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TECHNION RESEARCH AND DEVELOPMENT FOUNDATION – HAIFA, ISRAEL

ON COMPACTIFICATION OF METRIC SPACES

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HAIFA, ISRAEL

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ABSTRACT

Let $f: X \to X^*$ be a homeomorphism of a metric separable space X into a compact metric space X^*, such that $\overline{f(X)} = X^*$. The pair (f, X^*) is then called a metric compactification of X. If X is an absolute G_δ-space (F_σ-space) (i.e. a G_δ set (F_σ-set) in some compact space), then X is said to be of the first kind (cf. [6]) if there exists a compactification (f, X^*) of X such that $X = \bigcap_{i=1}^{\infty} G_i$, where G_i are sets open in X^* and $\dim [Fr(G_i)] < \dim X$, $i = 1, 2, \ldots$ ($Fr(G_i)$ - being the boundary of G_i and $\dim X$ -- the dimension of X). An absolute G_δ-space, (F_σ-space) which is not of the first kind is said to be of the second kind. In the present study spaces X which are both absolute F_σ and absolute G_δ-spaces of the second kind are constructed for any positive finite dimension, a problem related to one of A. Lelek in [11] is solved and a sufficient condition on X is given, under which $\dim [X^* - f(X)] \geq 1$ for any compactification (f, X^*) of X. It is noted also, that an analogous condition assures $\dim [X^* - f(X)] \geq n$.

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INTRODUCTION

Let $f: X \to X^*$ be a homeomorphism of a metric separable space X into a compact metric space X^* such that $\overline{f(X)} = X^*$. The pair (f, X^*) is then called a metric compactification of the metric space X. It is known[1] that for each metric separable space X there exists a homeomorphism $f: X \to J^{\aleph_0}$ of X into the Hilbert cube $J^{\aleph_0}$. Thus denoting $X^* = \overline{f(X)}$ (the closure of $f(X)$ in $J^{\aleph_0}$) we obtain a compactification (f, X^*) of X. It can be shown[2] that there always exists a compactification (f, X^*) such that $\dim X^* \leq \dim X$ where $\dim X$ denotes the dimension of X in the sense of Menger-Urysohn[3]. What can be said about the dimension $\dim (X^* - f(X))$ of the set $X^* - f(X)$ is considered in the present study. This question is closely related to some results obtained by B. Knaster in [6] and A. Lelek in [11][4].

I. SOME COMPACTIFICATIONS OF METRIC SPACES

I.1. Let X be a given topological space. Let $X^* = X \cup (x^*)$, where $x^* \notin X$ is an additional point, and let us define the topology in X^* by taking as open sets all sets open in X and all subsets U of X^*, such that $X^* - U$ is a closed compact subset of X. Then, the theorem of Alexndroff states:

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1) S. [8], p. 119, Theorem 1.
2) S. [4], p. 65, Theorem V, 6. Also [9], p. 72.
3) S. [4], p. 10 and 24. Also [8], p. 162.
4) I learned recently that some problems considered in the present study have been solved by Lelek in an entirely different way. (not published).

(1) The space X^* is a compact topological space and X^* is a Hausdorff space if and only if X is a locally compact Hausdorff space [5]

The space X^* is called the one-point compactification of the space X.

A topological embedding is usually allowed rather than insist that X actually be a subset of X^*. Thus by a compactification of a space X a pair (f, X^*) is understood, such that $f: X \to X^*$ is a homeomorphism of X into a compact space X^* and $\overline{f(X)} = X^*$ (i.e. the image $f(X)$ of X is dense in X^*). In this sense the one-point compactification of a non compact space X is a pair (i, X^*) where $i: X \to X^*$ is the identity mapping and $\overline{i(X)} = X^* = X \cup (x^*)$.

Another compactification of a topological space X is the Stone-Čech compactification $(e, \beta(X))$[6].
This compactification is defined as follows:

Let us take the set $F(X)$ of all continuous functions $f: X \to J$ mapping X into the interval $J = [0,1]$ and the product $J^{F(X)}$ with the Tychonoff topology. Let us define the mapping $e: X \to J^{F(X)}$ by correlating with each point $x \in X$ the point $e(x)$ whose f-th coordinate is $f(x)$, for each $f \in F(X)$. The mapping $e(x)$ is a continuous mapping of X into $J^{F(X)}$, and in the case when X is a completely regular T_1 – space it turns out to be a homeomorphism. In this case we define $\beta(X)$ by $\beta(X) = \overline{e(X)}$ and the pair $(e, \beta(X))$ is called the Stone-Čech compactification of X.

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5) S. [5], p. 150, also [3], p. 73.
6) S. [5], p. 152. For properties of the Stone-Čech compactification, see also [2] and [13].
7) S. [5], p. 153.

Let us note that:

(2) If $(e, \beta(X))$ is the Stone-Čech compactification of a completely regular T_1-space X and $f: X \to Y$ is a continuous mapping of X into a compact Hausdorff space Y, then $f[e^{-1}(x)]$ has a continuous extension on $\beta(X)$ into Y.[7)]

Numerous other compactifications are constructed for various purposes. One of the, used in the dimension theory, is the Wallman compactification $(\Phi, w(X))$. It turns out to be topologically equivalent to the Stone-Cech compactification, if $w(X)$ is a Hausdorff space[8)].

I.2. Considering the one-point compactification (i, X^*) of a metric space, we note that the space X^* is generally not a metric space. For instance, if X is a metric space which is not locally compact, then by (1) X^* cannot be a metric space (since every metric space is a Hausdorff space). Thus if we seek for a given metric space X, a compactification (f, X^*), where X^* is also a metric space, we generally cannot achieve this, by merely adding a single point and should provide for the set $X^* - f(X)$ to contain more than one point.

