

401 893

63-3-2

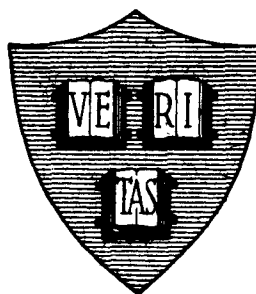
①

Office of Naval Research

Contract Nonr-1866 (32)

NR 371-016

SOLVING EQUATIONS OF UNIFORM FLOW



By

William F. Pickard

January 15, 1963

Technical Report No. 400

Cruft Laboratory
Harvard University
Cambridge, Massachusetts

ASTIA
RECEIVED
MAY 3 1963
ASTIA

AD No. 401 893
ASTIA FILE COPY

\$ 2.60

Office of Naval Research

(12) Contract Nonr-1866(32)^{el}, Proj. ↑
NR - 371 - 016

(7) Technical Report

(6) SOLVING THE EQUATIONS OF UNIFORM FLOW

(8) by
William F. Pickard

(9) January 15, 1963

The research reported in this document was supported by Grant 21869 of the National Science Foundation. Publication was made possible through support extended to Cruft Laboratory, Harvard University, by the Navy Department (Office of Naval Research), the Signal Corps of the U. S. Army, and the U. S. Air Force under ONR Contract Nonr-1866(32). Reproduction in whole or in part is permitted for any purpose of the United States Government.

(10) 23p. incl.
illus., 4 refs

(11) Technical Report No. 400

Cruft Laboratory

Harvard University

Cambridge, Massachusetts

(5) 234 300

(13) NA

(14) Incl.

(15) NA

OK

TR400

SOLVING THE EQUATIONS OF UNIFORM FLOW

by

William F. Pickard

Cruft Laboratory

Division of Engineering and Applied Physics

Harvard University, Cambridge, Massachusetts

ABSTRACT

↙ The problem of solving the equation for uniform flow in an open channel is reduced to that of evaluating three simple integrals. Practical methods of computing these integrals are discussed. ↗

(A)

SOLVING THE EQUATIONS OF UNIFORM FLOW*

1. Introduction to the Problem

The equation for uniform flow in an open channel is**

$$\frac{dy}{dx} = \frac{S_o - S_f}{1 - Z_c^2/Z^2} \quad [1]$$

where y is the stream depth, x is the distance along the channel bed, Z_c is the section factor at critical flow and Z is the section factor of the actual flow, and S_o is the slope of the stream bed; S_f is the energy grade line and is given by

$$S_f = Q^2/K^2 \quad [2]$$

where Q is the flowrate and K is the conveyance.

The section factor is given by (Ref. 1, Ch. 4)

$$Z^2 = C_z y^M \quad [3]$$

where C_z and M are positive constants.

The conveyance is given by (Ref. 1, Ch. 6)

$$K^2 = C_k y^N \quad [4]$$

where C_k and N are positive constants.

- - - - -

* This research was supported by a grant from the National Science Foundation.

** The reference for this equation and for the definitions and properties of the section factor and the conveyance is the treatise of Chow (1) on open channel hydraulics. The reference also gives the reader an excellent introduction to the physics of the problem and discusses previously devised methods of solving it.

[2], [3], and [4] can be substituted into [1] to yield

$$dx = \frac{1 - (y_c/y)^M}{S_o - (Q^2/C_k)y^{-N}} dy \quad [5]$$

where the critical depth y_c is given by

$$y_c = (Z_c^2/C_z)^{1/M} \quad [5']$$

In most applications [5] is integrated to give the distance δ between two depths y_1 and y_2 :

$$\delta = \int_{y_1}^{y_2} \frac{1 - (y_c/y)^M}{S_o - (Q^2/C_k)y^{-N}} dy \quad y_2 \geq y_1 \quad [6]$$

The only mathematical consideration to be observed in taking the integral of [6] is that no zero of the denominator of the integrand be permitted to occur over the range $(y_1, y_2)^*$. [6] can be given the more convenient form

$$\delta = B(M, N; S_o, y_c, Q^2/C_k; y_2) - B(M, N; S_o, y_c, Q^2/C_k; y_1) \quad [7]$$

where

$$B(M, N; S_o, y_c, Q^2/C_k; y) = \int \frac{1 - (y_c/y)^M}{S_o - (Q^2/C_k)y^{-N}} dy \quad [7']$$

It is convenient to study the function B .

*Physically, this would correspond to prohibiting the flow profile in a uniform channel from smoothly crossing the normal depth. This is in accord with experimental fact.

2. The Reduction of $B(M, N; S_o, y_c, Q^2/C_k; y)$ to

Its Four Normal Forms

To evaluate the B-function it is convenient to divide the fifteen hydraulic profiles into five classes.

Class I: M2, M3, S3, C3.

In this class $S_o > 0$ and $\begin{cases} y_n > y_1 \\ y_n > y_2 \end{cases}$ where the normal depth y_n is given by

$$y_n = \left(\frac{Q^2}{C_k S_o} \right)^{1/N} \quad [8]$$

The substitution of [8] into [7'] yields

$$B_I(y) = \frac{1}{S_o} \int \frac{1 - (y_c/y_n)^M (y_n/y)^M}{1 - (y_n/y)^N} dy \quad [9]$$

$0 < y < y_n < \infty$

which, subject to the substitution

$$\lambda_1 = (y_n/y)^{-N} \quad [10]$$

becomes

$$B_I(y) = - \frac{y_n}{N S_o} \int \frac{\lambda_1^{1/N} - (y_c/y_n)^M \lambda_1^{\frac{1-M}{N}}}{1 - \lambda_1} d\lambda_1 \quad [11]$$

Class II: M1, S1, S2, C1.

In this class $S_o > 0$ and $\begin{cases} y_n < y_1 \\ y_n < y_2 \end{cases}$ where y_n is given by [8].

