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NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS: AN ANNOTATED BIBLIOGRAPHY

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Compiled by
GEORGE R. EVANS

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Lockheed

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NOTICE

ABSTRACT

The primary area of interest of the search is the existence and uniqueness of solution to ordinary differential equations for the following two classes of problems:

1. Initial Value Problems
2. Boundary Value Problems

Also included are actual construction of solutions.

Excluded are stability of solutions and equations with periodic solutions, or equations with periodic coefficients.

Arrangement is chronological and the period covered is 1949–October 1962. It is suggested that for literature prior to 1949 the bibliography by da Silva Dias be consulted (Citation No. 18).

Search completed October 1962.

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INTRODUCTION

The growing necessity for scientists and technologists to deal with nonlinear phenomena has provided motivation for a vast effort in the study of systems of nonlinear differential equations. Published results are scattered through dozens of scientific journals, both foreign and domestic. There has been some effort by prominent investigators such as Lefschetz in the U. S. to keep abreast of the literature by compiling bibliographies and reviews of the more important papers. For the most part, these reviews have been confined to considerations such as stability which are of great importance in nonlinear mechanics and in communication and control theory, where initial value problems are common.

Compilation of the present bibliography was motivated by an interest in problems of fluid mechanics, where one often has to deal with boundary value problems in which one or more boundary conditions are applied at critical points of the system of differential equations. Examination of the pattern of integral curves in the neighborhood of the critical points is a powerful method of proving existence and uniqueness of solutions to such problems. The present bibliography is therefore oriented more toward the areas dealing with the detailed topological structure of integral curves and their asymptotic behavior near critical points of familiar two-dimensional systems of ordinary differential equations and of systems of higher dimension.

A number of papers also deal with proofs of existence and uniqueness of solutions near regular points. By the nature of the subject, the papers listed are predominantly mathematical in tone, although some are clearly written for the engineer or physicist.

Table of Contents

1. Al'muhamedov, M. I.
On conditions for the existence of stable and unstable centers. DOKLADY AKAD. NAUK SSSR. N.S. 67:961–964 (1949). (In Russian)

This paper contains various necessary and sufficient conditions for a center for the differential equation in polar coordinates

$$\frac{d\rho}{d\omega} = \frac{\rho^2\varphi_1 + \rho^3\varphi_2 + \ldots + \rho^{n+1}\varphi_n}{1 + \rho\psi_1 + \rho^2\psi_2 + \ldots + \rho^n\psi_n} \quad (1)$$

where ρ_s, ψ_s are polynomials in $\sin\omega$, $\cos\omega$. In particular one may reduce to (1) the equation

$$dy/dx = - \{x + P_n(x, y)\}/\{y + Q_n(x, y)\}, \quad (2)$$

where P_n, Q_n are polynomials of degree n beginning with terms of degree at least two. It is well known that for (1) the origin is a focus or a center. To distinguish between them the author endeavors to find a closed curve γ, (3) $\gamma + \gamma^2\theta_1 + \ldots + \gamma^{m+1}\theta_m = \epsilon$, where γ is the polar distance, the θ_s polynomials in $\sin\omega$, $\cos\omega$, ϵ a small positive constant, and where along γ the difference $\rho'(\omega) - \gamma'(\omega)$ has a fixed sign. Thus γ is a curve without contact in the sense of Poincaré. If a suitable curve γ exists the origin is a focus; otherwise it is a center. Furthermore the number of conditions necessary to determine which of the two occurs is finite. The author's results include as special cases those of Poincaré [Oeuvres, v. 1, p. 95] and Liapounoff [Problème Générale de la Stabilité du Mouvement, reprinted as Ann. of Math. Studies, no. 17, Princeton University Press, 1947].

2. Barbuti, Ugo.
Una proprietà che caratterizze l unicità della soluzione delle equazioni differenziali ordinarie del primo ordine. ATTI ACCAD. NAZ. LINCEI. REND. CL. SCI. FIS. MAT. NAT. 6(8):298–303 (1949) (In Italian)

The author gives a necessary and sufficient condition that a solution $y = f(x)$ of the differential equation $y' = F(x', y)$ is unique. Although the condition itself is too detailed to state here, the proof is quite short, and the result may be easily shown to include known criteria for unicity due to Peano and Perron.

3. El'sin, M. I.
The phase method and the classical method of comparison. DOKLADY AKAD. NAUK SSSR N.S. 68:813–816 (1949). (In Russian)

Generalizing the transformation which removes the coefficient of x' in (1): $x'' + p(t)x' + q(t)x = 0$ the author obtains the "characteristic operator"

$$J(\theta, p, q) = (\theta - p/2)' + \theta^2 + q - p^2/4$$

previously considered [same vol., 221–224 (1949)]. This operator is then used to deduce results concerning the oscillation of the solutions of (1).

4. Hukuhara, Masuo
On singular points of the ordinary differential equation of first order. MEM. FAC. SCI. KYŪSYŪ UNIV. A. 4:9–21 (1949). (In Esperanto)

Given an equation of the form $x^{a+1}y' = f(x, y)y$, a being a positive integer, $f(x, y)$ regular in the neighborhood of $(0, 0)$, $f(0, 0) \neq 0$, and the formal series satisfying the equation, an important problem is that of establishing the region of convergence of the series. This the author accomplishes under certain assumptions too detailed to relate here, by a combination of analysis and the method of the Tychonoff fixed-point theorem.

5. Hukuhara, Masuo
On the expansion of the solution of differential equations in the neighborhood of their singular point. MEM. FAC. SCI. KYŪSYŪ UNIV. A. 4:1–7 (1949). (In Esperanto)

The problem is that of finding formal solutions of the equations $xy' = y^{m+1}z^n f(x, y, z)$, $xz' = y^m z^{n+1} g(x, y, z)$, where $f(x, y, z)$, $g(x, y, z)$ are regular in the neighborhood of $x = y = z = 0$ and $f(0, 0, 0)$, $g(0, 0, 0)$ are both nonzero, having the form $y = u(1 + \Sigma' p_{ijk}x^i u^j v^k)$, $z = v(1 + \Sigma' q_{ijk}x^i u^j v^k)$. Several different cases occur depending upon whether a/b is rational or irrational, positive or negative rational.

6. Hukuhara, Masuo

Sur la généralisation des théorèmes de M. J. Malmquist. J. FAC. SCI. UNIV. TOKYO. SECT. I. 6:77–84 (1949)

J. Malmquist has treated the equation $x^{\alpha}y^{\beta}y' = f(x, y)$, where α, β are positive integers, in the neighborhood of $(0, 0)$ [Acta Math. 74, 175–196 (1941)]. The present paper gives existence theorems generalizing his results to the case where α, β are real constants satisfying simply $1 \geq \alpha, -1 < \beta$.

7. LaSalle, J.

Uniqueness theorems and successive approximations. ANN. OF MATH. (2)50:722–730 (1949).

The author considers the system

$$y_i' = f_i(x, y_1, \ldots, y_n) = f_i(x, y)$$

in one independent and n dependent variables. Let G be the class of functions $g(x)$ continuous on $0 \leqq x \leqq a$ and vanishing for $x = 0$. Let the f's be continuous in a region defined by $0 \leqq x \leqq a$, $l_i(x) \leqq y_i \leqq u_i(x)$, where $l_i(x)$, $u_i(x)$ belong to G. An approximating sequence $T^m(g(x))$ is generated by replacing y by a vector $g(x)$ with components from G and integrating $f(\sigma, g(\sigma))$. If T^m converges uniformly to a solution, $g(x)$ is called a zero approximation. If the f's are non-decreasing functions of the y's and if $l_i'(x) \leqq f_i[x, l(x)]$, $f_i[x, u(x)] \leqq u_i'(x)$, then $l(x)$ and $u(x)$ are zero approximations. For the case where the f's satisfy a generalized Lipschitz condition of considerable complexity the paper proves a uniqueness theorem which includes those of Osgood, Montel and Nagumo and also a theorem which specifies a subset of G composed of zero approximations and generalizes a theorem of Wintner [Amer. J. Math. 68, 13–19 (1946)]. For the case of more than one solution a result which leads to bounds on the difference of solutions is given.

8. Ważewski, Tadeusz
Sur les intégrales d'un système d'équations différentielles tangentes aux hyperplans caractéristiques issue du point singulier.
ANN. SOC. POLON. MATH. 21(1948),277–297 (1949)

On trouvera dans ce travail un théorème très général relatif aux solutions du système $du_i/dt = \varphi^i(u_1, \ldots, u_n)$ (où $i = 1, \ldots, n$) quand on suppose les φ nulles pour $u_1 = \ldots = u_n = 0$, continues au voisinage de ce point (Θ), et possédant la différentielle de Stoltz en (Θ). Pour donner une idée de ce théorème dans un cas particulier, supposons que les racines caractéristiques de la matrice $\| \varphi^i_{u_j} \|$ soient réelles et distinctes: $s_1 < \ldots < s_n$, et désignons par $d_1, \ldots, d_n$ les "droites caractéristques" correspondant respectivement à ces racines; si pour un certain i on a $s_i < 0$, alors la totalité des intégrales qui pour $t = +\infty$ sont en Θ tangentes à d_i se projette sur le plan passant par $d_1, \ldots, d_i$ comme un ensemble a i dimensions. Le théorème général est démontré en utilisant un hypothèse d'unicité des solutions correspondant à certaines données; mais on peut se passer de cette hypothèse en recourant à un principe topologique indiqué par l'auteur dans deux publications antérieures [also in Ann. 20 (1947), 279–313 (1948); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 210–215 (1947].

9. Dramba, C.
Sur les multiplicités singulières des systèmes différentielles. ACAD. REPUB. POP. ROMANE. STUD. CERC. MAT. 1:162–168 (1950). (In Romanian. Russian and French summaries)

The author considers differential equations

$$dx_i = A_i(x_1, \ldots, x_n)dt \quad (i = 1, \ldots, n).$$

It is assumed that the functional determinant of the functions A_i has rank $n - k$ and that all functions A_i vanish at the origin; there is then in general a k-dimensional variety M of singular points through the origin. It is shown how the solutions can be expressed in terms of k constants, which locate position on M and $n - k$ constants, which describe integral surfaces through the singular points.

10. Duff, G. F. D.

Limit cycles of systems of the second order.

PROC. NAT. ACAD. SCI. U. S. A. 36:749–752 (1950).

The limit cycles of the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ are studied by the device of imbedding this system in what the author calls a "complete family" of systems $\dot{x} = P(x, y, \alpha)$, $\dot{y} = Q(x, y, \alpha)$. Such a family can essentially be characterised by the properties that $P(x, y, \alpha + \pi) = -P(x, y, \alpha)$, $Q(x, y, \alpha + \pi) = -Q(x, y, \alpha)$, and that the vector with components P, Q turns in the positive sense as α increases. The following theorems concerning complete families are stated. (1) Limit cycles corresponding to different values of α do not intersect. (2) Limit cycles which are completely stable or completely unstable expand or contract monotonically as α varies in a fixed sense, while semistable cycles either split into two diverging cycles or disappear. (3) Each elementary critical point of positive determinant either generates or absorbs exactly one cycle in the range $0 \leqq \alpha \leqq \pi$. It is then shown how these theorems and some of their consequences can be used to gain information on the location, or on the nonoccurrence, of limit cycles in certain regions of the (x, y)-plane. The proofs are only partly sketched.

11. El'sgol'c, L. E.

An estimate for the number of singular points of a dynamical system defined on a manifold.

MAT. SBORNIK N. S. 26(68):215–223 (1950)

AMER. MATH. SOC. TRANS. NO. 68, 1952, 14p.

Let M^2 be a compact two-dimensional manifold and $dx_i/dt = X_i(x_1, x_2)$, $i = 1, 2$, a dynamical system over M^2, where the X are continuous in the local coordinates x_i and are such that the problem of Cauchy has a unique solution over M^2. It is also assumed that there are only a finite number of singular points (points where $X_1 = X_2 = 0$). The algebraic number of the singular points (sum of their indices) is $\chi(M^2)$, the Euler characteristic. When the vector (X_1, X_2) is a gradient more accurate information has been given for both the algebraic and actual number of the singular points, and this even for a compact M^n [M. Morse, The Calculus of Variations in the Large, Amer. Math. Soc. Colloquium Publ., v. 18, New York, 1934; Lusternik and Schnirelmann, Méthodes topologiques dans les problèmes variationnels I, Actualités Sci. Ind., no. 188, Hermann, Paris, 1934; Froloff and the author, Rec. Math [Mat. Sbornik] (1) 42, 637–642 (1935)]. The author gives a considerable extension of these results on the assumption that the α- and ω-sets of the system are made up of the simplest possible singular points in finite number: stable or unstable foci or nodes and saddle points. The general method consists in surrounding each unstable node or focus with a Jordan curve without contact. Assuming that the curves are all crossed outwards at time t_0, as t increases from t_0 and the paths are followed there results an isotopic deformation of the neighborhoods of the unstable points in question until there is merely left an arbitrarily small neighborhood of the stable foci or nodes and of the saddle points with the trajectories departing from them. The resulting decomposition of M^2 yields the inequalities of the paper. As an application, if M^2 is a projective plane or a torus there are at least three distinct singular points. The same method is applied to the case where besides the singular points (still of the same simple type) there is a stable or unstable limit-cycle. Under similar assumptions an extension is made to any M^n under the restriction that no trajectory joins two generalized saddle points (points which are conditionally stable).

12. Erugin, N. P.

Note on the integration in finite form of a system of two equations. AKAD. NAUK SSSR. PRIKL. MAT. MEH. 14:315 (1950). (In Russian)

The author points out that the system of two real equations $dx/dt = u(x, y)$, $dy/dt = v(x, y)$ can be reduced to the complex equation $dz/dt = F(z)$, $z = x + iy$, if u and v satisfy the Cauchy-Riemann conditions, and illustrates this by means of the equation $dz/dt = az + bz^2$. This method may be quite useful in obtaining families of typical solutions.

13. Obmorsev, A. N.

Investigation of phase trajectories at infinity. AKAD. NAUK SSSR. PRIKL. MAT. MEH. 14:383–390 (1950). (In Russian)

The Poincare system (1) $dx/dt = P(x, y)$, $dy/dt = Q(x, y)$ is investigated for a limit-cycle at infinity in the following manner. The phase plane is projected onto a hemisphere H of radius one tangent to the plane at the origin; then H is projected orthogonally on the plane. If ρ, θ are the polar coordinates, this yields the transformation $x = \rho(1 - \rho^2)^{-1/2} \cos\theta$, $y = \rho(1 - \rho^2)^{-1/2} \sin\theta$, and it leads from (1) to (2) $dp/d\theta = \rho(1 + \rho)(1 - \rho)^m \Phi(\rho, \Psi)/\Psi(\rho, \theta)$ with Φ, Ψ analytic and not divisible by $1 - \rho$. Infinity is now imaged into C, the circle of radius one. If $\Psi \neq 0$ on C and $m > 0$, C is a limit-cycle. If $m = 0$, C is not a trajectory. Finally if $m < 0$, C is a closed curve without contact. Stability and the existence of limit cycles approaching C are discussed. Application is made to the study of the trajectories of the system (1) corresponding to the following equation derived from the coupling of a series generator with an independently excited motor: $\ddot{x} - \mu((1 + x^2)^{-1} - v)\dot{x} + x = 0$. [References, besides the classical writings of Poincaré, Bendixson, and Liapounov: Petrovski, Rec. Math. [Mat. Sbornik] (1) 41, 107–155 (1934); von Mises, Compositio Math. 6, 203–220 (1938); Lefschetz, Lectures on Differential Equations, Annals of Mathematics Studies, no. 14, Princeton University Press, 1946, p. 142].

14. Tartakovskii, V.
Explicit formulas for the local expansions of solutions of a system of ordinary differential equations. DOKLADY AKAD. SSSR N.S. 72:633–636 (1950). (In Russian)

The author describes a new method for solving in explicit form (locally) a system (1) $dx_i/dt = f_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$, where the f_i are holomorphic at the origin. Let the monomials $x_1^{k_1} \ldots x_n^{k_n}$, $k_i \quad 0$, be viewed as the (countable) components of a row-vector y. Then (1) is equivalent to the linear equation (2) $dy/dt = yA$, where A is an infinite constant matrix such that in each row all but a finite number of elements are zero. The solution is based on the lemma: If $(A^v)^{(q)}(p)$ is the matrix of the elements of A^v in the rows $p_1, \ldots, p_n$ and columns $q_1, \ldots, q_n$ and if $p = p_1 + \ldots + p_n$, $q = q_1 + \ldots + q_n$, then there exist positive constants N, r such that for every positive integer v,

$$|(A^v)^{(q)}(p)| \leqq qr^q(2/r)^p N^v \cdot (v - 1)!.$$

The explicit form of the solution of (2) is $y = y_0 e^{At}$, when y_0 is the initial value of y. No proof of the lemma is given and the derivation of the solution is only outlined. The author states that his method yields a new proof of Cauchy's existence theorem.

15. Yoshizawa, Taro, and Hayashi, Kyuzo
On the uniqueness of solutions of a system of ordinary differential equations. MEM. COLL. SCI. UNIV. KYOTO (SER. A) 26:19–29 (1950).

Let $dy_i/dx = f_i(x, y_1, \ldots, y_n)$, $i = 1, \ldots, n$, be a system of ordinary differential equations which for $0 \leqq x \leqq a$ admits the solution $y_1 = \ldots = y_n = 0$. Let S_0 and S_1 be neighborhoods of the points $x = 0$, $y_i = 0$ and $x = a$, $y_i = 0$, respecitvely. Then a necessary and sufficient condition is given for the above solution $y_1 = \ldots = Y_n = 0$ to be the only one which passes through points of S_0 and S_1. The proof is based on a generalization of a function introduced by Okamura [same Mem. Ser. A. 23, 225–231 (1941)] in connection with the treatment of uniqueness questions. Applications to differential equations of higher order and some generalizations are given.

16. Zwirner, Giuseppe
Criteri di esistenza per un problema al contorno relativo all'equazione $y' = f(x, y; \lambda)$. REND. SEM. MAT. UNIV. PADOVA 19:141–158 (1950).

The author gives six different sets of conditions under which the boundary value problem $y'(x) = f(x, y(x); \lambda)$, $y(x_0) = y_0$, $y(x_1) = y_1$, has at least one solution. These sets of conditions are complicated, and unsuited for restatement in a review. An existence theorem for the boundary value problem $y'(x) = f(x, y(x); \lambda)$, $y(x_0) = y_0$, $y'(x_1) = v$, is given also.

17. Boothby, William M.
The topology of regular curve families with multiple saddle points. AMER. J. MATH. 73:405–438 (1951).

W. Kaplan [Duke Math. J. 7, 154–185 (1940)] has shown that any regular curve family filling the plane is the family of level curves for some continuous real function defined over the plane and without relative extrema. In the present paper this theorem is used as a lemma to establish the same theorem in case the curve family is regular except for singularities of the multiple saddle point type. The author announces that the results of this paper will be used in a later paper to extend, to curve families that are regular except for singularities of the multiple saddle point type, other results due to Kaplan [Trans. Amer. Math. Soc. 63, 514–522 (1948); Lectures in Topology, University of Michigan Press, Ann Arbor, Mich., 1941, pp. 299–301] connecting regular curve families with harmonic functions and with the family of solutions of a system of differential equations.

18. da Silva Dias, C. L.
Bibliography of theorems of existence, uniqueness, and dependence upon parameters for ordinary differential equations and systems of equations. BOL. SOC. MAT. SÃO PAULO 4(1949):31–62 (1951). (In Portuguese)

The bibliography is arranged chronologically and spans the interval 1868 to 1950.