In the present study we confine ourselves to metric compactifications (f, X^*) of metric separable spaces X only. This means the assumption that X is a separable metric space and X^* a metric space. As already noted, the one-point compactification is generally not a metric compactification. Let us show that an analogous statement holds for the Stone-Čech compactification $(e, \beta(X))$. This will be

7) S. [5], p. 153.

8) Ibidem, p. 168. For properties of the Wallman compactification, [15].

shown by the following

<u>Theorem 1</u>. If X is a non compact metric space and $(e, \beta(X))$ the Stone-Čech compactification of X, then $\beta(X)$ is not a metric space.

<u>Proof</u>. Suppose, to the contrary, that $\beta(X)$ is a metric space. Let $e(X)$ be the image of X in $\beta(X)$. Since X is not compact, there exists a sequence $A = \{a_n\}_{n=1,2,\dots}$ of points $a_n \in X$ which does not contain any convergent subsequence. Consider the points $e(a_n) = b_n$. Since $\beta(X)$ is compact and metric, the sequence $\{b_n\}_{n=1,2,\dots}$ contains a convergent subsequence $\{b'_n\} \subset \{b_n\}$. Let $b'_n \to b \in \beta(X)$ and consider the points $a'_n = e^{-1}(b'_n)$. By $A' = \{a'_n\} \subset A$ the sequence A' does not contain any convergent subsequence. Therefore A' is a closed subset of X. Let us define the real function $f: A' \to J = [0,1]$ by $f(a'_n) = \begin{cases} 0 \text{ for } n=2k \\ 1 \text{ for } n=2k-1 \end{cases} k = 1,2,\dots$.

Since A' does not contain any convergent subsequence, the function $f: A' \to J$ is continuous; and since A' is a closed subset of the metric space X, we can, using Tietze's extension theorem [9], extend this function, to a continuous function $f: X \to J$ (the extended function is denoted also by f). By (2), the function fe^{-1} has then a continuous extension $\tilde{f}$ on the whole of $\beta(X)$. But since $\tilde{f}(b'_n) = fe^{-1}(b'_n) = f(a'_n) = \begin{cases} 0 \text{ for } n=2k \\ 1 \text{ for } n=2k-1 \end{cases}$ and $b'_n \to b$ the function $\tilde{f}$ cannot be continuous at the point b. This contradiction shows that $\beta(X)$ is not a metric space.

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9) S. [8], p. 117.

Remark 1. Since, as noted at the end of Section I.1, the Wallman compactification $(\Phi, w(X))$ is in case of Hausdorff space $w(X)$ topologically equivalent to that of Stone-Čech it follows by Theorem 1 that if X is a non-compact metric space, then the space $w(X)$ is not a metric space.

II. PROBLEMS ON COMPACTIFICATIONS

II.1. The results of Section I indicate that metric compactifications of metric spaces are generally neither the Stone-Čech nor the one-point compactification. Now, since for metric compactifications the set $X^* - f(X)$ generally contains more than one point, there arises a problem of finding the structure of this set for some classes of metric spaces X. For example the following questions can be put:

(a) Is it always possible to find a compactification (f, X^*) of X such that $X^* - f(X)$ would be countable?

(b) Is it always possible to find a compactification (f, X^*) such that $\dim [X^* - f(X)] < \dim X$?

Regarding question (a), it is known that each space which does not contain a subset dense in itself, has a compactification (f, X^*) such that $X^* - f(X)$ is countable[10]. On the other hand, it is easily seen that for each compactification of the set X of rational numbers the set $X^* - f(X)$ is uncountable.

Indeed, since $f: X \to X^*$ is a homeomorphism, each point of $f(X)$ is a limit point and therefore X^* is perfect. Hence X^* is uncountable[11].

10) S. [7], p. 194, IV.

11) S. [3], p. 98.

Regarding (b), it is known, that for each space X, there exists a compactification (f, X^*) such that $\dim X^* = \dim X$ and thus $\dim [X^* - f(X)] \leq \dim X$. Easy examples show that in many cases this weak inequality $\leq$ can be replaced the strong $<$. It suffices, for example to take any n-dimensional cube J^n; $n = 1, 2, \ldots$ and any point $p \in J^n$. The set $X = J^n - (p)$ can be compactified by adding this single point. We then have $X^* = J^n$ and $\dim [X^* - f(X)] = \dim (p) = 0 < \dim X$, where $f = i$ is the identity mapping. On the other hand, it is not always possible to achieve the strong inequality $\dim (X^* - f(X)) < \dim X$. Indeed, for a 0-dimensional space X, $\dim (X^* - f(X)) < \dim X = 0$ means that $X^* - f(X)$ is empty and hence X is compact. It follows that for a 0-dimensional non compact space X this strong inequality is impossible. The problem of finding examples of n-dimensional spaces $X, n > 0$ of a simple topological structure for which $\dim [X^* - f(X)] < \dim X$ does not hold for any compactification (f, X^*) of X is more complicated. More precisely, this problem may be formulated as follows:

(c) Let X be a given n-dimensional space and $k \leq n$ an integer. Under what conditions on X shall we have $\dim [X^* - f(X)] \geq k$ for each compactification (f, X^*) of X?

II.2. B. Knaster discovered in [6] that there exist two kinds of absolute G_δ - spaces (also called G_δ-spaces in compact spaces or topologically complete spaces). Their definition is:

An absolute G_δ - space is said to be of the first kind, if there exists a compactification (f, X^*) such that $f(X) = \bigcap_{i=1}^{\infty} G_i$ and $\dim [Fr(G_i)] < \dim X$, where G_i $i = 1,2,\ldots$ are sets open in X^* and

$Fr(G_i)$ denotes the boundary of G_i in X^*. An absolute G_δ-space is said to be of the second kind if it is not of the first kind.

