT-hT, lT-fT)] \right. \\
& \left. - \phi_{wc_d r}(kT, lT-hT) - \phi_{wc_d r}(lT, kT-hT) \right\} \Big|_{kT=lT=0} = 0. \quad (4.11)
\end{aligned}$$

In order for the extremum determined by (4.7) to be a minimum, it is sufficient that

$$\left. \frac{\partial^2 \Delta C}{\partial \delta^2} \right|_{\delta=0} = 0. \quad (4.12)$$

The second partial derivative of the first variation of C with respect to δ is

$$\left. \frac{\partial^2 \Delta C}{\partial \delta^2} \right|_{\delta=0} = 2 \sum_{h=-\infty}^{\infty} \lambda(hT) \sum_{f=-\infty}^{\infty} \lambda(fT) \phi_{wrr}(kT-hT, lT-fT) \Big|_{kT=lT=0}. \quad (4.13)$$

Writing this result with ϕ_{wrr} expanded yields

$$\begin{aligned} \left. \frac{\partial^2 \Delta C}{\partial \delta^2} \right|_{\delta=0} &= 2 \sum_{h=-\infty}^{\infty} \lambda(hT) \sum_{f=-\infty}^{\infty} \lim_{N \rightarrow \infty} \sum_{n=-N}^N \langle\langle w(i, j; NT; nT) \\ &\quad \cdot r(nT-hT)r(nT-fT) \rangle\rangle_j \lambda. \end{aligned} \quad (4.14)$$

Changing the order of operations yields

$$\begin{aligned} \left. \frac{\partial^2 \Delta C}{\partial \delta^2} \right|_{\delta=0} &= 2 \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \langle\langle w(i, j; NT; nT) \\ &\quad \cdot \sum_{h=-\infty}^{\infty} \lambda(hT)r(nT-hT) \sum_{f=-\infty}^{\infty} \lambda(fT)r(nT-fT) \rangle\rangle_j \lambda \end{aligned} \quad (4.15)$$

Equation (4.15) is of the form

$$\left. \frac{\partial^2 \Delta C}{\partial \delta^2} \right|_{\delta=0} = 2 \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N w(i, j; NT; nT) m^2(nT), \quad (4.16)$$

where $m(nT)$ is the response of a system with weighting sequence $\lambda(kT)$ to an input $r(nT)$. In order for the condition of equation (4.12) to be satisfied it is sufficient, but not necessary that

$$w(i, j; NT; nT) > 0, \quad -\infty \leq nT \leq \infty. \quad (4.17)$$

This restriction does not place an especially severe limitation on the choice of the time weighting function $w(i, j; NT; nT)$. It does, however, exclude certain weighting functions which may otherwise be acceptable.

With a choice of time weighting function which satisfies equation (4.17), the solution for the optimum system can be carried out. The function which minimizes the system error must cause the first derivative of the variation of the criterion to be zero for an arbitrary selection of $\lambda(hT)$, subject, of course, to the restriction of equation (4.3), so

$$\left\{ \sum_{f=-\infty}^{\infty} g(fT) [\phi_{wrr}(1T-hT, kT-fT) + \phi_{wrr}(kT-hT, 1T-fT)] - \phi_{wc_d r}(kT, 1T-hT) - \phi_{wc_d r}(1T, kT-hT) \right\} \bigg|_{\substack{kT=1T=0 \\ 0 \leq hT \leq NT}} = 0 \quad (4.18)$$

Taking the two-dimensional z transform of equation (4.18) yields

$$\left\{ \sum_{l=-\infty}^{\infty} z_1^{-l} \sum_{k=-\infty}^{\infty} z_2^{-k} \sum_{f=-\infty}^{\infty} g(fT) [\phi_{wrr}(lT-hT, kT-fT) + \phi_{wrr}(kT-hT, lT-fT)] \right. \\ \left. - \sum_{l=-\infty}^{\infty} z_1^{-l} \sum_{k=-\infty}^{\infty} z_2^{-k} [\phi_{wcd^r}(kT, lT-hT) - \phi_{wcd^r}(lT, kT-hT)] \right\} \Bigg|_{\substack{kT=lT=0 \\ 0 \leq hT \leq NT}} = 0 \quad (4.19)$$

