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THE ATYPICAL ZEROS OF A CLASS OF ENTIRE FUNCTIONS

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ABSTRACT

A homogeneous differential-difference-integral equation with constant coefficients and a convolution integral has an exponential solution $\varphi(t) = e^{zt}$ if and only if z is the zero of a certain entire function. This paper is concerned with the asymptotic distribution of the zeros of such entire functions.

THE ATYPICAL ZEROS OF A CLASS OF ENTIRE FUNCTIONS

P. M. Anselone and R. P. Boas, Jr.

This note is concerned with the distribution of the zeros of certain entire functions which appear in the study of difference-integral and differential-difference-integral equations (cf. [1], [2], [4]). We shall deal first with a typical problem and generalize later. The equation

$$\varphi(t-1) - \varphi(t-2) = \int_{t-1}^{t} K(t-s)\,\varphi(s)\,ds = \int_0^1 K(s)\varphi(t-s)\,ds ,$$

with $K(s)$ integrable, has an exponential solution $\varphi(t) = e^{zt}$ if and only if z is a zero of the <u>characteristic function</u>

$$\Psi(z) = e^{-z} - e^{-2z} - \int_0^1 e^{-zs}K(s)\,ds = e^{-z}[1 - e^{-z} - e^{z}\int_0^1 e^{-zs}K(s)\,ds] .$$

Exponential solutions are important, for example, in a Laplace transform treatment of the equation.

Since $\Psi(z)$ is an entire function of exponential type and $\Psi(iy)$ is bounded, the zeros $r_n e^{i\theta_n}$, $n = 1, 2, \ldots,$ of $\Psi(z)$ satisfy $\sum r_n^{-1} |\cos \theta_n| < \infty$ (cf., e.g., [3], ch. 8). This suggests that the zeros are concentrated near the imaginary axis. In particular, by the Riemann-Lebesgue lemma and Rouché's

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theorem, $e^{z}\Psi(z)$, and hence $\Psi(z)$, have zeros $2\pi in + o(1)$ as $n \to \pm \infty$. It may, therefore, seem surprising that for a rather extensive class of functions $K(s)$ there are zeros of $\Psi(z)$ of arbitrarily large real part.

We shall prove the following theorem.

Theorem. Let $K(s)$ be absolutely continuous, $K(0) = \gamma \neq 0$, and $K'(s)$ essentially bounded. Let

$$\Psi(z) = e^{-z}[1 + h(z) - e^{z} \int_0^1 e^{-zs} K(s)\, ds] ,$$

where

$$h(z) \to 0 \text{ uniformly in } y \text{ as } x \to \infty .$$

Then $\Psi(z)$ has zeros z_n, $n = \pm N, \pm(N+1), \ldots,$ such that

$$z_n = \log |n| + 2\pi in + O(1) .$$

(After proving the theorem we shall indicate how the condition $K(0) \neq 0$ can be relaxed.)

We cannot apply the usual Abelian theorems for Laplace transforms since we require an estimate valid near the imaginary axis. Instead, the proof is based on the

<u>Lemma.</u> If $A \neq 0$, then $z - Ae^z$ has zeros ζ_n, $n = \pm 1, \pm 2, \ldots$, such that

$$\zeta_n = \log |n| + 2\pi i n + O(1) .$$

If $z - Ae^z = 0$ and $z = -w$, then $we^w + A = 0$. The asymptotic distribution of the zeros of $we^w + A$ (in a form more precise than we require) is given by Wright [5, p. 199]. The lemma follows.

We now establish the theorem. Integrate by parts to obtain

$$\int_0^1 e^{-zs} K(s)\, ds = z^{-1}[\gamma - e^{-z}K(1) + B(z)]$$

where $\gamma = K(0)$ and

$$B(z) = \int_0^1 e^{-zs} K'(s)\, ds = O(1/x) , \qquad x > 0 .$$

Therefore,

$$ze^z \Psi(z) = z - \gamma e^z + zh(z) + K(1) - e^z B(z) .$$

Let $\zeta_n = \gamma e^{\zeta_n}$ as in Lemma 1 with $A = \gamma$. Let $z = \zeta_n + \delta$, where δ is a fixed small complex number. Then

$$z - \gamma e^{z} = \zeta_n + \delta - \gamma e^{\zeta_n + \delta} = \zeta_n + \delta - \zeta_n e^{\delta} = \zeta_n(1 - e^{\delta}) + \delta ,$$

$$zh(z) = (\zeta_n + \delta)\, o(1) = o(|\zeta_n|) = o(|z - \gamma e^{z}|) ,$$

$$e^{z}B(z) = e^{\zeta_n + \delta} B(\zeta_n + \delta) = \gamma^{-1} e^{\delta} \zeta_n B(\zeta_n + \delta) = \zeta_n o(1) = o(|\zeta_n|) = o(|z - \gamma e^{z}|) .$$