It was shown by Lelek[12] that

(3) An absolute G_δ-space of finite dimension is of the first kind, if and only if there exists a compactification (f, X^*) of X such that $\dim [X^* - f(X)] < \dim X$.

Now, it was shown in [6] that the Cartesian product $N \times J$, where N is the set of irrational numbers in the interval $J = [0,1]$, is an absolute G_δ-space of the second kind. It was further proved in [11], that if Z is any compact space with $\dim Z = n \geq 0$, then the space $X = N \times Z$ is an absolute G_δ-space of the second kind. These results provide a solution of problem (c) for $n = k$ in the class of finite dimensional absolute G_δ-spaces. The sequel will i.a. include a solution of the following problems:

(a_1) Does there exist, for any positive finite dimension $n = 1,2,\dots$, a finite dimensional space X, which is both an absolute F_σ and G_δ-space of the second kind?

(a_2) Is it true that each absolute G_δ-space X of the second kind, having a positive finite dimension, n, contains a topological image of a set of the form $N \times Z$, where N is the set of irrational numbers of the interval $J = [0,1]$ and $\dim Z = \dim X$?

(a_3) Problem (c), for the case $k = 1$

12) S. [11], p. 31, Theorem 1.

and finally

(a_4) Construction of a weakly infinite dimensional absolute F_σ and G_δ-space of the first kind, such that for each compactification (f, X^*) there is $\dim (X^* - f(X)) = \infty$.[13]

Before proceeding with a solution of problems (a_1) - (a_4), we quote in the next section some facts on coverings.

III. COVERINGS

By covering of a space Y, a family $\mathcal{G} = \{G_i\}$ of sets G_i is understood such that $Y = \bigvee_i G_i$. If G_i are open (closed) sets the covering is called open (closed). If the diameters $\delta(G_i)$ of all G_i are $< \epsilon$, $\mathcal{G}$ is called an ϵ-covering and if $\mathcal{G}$ is finite - a finite covering.

$d_n(Y)$ denotes the infinum of all numbers $\epsilon > 0$ such that there exists a finite open ϵ- covering of Y satisfying

(4) $G_{i_0} \cap G_{i_1} \cap \ldots \cap G_{i_n} = 0$, for any set of $n+1$ indices $i_0 < i_1 < \ldots < i_n$ (i.e., such that the intersection of any $n+1$ different sets G_i is empty).

It is known that for finite coverings of a space Y the existence of an open ϵ- covering satisfying (4) is equivalent to that of a closed ϵ- covering satisfying (4), and that for a compact space Y,

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13) A space is called weakly infinite-dimensional if it is a union of a sequence of finite dimensional spaces X_k, with $\dim X_k \to \infty$; for $k \to \infty$.

dim Y $\leq$ n if and only if $d_{n+1}(Y) = 0$.[14] Let us now prove a property of the Lebesgue number λ of a finite covering.

(5) Let $F_0, F_1, \ldots, F_m$ be a finite family of closed subsets of a compact space Z.

Then there exists a number $\lambda > 0$ (the Lebesgue number of the family $(F_0, F_1, \ldots F_m)$) such that if a point $p \in Z$ is at distance $\leq \lambda$ from all the sets $F_{k_0}, F_{k_1}, \ldots, F_{k_n}$, these sets have a non-empty intersection.

<u>Proof</u>[15] Suppose the contrary. Then there exists a sequence of points $p_0, p_1, \ldots, p_n \in Z$, , $n = 0, 1, 2, \ldots$ and families $S_0 = (F_{k_0^0}, F_{k_1^0}, \ldots, F_{k_{n_0}^0}), \ldots, S_j = (F_{k_0^j}, \ldots, F_{k_{n_j}^j}), \ldots$, of sets such that the point p_j is at distance $\leq \frac{1}{j+1}$ from all the sets $F_{k_i^j}$ of the family S_j, but $\bigcap_{i=0}^{n_j} F_{k_i^j} = 0$. Since the number of different families S_j, $j = 0, 1, \ldots$ constructed from a given finite family of sets $\{F_k\}_{k=0,1,\ldots m}$ is finite, some family – say S_o – must appear in the sequence $\{S_j\}_{j=0,1,\ldots}$ an infinite number of times. Thus there exists a subsequence $\{p'_n\} \subset \{p_n\}$ such that p'_n is at distance $\leq \frac{1}{n+1}$ from all the sets $F_{k_o^o}, \ldots, F_{k_{n_o}^o}$ of S_o. Since Z is compact, the sequence $\{p'_n\}$ contains a convergent subsequence to some point $p \in Z$. Denoting this subsequence by $\{p'_n\}$, we have $p'_n \to p \in Z$. Now, by $\rho(p'_n, F_{k_i^o}) \leq \frac{1}{n+1}$ for $i = 0, 1, \ldots, n_o$ and every $n = 0, 1, \ldots$ and by $p'_n \to p$ we have $\rho(p, F_{k_i^o}) = 0$. Since F_i are closed sets, it follows that $p \in F_{k_i^o}$, $i = 0, 1, \ldots, n_o$ which is incom-

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14) S. [9], p. 60.

15) This is a standard proof and is given here for the sake of completeness only.

patible with the fact that $\bigcap_{i=0}^{s_0} F_{k_i^0} = 0$ (by the definition of S_j).