The substitution

$$\lambda_2 = (y_n/y)^N \quad [12]$$

reduces [9] to

$$B_{II}(y) = -\frac{y_n}{NS_o} \int \frac{\lambda_2^{-\frac{1+N}{N}} - (y_c/y_n)^M \lambda_2^{\frac{M-1-N}{N}}}{1 - \lambda_2} d\lambda_2 \quad [13]$$

$$\infty > y > y_n > 0.$$

Class III: H2, H3.

In this class $S_o = 0$. Thus [7'] reduces to

$$B_{III}(y) = \frac{-1}{(Q^2/C_k)} \int \left[y^N - y_c^M y^{N-M} \right] dy \quad 0 < y < \infty \quad [14]$$

Class IV: A2, A3.

In this class $S_o < 0$ and [7'] reduces to

$$B_{IV}(y) = \frac{1}{S_o} \int \frac{1 - (y_c/y_a)^M (y_a/y)^M}{1 + (y_a/y)^N} dy \quad [15]$$

where the adverse depth y_a is given by

$$y_a = \left| \frac{Q^2}{S_o C_k} \right|^{1/N} \quad [15']$$

Suppose $y_a/y \geq 1$. Define

$$\lambda_3 = (y_a/y)^{-N} \quad [16]$$

Then

$$B_{IV}(y) = \frac{y_a}{NS_o} \int \frac{(y_a/y)^{-N} \lambda_3^{1/N} - (y_c/y_a)^M \lambda_3^{\frac{1-M}{N}}}{1 + \lambda_3} d\lambda_3 \quad [17]$$

$$0 < y \leq y_a < \infty$$

Suppose $y_a/y < 1$. Define

$$\lambda_4 = (y_a/y)^N \quad [18]$$

Then

$$B_{IV}(y) = B_{IV}(y_a) - \frac{y_a}{NS_0} \int_1^{(y_a/y)^N - \frac{1-N}{N} - (y_a/y_a)^M \lambda_4^{\frac{M-1-N}{N}}} \frac{d\lambda_4}{1 + \lambda_4} \quad [19]$$

$$y_a \leq y < \infty$$

Class V: H1, A1; C2.

H1 and A1 are not physically realizable (Ref. 1, Ch. 9). C2 is the trivial case $y_c = y = y_n$; it is unstable.

The most interesting fact about the integrals thus far presented is that they can be simply expressed in terms of three fundamental functions, the backwater integrals:

$$\mathcal{G}_\gamma^{(1)}(z) = \int \frac{z^{\gamma-1}}{1-z} dz \quad 0 < z < 1 \quad [20a]$$

$$\mathcal{G}_\gamma^{(2)}(z) = \int z^{\gamma-1} dz \quad 0 < z < \infty \quad [20b]$$

$$\mathcal{G}_\gamma^{(3)}(z) = \int \frac{z^{\gamma-1}}{1+z} dz \quad 0 < z < 1 \quad [20c]$$

One can thus reduce the B's to

$$B_I(y) = - \frac{y_n}{NS_0} \left[\frac{\mathcal{G}_{1+N}^{(1)}(a_1)}{N} - (y_c/y_n)^M \frac{\mathcal{G}_{1-M+N}^{(1)}(a_1)}{N} \right] \quad [21]$$

$$a_1 = (y_n/y)^{-N} \quad [21']$$

$$B_{II}(y) = -\frac{y_n}{NS_0} \left[\mathcal{G}_{-1/N}^{(1)}(a_2) - (y_c/y_n)^M \mathcal{G}_{\frac{M-1}{N}}^{(1)}(a_2) \right] \quad [22]$$

$$a_2 = (y_n/y)^N \quad [22']$$

$$B_{III}(y) = \frac{-1}{(Q^2/C_k)} \left[\mathcal{G}_{N+1}^{(2)}(a_3) - y_c^M \mathcal{G}_{N-M+1}^{(2)}(a_3) \right] \quad [23]$$

$$a_3 = y \quad [23']$$

$$y_a \leq y \quad B_{IV}(y) = \frac{y_a}{NS_0} \left[\mathcal{G}_{\frac{1+N}{N}}^{(3)}(a_4) - (y_c/y_a)^M \mathcal{G}_{\frac{1-M+N}{N}}^{(3)}(a_4) \right] \quad [24]$$

$$a_4 = (y_a/y)^{-N} \quad [24']$$

$$y_a \geq y \quad = B_{IV}(y_a) - \frac{y_a}{NS_0} \left[\mathcal{G}_{-\frac{1}{N}}^{(3)}(\lambda_4) - (y_a/y_a)^M \mathcal{G}_{\frac{M-1}{N}}^{(3)}(\lambda_4) \right] \quad [24'']$$

In the next section methods of evaluating the $\mathcal{G}_{\gamma}^{(l)}(z)$ will be discussed.

3. Evaluation of the Backwater Integrals

The evaluation of the backwater integrals for all values of γ presents no intrinsic difficulty. However, it can become rather involved, especially for $\gamma = -m$ ($m = 0, 1, 2, \dots$), and presents practical problems in computation. For the purposes of this study it is sufficient, for $\mathcal{G}_{\gamma}^{(1)}$ and $\mathcal{G}_{\gamma}^{(3)}$, to consider only those portions of the γ -axis covered by $\frac{1+N}{N}$, $\frac{1-M+N}{N}$, $-\frac{1}{N}$, and $\frac{M-1}{N}$; or, for $\mathcal{G}_{\gamma}^{(2)}$, by $1+N$ and $1-M+N$. One can first note that

for an open channel it generally happens that (1, p. 132) $2 \leq N < 6$;
hence

$$-\frac{1}{2} \leq -\frac{1}{N} < -1/6 \quad [25a]$$

$$3/2 \geq \frac{1+N}{N} > 7/6 \quad [25b]$$

Similarly, using the values of M calculated by Chow (1, p. 67), one has

$$1 > \frac{M-1}{N} > 1/2 \quad [25c]$$

$$0 < \frac{1-M+N}{N} < 1/2 \quad [25d]$$

Thus, for the evaluation of $\mathcal{G}_{\gamma}^{(1)}$ and $\mathcal{G}_{\gamma}^{(3)}$, the ranges of γ which are of interest are

$$-\frac{1}{2} \leq \gamma < 0 \quad [26a]$$

$$0 < \gamma \leq 3/2 \quad [26b]$$

For the evaluation of $\mathcal{G}_{\gamma}^{(2)}$ the γ -range of interest is

$$0 < \gamma < 7 \quad [27]$$

The restrictions placed upon γ by [26] and [27] preclude the existence of logarithmic terms in any of the $\mathcal{G}_{\gamma}^{(i)}(z)$. Hence, it is convenient to expand the integrands of [20a] and [20c] in infinite series and, uniform continuity being seen to obtain, to integrate these series term by term:

$$\mathcal{G}_{\gamma}^{(1)}(z) = \sum_{n=0}^{\infty} \left(\pm 1\right)^n \frac{1}{n+\gamma} z^{n+\gamma} \quad [28]$$