Changing the order of summation yields

$$\left\{ \sum_{f=-\infty}^{\infty} g(fT) [\mathbb{E}_{wrr}(z_1, z_2) z_1^{-hT} z_2^{-fT} + \mathbb{E}_{wrr}(z_2, z_1) z_2^{-hT} z_1^{-fT}] \right. \\ \left. - \mathbb{E}_{wcd^r}(z_1, z_2) z_1^{-hT} - \mathbb{E}_{wcd^r}(z_1, z_2) z_2^{-hT} \right\} \Bigg|_{0 \leq hT \leq NT} = 0 \quad (4.20)$$

which may be simplified to

$$\left\{ [G(z_2) \mathbb{E}_{wrr}(z_1, z_2) - \mathbb{E}_{wcd^r}(z_2, z_1)] z_1^{-hT} \right. \\ \left. + [G(z_1) \mathbb{E}_{wrr}(z_2, z_1) - \mathbb{E}_{wcd^r}(z_1, z_2)] z_2^{-hT} \right\} \Bigg|_{0 \leq hT \leq NT} = 0. \quad (4.21)$$

The inverse two-dimensional z transform of equation (4.21) is

$$\left\{ \frac{1}{4\pi^2} \oint_{\Lambda_2} \oint_{\Lambda_1} [G(z_2) \mathbb{E}_{wrr}(z_1, z_2) - \mathbb{E}_{wcd^r}(z_2, z_1)] z_1^{-hT} z_1^{h-1} z_2^{f-1} dz_1 dz_2 \right.$$

$$\begin{aligned}
& + \frac{1}{4\pi^2} \int_{\Lambda_2} \int_{\Lambda_1} [G(z_1) \mathbb{I}_{wrr}(z_2, z_1) \\
& - \mathbb{I}_{wc_d r}(z_1, z_2)] z_2^{-hT} z_1^{h-1} z_2^{f-1} dz_1 dz_2 \Bigg|_{0 \leq hT \leq NT} = 0
\end{aligned} \tag{4.22}$$

It can be shown that the first double integral cannot be the negative of the second. The only admissible solution to this equation is, therefore,

$$G(z_2) \mathbb{I}_{wrr}(z_1, z_2) - \mathbb{I}_{wc_d r}(z_2, z_1) = 0 \tag{4.23}$$

and

$$G(z_1) \mathbb{I}_{wrr}(z_2, z_1) - \mathbb{I}_{wc_d r}(z_1, z_2) = 0 \tag{4.24}$$

in the region $0 \leq hT \leq NT$. Solving for $G(z_1)$ yields a solution for the optimum pulse-transfer function of extremely simple form:

$$G(z_1) = \frac{\mathbb{I}_{wc_d r}(z_1, z_2)}{\mathbb{I}_{wrr}(z_2, z_1)} \tag{4.25}$$

4.2 Physical Realizability of Solution

Equation (4.25) represents the optimum pulse-transfer function for the system in the sense of the criterion defined by equation (2.4), without regard for physical realizability. The result assumes that the weighted pulse-spectral density of the

reference input, and the weighted cross-pulse-spectral density for the desired output and the reference input are known.

Most functions of physical origin possess Laplace transforms and it is known that a z transform exists for any function which has a Laplace transform. The optimum pulse-transfer function may therefore be found if the weighted autocorrelation sequence of the reference input, and the weighted cross-correlation sequence between the desired output and the reference input are known, rather than the corresponding pulse-spectral densities.

Two conditions must be satisfied in order for the solution, equation (4.25), to be physically realizable. They are

$$g(kT) = 0 \quad \text{for } kT < 0 \quad (4.26)$$

and

$$\lim_{kT \rightarrow \infty} g(kT) = 0 \quad (4.27)$$

For these conditions to be satisfied $G(z_1)$, which is in the form of a rational polynomial in z_1 , must exhibit no positive power of z_1 when expanded in a power series, and must have no poles outside the unit circle in the complex z_1 plane.

If there are one or more poles of $G(z_1)$ present outside the unit circle a heuristic argument similar to that given by Bode

and Shannon¹ may be used to find a form of $G(z_1)$ which is physically realizable.