It follows by (5) that

(6) If Y is a closed subset of a compact space Z and $Y \subset \bigcup_{k=0}^{m} F_k$, where F_k are closed sets such that any different $n+1$ of them have an empty intersection; then, replacing each F_k by its ϵ-neighborhood[16] $G_k = S(F_k, \epsilon)$ (in Z) with $2\epsilon < \lambda$ we get an open (in Z) covering $\mathcal{G} = \{G_k\}$ of the set Y, such that for the family $\{\bar{G}_k\}$ of closures of G_k, any $n+1$ different sets $\bar{G}_k$ have also an empty intersection[17].

Another consequence of (5) is;

(7) If the closed sets $F_0, F_1, \ldots, F_m$ in a compact space Z have an empty intersection: $\bigcap_{k=0}^{m} F_k = 0$, then, there exists a number $\epsilon > 0$ such that no set of diameter $\leq \epsilon$ has a non empty intersection with each of the sets $F_0, F_1, \ldots, F_m$.

Indeed, it suffices to take $\epsilon = \dfrac{\lambda}{2}$ and to apply (5).

We shall now give some properties of coverings of simplexes.

Let $\sigma^s = (p_0, \ldots, p_s)$ be a closed s-dimensional simplex with vertices $p_0, p_1, \ldots, p_s$ in the Euclidean s-dimensional space E^s and let $f: \sigma^s \to Z$ be a homeomorphism of σ^s into a space Z. Let $\sigma^{s-1,i}$ denote the $(s-1)$ dimensional closed face of σ^s opposite to the vertex $p_i \in \sigma^s$, i.e.

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16) An ϵ-neighborhood of a set F is by definition the union over all $p \in F$ of the sets $S_p = \{z;\ \rho(p, z) < \epsilon;\ z \in Z\}$

17) For a proof of (6) see also [14], p. 414, Lemma 2 and [10], p. 257.

$\sigma^{s-1,i} = (p_o, \ldots, p_{i-1}, p_{i+1}, \ldots, p_s)$ $i = 0,1, \ldots, s$, and let $\tau^s = f(\sigma^s)$ and $\tau^{s-1,i} = f(\sigma^{s-1,i})$. Then τ^s is a curvilinear simplex with vertices $q_i = f(p_i)$ and $(s-1)$-dimensional faces $\tau^{s-1,i}$, $i = 0,1,\ldots, s$. Since f is a homeomorphism and $\bigcap_{i=o}^{s} \sigma^{s-1,i} = 0$, we have that $\bigcap_{i=o}^{s} \tau^{s-1,i} = 0$. Thus applying (7) with $m = s$ to the closed sets $F_i = \tau^{s-1,i}$, we find that there exists a number $\epsilon > 0$ such that no set with diameter $\leq \epsilon$ intersects each of the faces $\tau^{s-1,i}$.

Let now $\epsilon > 0$ be this number and let us show that

(8) Let $\epsilon > 0$ be a number such that no set with diameter $\leq \epsilon$ intersects each face $\tau^{s-1,i}$. Let further $\tau^s = \bigcup_{k=o}^{m} F_k$, where F_k are closed sets with diameters $\delta(F_k) \leq \epsilon$, $k = 0,1,\ldots m$. Then some $s+1$ sets $F_{k_o}, \ldots, F_{k_s}$ have a non empty intersection.

Since $\delta(F_k) \leq \epsilon$, no F_k containing a vertex q_j of τ^s intersects the face $\tau^{s-1,j}$ opposite to q_j. Since f is one-to-one, no set $f^{-1}(F_k)$ containing a vertex p_j of σ^s intersects the face $\sigma^{s-1,j}$ opposite to p_j. Now, the sets $f^{-1}(F_k)$ $k = 0,1,\ldots m$ cover the simplex σ^s and are closed, since f is continuous. Thus applying the same procedure as in the proof of [2,24] in [1], p. 194 we obtain that some $s+1$ sets $f^{-1}(F_{k_j})$, $j = 0,1,\ldots s$ have a non empty intersection. Hence also the sets F_{k_j}, $j = 0,1\ldots s$ have a non empty intersection.

IV. THE SOLUTION OF PROBLEMS FORMULATED IN II

IV. 1. An n-dimensional absolute F_σ and G_δ-space X and its properties.

Let $\sigma^n = (p_0, p_1, \dots p_n)$ be the n-dimensional closed simplex in the n-dimensional Euclidean space E^n with vertices $p_o = \underbrace{(0,0,\dots,0)}_{n}$ and $p_i = (0,\dots 0, \underset{i}{1}, 0,\dots 0)$, $i = 1,2,\dots,n$ (i.e. p_i is the point in E^n whose i-th coordinate is 1 and all other coordinates are 0). Let $A = \{a_j\}$ $j = 1,2,\dots$ be the sequence of points of the form $a_j = \frac{1}{j}$, $j = 1,2,\dots$ on the real axes E^1 and let $a_o = 0 \in E^1$. Denote by $Fr(\sigma^n) = \bigcup_{i=0}^{n} \sigma^{n-1,i}$ the boundary of the simplex σ^n.

Define

(9) $X = (A \times \sigma^n) \cup [(a_o) \times Fr(\sigma^n)]$

We have $X \subset E^{n+1}$ and the closure $\bar{X}$ of X in E^{n+1} is $\bar{X} = (A \times \sigma^n) \cup [(a_o) \times \sigma^n] = [A \cup (a_o)] \times \sigma^n$. Since $\bar{X}$ is a compact subset of E^{n+1} (as a product of two compact spaces $A \cup (a_o)$ and σ^n), $\bar{X}$ is a compact space, and since X can be written as a union $[(a_o) \times Fr(\sigma^n)] \cup [\bigcup_{j=1}^{\infty} (a_j) \times \sigma^n]$ of a countable number of compact sets, it follows that X is an absolute F_σ space.