It happens that the series in [28] can be summed at once (2) to yield

$$\mathbb{G}_{\gamma}^{(1)}(z) = \frac{z^{\gamma}}{\gamma} F(1, \gamma; \gamma+1; \pm z) \quad [29]$$

where F is an ordinary hypergeometric function. This fact, however, is of negligible value computationally and it is better to sum [28] by less direct methods.

[28] can be reduced at once to

$$\mathbb{G}_{\gamma}^{(1)}(z) = z^{\gamma} \left[\frac{1}{\gamma} + S_{\gamma}(\pm z) \right] \quad [30]$$

where

$$S_{\gamma}(\pm z) = \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n + \gamma} z^n \quad [30']$$

By utilizing the relation (3)

$$Li_p(\pm z) = \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^p} z^n \quad [31]$$

where Li_p is the polylogarithm function (3) of order p , it is possible to express [30'] as

$$S_{\gamma}(\pm z) = \sum_{k=1}^K (-\gamma)^{k-1} Li_k(\pm z) + (-\gamma)^K \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^K (n + \gamma)} z^n \quad [32]$$

The Eq. [32] can itself be simplified by noting that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^K (n+\gamma)} z^n &= z^{-\gamma} \int_0^z \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^K} (z')^{n+\gamma-1} dz' \\ &= z^{-\gamma} \int_0^z (z')^{\gamma-1} \text{Li}_K(\pm z') dz' \end{aligned} \quad [33]$$

Let a polynomial approximation to Li_K be formed:

$$\text{Li}_K(\pm z) = \sum_{j=1}^J {}^J_K c_j (\pm) z^j + {}^K R_J^{(\pm)}(z) \quad [34]$$

Then

$$S_{\gamma}(\pm z) = \sum_{j=1}^J {}^J_0 c_j (\pm) \frac{1}{j+\gamma} z^j + z^{-\gamma} \int_0^z (z')^{\gamma-1} {}^K R_J^{(\pm)}(z') dz' \quad K=0 \quad [35a]$$

$$\begin{aligned} &= \sum_{k=1}^{K-1} (-\gamma)^{k-1} \text{Li}_k(\pm z) + (-\gamma)^{K-1} \sum_{j=1}^J {}^J_K c_j (\pm) \frac{1}{j+\gamma} z^j \\ &+ (-\gamma)^{K-1} {}^K R_J^{(\pm)}(z) + (-\gamma)^{K-\gamma} \int_0^z (z')^{\gamma-1} {}^K R_J^{(\pm)}(z') dz' \quad K \geq 1. \end{aligned} \quad [35b]$$

The choice of suitable K and J is not simple, since a balance must be drawn between the difficulties of computing the polylogarithm functions as opposed to those of summing series. In addition, $S_{\gamma}(\pm z)$ has a singularity at $z=1$ which adversely affects the convergence of the series which obtain for $K \geq 0$ and makes it difficult to obtain suitable approximating polynomials.

To insure that the integrals in [33] were well behaved it was decided to approximate $\frac{Li_k(+z)}{(+z)}$ and then to represent $K R_J^{(+)}(z')$ by

$$K R_J^{(+)}(z') = z' K E_J^{(+)}(z') \quad [36]$$

where $K E_J^{(+)}(z)$ is the error in $\frac{Li_K(+z)}{(+z)}$. If then, the maximum value of $|K E_J^{(+)}(z)|$ over $(0, z)$ is $K M_J^{(+)}(z)$ the error terms are, in absolute value, less than

$$\frac{z}{\gamma+1} {}_0 M_J^{(+)}(z) \quad [37a]$$

for [35a] and less than

$$\frac{|\gamma|^{K-1}}{\gamma+1} z K M_J^{(+)}(z) \quad [37b]$$

for [35b].

The construction of the polynomial approximations is discussed in detail in Appendix I; it is sufficient to note the results here. For integrals of the third kind $K = 1$ was found to be adequate:

$${}_1 M_9^{(-)} < 5 \times 10^{-7} \quad [38.0]$$

$${}_1 c_1^{(-)} = -1.0000 \quad 0000 \quad [38.1]$$

$${}_1 c_2^{(-)} = 0.4999 \quad 8946 \quad [38.2]$$

$${}_1 c_3^{(-)} = -0.3330 \quad 7718 \quad [38.3]$$

$${}_1 c_4^{(-)} = 0.2476 \quad 3567 \quad [38.4]$$

$${}_1 c_5^{(-)} = -0.1883 \quad 5208 \quad [38.5]$$

$${}_1^9 c_6^{(-)} = 0.1315 \quad 6190 \quad [38.6]$$

$${}_1^9 c_7^{(-)} = -0.0729 \quad 0260 \quad [38.7]$$

$${}_1^9 c_8^{(-)} = 0.0265 \quad 6504 \quad [38.8]$$

$${}_1^9 c_9^{(-)} = -0.0045 \quad 6742 \quad [38.9]$$

Thus,

$$\mathcal{G}_\gamma^{(3)}(z) = z^\gamma \left[\frac{1}{\gamma} + \sum_{j=1}^9 {}_1^9 c_j^{(-)} \frac{1}{j+\gamma} z^j + \mathcal{O}(5 \times 10^{-7} \times \frac{z}{\gamma+1}) \right]. \quad [39]$$