19. Donskaya, L. I.
On the structure of the solutions of a system of three linear differential equations in the neighborhood of the irregular singular point $t = \infty$. DOKLADY AKAD. NAUK SSSR NS 80:321–324 (1951). (In Russian)

In slightly different notation, the homogeneous system $x' = Px\,(' = d/dt)$ is investigated, where $P = \Sigma_{m=0}^{\infty} P^{(m)} t^{-m}$ is a three-by-three matrix, t and $P^{(m)}$ are real, and $P^{(0)}$ is in canonical form. The detailed description of solution matrices of $x' = Px$ near $t = \infty$ is given in terms of certain exponentials, powers of t, and uniformly convergent series.

20. Dramba, C.
Sur la distribution des trajectoires autour d'un point singulier isolé. ACAD. REPUB. POP. ROMANE. BUL. STI. SECT. STI. MAT. FIZ. 3:333–340 (1951). (In Romanian. Russian and French summaries).

This note gives results about the number of tangents to integral curves at an isolated singular point of the system $dx_1 : \ldots : dx_n = X_1 : \ldots : X_n$, particularly in the case where the characteristic roots of $\| \partial X_i / \partial x_i \|$ are real.

21. Germay, R. H.
Extension d'un théorème de Poincaré aux équations récurro-différentielles de forme normale dépendant d'un paramètre variable. BULL. SOC. ROY. SCI. LIEGE 20:678–684 (1951).

By a classical method, involving the use of dominating functions, the author proves an existence and uniqueness theorem for solutions of a system of differential equations of the form

$$\frac{dx_n}{dt} = f_n(t, x_n, x_{n+1}, \lambda) \quad (n = 1, 2, 3, \ldots).$$

The functions $f_n(t, u, v, \lambda)$ are assumed to be representable by power series in u, v, λ, with coefficients which are continuous functions of t, in region defined by relations of the form

$$|u| \leqq \rho, \quad |v| \leqq \rho, \quad |\lambda| \leqq \rho, \quad t_0 \leqq t \leqq t_1 .$$

It is also assumed that the functions $|f_n|$ possess a common upper bound in this region.

22. Hayashi, Kyuzo, and Yoshizawa, Taro
New treatise of solutions of a system of ordinary differential equations and its application to the uniqueness theorems. MEM. COLL. SCI. UNIV. KYOTO SER. A. MATH 26:225–233 (1951).

The authors consider the system of ordinary differential equations

$$\frac{dy_i}{dx} = f_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{1}$$

in the domain G defined by $0 \leqq x \leqq a$, $|y_i| \leqq b_i$. The assumptions on (1) are those of Carathéodory [Vorlesungen uber reelle Funktionen, 2. Aufl., Teubner, Leipzig-Berlin, 1927, pp. 665–672]. With the notation $y = (y_1 \ldots y_n)$ let $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$ be points of $G\,(x_Q > x_P)$. Let $\epsilon\,\mathfrak{Y}_{PQ}$ be the set of all functions $y(x)$ which are absolutely continuous in $x_P \leqq x \leqq x_Q$ and such that the curve $y = y(x)$ lies in G and contains P and Q. Define

$$D(P, Q) = \inf_{y(x) \in \mathfrak{Y}_{PQ}} \int |y'(x) - f(x, y)|\,dx \qquad \text{if} \quad x_Q > x_P$$

$$= |y_P - y_Q| \qquad \text{if} \quad x_Q = x_P$$

$$= D(Q, P) \qquad \text{if} \quad x_Q < x_P .$$

The main theorem of the paper is then that $D(P, Q) = 0$ is the necessary and sufficient condition for Q and P to be on the same solution of (1). In addition some other properties of $D(P, Q)$ are proved and, as an application, a uniqueness theorem for (1) is stated.

23. Katō, Tizuko

On singular points of ordinary differential equations of the first order. NAT. SCI. REP. OCHANOMIZU UNIV. 1:17–21 (1951).

Hukuhara has shown [Mem. Fac. Engineering, Kyusyu Imp. Univ. 8, 203–247 (1937), p. 233] that the solution of $xy' = yF(x, y)$, under suitable assumptions and in a suitable region, can be expressed as $y = \sum \psi_k(u)x^k$, where u satisfies the equation $xu' = u^{n+1}(a + bu^n)$ with a and b constants, $a \neq 0$, and n a positive integer. The object here is to prove this theorem with the aid of a recent theorem of Hukuhara's on fixed points [cf., presumably, Jap. J. Math. 20, 1–4 (1950)].

24. Leontovic, E.

On the generation of limit cycles from separatrices. DOKLADY AKAD. NAUK SSSR N.S. 78:641–644 (1951). (In Russian)

Let the system (1) $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ be of class $N > 0$ in a certain plane region G. In G let the Jacobian $\Delta = D(P, Q)/D(x, y)$ be $\neq 0$ at all the critical points of (1). Consider in G the modified system (2) $\dot{x} = P + p$, $\dot{y} = Q + q$ where p, q and their first partials are arbitrarily small. It is known that upon passing from (1) to (2) new limit-cycles may arise: (a) from a complicated focus; (b) from a closed trajectory with zero characteristic number; (c) from a closed curve C which is a polygon whose sides are separatrices and whose vertices are saddle points. The first two cases have been considered more than once [see Andronov and Pontrjagin, C. R. (Doklady) Acad. Sci. URSS (N.S.) 14, 247–250 (1937); Andronov and Leontovič, ibid. (N.S.) 21, 423–426 (1938); Učenye Zapiski Gor'kovskogo Gosudarstv. Univ. 6, 3–24 (1939)]. The author discusses here (theorems stated without proofs) the case of a single separatrix S from a saddle point M back to S and obtains sufficient conditions for the generation of a preassigned number of limit-cycles in the neighborhood of S. [Further relevant reference as regards the analysis: H. Dulac, Bull. Soc. Math. France 51, 45–188 (1923).].

Reizin, L. E.

The behavior of the integral curves of a system of three differential equations in the neighborhood of a singular point. LATVIJAS PSR ZINATNU AKAD. VESTIS 2(43):333–346 (1951) AMER. MATH. SOC. TRANSL. (2) 1:239–252 (1955)

An analysis is made of the solutions of differential equations $x_i = F_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ near the singular point $0:(0, 0, 0)$. It is assumed that the F_i

differ from homogeneous polynomials of a fixed degree m by terms which are $0(r^m)$, where r is distance from O. The equations are rewritten in spherical coordinates r, ϕ, in the form:

$$\dot{r} = r^m R(\phi, \theta) + h(r, \phi, \theta), \quad r\phi = r^m \Phi(\phi, \theta) + f(r, \phi, \theta),$$

$$r \sin \phi\theta = r^m \Theta(\phi, \theta) + g(r, \phi, \theta).$$

An exceptional direction at O is defined to be one for which the corresponding (ϕ, θ) are such that Φ and Θ are 0. The following theorems are proved. 1. At least one exceptional direction exists. 2. If $\Phi \equiv \Theta \equiv 0$ and a continuous function $B(r)$ exists, whose integral from O to some positive r_0 exists, such that $|f| \leqq B(r)$ and $|g \csc \phi| \leqq B(r)$ for $r < r_0$, then all integral curves near O end at O with a definite direction and there is an integral curve for every direction at O. If further the functions hB^{-1}, fB^{-1} and $gB^{-1} \csc \phi$ satisfy Lipschitz conditions with respect to ϕ and θ, then there is precisely one curve for each direction. 3. If the direction of the z-axis is an isolated exceptional direction and $K = \Phi/R > 0$ in a sufficiently narrow conical region $S: 0 \leqq \phi \leqq \delta$, $0 < r \leqq r_0$ about the zΘaxis, then all solutions entering S for $r > 0$ continue in S and end at 0 with the z-axis as tangent. 4. If $K < 0$ in the set S, then there exists at least one solution entering S for $r = r_0$ and ending at 0 with the z-axis as tangent but there also exist other solutions entering S for $r = r_0$ and then leaving S for $r < r_0$.

26. Saharnikov, N. A.
A qualitative picture of the behavior of a trajectory near the boundary of a region of stability containing a singular point in the form of a center. AKAD. NAUK SSSR. PRIKL. MAT. MEH. 15:349–354 (1951). (In Russian)

The following theorem is proved: If A is a center of the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ (P, Q entire functions) and if all other singular points are simple, then either (1) all trajectories are closed cuvers (i.e. trajectories corresponding to periodic solutions) surrounding A; or (2) the boundary of the region swept by all the closed trajectories surrounding A consists of trajectories (a) passing through ∞, r (b) passing through at least one singular point, all the singular points on the boundary being necessarily of the saddle type.

27. Sansone, Giovanni
Equazioni differenziali nel campo reale: comportamento asintotico delle soluzioni; punti singolari; soluzioni periodiche e valutazione del periodo. UNIV. ROMA. IST. NAZ. ALTA MAT. REND. MAT. E APPL 10(5):265–289 (1951).

This is an expository article which summarizes, in a clear and convenient way, many known results concerning solutions of ordinary differential equations in the real domain. Among the topics discussed are the asymptotic behaviors of solutions of linear equations, the properties of solutions in the neighborhoods of singular points, and the existence and properties of periodic solutions.

28. Sestakov, A. A.
On the behavior of the integral curves of a system of n differential equations ($n \geq 3$) near to a singular point of higher order. DOKLADY AKAD. NAUK SSSR N.S. 79:205–208 (1951). (In Russian)

Let $x_s = X_s(x_1, \ldots, x_n)$, where the X_s vanish at the origin and are holomorphic in its neighborhood; the developments may begin with terms of degree higher than one. The author proves the existence of families of integral curves given by equations $x_i = (a_i + z_{i-1})t^{p_i}$ where the p_i are integers, a_i constants and $z_i \to 0$; the precise statement is too lengthy to be formulated here. There are several printing errors.

29. Erugin, N. P.
Construction of the whole set of systems of differential equations having a given integral curve. AKAD. NAUK SSSR. PRIKL. MAT. MEH. 16:659–670 (1952). (In Russian)

Questions of the following nature are dealt with: Under what conditions does the system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y) \tag{1}$$

have $w(x, y) = 0$ as trajectory. The answer is that the system must have the form

$$\dot{x} = F_1(w, x, y) - \frac{\partial w}{\partial y} M(x, y), \quad \dot{y} = F_2(w, x, y) + \frac{\partial w}{\partial x} M(x, y),$$

where $F_i(0, x, y) = 0$, and M is an arbitrary function. If (1) is to have the trajectories $w_1(x, y) = 0$, $w_2(x, y) = 0$, then it must be of the form

$$x = \frac{\partial w_2}{\partial y} F_1 - \frac{\partial w_1}{\partial y} F_2, \quad y = -\frac{\partial w_2}{\partial x} F_1 + \frac{\partial w_1}{\partial x} F_2,$$

where the F_i behave as before. Various special cases are considered. The extension to a system

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z)$$

with assigned trajectory $D_1(x, y, z) = 0$, $D_2(x, y, z) = 0$ is likewise taken up and it is shown that then

$$P = p(D_1, D_2, x, y, z) + \frac{\partial(D_1, D_2)}{\partial(y, z)} M(x, y, z),$$

with $p(0, 0, x, y, z) = 0$, and similar formulas for Q and R.

30. Haas, Felix.
A theorem about characteristics of differential equations on closed manifolds. PROC. NAT. ACAD. SCI. U. S. A. 38:1044–1047 (1952).

Let V denote a vector field which satisfies a Lipschitz condition on a closed orientable 2-manifold M. Let C^+ denote a positive semi-characteristic of the linear differential equation belonging to V and let C denote the set of ω-limit points of C^+. The author sketches a proof of the following theorem which is to be proved in detail elsewhere: If V has at most a denumerable number of singular points and if there exists a C^+ such that C contains no singular points, then either M is a torus and V has no singular points or C is nowhere dense on M.

31. Inaba, Mituo

A theorem on fixed points and its application to the theory of differential equations. KUMAMOTO J. SCI. SER. A. 1, 1:13–16 (1952).

Let K be a convex compact subset of a Banach space B, and φ a continuous map of K into itself depending on a parameter σ, where σ denotes points in a second Banach space B. For fixed σ, let M_σ denote the set of fixed points of φ. The main theorem states sufficient conditions for the "continuous" dependence of M_σ on σ. As an application, a theorem concerning the "continuous" dependence of the solutions $x_i = x_i(t)$ $(i = 1, 2, \ldots, n)$ of the system of ordinary differential equations

(*) $$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n, t) \quad (i = 1, 2, \ldots, n)$$

on the initial conditions and on the f_i is proved. This theorem does not suppose that the solution of (*) is unique.

32. Lefschetz, Solomon

Notes on differential equations. In CONTRIBUTIONS TO THE THEORY OF NONLINEAR OSCILLATIONS. Princeton, Princeton University Press, 1952. II:61–73

Part I: The topological structure of trajectories of an analytical system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ near a critical point $(P = Q = 0)$ is analyzed by elementary geometrical methods. Similar methods have already been used, e.g., by E. R. Lonn [Math. Z. 44, 507–530 (1938)]. Part II: The topological structure in the large of the trajectories of $\dot{x} = y - \lambda(x^3/3 - x)$, $\dot{y} = x$ (equivalent to van der Pol's equation $\ddot{x} + \lambda(x^2 - 1)\dot{x} + x = 0$) is completely described, by using the methods of Poincaré. The discussion includes a proof that every trajectory except $x = y = 0$ tends to the unique limit cycle as $t \to \infty$.

33. Stebakov, S. A.
Qualitative investigation of the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ by means of isoclines. DOKLADY AKAD. NAUK SSSR N.S. 82:677–680 (1952). (In Russian)

The autonomous system $x' = P(x, y)$, $y' = Q(x, y)$ ($' = d/dt$) is considered in the neighborhood of an isolated singular point. By constructing certain polygonal paths and considering the direction field defined by the above system it is shown that some qualitative results can be stated concerning the solution paths of the system near the singular point. The constructions do not seem to differ much from the standard ones used to prove the Poincaré-Bendixson theorem.

34. Andreev, A. F.
Solution of the problem of the center and the focus in one case. AKAD. NAUK SSSR PRIKL. MAT. MEH. 17:333–338 (1953). (In Russian)

Lyapunov in a paper attached to the Russian edition of his fundamental mémoire, General problem of the stability of motion [Gostehizdat, Moscow-Leningrad, 1950] considered the nature of the critical point at the origin of the planar system

$$\begin{aligned} \dot{x} &= y + Ax^3 + Bx^2y + Cxy^2 + Dy^3, \\ \dot{y} &= Kx^3 + Lx^2y + Mxy^2 + Ny^3. \end{aligned} \tag{1}$$

He showed among other results that if

$$K < 0, \quad 3A + L = 0, \tag{2}$$

the points is a center or a focus. The author shows that a necessary and sufficient condition for a center are the conditions (2) and in addition

$$\begin{aligned} 2A(B + M) + K(C + 3N) &= 0, \\ (B + M)(2A^3 - AKM - K^2N) &= 0. \end{aligned} \tag{3}$$

35. Besicovitch, A. S.
On the existence theorem for the differential equation $dy/dx = \phi(x, y)$. J. LONDON MATH. SOC. 28:110–112 (1953).

The differential equation $y' = f(x, y)$ is known to have a solution if f is measurable with respect to x for every fixed y and continuous with respect to y. This paper shows that measurability cannot replace continuity in the condition by constructing an f for which the equation has no solution.

36. Bishop, R. E. D.
The phase-plane construction in problems of elastic impact. ENGINEER, (LOND.) 196(5089):168–170, Aug 1953

Author considers a prismatic bar of arbitrary length loaded at one end of an axial force, which is a known function of time. Other end is free. Particle displacements are calculated in two ways: well-known wave solution and normal mode solution [see Timoshenko, "Vibration problems in engineering," p. 314]. Differential equation in the latter approach is solved graphically by means of phase-plane constructions.

37. Hartman, Philip, and Wintner, Aurel
On the behavior of the solutions of real binary differential systems at singular points. AMER. J. MATH. 75:117–126 (1953)

Let $\Gamma: x = x(t)$, $y = y(t)$ be a solution path of $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ tending to the critical point

$$x = y = 0 \quad (f(0,0) = g(0,0) = 0),$$

and let $\theta(t) = \tan^{-1} y(t)/x(t)$, $\psi(t) = \tan^{-1} \dot{y}(t)/\dot{x}(t)$. The questions whether (1) $\lim \theta(t) = \theta_0$ exists, (2) $\lim \psi(t) = \theta_0 \pmod{\pi}$, are investigated. Assumptions on f, g, apart from continuity, are: (a) there is a periodic dense open set A such that $\lim_{r \to 0} r^{-1} f(r \cos\theta, r \sin\theta) = F(\theta)$, $\lim_{r \to 0} r^{-1} g(r \cos\theta, r \sin\theta) = G(\theta)$ exist uniformly in any compact subset of A, and $F^2(\theta) + G^2(\theta) > 0$ in A; (b) the subset A_0 of A at which $G(\theta) \cos\theta - F(\theta) \sin\theta = J(\theta)$ vanishes has no cluster points in A. It is shown that: (i) If (1) holds, then θ_0 is either in A_0 or not in A; (ii) if (1) does not hold, then $|\theta(t)| \to \infty$; (iii) if $J(\theta)$ changes sign at $\theta = \theta_0$ in A_0, then at least one solution path Γ satisfies (1); (iv if $J(\theta)$ takes both positive and negative values, then (1) holds for some θ_0) for every solution path Γ. These results are applied to the special case

$$f(x, y) = \alpha x + \beta y + 0(r), \quad g(x, y) = \gamma x + \delta y + 0(r).$$

38. Kestin, J., and Zaremba, S. K.
Geometrical methods in the analysis of ordinary differential equations. Introduction to non-linear mechanics. APPL. SCI. RESEARCH B. 3:149–189 (1953).

This is an exposition of the qualitative theory of the real, plane, differential system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, including such topics as: (a) behavior of solutions near a singular (critical) point, both elementary and of higher order; (b) Kronecker-Poincaré index of an isolated critical point; (c) Poincaré and Bendixson criteria for the existence of periodic solutions; (d) extension of the plane differential system to the compact sphere or the projective plane. Several examples referring to one-dimensional adiabatic gas flow are analysed by these classical techniques.

39. Kimura, Toshifusa
Sur une généralisation d'un théorème de Malmquist. COMMENT. MATH. UNIV. ST. PAUL. 2:23–28 (1953).

The author generalizes a theorem of Malmquist [Acta Math. 36, 297–343 (1913)] relative to the equation (1) $dy/dx = P(x, y)/Q(x, y)$, where P, Q are polynomials in x, y. The following is established. (I) Consider the equation (2) $x^{\sigma+1}\, dy/dx = P/Q$, where σ is an integer ($\geqq 0$) and P, Q are polynomials in y whose coefficients are analytic in x at $x = 0$; if S is the set of accumulation of a solution $y(x)$ (near $x = 0$), then S consists of just a single point or S is the whole plane. (II) Every solution of (1) takes on all the values, except for a finite number of values, in an arbitrary neighborhood of the essentially singular point. (III) Under conditions as in (I), let $P = P_0(x)\,\Pi_1^p(y - P_i(x))$, $Q = Q_0(x)\,\Pi_1^q(y - Q_i(x))$, $\alpha_i = P_i(0)$, $\beta_i = Q_i(0)$, and assume that there exists a β_i distinct from all the α_i; if a solution $y(x)$ of (2) has no critical point in a vicinity of the origin, except at the origin, then the latter is an ordinary singular point.

40. Ku, Y. H.

A method for solving third and higher order nonlinear differential equations. J. FRANKLIN INST. 256,3:229–244 Sep 1953

Author gives a general method for solving nonlinear differential equation of the form $x^{(n)} + \Phi_{n-1} x^{(n-1)} + \ldots + \Phi_2 x^{(2)} + f(x^{(1)}, x) + f_1(x) = F(t)$ where $x^{(i)}$ is the ith derivative with respect to t: Φ_i is a nonlinear function of $x^{(1)}, x^{(i-1)}, \ldots$ and eventually t.