On the other hand the set $\bar{X} - X$ equals the interior of the simplex $(a_o) \times \sigma^n$. Since this interior is a union of compact sets, the set $\bar{X} - X$ is an F_σ set and therefore X is a G_δ - set in $\bar{X}$. It follows that

(b_1) The set X defined in (9) is both an absolute F_σ and G_δ-space. Evidently, dim X = n.

We shall now show that

(b_1') For each compactification (f, X^*) of X there is $\dim [X^* - f(X)] \geq \dim X = n$.

Indeed, suppose to the contrary that $\dim [X^* - f(X)] \leq n - 1 < \dim X$ and take the sets $r_j^n = f[(a_j) \times \sigma^n]$, $r_j^{n-1,i} = f[(a_j) \times \sigma^{n-1,i}]$ $i = 0,1,\ldots,n$, $j = 0,1,\ldots$ By $a_j \to a_o$ for $j \to \infty$, we have that for every $i = 0,1,\ldots n$, $\operatorname{dist}\{[(a_j) \times \sigma^{n-1,i}], [(a_o) \times \sigma^{n-1,i}]\} \to 0$ if $j \to \infty$, where $\operatorname{dist}(A,B) = \max[\sup_{x \in A} \rho(x,B), \sup_{x \in B} \rho(A,x)]$ is the distance of the sets A and B in the sense of Hausdorff[18]. Since $f: X \to X^*$ is a homeomorphism, and $[A \cup (a_0)] \times Fr(\sigma^n)$ is compact it follows that

(10) $\operatorname{dist}(r_j^{n-1,i}, r_o^{n-1,i}) \to 0$ for $j \to \infty$ and each $i = 0,1,\ldots n$.

Now the space X^* being compact, there exists a subsequence $\{j'\}$ of $\{j\}$ such that the sequence of sets $\{r_{j'}^n\}$ converges to a continuum $C \subset X^*$[19]. Writing j instead of j', we have $\operatorname{dist}(r_j^n, C) \to 0$ for $j \to \infty$. Since f is one-to-one, it follows that $C \cap [\bigcup_{j=1}^{\infty} r_j^n] = 0$, and since the set $\bigcup_{i=o}^{n} r_o^{n-1,i}$ is an $(n-1)$-dimensional compact subsets of C, we have by the assumption $\dim [X^* - f(X)] \leq n - 1$ and Corollary 1, in [4], p. 32, that $\dim C \leq n - 1$. Thus by the definition of $d_n(Y)$ (Cf. section III) we obtain $d_n(C) = 0$. Hence, by (6), there exists for every $\epsilon > 0$ an ϵ-covering of C by sets G_k open in X^*, $k = 0,1,\ldots m$ such that

(11) $\bar{G}_{k_o} \cap \bar{G}_{k_1} \cap \ldots \cap \bar{G}_{k_n} = 0$ for any set of subscripts $k_o < k_1 < \ldots \leq k_n$.

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18) S. [8], p. 106

19) S. [9], p. 110. Also [16], p. 11.

Now, since $\bigcap_{i=0}^{n} r_0^{n-1,i} = 0$, we can by (7), choose for this covering an ϵ so small that no $\bar{G}_k$ intersects each set $r_0^{n-1,i}$. Hence by (10) no set $\bar{G}_k$ intersects all the faces $r_j^{n-1,i}$, $i = 0,1,\dots n$ for sufficiently large j. Let $G = \bigcup_{k=0}^{m} G_k$. By $C \subset G$ and dist $(r_j^n, C) \to 0$ for $j \to \infty$ there exists a j_o such that $r_j^n \subset G$ for $j \geq j_o$. Fixing any $j \geq j_o$, we find that the sets $F_k = r_j^n \cap \bar{G}_k$ $k = 0,1,\dots,m$ satisfy the assumptions of (8) with s replaced by n and r by r_j. Hence by (8) some n+1 sets $F_{k_o}, \dots, F_{k_n}$, and therefore also the sets $\bar{G}_{k_o}, \dots \bar{G}_{k_n}$ have a non empty intersection, which is incompatible with (11). Thus (b'_1) is proved.

By (b_1), (b'_1) and (3) we obtain

Theorem 2. The set X defined in (9) is both an absolute F_σ and G_δ-space of the second kind and of dimension n.

This theorem gives an answer to problem (a_1).

IV. 2. On a problem of A. Lelek.

The following problem p.313 in [11], p. 34 was formulated by Lelek.

Does there exist, for each absolute G_δ-space X of the second kind with finite, positive dimension, a compact space Z with positive dimension, such that X contains a topological image of the set $N \times Z$ (N being the set of irrational numbers of the interval $J = [0,1]$) ?

A negative answer to this question was given in [12]. Now it is easily seen that a negative answer to problem (a_2) posed in section II contains as a special case, a negative answer to that of Lelek. (It suffices to take in (a_2) $n = \dim X = 1$). We now proceed to prove that the answer to (a_2) is negative.

Indeed, let X be the space defined in (9). We shall show that there does not exist a space Z with $\dim Z = \dim X = n$ such that $N \times Z$ has a topological image in X.