For the integrals of the second kind

$$\mathcal{G}_\gamma^{(2)}(z) = \frac{1}{\gamma} z^\gamma \quad [40]$$

For integrals of the first kind, unfortunately, it proved necessary to go to $K = 4$ to obtain a polynomial approximation which converged with sufficient rapidity:

$${}_4M_9^{(+)} < 7 \times 10^{-7} \times (1 - \ln(1 - z)) \quad [41.0]$$

(This odd weighting was chosen so that the relative accuracy of the approximation would not be greater than necessary near the singularity of $\text{Li}_1(z) = \ln(1-z)$ at $z = 1$.)

$${}_4^9 c_1^{(+)} = 1.0000 \quad 0046 \quad [41.1]$$

$${}_4^9 c_2^{(+)} = 0.0624 \quad 1793 \quad [41.2]$$

$${}_4^9 c_3^{(+)} = 0.0143 \quad 7767 \quad [41.3]$$

$${}_4^9 c_4^{(+)} = -0.0150 \quad 2213 \quad [41.4]$$

$${}_4^9 c_5^{(+)} = 0.0873 \quad 5126 \quad [41.5]$$

$${}_4^9 c_6^{(+)} = -0.2087 \quad 8801 \quad [41.6]$$

$${}_4^9 c_7^{(+)} = 0.2831 \quad 2643 \quad [41.7]$$

$${}_4^9 c_8^{(+)} = -0.1984 \quad 0826 \quad [41.8]$$

$${}_4^9 c_9^{(+)} = 0.0572 \quad 7062 \quad [41.9]$$

Hence,

$$\begin{aligned} \mathcal{B}_\gamma^{(1)}(z) = z^\gamma & \left[\frac{1}{\gamma} + \text{Li}_1(z) - \gamma \text{Li}_2(z) + \gamma^2 \text{Li}_3(z) \right. \\ & - \gamma^3 \sum_{j=1}^9 {}_4^9 c_j^{(+)} \frac{j}{j+\gamma} z^j \\ & \left. + \mathcal{O}(7 \times 10^{-7} \times \frac{\gamma^3}{\gamma+1} \times z \times (1 - \ln(1-z))) \right] \quad [42] \end{aligned}$$

Since the polylogarithms have been extensively tabulated(4), the forms [39], [40], and [42] provide a convenient way of computing the $\mathcal{B}_\gamma^l(z)$ by hand. The attainable accuracy will, of course, vary with the choice of l , γ , and z : it is believed that if eight figures are carried, the relative error in the answer will normally be less than 1 part in 10^5 .

The Eqs. [39] and [40] are well suited to use with digital computers. [42] is not quite as convenient for machine use since it requires the computation of three polylogarithm functions which can in turn necessitate the computation

of five logarithms plus the summation of two slowly convergent series:

it is advisable to compute $\mathcal{B}_\gamma^{(3)}(z)$ directly from [30] when the series for $S_\gamma(z)$ converges with suitable rapidity. From [30']

$$S_\gamma(z) = \sum_{n=1}^{\infty} \frac{z^n}{n+\gamma} \quad \begin{array}{l} |z| < 1 \\ -1/2 < \gamma \leq 3/2 \end{array} \quad [43]$$

To determine the z -range over which [43] is useful it is convenient to rewrite it as

$$S_\gamma(z) = \sum_{n=1}^N \frac{z^n}{n+\gamma} + T_N(z) \quad [44]$$

where the truncation error $T_N(z)$ is given by

$$T_N(z) = \sum_{n=N+1}^{\infty} \frac{z^n}{n+\gamma} \quad [44']$$

An application of Cauchy's integral method to [44'] yields

$$|T_N(z)| \leq |z|^N \sum_{n=1}^{\infty} \frac{|z|^n}{(N-1)+n} \leq |z|^N \int_0^{\infty} \frac{|z|^t}{(N-1)+t} dt \quad [45]$$

which, subject to the substitutions

$$|z| = e^{-p} \quad p > 0 \quad [46a]$$

$$(N-1) = \lambda \quad [46b]$$

becomes

$$|T_N(z)| \leq e^{-Np} \int_0^{\infty} \frac{e^{-pt}}{\lambda + t} dt \quad \begin{matrix} |z| < 1 \\ p > 0 \end{matrix} \quad [47]$$

Hence,

$$|T_N(z)| \leq e^{-p} \left| \text{Ei}(-\lambda p) \right| \quad [48]$$

where

$$\text{Ei}(\omega) = \int_{-\infty}^{\omega} \frac{e^t}{t} dt \quad [48']$$

Thus, the relative error $R_N(z)$ in the sum is, for $z \geq 0$, bounded by

$$|R_N(z)| \leq \frac{5}{2} \left| \text{Ei}(-\lambda p) \right| \quad [49]$$

Taking $|R_N(z)| < 5 \times 10^{-8}$ ($\lambda p \doteq 15$), one sees that $N = 64$ will suffice for $z \leq 0.75$. Hence, for use on digital machines it is suggested that the formula

$$\mathcal{G}_{\gamma}^{(1)}(z) = z^{\gamma} \sum_{n=0}^{64} \frac{z^n}{n+\gamma} \quad \begin{matrix} 0 < z \leq 3/4 \\ -\frac{1}{2} \leq \gamma \leq 3/2 \\ \gamma \neq 0 \end{matrix} \quad [50]$$

be employed when possible.

Rather than use the methods outlined above to produce extensive tables of the $\mathcal{G}_{\gamma}^{(l)}(z)$, which would, in this era of digital machinery, be obsolescent from birth, the equations [39], [40], [42], and [50] were combined to produce a single subprogram for the IBM 7090 computer. Copies of this subprogram can be obtained by writing the author; a complete description of it is given in Appendix II.

4. Conclusions

It was shown that the problem of integrating the differential equation governing the uniform flow of liquid in an open channel can be reduced to the evaluation of three transcendental functions. This latter problem was shown to reduce to that of computing certain simple algebraic quantities and calculating various low-order polylogarithm functions.