Let the ordinary pulse-spectral density of the reference input be written as

$$\Phi_{rr}(z_1, z_2) = \Phi_{rr}^i(z_1, z_2) \Phi_{rr}^o(z_1, z_2) \quad (4.28)$$

where $\Phi_{rr}^i(z_1, z_2)$ has poles only inside the unit circles in the z_1 and z_2 planes and $\Phi_{rr}^o(z_1, z_2)$ has poles only outside the unit circles. The input $r(nT)$ is first converted to a sequence of sampled white noise by passing it through a unit with the physically realizable transfer function

$$G_1(z_1, z_2) = \frac{1}{\Phi_{rr}^i(z_1, z_2)} \quad (4.29)$$

The white noise must then be passed through a physically unrealizable unit with the transfer function

$$G_2(z_1, z_2) = \frac{G(z_1)}{\Phi_{rr}^o(z_1, z_2)} \quad (4.30)$$

to obtain the optimum response. This optimum response is the sum of two statistically independent components: 1) a completely predictable component due to all of the samples of white noise which have previously occurred and the pulse (if one exists) that is simultaneously occurring, and 2) a completely unpredictable

component due to pulses of white noise which will occur in the future.

On the average, the best physically realizable measure of the second of these components is zero. Therefore, the optimum physically realizable pulse-transfer function is obtained by making

$$g(kT) = \begin{cases} 0, & kT < 0 \\ z^{-1} [\mathbb{I}_{rr}^{-1}(z_1, z_2) G(z_1)], & kT > 0. \end{cases} \quad (4.31)$$

By combining equations (4.30) and (4.25) it is then found that the optimum physically realizable pulse-transfer function is

$$G(z_1)_r = \frac{1}{\mathbb{I}_{rr}^{-1}(z_1, z_2)} \sum_{k=0}^{\infty} z_1^{-k} \frac{1}{2\pi j} \oint_{\lambda_1} \frac{\mathbb{I}_{rr}^{-1}(z_1, z_2) \mathbb{I}_{wc_d r}(z_1, z_2)}{\mathbb{I}_{wrr}(z_2, z_1)} z_1^{k-1} dz_1. \quad (4.32)$$

Assuming the system plant is prescribed in physical form, knowledge of the optimum physically realizable pulse-transfer function of the system allows synthesis of the optimum compensator for the system by standard methods.

4.3 Time-Varying Systems

This section is concerned with systems in which the plant varies as a function of time. Since the method of solution of Section 4.1 used the artifice of converting to the complex z -plane and thence back to the time domain, this method may not be applied to a system with time dependent components. A method of solution will be described in this section which requires only time-domain operations.

The procedure is similar to that described by Cruz⁵ which makes use of a matrix characterization of the system in the time domain. In particular, the system is described by a transmission matrix presented by Friedland.⁶ A linear system (either time-invariant or time-varying) may be characterized by a lower triangular matrix. For example, if a system has an input $x(k)$, a weighting sequence $g(n,k)$ and an output $y(n)$, the output matrix may be written in terms of the input matrix and the system transmission matrix as follows.

$$Y = GX \quad (4.33)$$

or

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n) \end{bmatrix} = \begin{bmatrix} g(0,0) & 0 & \dots & 0 \\ g(1,0) & g(1,1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g(n,0) & g(n,1) & \dots & g(n,n) \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(n) \end{bmatrix} \quad (4.34)$$

All systems which are non-predicting, and therefore physically realizable, have transmission matrices of the form of the matrix G above. Note particularly that $g(n,k) = 0$ for $k > n$.

Since in the present investigation, the output is of interest at sampling instants only, the signals and components of the system may be described conveniently and completely by matrices. Let R be defined as the reference input matrix, S as the signal input matrix, and N as the noise matrix. Equation (3.8) can be

written in matrix form as

$$R = S + N \quad (4.35)$$

or

$$\begin{bmatrix} r(i,j;0) \\ r(i,j;T) \\ \vdots \\ r(i,j;nT) \end{bmatrix} = \begin{bmatrix} s(i,j;0) \\ s(i,j;T) \\ \vdots \\ s(i,j;nT) \end{bmatrix} + \begin{bmatrix} n(i,j;0) \\ n(i,j;T) \\ \vdots \\ n(i,j;nT) \end{bmatrix} \quad (4.36)$$

The system error matrix is defined as

$$E = \begin{bmatrix} e(i,j;0) \\ e(i,j;T) \\ \vdots \\ e(i,j;nT) \end{bmatrix} \quad (4.37)$$

The actual system output matrix is defined as

$$C_a = \begin{bmatrix} c_a(i,j;0) \\ c_a(i,j;T) \\ \vdots \\ c_a(i,j;nT) \end{bmatrix} \quad (4.38)$$

The desired system output matrix is defined as

$$C_d = \begin{bmatrix} c_d(i,j;0) \\ c_d(i,j;T) \\ \vdots \\ c_d(i,j;nT) \end{bmatrix} \quad (4.39)$$