Suppose, to the contrary, that such a space Z exists and let $h: N \times Z \to X$ be a homeomorphism of $N \times Z$ into X. Fix a point $\xi \in N$. Then the n-dimensional space $(\xi) \times Z$ has a topological image in X. Now X being a countable union of compact disjoint sets $(a_j) \times \sigma^n$ and $(a_0) \times \mathrm{Fr}(\sigma^n)$, $j = 1,2,\ldots$ and $(\xi) \times Z$ being n-dimensional, it follows that $h[(\xi) \times Z]$ has an n-dimensional intersection with some set $(a_{j(\xi)}) \times \sigma^n$ [19']. This intersection, as n-dimensional subset of σ^n, contains an open subset of $(a_{j(\xi)}) \times \sigma^n$ [20]. Since h is one-to-one, the sets $h[(\xi) \times Z]$ and $h[(\xi') \times Z]$ are disjoint for $\xi \neq \xi'$, $\xi, \xi' \in N$ and since N is uncountable, we get an uncountable family of disjoint open sets contained in X, which is impossible.

IV. 3. A theorem on compactification.

We shall now prove a theorem with help of which it will be possible to construct for any $n = 1, \ldots \aleph_0$, a n-dimensional space X which is not locally compact at a single point and such that for each

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19') This is a consequence of the Sum Theorem for Dimension n, Cf. [4], p. 30.

20) This follows easily from Theorem IV, 3 in [4], p. 44.

compactification (f, X^*) of X there is $\dim (X^* - f(X)) \geq 1$.

Theorem 3. Suppose that the space X contains a sequence $\{C_i\}_{i=1,2,\ldots}$ of continua C_i and a point p such that

(c_1) the sets C_i are closed and open in the union $\bigcup_{i=1}^{\infty} C_i$ and disjoint $C_i \cap C_j = 0$ for $i \neq j$

(c_2) there exists a number $\delta > 0$, such that for each $i = 1,2,\ldots$ the diameters $\delta(C_i) \geq \delta$

and

(c_3) $\overline{\bigcup_{i=1}^{\infty} C_i} - \bigcup_{i=1}^{\infty} C_i = (p)$.

Then X is not locally compact at the point p, and for each compactification (f, X^*) of X there is $\dim (X^* - f(X)) \geq 1$.

Proof. Let U_p be an arbitrary neighborhood containing the point p. We have to show that the closure $\bar{U}_p$ is not compact. By (c_3) there exists a sequence of points $p_i \in \bigcup_{i=1}^{\infty} C_i$ such that $p_i \to p$ for $i \to \infty$ and such that the sequence $\{p_i\}_{i=1,2,\ldots}$ has only a finite number of points in common with each C_i. Thus we may assume, that for each $i = 1,2,\ldots$ there is $p_i \in C_i$. Let $S = S(p, r)$ be a spherical neighborhood of p with radius $r < \frac{\delta}{2}$ contained in U_p. By $p_i \to p$, the sets $C_i \cap S$ are not empty for i sufficiently large and since C_i are connected, we get, by (c_2), that for these i there is $C_i \cap Fr(S) \neq 0$, where $Fr(S) = \{q; \rho(p,q) = r, q \in X\}$ is the boundary of S. Choose from each such set $C_i \cap Fr(S)$ a point q_i and consider the sequence $\{q_i\}$. Since $\bar{S} \subset \bar{U}_p$, we have $\{q_i\} \subset \bar{U}_p$ and since $q_i \in Fr(S)$, there is $\rho(q_i, p) = r > 0$. Now, by $q_i \in C_i$ for i sufficiently large, (c_1) and (c_3), any convergent subsequence of $\{q_i\}$ tends to p, which is impossible by $\rho(q_i, p) = r > 0$. Thus $\bar{U}_p$ is not compact. It remains to show that if (f, X^*) is any compactification of X, then $\dim[X^* - f(X)] \geq 1$. For this purpose let us consider the sets $X_1 = \bigcup_{i=1}^{\infty} C_i \cup (p)$ and $f(X_1)$. The closure $\overline{f(X_1)} = X_1^* \subset X^*$ is a compactification of X_1. Let y be any point of $X_1^* - f(X_1)$. Then the point $y \notin f(X)$. Indeed, if there would exist a point $x \in X$ such that $y = f(x)$ then there would be $x \notin X_1$, since f is one-to-one. Now by $y \in \overline{f(X_1)}$ there exists a sequence of points $x_n \in X_1$ such that $f(x_n) \to y$. Thus by the continuity of f^{-1} it should be $x_n \to x \in X - X_1$. But by (c_2) the set X_1 is closed in X, and since $x_n \in X_1$ it follows that $x \in X_1$. This contradiction shows that $y \notin f(X)$. Thus

(12) $[X_1^* - f(X_1)] \cap f(X) = [\bar{X}_1 - f(X_1)] \cap f(X) = 0$

Let us take further $r < \frac{\delta}{2}$ and construct (analogously with the first part of the proof) points $p_i \to p$, $p_i \in C_i$ and $q_i \in C_i$, such that $\rho(p, q_i) = r > 0$ for i sufficiently large. Since $X_1^* = \overline{f(X_1)}$ is compact and $f(C_i) \subset X_1^*$ we can choose a subsequence of the sequence $\{f(C_i)\}$ of continua converging to some continuum C[21]. Denoting the subscripts of this subsequence by i we have therefore that dist $[f(C_i), C] \to 0$ for $i \to \infty$. Now, by $p_i \to p$, $p_i \in C_i$, it follows that C contains the point f(p). If C would reduce to this point f(p), then by $q_i \in C_i$ there would be $f(q_i) \to f(p)$ and since f^{-1} is continuous there would also be $q_i \to p$, in contradiction to $\rho(p, q_i) = r > 0$. It follows that C contains at least two points, and since it is a continuum we have dim $C \geq 1$. Therefore $\dim[C - (f(p))] \geq 1$.