The solution is constructed from generalized phase-plane equations, which are derived from the given differential equations. The method is applied to a third-, fourth-, and sixth-order equation.

41. Saito, Tosiya

Sur les solutions autour d'un point singulier fixe des équations différentielles du premier ordre. KŌDAI MATH. SEM. REP. 1953:121–126 (1953).

The equation (E) $Q(x, y)y' = yP(x, y)$ is studied in the neighborhood of an isolated fixed singularity at $x = 0$. It is assumed that $P = \Sigma_{k=0}^{m} a_k(x) y^k$ and $Q = \Sigma_{k=0}^{n} b_k(x) y^k$ where a_k and b_k are analytic and single valued, while, for each x, Q and yP are relatively prime. Let x_0 be a regular point of (E) and let $\varphi(x; x_0, y_0)$ satisfy (E) and have $\varphi(x_0; x_0, y_0) = y_0$. Let A denote an annular region centered at $x = 0$ and containing no singular point. If $|y_0|$ is sufficiently small and the residue r of $a_0(x)/b_0(x)$ at $x = 0$ is irrational, then in A each branch of $\varphi(x; x_0, y_0)$ is expressible in the form $\Sigma_{k=1}^{\infty} x^{rk} \nu_k(x)$ where each $\nu_k(x)$ is analytic and single-valued in A. The functions $\nu_k(x)$ can be obtained by quadratures.

42. Bishop, R. E. D.

On the graphical solution of transient vibration problems. INSTN. MECH. ENGRS. PROC. 168,10:299–322 (1954)

Paper is a general treatment with many examples of the "phase-plane" method of treating vibration problems, i.e., the method whereby displacement is plotted against velocity in a diagram. The use of this method for nonlineary problems is well known. The author applies it to transient and forced vibrations, to the deflections of columns with end loads as well as lateral loads, and to bending-moment distributions in beams. The discussions by others and the rebuttal by the author are a valuable extension of the paper.

43. Eckman, D. P.

Phase-plane analysis – a general method of solution for two-position process control

TRANS. ASME 76, 1:109–116, Jan 1954.

Author outlines procedure of solution of two-position process-control problems characterized by second-order differential equations through use of phase-plane analysis. Details, such as graphical exhibition of amplitude of steady-station oscillation, are effected for five particular cases. Author comments on Ziegler-Nichols' approximation and illustrates its application by three examples.

44. Gubar, N. A.

A characteristic of composite singular points of a system of two differential equations by means of simple singular points of neighboring systems.

DOKLADY AKAD. NAUK SSSR N. S. 94:435–438 (1954). (In Russian)

This paper deals with a real system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y) \tag{1}$$

where P, Q are analytic in a certain domain G and without common factor in G. Complicated singular points in G are studied and one supposes that at any such point

$$|P_{x'}| + |P_{y'}| + |Q_{x'}| + |Q_{y'}| \neq 0. \tag{2}$$

[Reviewer's remark: (2) restricts the singular points to types such that in the expansions of P, Q about any such point some terms of the first degree are present.] Let there be associated with (1) a system

$$\dot{x} = P + p(x,y), \quad \dot{y} = Q + q(x,y) \tag{3}$$

with p, q analytic in G. We say that p, q are ϵ-increments of rank r whenever they and their partials of order $\leqq r$ remain $< \epsilon$ in absolute value in G. The point M of G is said to be of multiplicity m for (1) whenever there exist positive ϵ_0, δ_0 such that: (a) for all ϵ_0-increments of rank m the system (3) has no more than m singular points in the δ_0-neighborhood of M; (b) for any $\epsilon < \epsilon_0$ and $\delta < \delta_0$, there exist ϵ-increments of rank m such that correspondingly in the δ-neighborhood of M, (3) has m ordinary singular points (node, focus, saddle point).

Using these definitions the author states a number of rather complicated theorems which describe the various possible phase-portraits about M. One half of these theorems corresponding to a single characteristic root at M are consequences of the work of Bendixson [Acta Math. 24, 1–30 (1901)]. The other half (both characteristic roots zero) are new.

45. Nemyckii, V. V.
Some problems of the qualitative theory of differential equations. (Survey of contemporary literature.) USPEHI MATEM. NAUK N.S. 9, 3(61):39–56 (1954). (In Russian)

This paper provides an excellent bird's eye view of the work done on the subject under discussion during the last five years. About 40% of the papers under review (totalling 58) refer to work done outside the USSR. The emphasis is still on the whole on Soviet research, which is rather natural under the circumstances, and indeed enhances the value for the non-Soviet reader. The various and well known topics of systems of the second order are discussed. About half of the paper is devoted, however, to a description of contributions to systems of higher order.

46. Otrokov, N. F.
On the number of limit cycles of a differential equation in the neighborhood of a singular point. MAT. SBORNIK N.S. 34(76):127–144 (1954). (In Russian)

The author considers the problem of finding the number of limit cycles in the neighborhood of a singular point of an equation $dy/dx = Q_N(x, y)/P_N(x, y)$, where P_N, Q_N are polynomials of degree N. The number of essential coefficients of P, Q (after certain canonical reductions) is

$$m = (N - 1)(N + 4) + 1$$

and a one-one correspondence is thus established between equations and points in E_m. A point A_0 in E_m is called k-cyclic if: (i) there are numbers $\epsilon_0 > 0$, $\delta_0 > 0$ such that the equation corresponding to any point in the ϵ_0-neighborhood of A_0 does not have more than k limit cycles in the δ_0-neighborhood of the singular point $x = y = 0$; (ii) for any ϵ, δ, $0 < \epsilon < \epsilon_0$, $0 < \delta < \delta_0$, there are equations corresponding to points of the ϵ-neighborhood of A_0 which have k limit cycles in the δ-neighborhood of $x = y = 0$. The main result of the paper is that, for $N \geqq 6$, k-cyclic points exist with k at least equal to $1/2(N^2 + 5N - 14)$, N even, or $1/2(N^2 + 5N - 26)$, N odd.

47. Sarantopoulos, Spyridon
Sur l'existence des intégrales holomorphes des équations différentielles du premier ordre dans le cas singulier. BULL. SOC. MATH. GRÈCE 28:128–166 (1954).

L'A. considera l'equazione differenziale (1) $x^2 dy/dx = \alpha(x)y + x\varphi(x) + xy\sigma(x)$, dove $\varphi(x) = \Sigma_{\nu=0}\, \beta_\nu x^\nu$, $\sigma(x) = \Sigma_{\nu=0}^{\infty}\, \delta_{\nu+1} x^\nu$ sono olomorfe nell'intorno dell'origine, e supposto che (2) $y = \Sigma_{\nu=0}^{\infty}\, \gamma_{\nu+1} x^{\nu+1}$ sia una soluzione formale della (1), nel caso che δ_1 non sia intero positivo o nullo, esprime i coefficienti $\gamma_{\nu+1}$ con dei determinanti di cui studia in questa prima parte alcune proprietà che gli occorreranno per dare le condizioni perchè la (2) rappresenti una funzione olomorfa.

48. Sarantopoulos, Spyridon
Sur l'existence des intégrales holomorphes des équations différentielles du premier ordre dans le cas singulier. II. BULL. SOC. MATH. GRÈCE 29:1–24 (1954).

L'A. completa una precedente ricerca [si veda la precedente recensione] e dimostra successivamente che data l'equazione differenziale

$$x^{\mu+1} dy/dx = xy + x\varphi(x) + \delta x^\mu y ,$$

con μ intero, $\mu \geqq 1$, $\varphi(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \dots$, olomorfa in un intorno di $x = 0$, e supposto che δ non sia intero positivo, allora condizione necessaria e sufficiente perchè questa possieda un integrale $y(x)$ olomorfo in un intorno di $x = 0$, con $y(0) = 0$, è che risulti $\Phi_\nu(x) = 0\ (\nu = 0, 1, \dots, \mu - 1)$ essendo la trascendente intera $\Phi_\nu(x)$ definita della relazione

$$\Phi_\nu(x) = \sum_{\lambda=0}^{\infty} \beta_{\mu\lambda+\nu} x^\lambda / (\nu + 1 - \delta)(\nu + 1 + \mu - \delta) \dots (\nu + 1 + \mu(\lambda - 1) - \delta) .$$

Se poi δ è un numero intero positivo e $\delta = h + 1 + \mu(1 - 1)$, alla condizione $\Phi_h(x) = 0$ deve invece sostituirsi l'altra $_h(\ ) = 0$, dove

$$\varphi_h(x) = \sum_{k=0}^{\infty} \beta_{\mu(1+k)+h} x^k / k!\, \mu^k .$$

Tanto nell'uno che nell'altro caso il raggio di convergenza di $y(x)$ non è inferiore al raggio di convergenza di $\varphi(x)$.

49. Seth, B. R.

Generalized singular points with applications to flow problems. PROC. INDIAN ACAD. SCI (A) 40(1):25–36, Jul 1954.

In many cases, the solution of a boundary-value problem can be obtained by an appropriate isolated singularity. For example, in treating potential flow he appears to claim that the flow induced by the translation or rotation of a given solid in an infinite liquid is the same as that due to a potential obtained by differentiation of the gravitational potential of the same solid, regarded as homogeneous.

50. Skackov, B. N.

Qualitative picture of the behavior of the integral curves in the neighborhood of a singular point in one case. VESTNIK LENINGRAD. UNIV. 9 (8):65–69 (1954) (In Russian)

Take a real three-dimensional system

$$x = Px + X(x) \tag{1}$$

where P is a constant matrix and the components of X are power series beginning with terms of degree at least two. Lyapunov has discussed the general case where P has one characteristic root zero and has singled out the special case where a regular transformation reduces (1) to the form

$$\dot{x} = \alpha x + \beta y + X, \quad \dot{y} = \gamma x + \delta y + Y, \quad \dot{z} = Z(x, y, z) \tag{2}$$

where the real parts of the characteristic roots of

$$\begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}$$

have the same signs, and the vector (X, Y, Z) behaves like X in (1) save that the three components vanish for $x = y = 0$. The z-axis is thus a line of singularities, there is a general solution $z = c + f(x, y, c)$ and the author discusses the character of the integral curves as the constant c varies.

51. Sugiyama, Shohei
Note on singularities of differential equations.
KODAI MATH. SEM. REP. 1954, 81–84 (1954).

Critical points of the system $\dot{x} = p(x, y)$, $\dot{y} = q(x, y)$ are studied. The system is written $\dot{z} = f(z, \bar{z})$ $(z = x + iy)$, $f(z, w)$ being assumed analytic near the critical point, which is taken as zero. If the series for $f(z, \bar{z})$ starts with nth degree terms, let $f_n(z, \bar{z})$ be the terms of nth degree. Then the index of the critical point at zero is $n - 2k$, where k is the number of zeros of $f_n(1, z)$ in $|z| < 1$. The equation is topologically equivalent to $\dot{z} = z^{n-2k}$ if $n - 2k \geqq 0$; otherwise to $z = \bar{z}^{2k-n}$. Infinity is handled by stereographic projection. The cases $n = 1, 2, 3,$ and 4 are considered separately. If $n = 1$, and $f_1(z, \bar{z}) = az + b\bar{z}$, the index at $z = 0$ is $+1$ if $|a| > |b|$, and is -1 if $|a| < |b|$. For $n > 1$, the nested-oval or multiple saddle-point form of the characteristics near the critical point, together with separatrices is derived.

52. Ura, Taro and Hirasawa, Yoshikazu
Sur les points singuliers des équations
différentielles admettant un invariant intégral.
PROC. JAPAN ACAD. 30:726–730 (1954).

It is shown that a system $x = X(x, y)$, $y = Y(x, y)$ having a positive invariant integral can have as isolated singularities (points at which $X = Y = 0$) only multiple saddle points or centers.

53. Zaremba, S. K.
Divergence of vector fields and differential
equations. AMER. J. MATH. 76:220–234
(1954).

Let $x = x(t)$, $y = y(t)$ be an integral curve of the system of equations (1): $dx/X(x, y) = dy/Y(x, y) = dt$; and let $x(t_i) = x_i$, $y(t_i) = y_i$, $i = 1, 2$. Let T_i be the orthogonal trajectory of the integral curves passing through the point (x_i, y_i); and let u_i be the parameter on T_i which vanishes at (x_i, y_i) and satisfies the equations

$$-dx/Y(x, y) = dy/X(x, y) = du_i/(X^2 + Y^2).$$

It is shown that du_2/du_1 is given, for $u_1 = 0$, by the formula

$$du_2/du_1 = \exp\int_{t_1}^{t_2} [X_x(x(t), y(t)) + Y_y(x(t), y(t))]\, dt.$$

This result is used to obtain simple proofs of various known theorems concerning the integral curves of the system (1); in particular, theorems concerning the existence and stability of limit cycles. A complicated new theorem is given which is a partial converse of Bendixson's theorem asserting the non-existence of closed integral curves in a simply connected domain in which $X_x(x, y) + Y_y(x, y)$ has a constant sign.

54. Andreev, A. F.
Investigation of the behaviour of the integral curves of a system of two differential equations in the neighbourhood of a singular point. AMER. MATH. SOC. TRANSL. 2(8):183–207 (1958).

This is a detailed study of the local phase-portrait at the origin of a real system

$$\frac{dx}{dt} = y + X(x, y), \quad \frac{dy}{dt} = Y(x, y), \tag{1}$$

where X, Y are real convergent power series beginning with terms of degree at least two. Every analytical system with the origin as isolated singularity, terms of the first degree not all zero but with both characteristic roots zero, may be reduced to the type (1) by a linear transformation of coordinates. Such a system was already investigated, strictly for stability, by Lyapunov [Mat. Sb. 17 (1893), 253–333 = supplement (pp. 369–449) to "General problem of the stability of motion, Gostehizdat, Moscow, 1950]. The author applies extensively the method of Frommer [Math. Ann. 99 (1928), 222–272] and obtains essentially eight phase-portraits. He also gives analytical expressions for the trajectories tending to the origin.

55. Bass, Robert W.
On the regular solutions at a point of singularity of a system of non-linear differential equations. AMER. J. MATH. 77:734–742 (1955).

The author constructs regular solutions of a system of non-linear ordinary differential equations near a certain type of irregular singular point. Let $f_i(z, w_1, \ldots, w_n)$ be n analytic functions of $n+1$ complex variables near $(z, w_1, \dots, w_n) = (0, 0, \dots, 0)$ and suppose $f_i(z, 0, 0, \dots, 0) \equiv 0$. Theorem. The system $z^{s_i} w_i'(z) = f_i(z, w_1, \dots, w_n)$ $(i = 1, 2, \dots, n)$ and $s = s_1 + s_2 + \dots + s_n < n$, has at least an $(n - s)$-parameter algebroid family of solutions regular at $z = 0$. The proof of this theorem, and of a slightly more general statement, use the method of comparison of coefficients. The resulting infinite set of non-linear equations is solved by means of Wintner's fixed-point theorem in an appropriate Hilbert space.

56. Coles, W. J.

Linear and Riccati systems. DUKE MATH. J.

22:333–338 (1955)

This paper is concerned with an ordinary linear vector differential equation

$$(*) \qquad x' = A(t)x, \; A(t) \equiv \| a_{ij}(t) \| \; (i, j = 1, \dots, n),$$

and associated Riccati systems of the form

$$(**) \quad y_i' = -y_i \sum_{k=1}^{n-1} a_{nk} y_k + \sum_{k=1}^{n-1} (a_{ik} - \delta_{ik} a_{nn}) y_k + a_{in} \; (i = 1, \dots, n-1);$$

in particular, (**) is satisfied by $y_i = x_i/x_n (i = 1, \dots, n-1)$, where $x \equiv (x_j(t))$ $(j = 1, \dots, n)$, is a solution of (*). Results on the equivalence of (*) and (**) are given, and there is presented a general solution of (**) in terms of $n+1$ particular solutions of this equation. Finally, it is shown how the knowledge of n solutions of (**) leads to a matrix $P(t)$ and a diagonal matrix $B(t)$ such that (*) is equivalent to $y = B(t)y$ under the transformation $y' = P(t)x$. The author notes relations between the derived results and those of Chiellini [Rend. Sem. Fac. Sci. Univ. Cagliari 18 (1948), 44–58] on Riccati systems.

57. Conte, Samuel D.

An equiconvergence theorem. INDUST. MATH. 6:27–28 (1955).

The author considers the system of linear differential equations $U' - [\lambda + q_1] V = 0$ and $V' + [\lambda + q_2] U = 0$, where λ is a complex parameter and q_1, q_2 are real functions of class C^1 on $[0, 1]$. The two-point boundary conditions are $U(0) \cos \alpha + V(0) \sin \alpha = 0$, $V(1) \cos \beta + V(1) \sin \beta = 0$. Let f_1, f_2 be integrable on $[0, 1]$. Then the author shows that the convergence of the expansions for f_1 and f_2 in terms of the eigenfunctions of the above system behaves like the convergence of the expansions of f_1, f_2 for the above system with q_1 and q_2 replaced by $q = 1/2(q_1 + q_2)$.

58. Corduneanu, C.
Sur un problème aux limites concernant les équations différentielles non-linéaires du second ordre. AN. STI. UNIV. "AL. I. CUZA" IASI. SECT. I. N.S. 1:11–16 (1955)
(In Romanian. Russian and French summaries)

Let $f(x, u, v)$ be continuous and possess continuous partial derivatives f_u, f_v in the slab $a \leqq x \leqq b$, $-\infty < u, v < \infty$; let $f_v \leqq 0$, $|f_u| < 2\{(b-a)^2 + 2|h|(b-a)\}^{-1}$ in the slab, and let f_v be bounded below in $a \leqq x \leqq b$, $-M \leqq u \leqq M$, $-\infty < v < \infty$ for each M. The author proves that the nonlinear differential equation $y'' = f(x, y, y')$ possesses a unique solution in $a \leqq x \leqq b$ satisfying the boundary conditions $y'(a) = A$, $y(b) + hy'(b) = B$, and shows the construction of this solution by successive approximations. {It seems to the reviewer that $h \geqq 0$ must be assumed for the general truth of the statement.}

59. Diaz, J. B., and Payne, L. E., editors
PROCEEDINGS, CONFERENCE ON DIFFERENTIAL EQUATIONS. UNIV. OF MARYLAND
College Park, Md., University of Maryland Book Store, 1956, 294p.

Differential systems with boundary conditions at more than two points, Whyburn, W. M.:

Paper considers ordinary differential systems of the form

$$y'_i = f_i(x, y_1, y_2, \dots, y_n), \quad (i = 1, 2, \dots, n)$$

in the real domain, with boundary conditions involving more than two points of an interval $a \leq x \leq b$. Boundary conditions involving "initial" values at more than one point are treated in some detail; brief references are made to conditions which involve more than the two end-points a and b, and to conditions which involve the end-point a and b, and to certain conditions at interior points.

Sturm-Liouville and beat equations whose eigen functions are ultra-spherical polynomials or associated Bessel functions, Bochner, S.: Solutions of the two equations

$$(1 - x^2) f_{xx} - (2y + 1) x f_x = f_t, \quad -1 \le x \le 1$$

$$f_{xx} + (2y/x) f_x = f_t, \qquad 0 \le x \le \infty$$

are studied for aribtrary real y.

Repeated branching through loss of stability, an example, Hopf, E.: Following up his earlier paper on a mathematical example displaying features of turbulence, Commun. Appl. Math. 1:303–328 Dec 1948, author treats an example of an integro-differential equation in two dependent and two independent variables (x, t) whose limiting solutions as $t \to \infty$ become more and more complicated in a discontinuous manner as a certain parameter decreases to zero through an infinite number of critical values.