The transmission matrices of the actual system and the desired system are defined as

$$G_a = \begin{bmatrix} g_a(0,0) & 0 & \vdots & 0 \\ g_a(T,0) & g_a(T,T) & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_a(nT,0) & g_a(nT,T) & \dots & g_a(nT,nT) \end{bmatrix} \quad (4.40)$$

and

$$G_d = \begin{bmatrix} g_d(0,0) & 0 & \vdots & 0 \\ g_d(T,0) & g_d(T,T) & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_d(nT,0) & g_d(nT,T) & \dots & g_d(nT,nT) \end{bmatrix} \quad (4.41)$$

Similarly the plant matrix and the compensator matrix are defined, respectively, as

$$P = \begin{bmatrix} p(0,0) & 0 & \vdots & 0 \\ p(T,0) & p(T,T) & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p(nT,0) & p(nT,T) & \dots & p(nT,nT) \end{bmatrix} \quad (4.42)$$

and

$$G_c = \begin{bmatrix} g_c(0,0) & 0 & \vdots & 0 \\ g_c(T,0) & g_c(T,T) & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_c(nT,0) & g_c(nT,T) & \dots & g_c(nT,nT) \end{bmatrix} \quad (4.43)$$

With reference to Figures 1 and 2, pages 8 and 10, and associated definitions, the following matrix relationships are evident.

$$E = C_d - C_a \quad (4.44)$$

$$C_d = G_d S \quad (4.45)$$

$$C_a = G_a R = G_a (S + N) \quad (4.46)$$

$$G_a = (I + PG_c)^{-1} PG_c \quad (4.47)$$

The optimization criterion, equation (2.4), may be expressed in matrix form as follows.

$$C = \langle \langle \langle E^t W E \rangle_n \rangle_j \rangle_i \quad (4.48)$$

The time-weighting matrix, W , is defined as

$$W = \begin{bmatrix} w(i,j;NT;0) & 0 & \dots & 0 \\ 0 & w(i,j;NT;T) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w(i,j;NT;nT) \end{bmatrix} \quad (4.49)$$

The argument of the right side of equation (4.48) may be expressed as

$$E^t W E = (C_d - C_a)^t W (C_d - C_a). \quad (4.50)$$

Expansion of this matrix yields

$$E^t WE = (G_d S - G_a R)^t W (G_d S - G_a R), \quad (4.51)$$

$$E^t WE = [S^t (G_d - G_a)^t - N^t G_a^t] W (G_d - G_a) S - G_a N], \quad (4.52)$$

and

$$\begin{aligned} E^t WE &= S^t (G_d - G_a)^t W (G_d - G_a) S - N^t G_a^t W G_a N \\ &\quad - S^t (G_d - G_a)^t W G_a N - N^t G_a^t W (G_d - G_a) S. \end{aligned} \quad (4.53)$$

If the signal and noise are statistically independent and if the noise is assumed to have zero mean, the last two terms drop out upon averaging. The criterion may therefore be written as

$$C = \langle \langle \langle S^t (G_d - G_a)^t W (G_d - G_a) S + N^t G_a^t W G_a N \rangle_n \rangle_j \rangle_i \quad (4.54)$$

Let the matrix $(G_d - G_a)$ be defined in terms of general elements as

$$(G_d - G_a) = G_e = [e_{pq}] \quad (4.55)$$

and the matrix G_a be defined as

$$G_a = [a_{pq}] \quad (4.56)$$

and the matrix W be defined as

$$W = [w_{pq}] \quad (4.57)$$

Let the matrix of the product $G_e^t W G_e$ be defined in terms of the elements of the product as

$$G_e^t W G_e = [\alpha_{pq}] \quad (4.58)$$

and the matrix of the product $G_a^t W G_a$ be defined as

$$G_a^t W G_a = [\beta_{pq}] \quad (4.59)$$

In each case above p and q are non-specific row and column indices respectively. The matrix product $G_e^t W G_e$ may be expressed in terms of the matrix elements as

$$\alpha_{pq} = \sum_{v=1}^{n+1} \sum_{u=1}^{n+1} e_{pu} w_{uv} e_{vq}, \quad \begin{matrix} p = 1 \dots n+1, \\ q = 1 \dots n+1. \end{matrix} \quad (4.60)$$