Therefore,

$$ze^{z}\Psi(z) = z - \gamma e^{z} + o(|z - \gamma e^{z}|) .$$

Consequently, by Rouché's theorem, $ze^{z}\Psi(z)$ and $z - \gamma e^{z}$ have the same number of zeros with $|z - \zeta_n| < |\delta|$, namely one, for each sufficiently large $|n|$. This establishes the theorem.

We can relax the requirement that $K(0) \neq 0$ to some extent. For a fixed positive integer m, let $K^{(m-1)}(s)$ be absolutely continuous, $K(0) = \ldots = K^{(m-2)}(0) = 0$, $K^{(m-1)}(0) = \gamma \neq 0$, and $K^{(m)}(s)$ essentially bounded. Then m integrations by parts yield

$$\int_0^1 e^{-zs} K(s)\, ds = \gamma z^{-m} - e^{-z}[z^{-1}K(1) + \ldots + z^{-m}K^{(m-1)}(1)] + z^{-m}B(z) ,$$

where $B(z) = O(1/x)$, $x > 0$, as above. Therefore

$$z^m e^z \Psi(z) = z^m - \gamma e^z + z^m h(z) + [z^{m-1}K(1) + \dots + K^{(m-1)}(1)] - e^z B(z) .$$

Consider $z^m - \gamma e^z$. If $\zeta = Ae^{\zeta}$ and $z = m\zeta$, then $\zeta^m = A^m e^{m\zeta}$ and $z^m = (mA)^m e^z$. Let $A = m^{-1}\gamma^{1/m}$ with the principal m^{th} root to obtain $z^m = \gamma e^z$. Therefore, $z^m - \gamma e^z$ has zeros $m\zeta_n$, $n = \pm 1, \pm 2, \dots$, where the ζ_n are as in the Lemma.

Essentially the same argument as in the proof of the theorem shows that $\Psi(z)$ has zeros z_{mn}, $n = \pm N, \pm (N+1), \dots$, such that

$$z_{mn} = m \log |n| + 2\pi imn + O(1) .$$

Again $\Psi(z)$ has zeros of arbitrarily large real part.

We now consider briefly a general differential-difference-integral equation of the form

$$\sum_{j=0}^{p} \sum_{k=0}^{q_j} a_{jk} \varphi^{(k)}(t - t_j) = \int_{t-c}^{t} K(t-s)\varphi(s)\,ds = \int_0^c K(s)\varphi(t-s)\,ds ,$$

where $K(s)$ is integrable, $t_1 < \dots < t_q$ and, without loss of generality, $a_{jq_j} \neq 0$ for $j = 0, 1, \dots, p$. An exponential function $\varphi(t) = e^{zt}$ satisfies the equation if and only if z is a zero of the characteristic function

$$\Psi(z) = \sum_{j=0}^{p} \sum_{k=0}^{q_j} a_{jk} z^k e^{-t_j z} - \int_0^c e^{-zs} K(s)\, ds .$$

Note that

$$z^{-q_0} e^{t_0 z} \Psi(z) = a_{0q_0} + h(z) - z^{-q_0} e^{t_0 z} \int_0^c e^{-zs} K(s)\, ds ,$$

where $a_{0q_0} \neq 0$ and

$$h(z) \to 0 \text{ uniformly in } y \text{ as } x \to \infty .$$

If $t_0 \leqq 0$, then $z^{-q_0} e^{t_0 z} \Psi(z) \to a_{0q_0}$ uniformly in y as $x \to \infty$, so that the zeros of $\Psi(z)$ are bounded above in real part. If $t_0 > 0$ then, under the conditions on $K(s)$ assumed above, $\Psi(z)$ has zeros of arbitrarily large real part. The proof is essentially the same as before.

Similarly, if $t_p \geqq c$, then the zeros of $\Psi(z)$ are bounded below in real part. If $t_p < c$ and $K(s)$ satisfies the above conditions with $s = 0$ replaced by $s = c$, then $\Psi(z)$ has zeros of arbitrarily small (negative, of large absolute value) real part.

With the foregoing results we are able to give a complete description of the asymptotic distribution of the zeros of $\Psi(z)$ for a large class of kernels $K(s)$.

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