No tables of backwater integrals were presented, since it was concluded that - in view of the extensive and rapidly growing use of digital computers - such tables would not be of widespread interest. Instead, a subprogram was written for use on an IBM 7090. With the aid of this subprogram the engineer can rapidly construct a program to calculate whatever flow profile he desires.

An area where considerable future progress can be made is that of the derivation of approximating polynomials for the several polylogarithm functions. A not unrealizable goal should be the polynomial approximation of $\text{Li}_2(z)$ which would reduce the evaluation of $\mathfrak{B}_\gamma^{(1)}(z)$ to the summation of a rather long series and the computation of the logarithm.

APPENDIX I

Construction of the Polynomial Approximations

The problem to be dealt with here is that of properly approximating certain transcendental functions. It was solved as follows.

Let $f(x)$ be a function which is to be approximated by a polynomial $P_J(x)$ over an interval $(0, 1)$. Let

$$D_J(x) = f(x) - P_J(x) \quad [A-1]$$

Further, let $g(x)$ be some positive definite function over $(0, 1)$, and let

$$Z_J(x) = D_J(x)/g(x) \quad [A-2]$$

be the error relative to the normalizing function $g(x)$. The problem is to adjust the coefficients a_j of

$$P_J(x) = \sum_{j=0}^J a_j x^j \quad [A-3]$$

so that $Z_J(x)$ falls within some desired bounds.

The method used is based upon that of Spitzbart and Macon (4). Let $Z_J(x)$ be specified over a set $\{x_p\}$ of $J+2$ distinct points as

$$Z_J(x_p) = \lambda_p^d \quad [A-4]$$

where the λ_p have been specified and d is some constant to be determined.

Then

$$f(x_p) - P_J(x_p) = d\lambda_p g(x_p) \quad [A-5]$$

d , a measure of the goodness of fit, and the coefficients a_j are determined by solving the system [A-5]. Let

$$A = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_J \\ d \end{pmatrix} \quad [A-6a]$$

$$F = \begin{pmatrix} f(x_0) \\ \vdots \\ f(x_{J+1}) \end{pmatrix} \quad [A-6b]$$

$$W = \begin{pmatrix} 1 & x_0 & \cdots & x_0^J & g(x_0) & \lambda_0 \\ 1 & x_1 & \cdots & x_1^J & g(x_1) & \lambda_1 \\ \vdots & \vdots & & \vdots & \vdots & \\ 1 & x_{J+1} & & x_{J+1}^J & g(x_{J+1}) & \lambda_{J+1} \end{pmatrix} \quad [A-6c]$$

With this abbreviated notation [A-5] can be rewritten as

$$WA = F \quad [A-7]$$

[A-7] will have a non-trivial solution, i.e., W^{-1} will exist, if and only if no polynomial of degree less than $J+1$ passes through the points

$$(x_p, \lambda_p g(x_p)) \quad (p = 0, 1, 2, \dots, J+1).$$

For the approximations desired here this condition was met by setting

$$\lambda_p = (-1)^p \quad (p = 0, 1, \dots, J) \quad [A-8a]$$

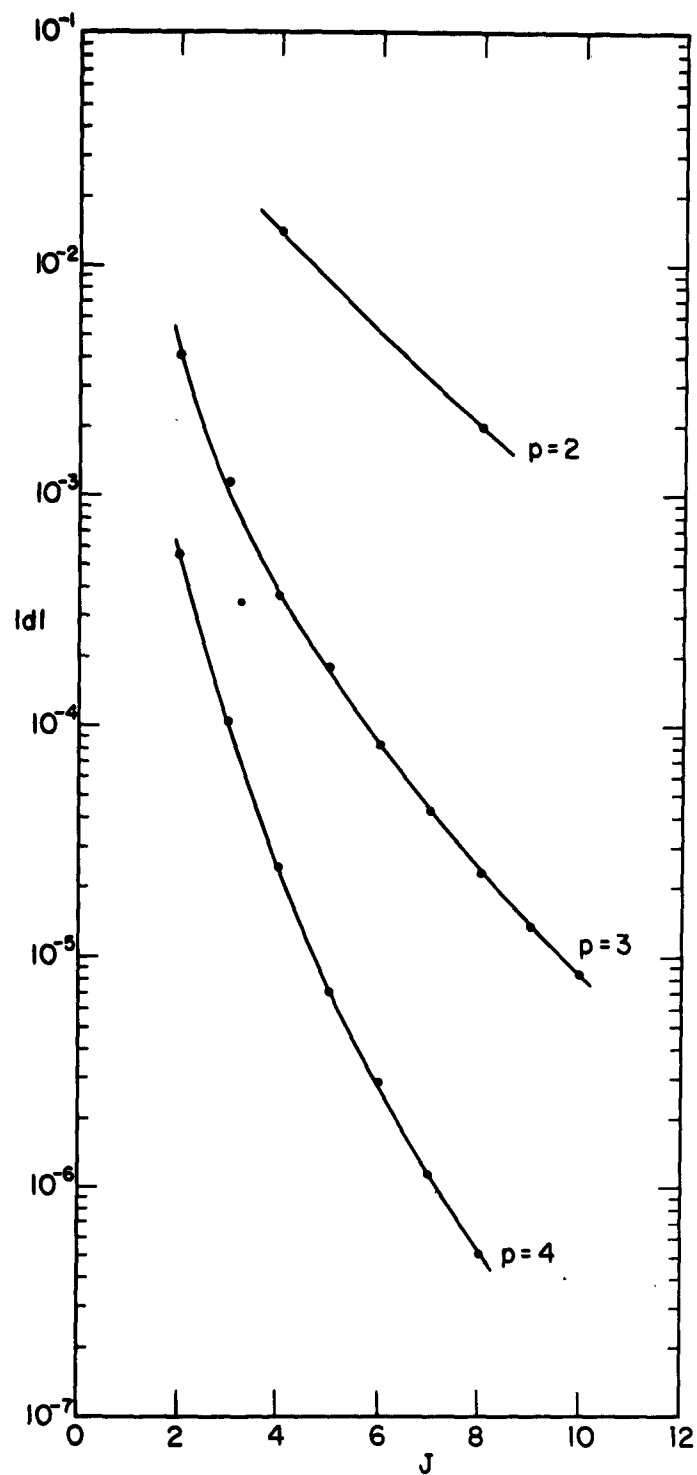


FIG. 1. POLYNOMIAL APPROXIMATION OF $Li_p(z)/z$.
 ERROR CRITERION $|d|$ AS A FUNCTION OF p AND
 THE ORDER OF APPROXIMATION J .