It should be noted that W is a diagonal matrix of order $n+1$, $[e_{vq}]$ is a lower triangular matrix of order $n+1$, $[e_{pu}] = [e_{vq}]^t$ and $[\alpha_{pq}]$ is real symmetric matrix of order $n+1$. Similarly for the product $G_a^t W G_a$,

$$\beta_{pq} = \sum_{v=1}^{n+1} \sum_{u=1}^{n+1} a_{pu}^e w_{uv} a_{vq}^e, \quad \begin{matrix} p = 1 \dots n+1, \\ q = 1 \dots n+1. \end{matrix} \quad (4.61)$$

Consider now the products $S_e^t G_e^t W_e^t S$ and $N_a^t G_a^t W_a^t N$. Both of these products are in the form of a one-element matrix, each being the product of a square matrix premultiplied by a row matrix and post-multiplied by a column matrix. In terms of the matrix elements the first of these products is

$$S_e^t G_e^t W_e^t S = \left[\sum_{y=1}^{n+1} \sum_{x=1}^{n+1} s_{px}^e a_{xy} s_{yq}^e \right]. \quad (4.62)$$

Since the S matrices consist of only one row or column, equation (4.62) may be written as

$$S_e^t G_e^t W_e^t S = \left[\sum_{y=1}^{n+1} \sum_{x=1}^{n+1} s_{1x}^e a_{xy} s_{y1}^e \right]. \quad (4.63)$$

Writing equation (4.63) so that the elements of the S matrices are displayed explicitly rather than by their position,

$$S_e^t G_e^t W_e^t S = \left[\sum_{y=1}^{n+1} \sum_{x=1}^{n+1} s(xT-T) a_{xy} s(yT-T) \right]. \quad (4.64)$$

Similarly,

$$N_a^t G_a^t W_a^t N = \left[\sum_{y=1}^{n+1} \sum_{x=1}^{n+1} n(xT-T) \beta_{xy} n(yT-T) \right]. \quad (4.65)$$

Equation (4.64) can be written

$$S^t G_e^t W G_e S = \left[\sum_{y=1}^{n+1} \sum_{x=1}^{n+1} \sum_{v=1}^{n+1} \sum_{u=1}^{n+1} s(xT-T) e^{g_{xu}} w_{uv} e^{g_{vy}} s(yT-T) \right]. \quad (4.66)$$

Commuting the elements of the summand yields

$$S^t G_e^t W G_e S = \left[\sum_{y=1}^{n+1} \sum_{x=1}^{n+1} \sum_{v=1}^{n+1} \sum_{u=1}^{n+1} e^{g_{xu}} e^{g_{vy}} s(xT-T) w_{uv} s(yT-T) \right]. \quad (4.67)$$

Similarly,

$$N^t G_a^t W G_a N = \left[\sum_{y=1}^{n+1} \sum_{x=1}^{n+1} \sum_{v=1}^{n+1} \sum_{u=1}^{n+1} a^{g_{xu}} a^{g_{vy}} n(xT-T) w_{uv} n(yT-T) \right]. \quad (4.68)$$

As before, the weighted statistical autocorrelation sequences of the signal and noise are defined, respectively, as

$$\phi_{wss}(xT, yT) = \langle \langle \langle s(xT-T) w(xT-T) s(yT-T) \rangle_n \rangle_j \rangle_i \quad (4.69)$$

and

$$\phi_{wnn}(xT, yT) = \langle \langle \langle n(xT-T) w(xT-T) n(yT-T) \rangle_n \rangle_j \rangle_i \quad (4.70)$$

Using equations (4.69) and (4.70), the criterion, equation (4.54), may now be written

$$C = \sum_{y=1}^{n+1} \sum_{x=1}^{n+1} \sum_{v=1}^{n+1} \sum_{u=1}^{n+1} [e^{g_{xu}} e^{g_{vy}} \phi_{wss}(xT, yT) + a^{g_{xu}} a^{g_{vy}} \phi_{wnn}(xT, yT)] \quad (4.71)$$

or

$$C = \sum_{y=1}^{n+1} \sum_{x=1}^{n+1} \sum_{v=1}^{n+1} \sum_{u=1}^{n+1} [(\xi_d - \xi_a)_{xu} (\xi_d - \xi_a)_{vy} \phi_{wss}(xT, yT) + \xi_{xu} \xi_{vy} \phi_{wnn}(xT, yT)]. \quad (4.72)$$