Problems related to characteristic surfaces, Riesz, M.: The solution of the wave equation for characteristic boundaries is studied by the author's method of analytical continuation of Riemann-Liouville integrals. Characteristic surfaces in m dimensions intersect themselves in $(m - 2)$-dimensional varieties called keels, and the solutions lead to interesting formulas of "curvatura integra" and sequences of special functions which are related to given characteristic surfaces in the way that different powers of the Lorentz distance are related to characteristic cones.

On the Euler-Poisson-Darboux equation, integral operators, and the method of descent, Diaz, J. B. and Ludford, G. S. S.: Paper commences with a self-contained account of the solution of Cauchy's problem for

$$\Delta u = u_{tt} + (k/t) u_t,$$

but for $-\infty < t < \infty$, instead of $t \ge 0$ as usual. A second section is devoted to the generation of solutions of linear hyperbolic equations in two independent variables by combining the ideas of Le-Roux and Bergman, and a third section shows how a reflection principle may be used to continue analytic solutions of elliptic equations.

On partial differential equations of mixed type, Protter, M. H.: After an extensive and valuable discussion of previous work on equations of mixed type, author proves existence and uniqueness theorems for the equation

$$K(y) b(x,y) u_{xx} - u_{yy} + a(x,y) u_x + b(x,y) u_y + c(x,y) u + f(x,y) = 0,$$

where $K(y)$ is monotone increasing with y, $K(0) = 0$, and $b(x,y) > 0$, for boundary values of u given on a segment of the x-axis and a characteristic curve through one of its end-points.

Some applications of Riesz's method, Copson, E. T.: Direct application of Riesz's method of analytical continuation is made to mixed initial-value problem for the three-dimensional wave equation arising from Burger's treatment [Nat. Aero. Res. Inst., Amsterdam, Rep. F. 157, 1954] of the half-plane diffraction problem, and to Cauchy's problem for the equation of damped waves, where it is shown that it is unnecessary to use Fremberg's kernal involving a Bessel function.

Discontinuity and representations of minimal surfaces, Chen, Y. W.: It is shown that the solutions of the equation for minimal surfaces when the boundary curves have points of tangential discontinuity behave differently from conformal mappings, and, due to the nonlinearity, may not even exist. How to modify the boundary conditions so that solutions exist is an important consideration in the hodograph method for compressible-flow problems.

Relations between different capacity concepts, Szego, G.: If C is the Newtonian capacity, and c is the logarithmic capacity of a closed domain D, then it is shown that

$$\pi C/2c \leq 1.07 .$$

The previous best value for the constant on the right is 1.12.

Some results on generalized axially symmetric potentials, Huber, A.: Paper gives some results concerning existence and nonexistence of solutions to problems of Dirichler's type for the equation

$$\Delta u + (k/x_n)\, u_{x_n} = 0 ,$$

some mean-value properties of the solutions, and some integral representation theorems. The last hold only for integer values of k, but the other results are established for nonintegral values.

On the numerical solution of the Dirichlet problem, Nehari, Z.: It is shown that some difficulties associated with the usual solution of Dirichlet's problem for Laplace's equation by series of orthonormal harmonic functions can be removed if the Dirichlet norm is replaced by suitable boundary norms. An alternative modification is to replace the set of harmonic functions by sets of functions which are not harmonic, but which have the correct boundary values. It is shown further that similar methods can be used to solve Dirichlet's problem for the equation

$$u_{xx} + u_{yy} = P(x, y)\, u ,$$

where $P(x, y)$ is positive and continuous.

Bounded or almost-periodic solutions of nonlinear differential systems, Ameria, L.: Nonlinear equations of the form

$$X' = F(t, X) \qquad [1]$$

are considered, where $X(t) = \{x_1(t), \ldots, x_n(t)\}$ is an unknown vector, and t is the independent variable. For suitable conditions on F, these equations possess an almost-periodic solution. The results are applied to the system of equations

$$X'' = -MX - \Phi(X') + F(t) \qquad [2]$$

which describe the motion of a point mass under the action of a linear elastic force $-MX$ with M constant and positive definite, a dissipative force $\Phi(x')$ (for sufficiently large $|X|$), and a continuous and bounded force $F(t)$. Then [2] possesses at least one bounded solution, and, if $F(t)$ is periodic, there is a periodic solution. Under more restrictive conditions on Φ, it is shown that the bounded solution is unique, that every other solution converges to this solution as $t \to \infty$, and that the bounded solution is almost-periodic if $F(t)$ is almost-periodic.

The extension of the Riemann mapping theorem to elliptic equations, Dressel, F. G., and Gergen, J. J.: Paper discusses past and recent results concerning the existence and uniqueness of pairs of mapping functions $u(x, y)$, $v(x, y)$ which give a 1:1 mapping of a Jordan domain D in the (x, y)-plane onto a Jordan domain Δ in the (u, v)-plane, and which satisfy linear elliptic equations of the form

$$au_x + bu_y = v_y, \quad cu_x + du_y = -v_x.$$

On the eigenfunctions of the membrane equation in a singular case, Pleijel, A.: The asymptotic behavior of the eigenfunctions of the equation

$$\Delta u + \lambda k(x, y, z)u = 0, \quad k(x, y, z) \geq 0,$$

which vanish on the boundary S of a finite domain V of (x, y, z)-space is studied for a subdomain V_0, with boundary S_0, in which k vanishes. If $\lambda_1 \leq \lambda_2 \leq \lambda_3$, etc., are the eigenvalues, and the corresponding eigenfunctions are ϕ_1, ϕ_2, ϕ_3, etc., orthonormalized so that

$$k\phi_m \phi_n \, dV = \delta_{mn},$$

then the principal result of the paper is that the series

$$\sum_{n=1}^{\infty} \phi_n^2(p)/\lambda_n$$

converges if $p \in V_0$, although it diverges if $p \in (V - V_0 - S_0)$; other series connected with Green's functions are also investigated.

An abstract formulation of the method of separation of variables, Friedman, B.: The method of separation of variables for partial differential equations is formulated in terms of direct products of Hilbert spaces. By this formulation we can determine the possible boundary conditions for which the method is applicable. Finally, a contour representation is given for the solution of some partial differential equations. (Author's abstract)

The beat equation and the Weierstrass transform, Widder, D. V.: Paper discusses the problems of inversion and representation when transformed function is bounded, in which case some technical difficulties of the general case are absent; the general case is treated in Hirschman and Widder's book, "The convolution transform."

Extensions of operational mathematics, Churchill, R. V.: If a function $F(x)$ satisfies a linear differential equation and prescribed boundary conditions at the ends of the interval (a, b), then suitable choice of the kernal K and the domain of s in the linear integral transformation

$$f(s) = \int_a^b K(x, s)\, F(x)\, dx$$

may lead to a simpler problem for $f(s)$, and properties of such transformations can be used to obtain properties of the solutions to such boundary-value problems as well as the solutions themselves. Primary purpose of paper is to determine the transformation when the differential equation and the boundary conditions are prescribed. Some speical cases are considered.

On general Sturm-Liouville operators, Feller, W.: The general form of linear differential operators of the second order which have certain specified local properties is derived, and some results on their semi-boundedness in Hilbert space, minimal solutions, behavior at boundaries, and Green's functions are obtained.

Heat transfer to Hagen-Poiseuille flows, Millsaps, K., and Pohlhausen, K.: The problem considered is the determination of the thermal profiles when a viscous liquid in laminar motion is flowing in a circular cylindrical tube, and the wall of the tube and the liquid are maintained at constant temperature up to a certain cross section at which the wall temperature is changed to a new value for the remainder of the tube. Original treatments by Nusselt and Graetz neglected dissipation due to viscosity and axial curvature of the thermal profiles, which are included in the present treatment since their effects are not negligible for liquids with small Prandtl numbers (such as liquid metals used in nuclear reactors). It is assumed that the physical properties of the liquids are constants, so the results hold only for small temperature changes. The resulting equation for the temperature leads to an eigenvalue problem, and extensive numerical and graphical results are given.

60. Friedrichs, K. O.

Asymptotic phenomena in mathematical physics.

BULL. AMER. MATH. SOC. 61:485–504 (1955)

The occurrence of discontinuities, shock, boundary layer, edge effect, skin effect, Stokes' phenomena, etc. are discussed from the physical and mathematical points of view in this excellent exposition based on the Gibbs lecture delivered by the author in 1954. Unsolved and partially solved problems are mentioned and a list of selected references is given.

61. Gomory, Ralph E.

Trajectories tending to a critical point in 3-space.

ANN. OF MATH. (2)61:140–153 (1955).

Consider $dv/dt = F(v)$, where $v(t)$ is a vector in real n-space which for $t \to \infty$ approaches a critical point P, say the origin; thus the differentiable vector-valued function $F(v)$ vanishes at $v = 0$. By an extension of the radius vector, one maps $E^n - P$ on the exterior of the unit sphere D' in E^n. The author considers the set $L(v)$ (or $L(v')$ when considered on D') of positive limit directions of the solution $v(t)$, by means of an associated autonomous differential system on D'. If $n = 3$ then $D' = S^2$ and the well-known structure of autonomous differential systems on S^2 can be interpreted to yield information about $L(v)$. Typical of the many results is the following: $L(v')$ either contains a critical point of the system on D' or else is a closed curve.

62. Hartman, Philip, and Wintner, Aurel

Asymptotic integrations of ordinary non-linear

differential equations. AMER. J. MATH

77:692–724 (1955).

Non-linear systems are considered of the form

$$(*) \qquad y' = JY + F(t, y) \quad , \quad y = (y', \dots, y^n) \quad (t \geqq t_0),$$

where J is a constant matrix and F is a continuous vector function for $t \geqq t_0$ and $|y| \leqq$ const., with $|F(t, y)|/|y| \to 0$ as $(t, y) \to (\infty, 0)$. No Lipschitz condition is required so that no local uniqueness is assumed. Asymptotic theorems are given which generalize previous ones of Poincaré for F analytic, of 0. Perron for F non-analytic under a Lipschitz condition, and of the same authors and of O. Dunkel for the linear case. There is no loss in generality in supposing that J has a normal form, i.e., (*) has the form

$$dy^{ji}/dt = \lambda y^{ji} + F_{ji} \quad , \quad dy^{jk}/dt = \lambda y^{jk} + y^{jk-1} + F_{jk}(k = 2, \ldots, h),$$

where $\lambda = \lambda(j)$, $h = h(j)$, $j = 1, \ldots, g, h(1) + \ldots + h(g) = n$. If $\mu(j) = \lambda(j)$, let $\mu^1 < \mu^2 < \ldots < \mu^f$ be the distinct numbers $\mu(j)$; for every $1 \leqq m \leqq f$ let q, p denote all integers with $\mu(q) = \mu^m$, $\mu(p) < \mu^m$; for every q, or p, let k denote all integers $1 \leqq k \leqq h(q)$, or $h(p)$. Finally let $L_m = \Sigma_q \Sigma_k |y^{qk}|^2$. (I) If $\mu = \mu^m < 0$, then, given $0 < \eta < 1$, there exist $\delta = \delta(\eta) > 0$, $T = T(\eta) > 0$, such that for all $t_0 \geqq T$ and set of $\Sigma_p h(p) + \Sigma_q h(q) = N < n$ numbers y_0, $y_0{}^{qk}$ with $\Sigma_p \Sigma_k |y_0{}^{pk}|^2 < \eta \Sigma_q \Sigma_k |y^{qk}|^2 < \eta\delta$, there exists a solution $y = y(t)$, $t \geqq t_0$, of (*) satisfying the partial set of initial conditions $y^{pk}(t_0) = y_0{}^{pk}$, $y^{qk}(t_0) = y_0{}^{qk}$, and $L_j = 0(L_m)$ if $j < m$, $\log |y(t)| = (\mu^m + 0(1))t$ as $t \to +\infty$. This asymptotic result on the "logarithmic scale" is then refined into the following one of actual asymptotic evaluation of a convenient integral $y(t)$. Let $h^* = \max h(q)$, $h_0 \geqq h_*$, $0 \leqq j_0 \leqq h_*$; Let $l(q)$, $k(q)$ be the least and the greatest integers such that $1 \leqq l(q) \leqq k(q) \leqq \min\,[h(q), h_0 - j_0]$, $h(q) - l(q) \geqq j_0$; let $y_{qk}(t)$ denote functions with $y_{qk}(t) = 0(e^{\mu t}t^{k-b})$ for $1 \leqq k \leqq k_0$, $y_{qk}(t) = e^{\lambda(q)}t^{k-k_0}/(k-k_0)! + 0(e^{\mu t}t^{k-b})$ for $k_0 \leqq k \leqq h(\ )$. (II) If $|F(t,y)| \leqq \phi(t, |y|)$ for $0 \leqq t < +\infty$, $|y| \leqq \alpha_0$, $\alpha_0 > 0$, where ϕ is a scalar function non-decreasing in $|y|$ for every t and $\int^\infty e^{-\mu t}t^{h_0-j_0-1} \phi(t, \alpha e^{\mu t}t^{j_0})\,dt < +\infty$ for every $0 < \alpha \leqq \alpha_0$, then given arbitrary constants C^{qk} not all zero, there exists a solution $y = y(t)$ of the form $y = \Sigma_q \Sigma_k C^{qk} y_{qk}(t)$ where Σ_k ranges over all $l(q) \leqq k \leqq k(q)$. Both statements (I) and (II) are shown to have partial converses and some more stringent forms.

63. Kimura, Toshifusa

Sur une généralisation d'un théorème de Malmquist. II. COMMENT. MATH. UNIV. ST. PAUL. 3:97–107 (1955).

[For part I see citation #39.] The author considers the equation (1) $dy/dx = R(x, y)$, where R is rational in y, with coefficients analytic at the origin. If (1) has an integral with the origin as an e.s.p. (essentially singular point), the equation has form (2) $x^{\sigma+1}\,dy/dx = P(x,y)/Q(x,y)$, where P, Q are polynomials in y, without common factor and not containing x as a factor; $\sigma \geqq 0$ is an integer. Results such as the following are proved. If there exists at least one zero γ of $Q(0, y)$ of multiplicity v, $v \geqq \mu(\geqq 0)$, where μ is the multiplicity of the zero γ of $P(0, y)$, then in an arbitrary vicinity of the e.s.p. $x = 0$ there is an infinity of movable critical points. The forms are given, to which (2) can be reduced by a homographic transformation when (2) is not of Riccati type and if (2) has an integral with origin for an e.s.p. in a vicinity of which this integral has no movable critical points.

64. Ku, Y. H.
Analysis of nonlinear system with more than one degree of freedom by means of space trajectories, J. FRANKLIN INST. 259,2:115–131, Feb 1955.

Paper outlines certain generalizations of the phase plane representation by considering Euclidian n-spaces with $x, \dot{x}, \ddot{x}, ..$ axes and by studying the phase trajectories in these spaces. Author gives examples of the three-phase spaces in the case of the van der Pol, Blasius, and some other equations. He considers also the cross sections by two-dimensional planes such as $(x, \dot{x})$, $(x, \ddot{x})$, $(\dot{x}, \ddot{x})$, .. which permit gaining insight into the behavior of the trajectory. Systems with several degrees of freedom are treated in a similar manner. The procedure is essentially graphical and consists in plotting the slopes of trajectories from the assumed initial conditions.

Author observes that there are two choices in carrying out his procedure. One of them deals directly with n differential equations of the first order (the equivalent system) with n dependent variables. The other one operates with one single differential equation of the nth order using derivatives as dependent variables. On its face there seems to be no difference between the two procedures by virtue of the very definition of the equivalent system, but author prefers to explain this point in a later paper.

65. Mikolajska, Z.
Sur l'allure asymptotique des intégrales des systèmes d'équations différentielles au voisinage d'un point asymptotiquement singulier. ANN. POLON. MATH 1:277–305 (1955).

This is another application of Ważewski's topological theory. Differential systems are considered

$$(1) \qquad dZ/dt = f(t, z), \quad Z = (z_1, \ldots, z_n) \quad (a < t < b),$$

where f is a real continuous vector function of t and Z in the open set Ω which is the produce of $a < t < b$, and of an open set θ of the n-dimensional real $(z_1, \ldots, z_n)$-space. If $z = \phi(t)$, $\alpha < t < \beta$, is an integral of (1) then we denote by $\phi([t_0, \epsilon])$ the subset of all points $z = \phi(t) \in \theta$ with $t_0 \leqq t < \beta$. An integral $\phi(t)$ is said to be asymptotic relative to the open subset ω θ or to the point $z_0 \in \theta$ provided $\beta = b$ and $\phi(t_0, b)$ ω, or $\beta = b$ and $\phi(t) \to b - 0$ (briefly, asymptotic rel. to ω, or rel. to z_0). A point z_0 is said to be asymptotically strongly singular (a.s.s.) provided there is a neighborhood ω of z_0 in θ such that the family of all integrals asymptotic rel. to ω is not empty and coincides with the family of all integrals asymptotic rel. to z_0. Both the following two problems are discussed: (a) sufficient conditions in order that a point z_0 be a.s.s.; (b) evaluations of the dimension of the family of the integrals which are asymptotic rel. to a point z_0 supposed to be a.s.s.. This is a theorem concerning problem (a), where z_0 is taken to be $z_0 = 0$ and is supposed a.s.s.. Suppose system (1) is written in the form

$$(2) \qquad dX/dt = F(t, X, Y), \qquad dY/dt = G(t, X, Y),$$

$$X = (x_1, \ldots, x_p), \quad Y = (y_1, \ldots, y_q), \quad p + q = n, \quad p \geqq 0, \quad q \geqq 0,$$

and suppose a uniqueness theorem holds. Given $\epsilon > 0$, $\delta > 0$, denote by S and E the sets $S = [|X| \leqq \gamma, |Y| \leqq \delta, t_0 < t < b]$, $E[|X| = \delta, t_0 < t < b]$ and by XF, YG the real scalar products of X and F, of Y and G. Suppose that for given $\gamma, \delta > 0$, we have $XF > 0$ in S and $YG < 0$ in E (where if $p = 0$ [$q = 0$] only the second [first] relation is considered). Then the family of the solutions asymptotic rel. to $z_0 = 0$ is at least q-dimensional [has at least one element if $q = 0$].

66. Petrovskii, I. G., and Landis, E. M.
On the number of limit cycles of the equation $dy/dx = M(x, y)/N(x, y)$, where the M and N are polynomials of second degree. DOKL. AKAD. NAUK SSSR N.S. 102:29–32 (1955). (In Russian)

Outline of proof that the real equation

$$(1) \qquad dy/dx = M(x, y)/N(x, y),$$

where M, N are real quadratic polynomials, has at most three limit-cycles. As Bautin has shown [Mat. Sb. N.S. 30(72), 181–196 (1952)] that there are systems (1) with three limit-cycles, three is the true upper bound of the number of limit-cycles.

The method of the authors rests upon the following most interesting consideration. Let R_4 be the complex projective completion of the space of the complex variables x, y (the infinite region is a complex line = a sphere). Let Φ denote a complete complex solution $y = \varphi(x)$ of (1) in R_4. One augments Φ by the poles and the branch-points of finite order of φ. An oriented Jordan curve on Φ is called a cycle. Lemma 1. A real oriented limit-cycle of (1) is a cycle not ~ 0 on the appropriate Φ. Two limit-cycles on the same Φ are not homologous to one another.

A cycle on Φ is said to be simple if its projections on the (complex) planes x, y are Jordan curves. Cycles $L_1, \ldots, L_k$ on various Φ's are said to be properly situated whenever no two of their projections on the x-plane intersect. Lemma 2. The number of limit-cycles of (1) does not exceed the maximum number of properly situated simple cycles that are neither ~ 0 nor to one another.