and

$$\lambda_{J+1} = \lambda_J . \quad [A-8b]$$

Of course, the formalism leading to [A-7] does not really solve the approximation problem completely, since it does not specify either $g(x)$ or the set $\{x_p\}$. The choice of $g(x)$ was left arbitrary and was varied from one $f(x)$ to another. The $\{x_p\}$ problem was solved by requiring the extremals of $Z_J(x)$ to coincide with $\{x_p\}$. For monotone $f(x)$ this leads to the danger of having only $J+1$ extremals, and in this case x_{J+1} was chosen to be 1, x_J being unchanged if $x=1$ was an extremal, or being placed between $x=x_{J-1}$ and $x=1$ if $x=1$ was an extremal. This choice of $\{x_p\}$ meant in practice that the set $\{x_p\}$ was found iteratively: a set $\{x_p^{(0)}\}$ was chosen, the a_j found, and a curve of $Z_N(x)$ computed; from this Z_N a new set of extremals, i.e., $\{x_p^{(1)}\}$, was found and the process repeated; the iteration was continued in this manner until convergence resulted or until $Z_N(x)$ and/or d became so small that computing error rendered further iteration unprofitable. This method of selecting the λ_p and the $\{x_p\}$ in effect adjusted $D_N(x)$ so that its extremals were of alternating sign and so that $|g(x)d|$ was an envelop for it.

A FORTRAN program embodying these features was written and run on the IBM 7090 computer at the Harvard University Computing Center. It was found that the W -matrices tended to be fairly poorly conditioned with the result that it was difficult to procure solutions for [A-7] when J exceeded ten.

Since $\mathcal{G}_\gamma^{(1)}(z)$ is singular at $z = 1$, the weighting function

$$g(x) = 1 - \ln(1-z) \quad [\text{A-9}]$$

was chosen, and approximations to $\text{Li}_p(z)/z$ for various J and $p = 2, 3, 4$ were generated. The results of these calculations are summarized in Fig. I.

The singularity at $z = 1$ in the p^{th} derivative of $\text{Li}_p(z)$ is seen to affect powerfully the goodness of fit. The case $p = 4$, $J = 8$ was chosen as adequate and the results presented in [41] were adopted. With an approximating program of higher accuracy it seems likely that sufficiently accurate results could be obtained with $p = 3$, $J > 16$. It seems probable that the obtaining of a good approximation for $p = 2$ will require a quite large value of J .

The fitting of $\mathcal{G}_\gamma^{(3)}(z)$ was much simpler, since it and its derivatives possess no singularity over $(0, 1)$. In this case a weight function equal approximately to unity was chosen and the results quoted in [38] obtained for $p = 1$, $J = 8$.

APPENDIX II

A Program for Computing the Backwater Integrals

A FAP coded FORTRAN type subprogram, named BKW, was written to compute the backwater integrals. It was subsequently debugged and tested on the IBM 7090 at the Harvard University Computing Center. Binary decks for this routine may be obtained by writing the author; these decks also contain the special subroutines AGLOG, AGPWR, and AGTLG called by BKW.

A listing of BKW is presented in Figs. IIa through IIe. The program for computing $\mathcal{B}_\gamma^{(l)}(z)$ is seen to be divided roughly into five major subsections. In the first subsection the program decides which of the three integrals ($l = 1, 2, 3$) is to be evaluated and transfers control to the appropriate subsection. There is a separate subsection (B1, B2, or B3) for each value of l . Finally, there is a subsection (ERR and/or RET) which controls returns to the calling program.

B1. This subsection first examines z : for $z \leq 0$ or $z \geq 1$, an error return (transfer to ERR) is specified, for $0 < z \leq 0.75$, an evaluation by the series [50] (transfer to SER1) is specified; for $0.75 < z < 1.00$, an evaluation by means of Eq. [42] is specified. The first step in the evaluation by either [50] or [42] is the examination of γ : if the γ specified equals a value held in storage (zero if this is the first pass through SER1 or the corresponding routine for large z , otherwise the value of γ on the last

pass through the appropriate routine), the values of $\frac{1}{n+\gamma}$ (or $\frac{9}{4} c_j^{(+)} \frac{j}{j+\gamma}$) are known to be computed and held in storage already so that $\mathcal{B}_\gamma^{(1)}(z)$ can be computed at once from [42] (or [50]) and control transferred to RET; if γ does not agree with the value held in storage, it is checked to make sure that it is within the prescribed range of values and, assuming that it passes this test (transfer to ERR otherwise), the appropriate constants are calculated and the integral evaluated as before. In the course of these calculations various special subroutines are utilized: AGLOG, a 27 bit accuracy, floating point logarithm routine; AGPWR, a 26 bit accuracy, floating point power routine; AGDLG, a routine for $\text{Li}_2(z)$; AGTLG, a routine for $\text{Li}_3(z)$.

B2. This subsection first obtains γ and z and tests to see that they lie within the allowed ranges. Failure of this test causes program control to be transferred to ERR. Success of this test results in the evaluation of $\mathcal{B}_\gamma^{(2)}(z)$ directly from [40]; control is then transferred to RET.

B3. This subsection is roughly similar to that for $\mathcal{B}_\gamma^{(1)}(z)$, except that the integrals for all allowable z are computed from the same formula. First, z is checked for range. Then γ is checked, and, if necessary, the constants $\frac{9}{4} c_j^{(-)} \frac{j}{j+\gamma}$ are recomputed. Finally, $\mathcal{B}_\gamma^{(3)}(z)$ is computed from [39], and control is transferred to RET.