21) S. [9], p. 110.

Now, by (c_1) we have $C \wedge f(C_i) = 0$ for each $i = 1,2, \ldots$ Therefore by $X_i^* \subset X^*$ and (12) it follows that $\dim [X^* - f(X)] \geq 1$. Theorem 3 is proved.

Remark 2. In a quite analogous way one could prove that

If the space X contains topologically the set defined by (9) and $\overline{A \times \sigma^n} - A \times \sigma^n = (a_o) \times Fr(\sigma^n)$, then for each compactification (f, X^*) of X there is $\dim [X^* - f(X)] \geq n$. (For $n = 2$, see Fig. 3).

Example 1. Let $X = (a_o) \cup [\bigcup_{j=1}^{\infty} (a_j) \times J]$ where $a_o = 0$ and $a_j = \frac{1}{2^{j-1}}$, $j = 1,2, \ldots$ are real numbers on the real axes and $J = [0, 1]$ (S. Fig. 1). This 1-dimensional space X is not locally compact at the single point $a_o = 0$, and by Theorem 3 $\dim [X^* - f(X)] \geq 1$ for any compactification (f, X^*) of X. It is also easily seen that X is an absolute F_σ and G_δ-space and thus, by (3) and $\dim X = 1$, we obtain that X is an absolute F_σ and G_δ-space of the second kind.

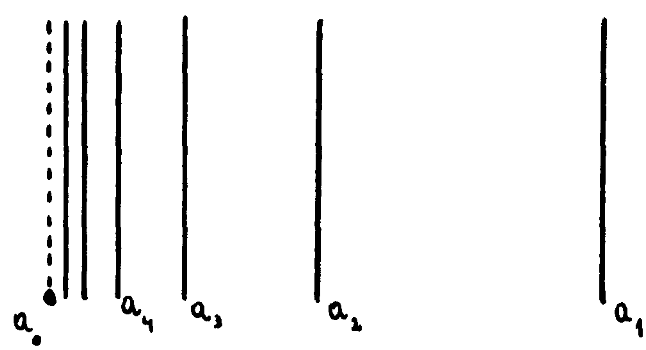
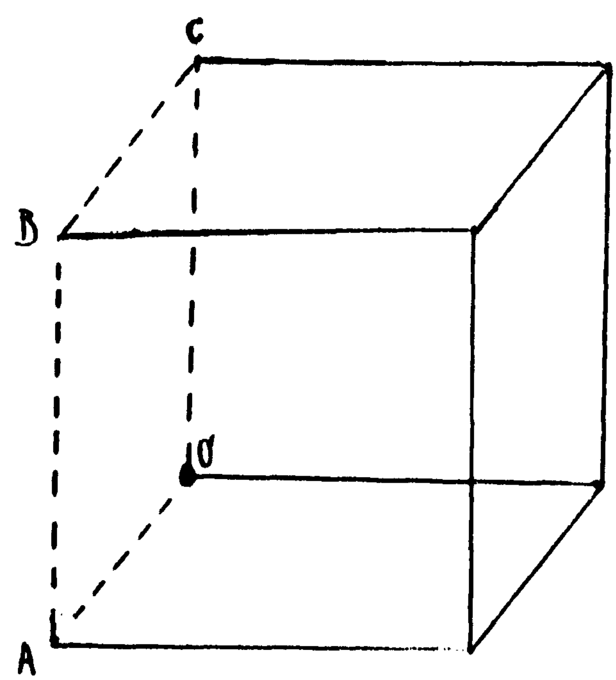


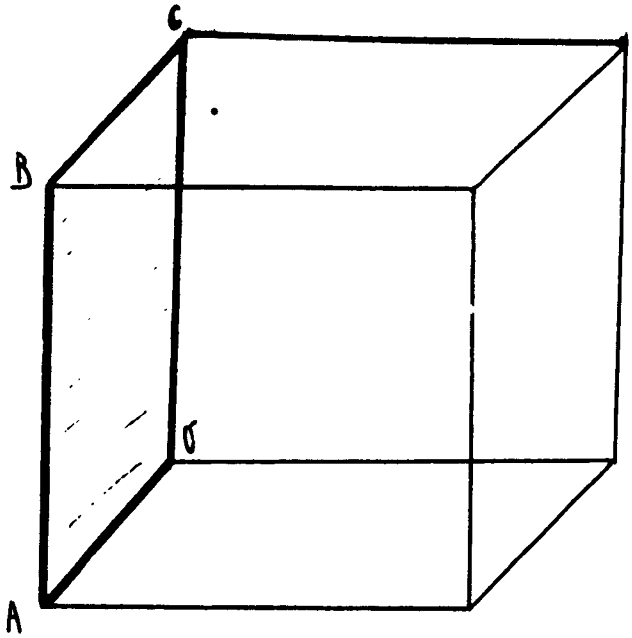
Fig. 1

<u>Example 2</u>. Let $n = 2,3,\dots \aleph_0$ and let $X = (J^n - X_1) \cup (0)$, where $X_1 = \{x;\ x = (x_1, x_2, \dots, x_n),$ $x_1 \neq 0;\ 0 \leq x_i \leq 1,$ for $i = 2,3,\dots n\}$ and $0 = \underbrace{(0, 0, \dots 0)}_{n}$ (If $n = \aleph_0$, J^n is the Hilbert cube). It is clear that $\dim X = n$, and that X is not locally compact at the single point $0 = \underbrace{(0,0,\dots 0)}_{n}$. It is also easy to construct a sequence C_i of continua in X, such that the assumption s of Theorem 3 be satisfied for the point $p = \underbrace{(0,0,\dots 0)}_{n}$. Hence $\dim\ [X^* - f(X)] \geq 1$ for any compactification (f, X^*) of X (for $n = 3$, see Fig. 2).