Using now continuity and limiting relations in the space of the coefficients of P, Q, the asserted result follows.

67. Terracini, A.
Aspetti proiettivi nella teoria delle equazioni differenziali. REND. SEM. MAT. MESSINA 1:115–119 (1955).

Let (u, v) denote curvilinear coordinates of a point P on a surface S. A set of ∞^2 curves of the cubic type is composed of the integral solutions of a second order differential equation of the form

$$v'' = A(u,v) + B(u,v)v' + C(u,v)v'^2 + D(u,v)v'^3. \tag{1}$$

Examples of such systems are the geodesics of the surface S, natural families, isogonal trajectories, and axial systems. The class of all sets of ∞^2 curves of the cubic type is topologically invariant, although no two such sets are topologically equivalent, in general. A system of ∞^3 curves of the type (F) consists of the integral solutions of a third order differential equation of the form

$$v''' = F(u,v,v') + G(u,v,v')v'' + H(u,v,v')v''^2. \tag{F}$$

The class of all such sets of the type (F) is topologically invariant. When F is identically zero, a system of ∞^3 curves of the type (F) is said to be of the type (G). The types (G) and (F) arose in the study of dynamical families of ∞^3 curves on a plane and also on a surface S. The type (G) is not topologically invariant but is projectively invariant when (u, v) are considered to be non-homogeneous projective coordinates of a point P on a plane S. The author develops certain interesting projective properties of the systems of curves of the cubic type and the types (F) and (G).

68. Boruvka, O.
Uber eine Verallgemeinerung der Eindeutigkeitssatze fur Integrale der Differentialgleichung $y' = f(x, y)$. ACTA FAC. RERUM NAT. UNIV COMENIAN. MATH. 1:155–167 (1956)
(In Czech and Russian summaries)

The author proves a uniqueness theorem for the ordinary differential equation $y' = f(x, y)$. The statement of the theorem is too lengthy to be reproduced here in full. However, loosely speaking, it goes as follows: Suppose we can construct continuous functions $\varphi(x;u,v)$ and $\Phi(x;u,v)$ satisfying a number of conditions, the most important of which are (1) $\varphi(x;u,v) = 0$ if $u = v$, >0 otherwise, (2) $\lim_{x\to\xi} \varphi(x;u(x),v(x)) = 0$ for arbitrary solutions $U(x)$, $v(x)$ of $y' = f(x,y)$ passing through (ξ,η), (3) $\varphi_x(x,u,v) + \varphi_u(x,u,v)\, f(x,u) + \varphi_v(x,u,v)\, f(x,v) \leqq \Phi(x;u,\varphi(x,u,v))$. (4) For each solution $u(x)$ of $y' = f(x,y)$ passing through (ξ,η), $z(x) \equiv 0$ is the only solution to $z' = \Phi(x,u(x),z)$ near $x = \xi$; then uniqueness holds for $y' = f(x,y)$ near (ξ,η). The theorem includes as special cases criteria of Peano, Tonnelli, Bompiani, Osgood and Tamarkin, and Lipschitz.

69. Burton, L. P.
Conditions which preclude the existence of critical solutions of an ordinary differential system. PROC. AMER. MATH. SOC. 7:791–795 (1956)

L'A. considera il sistema di equazioni

$$y_{i+1}' = f_{i+1}(x, y_i) \quad (i = 1, 2, \ldots, n; y_{n+1} = y_1)$$

nell'ipotesi che un numero dispari delle f_{i+1} siano funzioni descrescenti di y_i e le rimanenti crescenti. Se $\{y_i(x)\}$ e $\{z_i(x)\}$ sono due soluzioni del sistema definite nell'intervallo $x_0 < x < x_0 + a$, la prima verificante le condizioni iniziali: $y_i(x_0) = y_{i0}$, la seconda tale che il punto $(x_0, y_{10}, \ldots, y_{n0})$ sia di accumulazione per il suo diagramma, è impossibile che in ogni intervallo $x_0 < x < x_0 + b$ con $b < a$ sia per ogni $k: z_k < y_k$.

70. Cecik, V. A.
On a certain class of systems of ordinary differential equations with singularity. DOKL. AKAD. NAUK SSR N.S. 108:784–786 (1956) (In Russian)

The author investigates the system of differential equations

(1) $$y_k' = f_k(x, y_1, y_2, \ldots, y_n) \quad (k = 1, \ldots, n)$$

with initial conditions

(2) $$\lim_{x \to 0} y_k(x) = 0 (k = 1, \ldots, n).$$

The right hand members of (1) are assumed to be continuous functions of the variables for $x > 0$. For $x = 0$ they may be unbounded. Such systems are called singular. Here the author considers a system of singular differential equations, where the right hand members are not bounded by summable functions. It is also assumed that the functions f_k and f_k' are continuous in the region

$$D: 0 < x \leqq b, \ |y_i| \leqq a (i = 1, \ldots, n).$$

The following theorems are proven:
Theorem 1. Let the following inequalities be satisfied in region D:

1) $|f_k(x, y_1, \ldots, y_{k-1}, 0, y_{k+1}, \ldots, y_k)| \leqq \psi(x), \ (k = 1, \ldots, n);$

2) $$f_{k, y_k}'(x_1, y_1, \ldots, y_n) \leqq \psi(x) \quad (k = 1, \ldots, l),$$

where $\psi(x)$ is a continuous summable function, and $0 \leqq l \leqq n$;

3) $$f_{k, y_k}'(x, y_1, \ldots, y_n) \geqq \overline{\psi}_k(x) \quad (k = l + 1, \ldots, n),$$

where $\overline{\psi}_k(x)$ are positive non-summable functions. Then system (1) has at least one solution satisfying conditions (2).

Theorem 2. Let the condition of theorem 1 be satisfied and let $l < n$. Then the solution of problems (1), (2) is not unique. Many solutions in this case depend on $n - l$ parameters.

Theorem 3. Let the conditions of theorem 1 be satisfied and $l = n$. In the region D let there exist derivatives $f_{k, yi}'(i = 1, \ldots, n)$, which for $i \neq k$ satisfy the inequalities

$$|f_{k, y_i}'(x, y_1, \ldots, y_n)| \leqq \psi(x).$$

Then the system (1) has a single solution which satisfies the zero initial conditions (2).

The above results when applies to the single equation

(5) $$y' = f(x, y)$$

lead to the following theorems.

Theorem 4. If a continuous differentiable function $\varphi(x)$ ($\varphi(0) = 0$) exists, such that $f(x, \varphi(x))$ is summable; then, if $f_y'(x, y)$ has a summable function as an upper bound, equation (5) has a single solution satisfying the zero initial condition.

If $f_y'(x, y)$ is bounded below by a positive non-summable function, then equation (5) has a continuum of solutions.

Theorem 5. Let there be a continuous differentiable function $\overline{\varphi}(x)$ in the interval $[0, \epsilon]$ $(0 < \epsilon \leqq b)$, such that

1) $f_y'(x, y) > 0$ for $y > \overline{\varphi}(x)$, 2) $f(x, y) > \max \overline{\varphi}'(x)$,

3) $\varphi(0) = 0$, 4) $\int_0^\epsilon f(x, \varphi(x))\,dx = +\infty$. Then equation (5)

has no solution which satisfies the initial condition $y(0) = 0$.

For the general case the following theorem is stated:

Theorem 6. Let the conditions of theorem 3 be satisfied and let

$$y_k = y_k(x, x_0, y_{1,0}, \ldots, y_{n,0}) \quad (k = 1, \ldots, n)$$

be the solution of system (1) which passes through the point $x = x_0$, $y_k = y_{k,0}$ $(k = 1, \ldots, n)$. Then the functions $y_k = y_k(x, x_0, y_{1,0}, \ldots, y_{n,0})$ are continuous functions of the arguments in the region $0 < x_0 \leqq x \leqq b$, $|y_{k,0}| \leqq a$; $x_0 = 0$, $y_{n,0} = 0$ $(k = 1, \ldots, n)$.

71. Conti, Roberto
Sulla prolungabilita delle soluzioni di un sistema di equazioni differenziali ordinarie.
BOLL. UN. MAT. ITAL. (3), 11:510–514 (1956)

Consider the system of differential equations

(1) $$\dot{x} = f(t, x) \quad (a < t < b),$$

where $f(t, x)$ is a real continuous function defined in S: $a < t < b$, $0 \leqq \Sigma x_i^2 < +\infty$. Theorem: Let $\omega(t, u)$ be a real continuous function in S_1: $a < t < b$, $0 < u < +\infty$; let $u_0(t)$ be the maximum solution of the equation (2) $\dot{u} = \omega(t, u)$, $u(t_0) = u_0$ and let T^+ be the upper limit of the values of t for which $u_0(t)$ is defined. Let $V(t, x)$ be a real continuous, non-negative function defined in S and having a continuous first derivative in S. Suppose $V_x(t, x) f(t, x) + V_t(t, x) \leqq \omega(t, V(t, x))$ at all points $(t, x) \in S$, $V(t_0, x^0) = u_0$, and $\lim_{\|x\| \to +\infty} V(t, x) = +\infty$, $\|x\| = (\Sigma x_i^2)^{1/2}$. If $T^+ = b$ and the above hypotheses are satisfied, then the solution of system (1) with initial conditions $x(t_0) = x_0$ can be extended in t. This theorem generalizes a previous result of the author [same Boll. (3) 11 (1956), 344–349].

72. Corduneanu, C.
Quelques considérations concernant certains systèmes non linéaires d'équations différentielles. ACAD. R. P. ROMINE. FIL. IASI. STUD. CERC. STI. MAT. 7(2):13–32 (1956) (In Romanian. Russian and French summaries)

In the first part of this paper, the author investigates the system (*) $dx_i/dt = f_i(t, x_i) + g_i(t, x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ under the following conditions: (i) f_i and g_i are continuous for $t \geqq 0$ and for all real x_i; (ii) $f_i(t, 0) + g_i(t, 0, \ldots, 0)$ are bounded on $t \geqq 0$; (iii) there exist positive numbers m and M so that for each $i = 1, \ldots, n$ either $-M \leqq \partial f_i/\partial x_i \leqq -m$ or $m \leqq \partial f_i/\partial x_i \leqq M$; (iv) the g_i satisfy a Lipschitz condition, and the Lipschitz constant is $< m\, n$. Under these conditions the author establishes the existence, and some properties, of bounded solutions. He investigates the stability of these solutions, and extends his investigations to the case $-\infty < t < \infty$ and to the case where the x_i are restricted to a bounded region, $\Sigma |x_i| \leqq a$.

In the second part of the paper he investigates similarly the system obtained by replacing dx_i/dt in (*) by d^2x_i/dt^2.

73. Demidovic, B. P.
On the existence of a limiting regime of a certain non-linear system of ordinary differential equations. AMER. MATH. SOC. TRANSL. (2)18:151–161 (1961)

A finite symmetric matrix $A = (a_{ij})$ is positive [negative] definite whenever its characteristic roots λ_i [the $-\lambda_i$] are all positive. It is uniformly positive [negative] definite (= u.p.d. u.n.d.) whenever the λ_i [the $-\lambda_i$] are above a certain $h > 0$.

Consider a system

$$(1) \qquad \dot{X} = F(X) + G(t),$$

where X, F, G are real n-vectors, F is of class C^1 in the components $x_1, \ldots, x_n$ of X for all values of these variables and $G(t)$ is continuous with period T. The author discussed above all the existence of a periodic solution to which tend all the other solutions.

Let $f_i(X)$ be the components of F and $_i$ and n-vector with components ξ_{ij}. Consider the matrix $W = (f_{ix_j}(\ _i))$ and the related symmetrix matrix $F(F) = 1/2(W + W')$. Theorem 1. If $J(F)$ is u.p.d. or u.p.n. then (1) has a solution $X_0(t)$ of period T, and for $t \to \pm\infty$ all solutions tend to $X_0(t)$. Hence $X_0(t)$ is unique.

Suppose now that in (1) $G(t)$ is merely bounded for all t. Theorem 2. Under the same assumptions for F there exists then a bounded solution $X_1(t)$ to which all other solutions tend asymptotically and $X_1(t)$ is again unique.

74. Gubar', N. A.

Characterization of compound singular points of a system of two differential equations by means of rough singular points of closely related systems. AMER. MATH. SOC. TRANSL. (2)18:117–149 (1961)

The author studies the real analytical systems in two variables

$$(1) \qquad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

which are reducible, by a linear transformation of coordinates and suitable choice of time to one of the two types

$$(a) \qquad \dot{x} = X(x, y), \quad \dot{y} = y + Y(x, y),$$

$$\dot{x} = y + X(x, y), \quad \dot{y} = Y(x, y),$$

where X, Y are power series in x, y beginning with terms of degree at least two. His concern is the local phase portrait at the origin. The basic method utilized is to vary slightly P and Q in (1) to obtain

(2) $$\dot{x} = P + p, \quad \dot{y} = Q + q,$$

where p, q together with their partials of order $\leqq r$ are $< \epsilon$ in absolute value near the origin. There appear then a certain number of ordinary singularities: nodes, foci, saddle points (these are the "rough" singularities) near the origin. The detailed classification are obtained from an examination of the behavior of these ordinary singularities. The treatment is wholly analytical and altoghether too complicated for a detailed description. It may be said that systems of type (a) have been discussed in detail by Bendixson [Acta Math. 24 (1901), 1–88] and those of type (b) more recently by Andreev [Vestnik Leningrad. Univ. 10 (1955), 43–65; no. 8]. There is also on type (b) a recent Note by Barocio [Ann. of Math. Studies 36, Contributions to the theory of nonlinear oscillations, III. 127–135, 1956]. [Additional references: N. B. Haimov, Uc. Zap. Stalinabadsk. Inst., 3–30, 1952.]

75. Hajek, Otomar

Singularities of differential equations, I, II.

POKROKY MAT. FYS. ASTR. 1:551–559

(1956), 2:137–144 (1957) (In Czech)

By a 'singular point' of the equation (1) $dy/dx = F(x, y)$ (x, y being real, F a continuous function), the author means a point at which uniqueness does not hold. From the results proved follows for instance this assertion: let S be the set of singular points, S^+ [S^-] the set of points at which uniqueness fails with increase [decrease] in X, U^a the derived set of a set U; if $S \quad S^{+a} \quad S^{-a}$ is a denumerable set, then S has measure 0.

A characterization is given of set of functions $\{f_a\}$ with the property that there exists an equation (1) of which all the f_a are solutions. An example is constructed in which the closure of the set S is the y-axis. A large number of other results are proved, many of which are known or evident.

76. Herbst, Robert Taylor

The equivalence of linear and nonlinear

differential equations. PROC. AMER.

MATH. SOC. 7:95–97 (1956)

Thomas [same Proc. 3 (1952), 899–903] raised the question: What equations of order n have general solution expressible as $F(u_1, \ldots, u_n)$, where $u_1, \ldots, u_n$ constitute a variable set of solutions of a fixed linear differential equation? The author gives the complete answer to this question for linear homogeneous equations of the second order: If u, v are variable independent solutions with Wronskian w of the equation $Y'' - w(x)^{-1} w'(x) Y' + q(x) Y = 0'$ where w and q are given functions, then the equation $y'' - w^{-1} w' y' = f(y, y, ' w, q)$ has general solution $y = F(u, v)$ if and only if $f = -qZ(y) + A(y) y'^2 C(y)$, where Z, A, C satisfy $ZC' + (3 - AZ) C = 0$, $Z' - AZ = 1$.

77. Kamke, E.

DIFFERENTIAL EQUATIONS OF REAL FUNCTIONS [Differential gleichungen Reeller Funktionen] , Leipzig, Akademische Verlagsgescellschaft, 1956, 442p.

This third edition of the well-known treatise by Professor E. Kamke gives a systematic and up-to-date survey of the theory of the differential equations of real functions. The book is divided in two parts. The first part deals with ordinary differential equations whereas the second part deals with partial differential equations of the first and second order.

The material in the book is presented in a very clear and lucid manner and with mathematical rigor. Many well-chosen examples illustrate the chapters.

78. Lefschetz, Solomon

On a theorem of Bendixson. BOL. SOC. MAT. MEXICANA (2)1:13-27 (1956)

In his memoir on differential equations [Acta Math. 24 (1901), 1-88], Bendixson investigates systems of the form (*) $x' = X(x, y)$, $y' = Y(x, y)$ ($' = d/dt$) where X and Y are real and holomorphic at the origin, $X(0, 0) = Y(0, 0) = 0$, and the origin is an isolated critical point. Bendixson showed that by a finite sequence of quadratic transformations one may replace the study of the critical point at the origin by that of a finite set of ordinary points or isolated critical points where at least one characteristic root is non-zero: the so-called Bendixson systems. It is not easy to obtain the quadratic transformations in any practical instance. In the present paper the author gives a constructive process to reduce the critical point, in a finite number of setps, to ordinary or Bendixson types.

Let n be the lowest degree present in the series for X and Y, and let X_n and Y_n be the homogeneous polynomials of degree n composed of the terms of degree n in these series. Let a TO-curve by a characteristic of (*) tending to the origin in a definite direction. If a TO-curve is tangent to $ax + by = 0$, then $ax + by$ is a factor of $yX_n - xY_n$. The author shows first that if $ax + by$ is not a common factor of X_n

and Y_n, so that $ax + by = 0$ is not a common tangent to $X = 0$ and $Y = 0$ at the origin, it is possible to determine quite simply the TO-curves, if any, tangent to $ax + by = 0$ at the origin. The cases in which $yX_n - xY_n$ does or does not vanish identically are treated separately.

The case in which $ax + by = 0$ is a common tangent to $X = 0$ and $Y = 0$ at the origin is much more complicated. The author shows that by appropriate changes of variables it is possible to obtain a system with the following properties. X_n and Y_n each contain a term ay^n. When $X = 0$ and $Y = 0$ are solved for y in terms of x, each has the same numbers of real and complex branches, and each branch is represented with y equal to a series in powers of $x^{1/q}$ for some integer q. Neither axis is a common tangent $X = 0$ and $Y = 0$, and if $y = mx$ is a common tangent to $X = 0$ and $Y = 0$ have the same number of branches tangent to it. If of the branches of $X = 0$ tangent to $y = mx$, J have the partial series $mx + \ldots + \sigma x^{(p-1)/c}$ in common, then exactly J branches of $Y = 0$ will have these same terms in common. With this assumption, the author gives a complicated but straightforward analysis of the critical directions $y = mx$, covering all possible cases of TO-curves in such a direction.

The author shows that there is always an order for a TO-curve. The order is a number μ such that as $x \to 0$ along the curve, $|y|\ |x|^{-\mu+\epsilon} \to 0$ and $|y|\ |x|^{-\mu-\epsilon} \to +\infty$ for any $\epsilon > 0$. This justifies the process used by Briot and Bouquet [Ince, Ordinary differential equations, Longmans-Green, London, 1926, p. 297].

79. Sansone, G.; and Conti R.

EQUAZIONI DIFFERENZIALI NON LINEARI.

Edizioni Cremonese, Roma, 1956, 647p.