ERR and RET. ERR is an error routine which clears the accumulator and turns on sense light 4; it then transfers control to RET. RET resets the three index registers and returns control to the calling program.

* * *

USAGE. The appropriate calling sequence in FAP is

A	TSX	BKW, 4
	TSX	L
	TSX	GAMMA
	TSX	Z

The normal return is to A+4 within $\mathcal{B}_\gamma^{(1)}(z)$ in the accumulator; in case of an error in the range of z or γ , return is to A+4 but the accumulator will be clear and sense light 4 will be on. (Note that L must be a FORTRAN integer and that GAMMA and Z must be FORTRAN floating point numbers.) The appropriate calling sequence in FORTRAN is

$$Y = \text{BKW}(L, \text{GAMMA}, Z)$$

where Y is the value of the desired integral.

TIMING. This is quite variable. It runs from about 10 ms for the worst cases of $\mathcal{B}_\gamma^{(1)}(z)$ down to 2^+ ms for $\mathcal{B}_\gamma^{(3)}(z)$ and 2 ms for $\mathcal{B}_\gamma^{(2)}(z)$.

ACCURACY. Error is expected to be less than 1 part in 10^5 in most cases. For $\mathcal{B}_\gamma^{(1)}(z)$ ($0.75 < z < 1.00$) it is normally less than 1 part in 10^6 .

BKW	ENTRY	BKW	
	SXD	*-2,4	
	SXD	ERAS+1,1	SAVE INDICES
	SXD	ERAS+2,2	
	SXD	ERAS+4,4	
	CLA*	1,4	GET L
	CAS	CNST	
	TRA	*+2	
	TRA	B1	L = 1
	CAS	CNST+1	
	TRA	*+2	
	TRA	B2	L = 2
	CAS	CNST+2	
	TRA	*+2	
	TRA	B3	L = 3
	TRA	ERR	
B1	CLA*	2,4	GET AND SAVE GAMMA
	STO	ERAS	
	CLA*	3,4	GET Z
	TMI	ERR	Z NEGATIVE
	TZE	ERR	Z ZERO
	CAS	CNST+3	
	TRA	ERR	Z TOO LARGE
	TTR	*-1	
	STO	Z	Z OK. SAVE
	CAS	CNST+4	
	TRA	*+6	Z GT 0.75
	TTR	*+1	
	TSX	SER1,4	
	HTR	ERAS	
	HTR	Z	
	TRA	RET	
	CLA	ERAS	
	CAS	G1	
	TTR	*+2	
	TRA	*+2	OLD CONSTANTS SUFFICE
	TRA	CON1	GET NEW CONSTANTS FOR ASYMP EVAL
	CLA	CNST+3	BEGIN ASYMPTOTIC EVALUATION
	FDP	G1	
	STQ	TOT	
	CLA	CNST+3	
	FSR	Z	
	STO	ERAS	
	TSX	\$AGLOG,4	
	HTR	FRAS	
	HTR	ERAS	
	CHS		
	FAD	TOT	
	STO	TOT	LI1 INCLUDED
	TSX	\$AGDLG,4	
	HTR	Z	
	XCA		
	FMP	G1	
	CHS		
	FAD	TOT	
	STO	TOT	LI2 INCLUDED
	TSX	\$AGTLG,4	
	HTR	Z	

FIG. IIa LISTING OF BKW (PAGE 1 OF 5)

	XCA		
	FMP	G1	
	XCA		
	FMP	G1	
	FAD	TOT	
	STO	TOT	
	AXT	0,1	L13 INCLUDED
	PXD	0,0	START APPROXIMATING SERIES
	FAD	S1+8,1	
	XCA		
	FMP	Z	
	TXI	*+1,1,1	
	TXL	*-4,1,8	
	XCA		
	FMP	G1	
	XCA		
	FMP	G1	
	XCA		
	FMP	G1	
	CHS		
	FAD	TOT	
	STO	TOT	COMPLETE SERIES
	TSX	\$AGPWR,4	
	HTR	Z	
	HTR	G1	
	XCA		
	FMP	TOT	B1 COMPLETE
	TRA	RET	
SER1	SXD	ERAS+3,4	SAVE XR(4)
	CLA*	2,4	
	STO	Z	SAVE Z
	CLA*	1,4	GET GAMMA
	CAS	G1S	
	TTR	*+2	
	TRA	*+2	OLD CONSTANTS SUFFICE
	TRA	CON1S	NEW CONSTANTS NECESSARY
	AXT	0,4	BEGIN SERIES SUMMATION
	PXD	0,0	
	FAD	S1S+64,4	
	XCA		
	FMP	Z	
	TXI	*+1,4,1	
	TXL	*-4,4,63	
	FAD	S1S	
	STO	ERAS	
	TSX	\$AGPWR,4	
	HTR	Z	
	HTR	G1S	
	XCA		
	FMP	ERAS	
	LXD	ERAS+3,4	
	TRA	3,4	
CON1	TZE	ERR	CHECK RANGE OF GAMMA
	CAS	CNST+5	
	TRA	*+3	
	TTR	*-1	
	TRA	ERR	
	CAS	CNST+6	

FIG. IIb LISTING OF BKW (PAGE 2 OF 5)