In view of the fact that $[\xi_{xu}]$ and $[\xi_{xu}]$ are upper triangular matrices, and $[\xi_{vy}]$ and $[\xi_{vy}]$ are lower triangular matrices, equation (4.72) may be written as

$$C = \sum_{y=1}^{n+1} \sum_{x=1}^{n+1} \sum_{v=y}^{n+1} \sum_{u=x}^{n+1} [(\xi_d - \xi_a)_{xu} (\xi_d - \xi_a)_{vy} \phi_{wss}(xT, yT) + \xi_{xu} \xi_{vy} \phi_{wnn}(xT, yT)]. \quad (4.73)$$

The criterion is now in a form which can be optimized with respect to the actual system transmission matrix. This is done by taking the partial derivative of C with respect to ξ_{xu} and equating the result with zero in the usual manner.

$$\frac{\partial C}{\partial \xi_{xu}} = \sum_{y=1}^{n+1} \sum_{x=1}^{n+1} \sum_{v=y}^{n+1} \sum_{u=x}^{n+1} [(\xi_a - \xi_d)_{vy} \phi_{wss}(xT, yT) + \xi_{vy} \phi_{wnn}(xT, yT)] = 0 \quad (4.74)$$

This result is independent of u, and may be written as

$$\sum_{y=1}^{n+1} \sum_{x=1}^{n+1} \sum_{v=y}^{n+1} \left\{ \xi_{vy} [\phi_{wss}(xT, yT) + \phi_{wnn}(xT, yT)] - \xi_{vy} \phi_{wss}(xT, yT) \right\} = 0 \quad (4.75)$$

A solution for the elements of the optimum system transmission

matrix is therefore

$$a_{vy}^{g_{vy}} = \frac{d_{vy}^{g_{vy}} \phi_{wss}(xT, yT)}{\phi_{wss}(xT, yT) + \phi_{wnn}(xT, yT)}, \quad \begin{array}{l} x = 1 \dots n+1, \\ y = 1 \dots n+1, \\ v = y \dots n+1. \end{array} \quad (4.76)$$

Equation (4.76) is in the form of a set of algebraic equations which may be solved for the elements of the optimum system transmission matrix. Since the matrix is a lower triangular matrix of order $n+1$, the complete solution for its elements will require the solution of $\frac{1}{2} [(n+1)^2 + n+1] = \frac{1}{2}(n^2 + 3n+2)$ equations.

4.4 Physical Realizability of the Solution

The solution, equation (4.76), assumes the transmission matrix of the desired system, the weighted autocorrelation sequence of the signal, and the weighted autocorrelation sequence of the noise are known.

Salzer¹² has discussed the requirements for synthesizing a function by means of a digital computer with a linear program. Each variable in equation (4.76) is known a priori, and only the operations of addition, multiplication and division are necessary to solve for $a_{vy}^{g_{vy}}$, the general element of the optimum system transmission matrix. Since the plant is assumed to be in physical form, the compensation matrix may easily be computed and synthesized.

CHAPTER V

SUMMARY AND CONCLUSIONS

This thesis has presented a mean amplitude-set-time weighted error criterion for use in synthesizing optimum sampled-data systems. Procedures for application of the criterion to both time-invariant and time-varying systems were developed. In each case a physically realizable solution for the optimum system was derived.

Although the square of the system error was used in this thesis for the amplitude weighting function, it was pointed out that this selection is not the best in many cases. A similar problem arises in the selection of the time-weighting function and the subset-weighting parameter. There is no general procedure known which will take the place of a designer's experience in the selection of these weighting functions. Many factors, among them the characteristics and complexity of the system, the characteristics of the inputs to the system, and cost, must be considered in making the decision on the type of weighting function to be used. A logical study of these factors is a desirable subject for further work.

Several times in the process of solving for the optimum systems it was necessary that orders of summation or integration be interchanged. The necessary and sufficient condition for this operation to be permissible is that the summations or integrations involved be absolutely convergent in the region of interest. To show this rigorously would entail detailed, specific knowledge and consideration of every function of interest in the design of the system. It is sufficient for the purposes of this thesis to simply state that most summations and integrations arising in connection with physical systems are absolutely convergent.

Another operation which was performed without formal justification was the interchange of summations and averages. This operation is permissible if the average of the individual factors of the summand has a finite value.

This thesis was concerned only with completely sampled systems. An extension of the work herein to systems operating with both continuous and sampled data is surely possible through application of the modified z transform. Another approach would probably be necessary, however, if variable-rate sampled-data systems or digital-information systems were to be considered.

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