(In Italian)

Nel presente volume sono studiate, con mezzi prevalentemente geometrici, le proprietà asintotiche delle soluzioni dei sistemi di equazioni differenziali ordinarie di tipo normale

$$\text{(A)}\qquad \frac{dx_i}{dt} = f_i(t; x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n).$$

Il primo capitolo è dedicato ad una sintetica esposizione dei risultati di carattere generale relativi ai sistemi (A). La parte centrale del volume è invece dedicata al caso particolare dei sistemi

$$\text{(B)}\qquad \frac{dx}{dt} = f(t; x, y), \quad \frac{dy}{dt} = g(t; x, y),$$

che traducono i numerosi problemi suggeriti dalle scienze applicate corrispondenti a sistemi meccanici con un sol grado di libertà. In questa parte del volume trova posto una rielaborazione ed una rassegna, corredata da un'ampia bibliografia che giunge fino al 1955, di molti dei risultati più importanti da quelli classici fino ai più recenti. Ai sistemi (B) autonomi, vale a dire con f e g indipendenti da t, sono dedicati i capitoli II, III, IV, V, VI, secondo il seguente schema: Cap. II. Caso lineare, sistemi omogenei, caso analitico, problema del centro, singolarità all'infinito. Cap. III. Singolarità di Briot-Bouquet. Cap. IV. Teoria generale dello spazio delle fasi: insieme limite, cicli piani, punti singolari isolati e loro classificazione, indice, sistemi di vettori sul cilindro e sul toro. Cap. V. Problema della conservazione di determinate configurazioni nel piano delle fasi sotto l'azione di una perturbazione costante nel tempo e, in particolare, il problema della stabilità di struttura. Nel cap. VI sono prese in esame alcune delle più importanti equazioni della forma $d^2x/dt^2 = f(x, dx/dt)$, quali l'equazione del pendolo, l'equazione di van der Pol, l'equazione di Liénard e sue generalizzazioni, un'equazione della dinamica dei fili, ecc. Il cap. VII tratta invece i sistemi (B) non autonomi e costituisce una rassegna coordinata di numerosi risultati interessanti le applicazioni finora pubblicati solo in riviste di matematica e di tecnica. Nei due successivi capitoli viene ripreso lo studio dei sistemi (A) e precisamente nel cap. VIII viene tratteggiata la teoria dei sistemi lineari e nel cap. IX è invece esposta la teoria della stabilità dei sistemi (A) nelle diverse accezioni (secondo metodo di Liapunov, stabilità dei sistemi lineari, stabilità in prima approssimazione, equivalenza asintotica). Molti di questi risultati appaiono per la prima volta in un volume.

Questo volume si inserisce molto degnamente nella Collana delle Monografie del Consiglio Nazionale delle Ricerche italiono, e può essere consultato utilmente sia dai cultori di matematica pura che da quelli di matematica applicata.

80. Villari, Gaetano

Cicli limite e fusione di separatrici. ANNALI DI MATEMATICA; PURA ED APPLICATA 42(Series 4):259–277, 1956.

This paper investigates for a generalized Liénard equation

(1) $$\ddot{x}(t) + f(x, \lambda)\dot{x}(t) + g(x) = 0,$$

involving a nonnegative parameter λ, the generation of limit cycles, and in general, the nature of the dependence of the solutions on λ. Solutions are studied in the phase space (x, y), $x = x(t)$, $y = \dot{x}(t)$, under the hypotheses: (a) f is continuous in $-\infty < x < \infty$, $0 \leqq \lambda < \infty$, is even in x, and for some α_λ (depending on λ) satisfies $f(x, \lambda) \lesseqgtr \beta$. (Thus the trajectories in the phase space are symmetric about the origin.) Under these hypotheses it is shown that if λ is such that $\beta \leqq \alpha_\lambda$, then the system has no limit cycles, while for $\beta > \alpha_\lambda$ the system has at most one closed trajectory.

Under the additional assumptions (c) $f_x < 0$, $f_\lambda > 0$ for $x, \lambda > 0$, subscript denoting differentiation, (d_1) $\lim_{\lambda \to \infty} \alpha\lambda = \infty$, ($d_2$) $\lim_{\lambda \to 0+} \alpha\lambda = 0$, the author proves the following. There is a unique $\lambda^* > 0$ such that: 1. For $\lambda > \lambda^*$, (1) has no periodic solutions, and the trajectories in the phase space have a certain behavior-described by a diagram. 2. For $\lambda = \lambda^*$, (1) has no periodic solutions, and some trajectories in the phase space through singular points have fused together, as shown by another diagram. 3. For $\lambda < \lambda^*$, (1) has exactly one periodic solution (and its behavior as $\lambda \to \lambda^*$ is studied).

The proof involves a rather detailed and careful analysis of the trajectories in the phase space, which cannot be described here. It is related to E. Leontovic , Dokl. Akad. Nauk SSSR (N. S.) 78 (1951), 641–644.

81. Whyburn, William M.

Differential systems with boundary conditions at more than two points. In PROCEEDINGS OF THE CONFERENCE ON DIFFERENTIAL EQUATIONS (DEDICATED TO A. WEINSTEIN), College Park, Md. University of Maryland Book Store, 1956, p. 1–21

This paper is concerned with differential systems consisting of a set of n first order ordinary differential equations in $y_i(x)$, $(i = 1, \ldots, n)$, together with boundary conditions that may involve the values of these functions at more than two points of a fixed interval $X: a \leqq x \leqq b$. For systems involving "initial" values at more than one point there is given a brief survey of known results; in the case of a system of the form

$$(*) \quad y_n' = f_n(x, y), \quad y_i' = f_i(x, y) + g_i(x, y) y_{i+1} \quad (i = 1, \ldots, n - 1),$$

in $y = (y_1, \ldots, y_n)$ the author employs a theorem of R. Conti [Ann. Mat. Pura Appl. (4) 35 (1953), 155–182] to establish various results concerning the dependence of solutions on the initial points and initial values. There is established also an existence theorem for a system involving a set of differential equations somewhat more general than (*) and initial values at more than one point. The last two sections of the paper are devoted to a survey of the literature on systems consisting of a set of linear differential equations and boundary conditions involving the values of the $y_i(x)$ at the endpoints of X and also at the points of a given subset of $a < x < b$ of suitable character, with special attention to the case of discontinuous solutions restricted by interface conditions.

82. Zindler, R. E.
A note on L. E. Reizin's paper "Behavior of integral curves of systems of three differential equations near a singular point."
PROC. AMER. MATH. SOC. 7:283–289 (1956)

In the first part of the paper the author corrects a careless statement of hypotheses in Theorem 2 of a paper of Reizin′ [Latvijas PSR Zinātnu Akad. Vēstis 1951, no. 2(43), 333–346] . In the second part an alternative form of the hypotheses is proposed.

83. Matuda, Tizuko
On the behavior of the solutions of an ordinary differential equation of first order near a non-movable singularity. SUGAKU 8:139–148 (1956/57) (In Japanese)

A Japanese edition of the author's papers "Sur les points singuliers des équations differentielles ordinaires du premier ordre. I, II, III, IV, V" [Nat. Sci. Rep. Ochanomizu Univ. 2 (1951), 13–17: 4 (1953), 36–39; 5 (1954), 1–4, 175–177; 8 (1957), 1–6] together with some supplementary results.

84. Bihari, Imre.
On the unicity of the solutions of the ordinary first order differential equations. MAT. LAPOK 8:115–119 (1957) (In Hungarian. English and Russian summaries)

The author gives conditions under which if $u' = \omega(x, u)$ has more than one solution passing through the origin, then so does $y' = f(x, y)$; here f, ω are continuous and $\omega \geqq 0$. Among other things, it is required that $\omega(x, 0) = 0$, and that

$$|f(x, y) - f(x, \phi)| \geqq \omega(x, |y - \phi|),$$

where $\phi(x)$ is some solution of $y' = f(x, y)$ with $\phi(0) = 0$.

85. Boetti, Giovanni

Sopra una classe di equazioni differenziali ordinarie del primo ordine dotate di una singolarità mista. I. REND. MAT. E APPL. 5(16):207–220 (1957)

Consider $dy/dx = (ax + by + f(x, y))/g(x)$, $ab \neq 0$, f, g continuous in a neighborhood of the origin, $g(0) = 0$, $g \neq 0$ otherwise, $\int dx/g(x)$ divergent at the right and at the left of the origin. Theorem A: If $f \equiv 0$ and g does not change signs, the characteristic behaves as in a saddle point in one of the halfplanes bounded by the y-axis, as in a node in the other halfplane; if g changes sign at $x = 0$ the singularity is either a saddle point or a node. Theorem B: If f satisfies a Lipschitz condition and $|f| \leqq M|y|$ with $M < |b|$, the behavior is the same as in Theorem A.

86. Chan, Chan Khun

The existence and uniqueness of solutions of boundary problems for non-linear ordinary differential equations. DOKL. AKAD. NAUK SSSR N. S. 113:1227–1230 (1957) (In Russian)

The author states existence and uniqueness results for boundary value problems of the type:

(1) $y^{(n)} = \varphi(x, y, y', \ldots, y^{(n-1)})$, $y^{(k)}(a) = y_0^{(k)}$ $(k = 0, 1, \ldots, n-2)$, $y^{(i)}(b) = y_1^{(i)}$, where i is one of the numbers $0, 1, \ldots, n-2$; (2) $u'' = F(t, u, u')$, $u(a) = \alpha$, $u(b) = \beta$, where u is an n-dimensional vector; (3) $y'' = \varphi(x, y, y') + \alpha y$, α positive constant, $y(0) = y(1)$, $y'(0) = y'(1)$; (4) $w'' = f(x, w, w')$, $w(0) = w(1) = 0$.

It is stated that proofs depend on general implicit function theorems, continuation methods, and the Schauder-Leray result. An application is indicated to the problem

$$u_t = u_{xx} - f(x, u, u_x), \quad u(0, t) = u(1, t) = u(x, 0) = 0.$$

87. Chang, Li-ling

Topological structure of integral curves of the differential equation

$$\frac{dy}{dx} = \frac{ax^2 + bxy + cy^2 + \varphi(x,y)}{ex + fy + \psi(x,y)}$$

(when $\varphi = \psi = 0$ the fraction is irreducible) in the neighborhood of the origin. ADVANCEMENT IN MATH. 3:650–654 (1957) (In Chinese)

For the case $\epsilon > 0$, the author finds the criteria for the topological structure of integral curves in the neighborhood of the origin.

88. Chang, Die

Topological structure of integral curves of the differential equation

$$\frac{dy}{dx} = \frac{ax^3 + bx^2y + cxy^2 + dy^3}{a_1x^3 + b_1x^2y + c_1xy^2 + d_1y^3}$$

ADVANCEMENT IN MATH. 3:234–245 (1957) (In Chinese. English summary)

Author's summary: "The topological structure of integral curves are classified into fifteen classes with algebraic criteria."

89. Corduneanu, C.

Quelques remarques concernant certaines classes de systemes differentiels. AN. STI. UNIV. "AL. I. CUZA" IASI. SECT. I N.S. 3:37–44 (1957) (In Russian and Romanian summaries)

The paper contains extensions, to systems of ordinary differential equations of previous results for a single equation [C. Corduneanu, same An. 2 (1956), 33–52; C. R. Acad. Sci. Paris 245 (1957) 21–24].

Sharper results are also obtained in the case of a single equation. For example, suppose that $f(x,y)$, $\partial f(x,y)/\partial y$ are real valued and continuous, with $|f(x,y)|$ bounded, and that there exists a number $m > 0$ such that $\partial f/\partial y \leqq -m < 0$, all on $x \geqq 0$, $-\infty < y < +\infty$. Then there exists exactly one solution $y(x)$ of the initial value problem $dy/dx = f(x,y)$, $x \geqq 0$; $y(0) = y_0$, which is bounded in absolute value on $x \geqq 0$.

90. Ehrmann, Hans

Uber die Existenz der Losungen von Randwertaufgaben bei gewohnlichen nichtlinearen Differentialgleichungen zweiter Ordnung. MATH. ANN 134:167–194 (1957)

This paper is concerned with the existence of solutions to the boundary value problem (1) $y'' = f(x,y,y')$ $a_1y(x_0) + a_2y'(x_0) = a_0$, $b_1y(x_1) + b_2y'(x_1) = b_0$; where $a_0^2 + a_1^2 \neq 0$, and $b_0^2 + b_1^2 \neq 0$. The function $f(x,y,v)$ is required to be defined in the region G: $x_0 \leqq x \leqq x_1$, $-\infty < y < \infty$, $-\infty < v < \infty$ and to satisfy sufficient conditions for the existence of a continuously differentiable solution of the equation (2) $y'(x) - y'(\alpha) = \int_\alpha^x f(t, y(t), y'(t))dt$ for $x_0 \leqq \alpha < x_1$ which is also required to be continuous in α, $y(\alpha)$, and $y'(\alpha)$ for $x_0 \leqq \alpha < x_1$, $-\infty < y(\alpha) < \infty$, $-\infty < y'(\alpha) < \infty$. The author proves several existence theorems, all of which seem to follow relatively easily from the following main theorem: If, in addition to the above conditions on f, there exists a continuous function $h(y)$ defined for $-\infty < y < \infty$ and constants K_1, K_2, and K_3 such that for all $(x,y,v) \in G$ $|f(x,y,v) - h(y)| \leqq K_1|y| + K_2|v| + K_3$; and if for any $C > 0$ there exist positive constants ϵ, C, and Y such that if $x_0 \leqq x \leqq x_1$, $|y| \geqq Y$, and $|v| \leqq C|y|^{1+\epsilon}$, then $f(x,y,v)/y \leqq -C$; then there exists an infinite number of solutions of (2) with $\alpha = x_0$ which satisfy the boundary conditions of (1).

91. El'sgol'ts, L. E.

DIFFERENTIAL EQUATIONS. RUSSIAN MONOGRAPHS AND TEXTS ON ADVANCED MATHEMATICS AND PHYSICS, VOL. IV. New York, Gordon and Breach Publishers, Inc.; Delhi, India, Hindustan Publishing Corp. 1961, 360p.

From the original Russian edition. (Gosudarsty. Izdat. Tehn. - Teor. Lit., 1957). This textbook is intended for students in mathematics and physics in the state universities in the USSR. There are six chapters on the topics: differential equations of first order, of higher order, systems of equations, theory of stability, equations with deviating argument, partial equations of first order. There are exercises with solutions.

92. Hudai-Verenov, M. G.
Some theorems on limit cycles for the equation of Lienard. AMER. MATH. SOC. TRANSL. 2(18):163–171 (1961)

The equation considered is $\ddot{x} + f(x)\dot{x} + x = 0$, or rather the equivalent system

$$(*) \qquad \dot{x} = y, \qquad \dot{y} = -f(x)y - x.$$

Th. 1: Let there exist $x_1 < x_1^* < 0 < x_2^* < x_2$ such that: (a) $f(x)$ is monotonic on $[-\infty, x_1]$ and $[x_2, +\infty]$; (b) it is monotone increasing on $[x_1^*, 0]$, monotone decreasing on $[0, x_2^*]$, or vice versa; (c) $|f| < 2$ on $[x_1, x_1^*]$ and $[x_2^*, x_2]$. Then if (*) has limit cycles they are star-like (each vector from the origin intersects the limit-cycle in just one point).

From Th. 1 follows Th. 2 [Sansone, Univ. e Politec. Torino. Rend. Sem. Mat. 10 (1951), 155–171, Massera, Boll. Un. Mat. Ital. (3) 9 (1954), 367–369: If $f(0) > 0$, f is continuous monotone increasing for $x \leqq 0$, monotone decreasing for $x \geqq 0$, then there is at most one limit-cycle. Observe that "monotony" may be replaced above by "non-increasing" or "non-decreasing". For even $f(x)$ with four roots and going downwards beyond the roots, also under special analytical conditions, it is shown that there are at least two limit-cycles.

93. Iwano, Masahiro
Intégration analytique d'un système d'équations différentielles non linéaires dans le voisinage d'un point singulier. I. ANN. MAT. PURA APPL. 4(44):261–292 (1957)

The author considers the system of equations

$$x^{\sigma+1}\frac{dy_j}{dx} = f_j(x, y_1, \ldots, y_n) \qquad (j = 1, \ldots, n),$$

where the functions f_j are holomorphic in the domain $|x| < r$, $|y_1| < n, \ldots, |y_n| < n$, and applies methods of [same Ann. 19 (1940), 35–44; Mem. Fac. Sci. Kyūsyū Univ. A. 2 (1942), 125–137; 4 (1949), 9–21; 5 (1950), 61–63] to the study of solutions in the neighborhood of $x = 0$. The results generalize and sharpen results obtained previously, for the same or closely related systems of equations, by Hukuhara and by Malmquist [J. Malmquist, Acta Math. 73 (1940), 87–129; 74 (1941), 1–64, 109–128]. The analysis is extremely complicated, and not even the results can be detailed here. If the y's are written as suitable formal power series in x and new dependent variables $z_1, \ldots, z_n$, the system of equations is reduced to a form such that solutions belonging to a certain class can be obtained by quadratures. Thus there is determined a class of formal solutions of the original system, each of these solutions being a set of formal infinite series involving certain auxiliary functions of x and arbitrary constants. The principal novel problem considered by the author is that of determining sufficient conditions under which the formal series converge and yield actual solutions. It is shown that the series do converge provided the initial values of the y's are suitably restricted in absolute magnitude, and provided that x is restricted to a simply-connected region bounded by two rays from the origin and by an arc of a certain auxiliary curve. The proof as given involves some undesirable restrictive assumptions. The author indicates that these restrictions will be removed in another paper.

94. Magiros, Dem. G.
On the singularities of a system of differential equations, where the time figures explicitly.
PRAKT. AKAD. ATHENON 32:448–451 (1957)
(In Greek summary)

An abridged version of results later published elsewhere [Information and Control 1 (1958), 198–227]. (See citation No. 114).

95. Mitrinovitch, D. S.
Compléments au traité de Kamke. V
UNIV. BEOGRADU. PUBL. ELEKTROTEHN.
FAK. SER. MAT. FIZ. 11:10 (1957) (In Serbo-Croatian summary)

The author notes that the quation

$$y^{(n+k)} - f(x) L_n(y) = 0,$$

where $L_n(y) = x^n D^n(y/x)$, $D = d/dx$, can be reduced to one of order k by the substitution $z = L_n(y)$. He notes that many higher order equations can be reduced to familiar second order ones by this means.

He also notes that the equation $\Sigma_k^n = 1\, f_k(x)\, L_{n_k}(y) = 0$ can be reduced by the substitution $y = xz$ to one whose order is the maximum of the N_k's minus the minimum of the N_k's.

96. Pliś, A.

One-sided non-uniqueness in ordinary differential equations. BULL. ACAD. POLON. SCI. CL. III. 5:583–588 (1957) XLIX-L (Russian summary)

The author has succeeded in constructing an example of a system of two differential equations (*) $x_1 = f_1(t, x_1, x_2)$, $x_2 = f_2(t, x_1, x_2)$, where f_1 and f_2 are bounded continuous functions, defined throughout the entire real (t, x_1, x_2) -space R^3, such that each point of R^3 is a point of left-sided uniqueness and also a point of right-sided non-uniqueness for solutions of system (*).

To define right-sided non-uniqueness of solutions let P denote any point (t, x_1, x_2) in R^3 and set

$$c(s;P) = \max\, [\,(y_1(s) - z_1(s))^2 + (y_2(s) - z_2(s))^2\,]^{1/2}$$

as the curves $\{y_1(s), y_2(s)\}$ and $\{z_1(s), z_2(s)\}$ range over all possible integral curves of system (*) passing through point P. The point P is called a point of right-sided non-uniqueness for system (*), if for any positive number r there exists a number s, $t < s < t + r$, such that $c(s;P) > 0$. Otherwise P is a point of right-sided uniqueness. Left-sided uniqueness and non-uniqueness are analogously defined.