	TRA	ERR	
	TTR	*+1	
	STO	G1	SAVE NEW VALUE OF GAMMA
	AXT	0,1	SET UP NEW CONSTANTS
	CLA	CNST+7	
	STO	ERAS+5	
	CLA	ERAS+5	
	FAD	G1	
	STO	ERAS+6	
	CLA	ERAS+5	
	FDP	ERAS+6	
	FMP	C+8,1	
	STO	S1+8,1	
	CLA	ERAS+5	
	FSB	CNST+3	
	STO	ERAS+5	
	TXI	*+1,1,1	
	TXL	*-11,1,8	
	TRA	B1+21	RETURN TO B1 EVALUATION
CONIS	TZE	ERR	CHECK RANGE OF GAMMA
	CAS	CNST+5	
	TRA	*+3	
	TTR	*-1	
	TRA	ERR	
	CAS	CNST+6	
	TRA	ERR	
	TTR	*+1	
	STO	G1S	SAVE NEW VALUE OF GAMMA
	AXT	0,4	SET UP NEW CONSTANTS
	CLA	CNST+8	
	STO	ERAS+5	
	CLA	ERAS+5	
	FAD	G1S	
	STO	ERAS+6	
	CLA	CNST+3	
	FDP	ERAS+6	
	STQ	S1S+64,4	
	CLA	ERAS+5	
	FSB	CNST+3	
	STO	ERAS+5	
	TXI	*+1,4,1	
	TXL	*-10,4,64	
	TRA	SER1+8	RETURN TO SER1
ERR	PSE	100	TURN ON SENSE LIGHT 4
	PXD	0,0	CLEAR AC
	TRA	RET	PREPARE TO EXIT
RET	LXD	ERAS+1,1	
	LXD	ERAS+2,2	
	LXD	ERAS+4,4	
	TRA	4,4	GO BACK TO MAIN PROGRAM
B2	CLA*	2,4	OBTAIN GAMMA AND CHECK RANGE
	STO	G2	
	TXI	ERR	
	TZE	ERR	
	CAS	CNST+9	
	TRA	ERR	
	TTR	*+1	
	CLA*	2,4	OBTAIN Z AND CHECK RANGE

FIG. IIc LISTING OF BKW (PAGE 3 OF 5)

	TMI	ERR	
	TZE	ERR	
	STO	Z	
	TSX	\$AGPWR,4	Z**GAMMA
	HTR	Z	
	HTR	G2	
	FDP	G2	
	XCA		
	TRA	RET	
33	CLA*	2,4	GET GAMMA
	CAS	G3	
	TRA	*+2	
	TRA	*+2	OLD CONSTANTS SUFFICE
	TRA	CON3	NEW CONSTANTS NEEDED
	CLA*	3,4	GET AND TEST Z
	TMI	ERR	
	TZE	ERR	
	CAS	CNST+3	
	TRA	ERR	
	TTR	*+1	
	STO	Z	
	AXT	0,1	SUM SERIES
	PXD	0,0	
	FAD	S3+8,1	
	XCA		
	FMP	Z	
	TXI	*+1,1,1	
	TXL	*-4,1,8	
	STO	ERAS	
	CLA	CNST+3	
	FDP	G3	
	XCA		
	FAD	ERAS	
	STO	ERAS	FIND B3
	TSX	\$AGPWR,4	
	HTR	Z	
	HTR	G3	
	XCA		
	FMP	ERAS	
CON3	TRA	RET	
	TZE	ERR	TEST GAMMA
	CAS	CNST+5	
	TRA	*+3	
	TTR	*-1	
	TRA	ERR	
	CAS	CNST+6	
	TRA	ERR	
	TTR	*+1	
	STO	G3	GAMMA OK
	AXT	0,2	SET UP SERIES REVISION
	CLA	CNST+7	
	STO	ERAS+6	
	CLA	ERAS+6	REVISE
	FAD	G3	
	STO	ERAS	
	CLA	D+8,2	
	FDP	ERAS	
	FMP	ERAS+6	

FIG. II_d LISTING OF BKW (PAGE 4 OF 5)

	STO	S3+8,2	
	CLA	ERAS+6	
	FSR	CNST+3	
	STO	ERAS+6	
	TXI	*+1,2,1	
	TXL	*-11,2,8	
	TRA	33+5	
CNST	PZE	0,0,1	
	PZE	0,0,2	
	PZE	0,0,3	
	DEC	1.00	
	DEC	0.75	
	DEC	-0.50	
	DEC	1.50	
	DEC	9.00	
	DEC	64.00	
	DEC	7.00	
C	DEC	+0.100000046E+01	LI4 CONSTANTS
	DEC	+0.624179326E-01	
	DEC	+0.143776655E-01	
	DEC	-0.150221251E-01	
	DEC	+0.873512559E-01	
	DEC	-0.208788007E-00	
	DEC	+0.283126429E-00	
	DEC	-0.198408261E-00	
	DEC	+0.572706185E-01	
D	DEC	-0.999999993E+00	LI1 CONSTANTS
	DEC	+0.499989465E-00	
	DEC	-0.333077177E-00	
	DEC	+0.247635670E-00	
	DEC	-0.188352078E-00	
	DEC	+0.131561905E-00	
	DEC	-0.729025990E-01	
	DEC	+0.265650421E-01	
	DEC	-0.456741825E-02	
ERAS	BSS	7	
Z	BSS	1	
G1	BSS	1	
G1S	BSS	1	
G2	BSS	1	
G3	BSS	1	
TOT	BSS	1	
S1	BSS	9	
S1S	BSS	65	
S3	BSS	9	
	END		

FIG. IIe LISTING OF BKW (PAGE 5 OF 5)

For $\mathcal{G}_{\gamma}^{(3)}(z)$ it is normally less than 2 parts in 10^7 and often less than 2 parts in 10^8 . For $\mathcal{G}_{\gamma}^{(1)}(z)$ ($0.00 < z \leq 0.75$) the error is normally confined to the eighth figure. The error in $\mathcal{G}_{\gamma}^{(2)}(z)$ is normally less than 2 parts in 10^8 . Regions of high relative error can be expected in $\mathcal{G}_{\gamma}^{(1)}(z)$ for $\gamma < 0$ and values of z for which $\mathcal{G}_{\gamma}^{(1)}(z) \approx 0$; however, even here several good decimal places will normally be available.

REFERENCES

1. V. T. Chow, Open Channel Hydraulics. New York: McGraw-Hill Book Company, Inc., 1959.
2. A. Erdélyi (editor), Higher Transcendental Functions, Vol. I. New York: McGraw-Hill Book Company, Inc., 1953.
3. L. Lewin, Dilogarithms and Associated Functions. London: Macdonald and Co., 1958.
4. A. Spitzbart and N. Macon, Proc. Amer. Math. Soc. 9, 243-9, 1958.