For each compactification (f, X^*) of this full cube X excluding the full square OABC but including point O, $\dim\ [X^* - f(X)] \geq 1$.

Fig. 2.



According to Remark 2, for each compactification (f, X^*) of this full cube X excluding the interior of the square OABC (but including OA, AB, BC and CO) $\dim [X^* - f(X)] \geq 2$.

Fig. 3.

IV.4. A weakly infinite-dimensional absolute F_σ and G_δ-space

As stated in (3), a finite dimensional absolute G_δ-space X is of the first kind if and only if there exists a compactification (f, X^*) of X such that $\dim (X^* - f(X)) < \dim X$.

We shall now show that the above condition is not necessary for infinite dimensional spaces. More precisely, we shall construct an absolute F_σ and G_δ-space of the first kind which is weakly infinite-dimensional and such that for each compactification (f, X^*) of X, there is $\dim [X^* - f(X)] = \infty$. Let us take, for fixed n, the set of point $x_{n,m} = \frac{1}{2^n} + \frac{1}{2^m}$, $m = n+1, n+2, \dots$ on the real axes, and let $A_n = \bigcup_{m=n+1}^{\infty} (x_{n,m})$. Define $X_n = (A_n \times \sigma^n) \cup [(\frac{1}{2^n}) \times \mathrm{Fr}(\sigma^n)]$ where σ^n is a n-dimensional closed simplex with diameter $\delta(\sigma^n) = \frac{1}{2^n}$, and $\mathrm{Fr}\,(\sigma^n)$ is the boundary of σ^n. The set X is then defined by

(13) $X = \bigcup_{n=1}^{\infty} X_n$

The set X can be considered as a subset of the Hilbert cube $J^{\aleph_0}$, and the closure $\bar{X}$ equals $\bar{X} = \bigcup_{n=1}^{\infty} X_n \cup [\bigcup_{n=1}^{\infty} [(\frac{1}{2^n}) \times \mathrm{Int}\,(\sigma^n)] \cup (0)$ where $\mathrm{Int}\,\sigma^n = \sigma^n - \mathrm{Fr}\,(\sigma^n)$ and $0 = (0,0,\dots)$ is the point all whose coordinates are zero. It is also easily seen that $\bar{X}$ may be written in the form $\bigcup_{n=1}^{\infty} \tilde{X}_n \cup (0)$, where $\tilde{X}_n = [A_n \cup (\frac{1}{2^n})] \times \sigma^n$. Since $\bar{X}$ is a compact space and X is a countable union of compact sets, we find that X is an absolute F_σ-space. Further, we can write each set $(\frac{1}{2^n}) \times \mathrm{Int}\,(\sigma^n)$ as a union $\bigcup_{i=1}^{\infty} F_i^n$ of compact sets F_i^n, $i = 1,2,\dots$ Thus $\bar{X} - X = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F_i^n \cup (0)$ is an F_σ set and thus X is an absolute G_δ-space. Moreover, the sets $\bar{X} - [\bigcup_{n=1}^{s} \bigcup_{i=1}^{s} F_i^n \cup (0)] = G_s$ are open in $\bar{X}$, $\dim\ [\mathrm{Fr}\,(G_s)] \le s$ and $\bigcap_{s=1}^{\infty} G_s = X$. Hence, X is an absolute F_σ and G_δ-space of the first kind. By the definition of X, it follows that X is a weakly infinite-dimensional space i.e.

dim $X = \infty$.[22]

We shall now show that for each compactification (f, X^*) of X there is dim $[X^* - f(X)] = \infty$. For this purpose, let us note that the set X_n is homeomorphic with the space difined in (9), and hence by (b'_1) there is dim $[X_n^* - f(X_n)] \geq$ dim $X_n = n$ for each compactification (f, X_n^*) of X_n. Now it is easily seen that

(14) $\overline{f(X_n)} \cap f(X - X_n) = 0$

where $\overline{f(X_n)}$ is the closure of $f(X_n)$ in X^*.

Indeed, suppose to the contrary that the set in (14) is not empty and let $y \in \overline{f(X_n)} \cap f(X - X_n)$. We have $f(X - X_n) = \bigcup_{k \neq n} f(X_k)$. Then $y = f(x)$ where $x \in X_k$ for some $k \neq n$. Since $y \in \overline{f(X_n)}$, there exists a sequence $\{y_i\}_{i=1,2,\ldots}$ such that $y_i \to y$ and $y_i = f(x_i)$ with $x_i \in X_n$. Since f^{-1} is continuous, it follows by $y = f(x)$ that $x_i \to x$. This is impossible, since $x_i \in X_n$, $x \notin X_n$ and X_n is a closed (also open) set in X.

Now $X_n^* = \overline{f(X_n)}$ is a compactification of X_n and therefore, by dim $[X_n^* - f(X_n)] \geq$ dim $X_n = n$ and (14), we have that dim $[X^* - f(X)] \geq n$. Since n is arbitrary, it follows that dim $[X^* - f(X)] = \infty$.

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22) For weakly infinite-dimensional spaces X, dim $X = \omega$ is sometimes written instead of dim $X = \infty$.

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of the second kind are constructed for any positive finite dimension, a problem related to one of A. Lelek in [11] is solved and a sufficient condition on X is given, under which $\dim [X^* - f(X)] \geq 1$ for any compactification (f, X^*) of X. It is noted also, that an analogous condition assures $\dim [X^* - f(X)] \geq n$.

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