97. Santoro, Paolo

Studio qualitativo del sistema
$\dot{x} = ax^2 + bxy + cy^2\; f(x, y)$,
$\dot{y} = dx^2 + exy + hy^2 + g(x, y)$
nell'intorno del punto singolare $(0, 0)$.
BOLL. UN. MAT. ITAL. 3(12):566–590 (1957)

Consider the real system in the title where not all the constants a, b, c, d, e, h are zero, $f, g \in C^2$, $f = \rho^3\varphi$, $g = \rho^3\psi$, $\rho = (x^2 + y^2)^{1/2}$, $\varphi, \psi \in C^2$; it is assumed that $(0, 0)$ is an isolated singular point. The system without f, g is called a reduced system, and degenerate if the second-degree terms have a common real linear factor; the main purpose of this paper is to study this degenerate system. The singular point of a degenerate reduced system may be of 14 different (not affinely equivalent) types. For the complete system, the main result is that there always exists an integral curve which tends to the origin in a definite direction.

98. Sideriades, L.

Systèmes non linéaires du deuxième ordre.

J. PHYS. RADIUM 8(18):304–311 (1957)

The author gives a geometric description of the solutuions of two second order autonomous differential equations in two independent variables and illustrates his discussion on some oscillators of physical importance.

99. Utz, W. R.

Properties of solutions of certain second order nonlinear differential equations.

PROC. AMER. MATH. SOC. 8:1024–1028 (1957)

The nontrivial solutions $x(t)$ $(x(t) \not\equiv 0)$ of the linear differential equation $x'' + dx' + ex = 0$, where d and e are positive constants, are either oscillatory or monotonically approach zero as $t \to \infty$. Sufficient conditions are given that this be true of the nontrivial solutions of

$$x'' + f(x, x') + g(x) = 0.$$

It is assumed that $x(t) \equiv 0$ is a solution and that f and g are differentiable. It is shown to be sufficient, for example, that $f(x, x') \geqq 0$ for all (x, x'), $xg(x) > 0$ for all $x \neq 0$, and $\int_0^x g \to \infty$ as $x \to \infty$. By placing further restrictions on f and g it is shown that the amplitudes of all oscillatory solutions must decrease monotonically.

By restricting f to be of the form $f(x, x') = h(x)x'$, sufficient conditions are given that allow negative damping $(h(x) < 0)$. The results are simple consequences of the boundedness and uniqueness of the solutions.

100. Wintner, Aurel
On a principle of reciprocity between high- and low-frequency problems concerning linear differential equations of second order. QUART. APPL. MATH. 15:314–317 (1957)

For the differential equation (*) $d^2x/dt^2 + xf(t) = 0$, in which $f(t) > 0$ for $t > 0$, there are various theorems on the asymptotic behaviour of solutions for $t \sim \infty$. The author remarks, as an empirical fact, that there is a correspondence between theorems that concern (a) the case $f(t)$ large and (b) the case $f(t)$ small, for $t \sim \infty$; and he shows that the underlying fact is that (*) is converted into $d^2y/ds^2 + y/f(t) = 0$ by the substitution

$$s = \int f(t)\,dt, \quad \frac{dy}{ds}\frac{dx}{dt} = -xy,$$

which is significant in the asymptotic context provided $s \to \infty$ as $t \to \infty$. (The substitution is in fact a disguised form of $y = dx/dt$.)

101. Shintani, Hisayoshi
On the paths of an analytic two dimensional autonomous system in a neighborhood of an isolated critical point. J. SCI. HIROSHIMA UNIV. SER. A 21:209–218 (1957/58)

Consider the system (1) $\dot{x} = ax + by + f^*(x, y)$, $\dot{y} = cx + dy + g^*(x, y)$, where f^* and g^* are $o(r)$ as $r = (x^2 + y^2)^{1/2} \to 0$, $a^2 + b^2 + c^2 + d^2 \neq 0$, and both the roots of the characteristic equation are zero. When neither root is zero, the behavior of the paths of (1) near the origin is well known. When one root is zero, the behavior of the paths is also known [Bendixson, Acta Math. 24 (1901), 1–88]. The case of two zero roots has been investigated previously by Gubar [Dokl. Akad. Nauk SSSR 95 (1954), 435–438]. Andreev [Vestnik Leningrad Univ. 10 (1955), no. 8, 43–65] and Barocio [Contributions to the theory of nonlinear oscillations, vol. 3, pp. 127–135, Princeton Univ. Press, 1956]. In this paper the author obtains, essentially, seven distinct phase portraits for the paths of (1) near the origin. He uses a theorem due to Lonn [Math. Z. 44 (1938), 507–530] on the directions of approach to a critical point and a theorem of Keil [Jber. Deutsch. Math. Verein 57 (1955), 111–132] relating to the number of paths tending to the origin in certain directions. Though the author's methods differ from those of Gubar, Andreev, Barocio, and Keil, his results are the same.

102. Ura, Taro

The problem of the extension of characteristic curves and stability. I. SUGAKU 9:137–148 (1957/58) (In Japanese)

This is a preliminary article for part II (following citation). This half is an introduction to some basic concepts and theorems like the existence theorem and uniqueness theorems for ordinary differential equations. Later some set theoretical discussions are made about the extensions and limiting point sets of trajectories of points belonging to a region under discussion.

103. Ura, Taro

The problem of the extension of characteristic curves and stability. II. SUGAKU 9:218–235 (1957/58) (In Japanese)

This is a continuation of part I, above. The discussion is limited to the case of a two-dimensional region. At first the theory and classification of singular points as given by Poincaré and Bendixson is introduced. Later the successive extension of trajectories of points belonging to a given set is defined up to any ordinary number. Cantor's normal form on an ordinary number is essentially used.

Given an invariant region, the minimum of the ordinary numbers such that the extension of the orbit of the given invariant region differs from the original region, is proved to be a limiting number. When we write this number in the form of an exponent of ω, the exponent is called the order (or degree or rank, as the translation may vary) of the stability. Numerous examples are given to illustrate various situations.

104. Abian, Smbat; and Brown, Arthur B.
On the solution of the differential equation $f(x, y, y^{(1)}, \ldots, y^{(n)}) = 0$. BOLL. UN. MAT. ITAL. 3(13):383–394 (1958) (Italian summary)

This paper shows how to apply the method of successive substitutions directly to equations of the form given in the title, without solving for the nth derivative. It is not assumed that any partial derivatives of the function f exist. Four estimates (inequalities (53), (54), (55) and (56)) of the error of the mth approximation are given. (55) and (56) are corollaries of (53) and (54) respectively, but would seem likely to give smaller estimates of the error when $m > 2$. However, (53) and (54) might be useful at the outset of a computation to place a bound on the number of successive substitutions required.

105. Braier, Alfred
Méthode graphique pour l'étude des systèmes non linéaires dans le plan des phases. BUL. INST. POLITEHN. IASI NS (4)8,3–4:107–112 (1958) (1 insert) (Russian and Romanian summaries)

De l'introduction de l'auteur: "Dans cette note on expose une méthode graphique qui permet de construire les trajectoires du plan de phase de meme que les solutions des equations différentielles $\ddot{x} + f(x, \dot{x}) = F(t)$."

106. Conti, Roberto
Sistemi differenziali ordinari con condizioni lineari. ANN. MAT. PURA APPL. 4(46):109–130 (1958)

In this paper conditions are given which guarantee the existence of at least one solution of the nonlinear differential equation $dx/dt = A(t, x)x + a(t, x)$ satisfying the linear boundary condition $\int_{\gamma}^{0} dF(t)x(t) = c$, where $x = x(t)$ and $a(t, x)$ are real n-component vectors; c is a given real constant n-component vector; and $A(t, x)$ and the $F(t)$ of bounded variation are given n by n square matrices. The results obtained provide, as indicated, an extension of the work of W. M. Whyburn and others. Uniqueness of the solution is also established in certain special cases.

107. Cunningham, W. J.
INTRODUCTION TO NONLINEAR ANALYSIS.
McGRAW-HILL ELECTRICAL AND ELECTRONIC
ENGINEERING SERIES. New York-Toronto-
London, McGraw-Hill Book Co., 1958. 349p.

This book is on the non-linear ordinary differential equations of engineering interest. The treatment is nonrigorous but the author does indicate the limitations of the various methods considered. The reader is assumed to be familiar with electrical circuits, mechanical vibrations and linear differential equations with constant coefficients.

108. Fujita, Hiroichi
Oscillation represented by the third order
differential equation. II. PROC. FAC. ENGRG.
KEIO UNIV. 11:94–101 (1958)

From the third order differential equation

$$x''' + F(x'', x', x) = 0, \quad x' = dx/dt,$$

one secures the system $x' = y$, $y' = z$, $z' = -F(x, y, z)$ from which it follows that $dx/dy = y/z$ and $dz/dy = -F(x, y, z)/z$. The author, in order to remove z from the denominators, sets $E_1 = 1/2(x^2 + y^2)$, $E_2 = 1/2(y^2 + z^2)$ to secure the system $dE_1/dx = x + z$, $dE_2/dy = y - F(x, y, z)$. Advantages of discussing $x''' + F(x'', x', x) = 0$ in the $x - E_1$ and $y - E_2$ planes are given, and this approach is used in a study of a Colpittz oscillator equation $x''' + Ax'' + x' + (x - ex^3) = 0$. This paper is independent of the author's earlier paper of the same title in same Proc. 8 (1955), 61–67.

109. Ghizzeti, Aldo
Su una particolare equazione differenziale
ordinaria non lineare. ATTI ACCAD. NAZ.
LINCEI. REND. CL. SCI. FIS. MAT. NAT.
8(24):262–269 (1958)

Autonomous differential equations of the form, $x'' + f(x') + x = 0$, are considered, $f'(x') > 0$ and $x'f(x') > 0$ for $x' \neq 0$. Solutions of equations of this sort have the property that $x(t)$ and $x'(t) \to 0$ as $t \to \infty$. By elementary methods, separation theorems for the zeros of $x(t)$ and its derivatives are obtained, from which the author deduces criteria for distinguishing the oscillating case from the non-oscillating case. In the former case (where $x(t)$ has infinitely many zeros without being identically zero), a nomographic device is given for the rapid calculation of the solution for $t > 0$. But to set up the nomogram it is necessary to integrate the equation numerically over a limited interval under several initial conditions.

110. Hurewicz, Witold
LECTURES ON ORDINARY DIFFERENTIAL EQUATIONS. Cambridge, Mass., The Technology Press of the Massachusetts Institute of Technology; New York, John Wiley & Sons, Inc., 1958. 122p.

This excellent little book is essentially a reprint of notes by J. P. Brown from lectures given in 1943 by the late author at Brown University, to which have been added a preface by N. Levinson, the obituary "Witold Hurewicz, in memoriam" by S. Lefschetz [Bull. Amer. Math. Soc. 63 (1957), 77–82], and a short list of relevant books. It covers mainly existence theorems for first order scalar and vector equations, basic properties of linear vector equations, and two-dimensional non-linear autonomous systems. These topics are presented here in highly attractive form with particular emphasis throughout on geometric methods.

Chapter 1 deals with the scalar equation $y' = f(x, y)$. The fundamental uniqueness and existence theorems are proved by means of ϵ-approximate solutions and the Cauchy-Euler method. Theorems on the dependence of a solution on its initial values and on the continuation of a solution follow. The chapter closes with a discussion of Picard's method. In chapter 2 these results are transferred to the vector equation $\dot{x} = f(x, t)$, and the general theorem on continuity and differentiability of a solution depending on parameters is given. Chapter 3 is concerned with the basic properties of linear vector equations of first order and linear scalar equations of higher order. Reduction of order, Wronskian determinant, Green's function are discussed for the general case. Equations with constant coefficients are treated in detail. Chapter 4 is devoted to a careful discussion of the configurations near the origin of the characteristics of the linear and non-linear autonomous systems: $\dot{x} = P(x, y)$; $y = Q(x, y)$; $P(0, 0) = Q(0, 0) = 0$. Here, the author's thoroughly geometric treatment, particularly of the various non-linear cases, is especially attractive. The final Chapter 5 is mainly devoted to a proof of the Poincaré-Bendixson theorem on the existence of limit cycles. A brief discussion of orbital stability and the index of simple singularities rounds out the chapter.

111. Kooi, O

The method of successive approximations and uniqueness-theorem of Krasnoselskii and Krein in the theory of differential equations. NEDERL. AKAD. WETENSCH. PROC. SER. A 61 = INDAG. MATH. 20:322–327 (1958)

Krasnoselskii and Krein proved the uniqueness of a solution of $y' = f(x, y)$ through the point (x_0, y_0) under the conditions: $f(x, y)$ continuous for $0 \geqq x - x_0 \geqq a$, $|y - y_0| \leqq b$; and

$$(1) \qquad (x - x_0)|f(x, y_1) - f(x, y_2)| \geqq k|y_1 - y_2|;$$

$$(2) \qquad |f(x, y_1) - f(x, y_2)| \leqq p|y_1 - y_2|^{\alpha};$$

where $k > 0$, $\alpha > 0$, $0 < k(1 - \alpha) < 1$ [Uspehi Mat. Nauk. (N.S.) 11 (1956), no. 1(67), 209–213]. The author proves the existence as well as the uniqueness under a set of general conditions which include (1) and (2) as a special case.

112. Kozlov, E. M.

On reducing the order of a system of linear differential equations by its partial solution. DOPOVIDI AKAD. NAUK UKRAIN. RSR 918–928 (In Ukrainian. Russian and English summaries)

The paper deals with two methods enabling one to find partial solutions of the n-vector system $\dot{x} = A(t)x$ when one of the characteristics roots is near zero. Least squares are used to obtain initial conditions such that numerical integration yields a slowly varying partial solution. Then the order of the system is lowered by the number of linearly independent partial solutions obtained.

113. Leimanis, E., and Minorsky, N.
DYNAMICS AND NON-LINEAR MECHANICS.
SOME RECENT ADVANCES IN THE DYNAMICS
OF RIGID BODIES AND CELESTIAL MECHANICS;
THE THEORY OF OSCILLATIONS. New York,
John Wiley and Sons, Inc., (Surveys in Applied
Mathematics, Vol. II) 1958. 206p.

The first part of this book surveys modern analytical progress in the treatment of classical dynamical problems which are assuming increasing importance in today's technology. The second part outlines the current state of the art in the analysis of nonlinear oscillating systems and points the way to future developments in this field, which is receiving such great emphasis in control system engineering and other technical areas.

This monograph, published as part of a five-volume series, was prepared under the editorial direction of Applied Mechanics Reviews. Preparation of the material was under the sponsorship of the Office of Naval Research, U. S. Navy.

114. Magiros, Demetrios G.
Subharmonics of any order in nonlinear systems
of one degree of freedom: application to sub-
harmonics of order 1/3. INFORMATION AND
CONTROL 1:198–227 (1958)

This paper is an exposition of the classical small-parameter method in the spirit of Poincaré. The method is applied to the equation

$$\ddot{Q} + Q + \epsilon(k\dot{Q} - c_1Q + c_2Q^2 + c_3Q^3) = B \sin nt,$$

where solutions of period 2π are sought; the case $n = 3$ is studied with greater detail.

115. Nevanlinna, Rolf

Cauchy's polygon method. ARKHIMEDES 2:1–11 (1958) (In Finnish)

The author discusses the use of Cauchy's polygon method in the existence and uniqueness proofs of the solution of the differential equation $dy/dx = f(x, y)$, where $f(x, y)$ is continuous in $|x| \leqq \rho_x$ $|y| \leqq \rho_y$. It is assumed moreover that

(1) $$\int_0^{\rho_z} \frac{d\rho}{\psi(\rho)} = \infty$$

with $\psi(\rho) = \sup |f(x, y_1) - f(x, y_2)|$ for $|x| \leqq \rho_x$, $|y_i| \leqq \rho_y$ $(i = 1, 2)$, $|y_1 - y_2| \leqq \rho$. Then it is shown that Cauchy's polygon y $y_D(x)$ which corresponds to the subdivision D of $|x| \leqq \rho_x$ approaches a limiting function $y = y(x)$ with $y(0) = 0$ which satisfies $y' = f(x, y)$ for $|x| < \rho_x$. The uniqueness of the solution was shown by Osgood in 1899.

The generality of the theorem is in that (1) is trivially fulfilled if f satisfies the Lipschitz condition usually used in the theory of differential equations. Condition (1) is, more generally, satisfied by the function

$$\psi(\rho) = C\rho \log \frac{1}{\rho} \log \log \frac{1}{\rho} \ldots \log_n \frac{1}{\rho},$$

but replacing the last factor by $(\log_n 1/\rho)^{1+\epsilon}$ already suffices to make integral (1) convergent.

The author's approach leads to existence and uniqueness from a single starting point and gives an explicit estimate showing the rate of convergence of Cauchy's method. The argument can be generalized to finite and infinite systems of differential equations.

116. Parsons, D. H.

One-dimensional diffusion, with the diffusion coefficient a non-linear function of concentration. J. LONDON MATH. SOC. 33:246–251 (1958)

The initial value problem for the one-dimensional diffusion equation with semi-constant initial data is studied in cases when the diffusion coefficient D is related to the concentration c by the formulae $D = (\alpha + \beta c)^n$ or D $k \exp(\alpha + \beta c)$ $(\alpha, \beta, k, n$ constants$)$. The problem is reduced to the consideration of $(DE)p' + up + 2(m + u^2)p^2 + u(1 + 2m + u^2)p^3 = 0$, $m = 1/n$, for $p(u)$ on $(-\infty, \infty)$. The methods and results are similar to those in author's previous account for the case $n = 1$ [Quart. Appl. Math. 15 (1957), 298–303].

117. Picone, Mauro; and Gross, Wolf
Intervallo d'esistenza e limitazioni per la soluzione di un sistema di equazioni differenziali ordinarie. ATTI ACCAD. NAZ. LINCEI. REND. CL. SCI. FIS. MAT. NAT. 25:225–230 (1958)

The authors establish an existence, uniqueness and boundedness theorem for a system of ordinary differential equations satisfying certain special conditions. The method is that of successive approximations.

118. Plehotin, A. P.
Theorem on the existence and uniqueness of the solution of a boundary value problem for a system of ordinary differential equations. DOKL. AKAD. NAUK SSSR 123:613–615 (1958) (In Russian)

The system (1) $y = f(t, y)$ is considered for n-vectors y, together with a boundary condition

$$\text{(2)} \qquad \sum_{m=0}^{\mu} \alpha_m y(t_m) = b, \quad t_0 \leqq t_1 \leqq \dots \leqq t_\mu .$$

Here the α_m are constant n by n matrices, and b is a constant n-vector. The author states a set of rather complicated conditions which will guarantee the existence of a unique solution of (1), (2) for $t_0 \leqq t \leqq t_\mu$, which moreover satisfies a certain estimate in terms of an approximate solution.

119. Pliś, A.; and Ważewski, T.
A uniqueness condition with a standard differential equation without uniqueness property. BULL. ACAD. POLON. SCI. SÉR. SCI. MATH. ASTR. PHYS. 6:145–148 (1958)

In the system (1) $X' = F(t, X)$ a solution $X = X(t)$ $(\alpha < t < \beta)$ will be called unique to the right if, for any solution $Y(t)$ of (1) defined in $\gamma \leqq t \leqq \delta$, $\alpha < \gamma < \delta < \beta$, and satisfying $Y(\gamma) = X(\gamma)$, we have $Y(t) \equiv X(t)$, $\gamma \leqq t \leqq \delta$. The main theorem also requires the following definition: We say that a function $g(t, x)$ defined in a subset S of the half-plane $x > 0$ has property P if for every real number s there exists a sequence $x_n(t)$ of positive solutions of $x' = g(t, x)$ such that either (I) there exists $\epsilon > 0$ such that $\lim_{n \to \infty} x_n(t) = 0$ for $s \leqq t \leqq s + \epsilon$, or (II) there exists a sequence of numbers $t_n > s$, $t_n \to s$ as $n \to \infty$, and for each fixed n, $x_n(t) \to 0$ as $t \to t_n$ $(s < t < t_n)$.

Theorem: Suppose that the right hand members of the system (1), defined in the open set R, satisfy the inequality

$$|F(t, X) - F(t, Y)| < g(t, |X - Y|)$$

for $(t, X) \in R$, $(t, Y) \in R$, $(t, X - Y) \in S$, $|A| = (a_1^2 + a_2^2 + \ldots + a_n^2)^{1/2}$, where the function $g(t, x)$ has property P. Then every solution of (1) is unique to the right.

120. Pliss, V'. A.
Eiserman's problem in the case of three simultaneous differential equations. DOKL. AKAD. NAUK SSSR 121:422–425 (1958) (In Russian)

The author presents a number of conditions under which all the solutions of a nonlinear system of the form $dx/dt = Ax + f(x_1)b$, $x(0) = c$ approach zero as t approaches infinity, where x is an n-dimensional vector, A is a constant matrix, b is a constant vector, and $f(x_1)$ is a scalar function of one component of x, say x_1.

121. Reizin', L. E.
Behavior of the integral curves of a system of differential equations near a singular point in a space of more than one dimension. AMER. MATH. SOC. TRANSL. 2(18):173–186 (1961)

The original Russian article appeared in Latvijes. PSR Zinatu Akad. Vestis 1958, no. 3 (128), 107–120.

The system considered is

$$\dot{x}_i = X_i(x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, n), \tag{1}$$

where the X_i are forms of degree n with the origin as unique common zero. After some generalities, description of special directions of approach, and assertion that their number is finite, in fact, among the common solutions of

$$x_i X_j - x_j X_i = 0 \quad (i, j = 1, 2, \ldots, n), \tag{2}$$

the author projects on a sphere of constant radius centered at the origin and gives a differential system for the projected paths. The special directions correspond to some of the critical points of the projection.

After this the author turns his attention to the case $n = 3$. The projection may then be reduced to

$$\begin{aligned} u &= U = X_1(u, v, 1) - uX_3(u, v, 1) \\ v &= V = X_2(u, v, 1) - vX_3(u, v, 1) \\ u &= x_1/x_3, \quad v = x_2/x_3, \quad x_3 = 1. \end{aligned} \tag{3}$$

The indices of a point and closed curve (containing no critical point) are defined. The multiplicity $\mu(P)$ of a point P is its multiplicity as intersection of $U = V = 0$. Let $I(P)$ be the index of P. The following properties are proved: (a) $I(P) - \mu(P)$ is even; (b) $\mu(P) \leqq [I(P)]^2$; (c) if the system has the maximum number $m^2 + m + 1$ of critical points, then they are all simple and $1/2(m^2 + m) + 1$ have index $+1$ and the rest index -1; (d) $I(P) \leqq m$; (e) if $m + 1$ critical points are on the same line L, then the intervals of L between consecutive critical points are paths; (f) for $m = 2$ there are at most two foci or centers.

In a special case the author also investiages the (rather complicated) 3-space behavior corresponding to a critical point of (3).

122. Sidériadès, Lefteri
Méthodes topologiques dans un espace à trois dimensions. C.R. ACAD. SCI. PARIS 247:911–913 (1958)

The paper presents some general ideas on classification of critical points of a system $\dot{x}_i = X_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$.

123. Temple, G.
Linearization and delinearization. In PROC. INTERNAT. CONGRESS MATH. New York, Cambridge Univ. Press, 1960, p. 233–247.

This paper surveys briefly the problems of solving non-linear differential equations. Two methods are discussed. One is the method of linearization by which non-linear equations are forcibly reduced to an associated, approximate linear form. The other is the method of delinearization by which the non-linearities are partially restored. For each case, examples are cited.

124. Abian, Smbat; and Brown, Arthur B.
On the solution of simultaneous first order implicit differential equations. MATH. ANN. 137:9–16 (1959)

The authors prove by successive approximations an existence and a uniqueness theorem for the system

$$F_i(x, y_1, \ldots, y_n, y_1', \ldots, y_n') = 0 \quad (i = 1, \ldots, n).$$

The method does not require solving for the derivatives. Four appraisals of the error are given, two of which are independent of previous computational errors. {It appears that lemma 4 does not in fact apply to (47) unless $|c_i| \leqq V_i$, since $U_i(x) = b_i$ implies $U_i'(x) = 0$.}

125. Abian, Smbat; and Brown, Arthur B.
On the solution of the differential equation
$f(x, y, y') = 0$. AMER. MATH.
MONTHLY 66:192–199 (1959)

The authors establish local existence and uniqueness of a solution of $f(x, y, y') = 0$, $y(x_0) = y_0$, under conditions weaker than those usually assumed. A value z_0 is chosen so that $f(x_0, y_0, z_0)$ is small, and the following assumptions are made about $f(x, y, z)$ in a neighborhood of (x_0, y_0, z_0). f is assumed to be continuous and to satisfy a Lipschitz condition in y. Positive constants D_2 and D_3 are assumed to exist such that if $z_1 \neq z_2$,

$$D_2 \leqq (f(x, y, z_2) - f(x, y, z_1))/(z_2 - z_1) \leqq D_3.$$

The proof is by successive approximations, applied to $y' = F(x, y, y')$, where $F(x, y, y') = y' - kf(x, y, y')$ for some k chosen sufficiently small. The initial approximation $Y_1(x)$ is given by $Y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0, z_0)\,dt$. If $Y(x)$ is the solution, local estimates on $|Y(x) - Y_n(x)|$ are obtained which converge to zero as the nth power of a small quantity. The theorem is illustrated with an example which shows how the choice of z_0 can possibly determine to which of several solutions the process may converge.

126. Becker, H.
The reciprocal method and its application for
finding approximate solutions of nonlinear
differential equations . REGELUNGSTECH
7,5:168–170, May 1959 (In German)

The reversion method makes it possible to find the (approximate) solution of a non-linear differential equation by solving a sequence of linear differential equations. The basic relations of the method are given and their application is demonstrated in three examples.

127. Cecik, V. A.
Investigation of systems of ordinary differential equations with a singularity.
TRUDY MOSKOV. MAT. OBSC. 8:155–198 (1959) (In Russian)

The present paper concerns the difficult question of existence, uniqueness, continuous dependence upon parameters, and approximation, of solutions of real differential systems in a neighborhood of a singular point. Systems of the form

$$dy_k/dx = f_k(x, y_1, \ldots, y_n) \quad (k = 1, \ldots, n) \tag{1}$$

are investigated, where $x = 0$ is the singular point, and the real function f_k are supposed to be continuous in their arguments, together with their first order partial derivatives f'_{k,y_i} $(i, k = 1, \ldots, n)$ in a region D of the form $0 < x \leqq b$, $|y_i| \leqq a (i = 1, \ldots, n)$. Solutions $y_k(x)$ $(k = 1, \ldots, n)$ are sought satisfying system (1) in some neighborhood of $x = 0$, $0 < x \leqq c$ $(0 < c \leqq b)$, and the initial condition

$$y_k(0+) = 0 \quad (k = 1, \ldots, n). \tag{2}$$

The following is one of the existence theorems proved by the author. (I) If

$$|f_k(x, y_1, \ldots, y_{k-1}, 0, y_{k+1}, \ldots, y_n)| \leqq \psi(x) \quad (k = 1, \ldots, n),$$

$$|f'_{k,y_k}(x, y_1, \ldots, y_n)| \leqq \psi(x) \quad (k = 1, \ldots, p),$$

$$f'_{k,y_k}(x, y_1, \ldots, y_n) \geqq \Psi(x)$$

$$(k = p+1, \ldots, n, \text{ for some } p, 0 \leqq p \leqq n),$$

where $\psi(x)$, $\Psi(x)$ are non-negative continuous functions in $(0, b]$, ψ summable and Ψ nonsummable in $(0, b]$, then (1) has at least one solution $y(x)$ satisfying (2). In addition, the author proves that, for $p < n$, there is at least an $(n-p)$-parameter family of solutions $y(x)$ satisfying (2). For $n = 1$, the following negative theorem is proved. (II) If there is a continuously differentiable function $\Phi(x)$, $0 \leqq x \leqq b$, $\Phi(0) = 0$, such that $f_y'(x, y) > 0$ for $y > \Phi(x)$, $f(x, y) \geqq \max \Phi'(x)$, where max is taken for $0 \leqq x \leqq b$, and $\int_a^b f(x, \Phi(x))dx = +\infty$, then the equation $y' = f(x, y)$ has no solution $y(x)$ with $y(0+) = 0$. For $n \geqq 1$, $p = n$, theorem I does not assure uniqueness as is shown by examples. A uniqueness theorem was given by M. A. Krasnosel'skii and S. G. Krein [Voronez. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 2 (1956), 3–23] slightly improving a previous analogous result of L. Tonelli [Rend. Accad. Naz. Lincei (6), 1 (1925), 272–277]. The author proves the following uniqueness theorem. (III) If

$$|f_k(x, y_1, \ldots, y_{k-1}, 0, y_{k+1}, \ldots, y_n)| \leqq \psi(x) \quad (k = 1, \ldots, n),$$

$$f'_{k, y_k}(x, y_1, \ldots, y_n) \leqq \psi(x) \quad (k = 1, \ldots, n),$$

$$|f'_{k, y_i}(x, y_1, \ldots, y_n)| \leqq \psi(x) \quad (i \neq k; i, k = 1, \ldots, n),$$

everywhere in D, where $\psi(x)$ is a non-negative function in $(0, b]$, then (1) has one and only one solution $y(x)$ satisfying (2). Under the conditions of III, analogous theorems assure continuous dependence upon parameters. Finally, a method of successive approximations is proposed, and its convergence proved. Also, evaluations of the error after the mth approximation are given. Note that for the nth order differential equation

$$y^{(n)} = f(x, y', \ldots, y^{(n-1)}) \tag{3}$$

the following statement holds. (IV) If the functions $|f(x, y_1, \ldots, y_{n-1}, 0)|$, $|f_{y_k}'(x, y_1, \ldots, y_n)| (k = 1, \ldots, n-1)$, $f_{y_n}'(x, y_1, \ldots, y_n)$ are all $\leqq$ a given non-negative continuous function $\psi(x)$, summable in $(0, b]$, then there is one and only one solution $y(x)$ of (3) satisfying $y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0$.

128. Cesari, L.
ASYMPTOTIC BEHAVIOR AND STABILITY PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS (Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 16) Berlin, Springer-Verlag, 1959. 271p. (Paperbound)

The literature on title subject has grown extensively during the past few decades, and purpose of present volume is to "present many of the viewpoints and questions in a readable short report for which completeness is not claimed." Many proofs are sketched and details often omitted, but copious bibliographic notes and comments are sprinkled throughout the text. The bibliography itself takes up 69 pages. We have here a valuable guide to the literature and a source book for future research effort.

There are four chapters. The first deals with the various concepts of stability and linear systems with constant coefficients. Chapter 2 treats linear systems with variable coefficients and periodic coefficients, and the second-order linear differential equations and generalizations. Chapter 3, perhaps the most valuable, studies non-linear systems. Two sections are devoted respectively to the first and second methods of Lyapunov. A third section takes up analytical methods and a number of applied examples associated with the names of Rayleigh, van der Pol, Lienard and others are presented. The fourth section concerns analytic-topologic methods. General asymptotic developments is the subject of Chapter 4.

129. Coleman, Courtney
ASYMPTOTIC STABILITY IN 3-SPACE
RIAS, Inc., Baltimore, Maryland. AFOSR
Rept. no. TN-59-438, 1959. 12p. [Contract
AF49(638)382] . ASTIA AD-247 250

Reprint From: Contributions to The Theory of Nonlinear Oscillations, 5:257-268.

130. Coleman, Courtney
A certain class of integral curves in 3-space.
ANN. OF MATH. 2(69):678–685 (1959)

The author considers the system (*) $dx/dt = f(x)$, where f is a 3-dimensional vector whose components are homogeneous polynomials of degree $m \geqq 1$ in the components of x.

This homogeneity implies that the homothetic of an integral curve is also an integral curve. Things can then be reduced to the sphere S^2 and one thus gets a classification of the integral surfaces of (*) and a partial classification of the integral curves on those surfaces.

131. Iwano, Masahiro
Intégration analytique d'un système d'équations
différentielles non linéaires dans le voisinage
d'un point singulier. II. ANN. MAT. PURA
APPL. 4(47)91–149 (1959)

This is a direct continuation of a memoir by the same author, which has been reviewed previously [Ann. Mat. Pura Appl. (4) 44 (1957), 261–292] . The same system of differential equations is under consideration; and the author continues to seek conditions under which certain formal infinite series converge, and yield actual solutions of the differential equations. As in the case of the first memoir, the analysis and results are so complicated that no adequate description of them is possible here.

132. Kisynski, J.

Sur les équations différentielles dans les espaces de Banach. BULL. ACAD. POLON. SCI. SÉR. SCI. MATH. ASTR. PHYS. 7:381–385 (1959) (Russian summary, unbound insert)

Let $f(t, x)$ be a continuous function from $[0, a] \times E$ into E, where $a > 0$, and E is a Banach space. The author considers the differential equation (*) $x' = f(t, x)$, $x(0) = x_0$, and proves that a unique solution exists on $[0, a]$, and may be obtained by the method of successive approximations, with an arbitrary element of E as first approximation, provided the following two hypotheses hold. (1) For every $r > 0$, there exists a real-valued function $w_r(t, u)$ defined on $(0, a] \times [0, 2r]$, measurable in t for fixed u, continuous and non-decreasing in u for fixed t, $w_r(t, 0) = 0$ for almost all $t \in [0, a]$, $\int_\epsilon^a w_r(t, 2r)\,dt < \infty$, such that $u(t) \equiv 0$ is the unique continuous solution of $u(t) = \int_0^t w_r(\tau, u(\tau))\,d\tau$, $u'(0) = 0$, on $[0, \epsilon]$ when $0 < \epsilon < a$, and such that $\|f(t, x) - f(t, y)\| \leqq w_r(t, \|x - y\|)$, $0 < t \leqq a$, $\|x\|$, $\|y\| \leqq r$. (2) If $M(t, r) = \sup \|f(t, x)\|$, $\|x\| \leqq r$, and $\rho_0 > 0$ is arbitrary, there exists a continuous function defined on $[0, a]$ satisfying $\rho(t) = \int_0^t M(\tau, \rho(\tau))\,d\tau + \rho_0$.

The author mentions that (1) is needed not only to show the uniqueness, and the convergence of successive approximations, but also for the existence of the solution. There is a second theorem which states, under the same hypotheses, that the solution of (*) is a continuous function of the initial condition x_0.

133. Kuzmak, G. E.

On the computation of asymptotic solutions corresponding to nonclosed integral curves of the "standard" equation. DOKL. AKAD. NAUK SSSR 125:992–995 (1959) (In Russian)

The author investigates the equation

(*) $$y'' + \epsilon f(\epsilon t, y)y' + F(\epsilon t, y) = 0,$$

where $\epsilon > 0$ is a small parameter. The "standard" equation referred to in the title is

$$(**) \qquad \phi^2(\epsilon t)(\partial^2 y_0/\partial\omega^2) + F(\epsilon t, y_0) = 0,$$

where ϕ is determined by a complicated expression and ω is the integral of ϕ. In an earlier paper [Dokl. Akad. Nauk SSSR 120 (1958) 461–464] he considered the oscillatory case. Here he extends his method to the case where the integral curves in the phase plane y_0, $\partial y_0/\partial\omega$ are non-closed. The author shows that by an appropriate choice of the arbitrary functions that enter into the solution of (**) one may obtain functions y_0 and y_0' which are close (within terms of order ϵ^2) to the solution of (*) and its derivative respectively.

134. Langer, R. E.
BOUNDARY PROBLEMS IN DIFFERENTIAL EQUATIONS (Proceedings of a Symposium, Mathematics Research Center at University of Wisconsin, April 20–22, 1959), Madison, University of Wisconsin Press, 1960. 324p.

Book contains the following articles: (1) K. O. Friedrichs, Boundary problems of linear differential equations independent of type, where a unified theory is sketched for first-order systems with certain symmetry properties. (2) P. R. Garabedian, Numerical estimates of contraction and drag coefficients. Here a generalized vena contracta problem is considered. (3) P. Henrici, Complete systems of solutions for a class of singular elliptic partial differential equations. The article gives an extension for singularities of a theory by I. N. Vekua. (4) L. Collatz, Application of the theory of monotonic operators to boundary value problems. Several general estimates are derived. (5) J. B. Diaz, Upper and lower bounds for quadratic integrals and, at a point, for solutions of linear boundary value problems. (6) J. Schroder, Error estimates for boundary value problems using fixed-point theorems. (7) G. Fichera, On a unified theory of boundary value problems for elliptic-parabolic equations of second order. This article gives an interesting theory together with estimates and existence theorems for solutions. (8) R. S. Varga, Factorization and normalized iterative methods, where elliptic difference equations are considered.

(9) D. Young and L. Ehrlich, Some numerical studies of iterative methods for solving elliptic difference equations. (10) G. Birkhoff, Albedo functions for elliptic equations, concerning self-adjoint elliptic differential equations with integral boundary conditions as appear in reactor theory. (11) J. Douglas, Jr., A numerical method for analytic continuation. The problem considered is analytic continuation if the values on a circle are given only approximately. (12) W. T. Koiter, Stress distribution in an infinite elastic sheet with a doubly-periodic set of equal holes. (13) H. F. Buckner,

Some stress singularities and their computation by means of integral equations. Investigations are made for a rotor and a slab, both sharply notched. (14) I. N. Sneddon, Boundary value problems in thermoelasticity. (15) L. Fox, Some numerical experiments with eigenvalue problems in ordinary differential equations. (16) R. Bellman, Dynamic programming, invariant imbedding, and two-point boundary value problems. Some important approaches to multi-point boundary value problems are given. (17) R. Courant, Remarks about the Rayleigh-Ritz method. (18) B. A. Troesch, Free oscillations of a fluid in a container. (19) C. H. Wilcox, A variational method of computing the echo area of a lamina.

135. LaSalle, J. P., and Lefschetz, S.
RECENT SOVIET CONTRIBUTIONS TO
ORDINARY DIFFERENTIAL EQUATIONS AND
NONLINEAR MECHANICS. Rept. no. AFOSR TN 59-308,
Apr 1959. 47p. (RIAS TR 59-3) ASTIA AD-213 092

This report is an appraisal of recent Soviet contributions to differential equations and nonlinear mechanics. It contains a general appraisal of the significance and implications of Soviet research in this field. A somewhat nontechnical description is given of the major areas of research and of individual Soviet contributions. An appendix includes a more technical appraisal of the Soviet contributions. A mathematical abstract together with the names of the authors and exact references is given in this appendix for each of the major papers and books available to authors in 1958.

136. Lee, Wen-yung
The topological structure of the distribution
of the integral curves with one singular point
on the torus. ACTA MATH. SINICA 9:181–190
(1959) (In Chinese. English summary)

In this paper we consider the topological structure of the distribution of the integral curves with one singular point on the torus. First of all, we define three kinds of sectors. We consider the neighborhood of the singular point by separating it into sectors. Using the property of the torus and the fact that the sum of the index of the singular point is equal to zero, we obtain three classes of fundamental figures. We define five classes of operations. We prove that from the three classes of fundamental figures and the five classes of operations we can obtain all possible topological structure with one singular point on the torus. Finally, we prove the differentiability of the figure and consider also the case of a finite number of singular points on the torus.

137. Massera, J. L.; and Schäffer, J. J.
Linear differential equations and functional analysis. III. Lyapunov's second method in the case of conditional stability. ANN. OF MATH. 2(69):535–574 (1959)

In this third part of their work on linear differential equations

$$\dot{x} + A(t)x = 0 \tag{1}$$

in a Banach space X [for parts I, II see same Ann. (2) 67 (1958), 517–573; 69 (1959), 88–104], the authors take up the (very considerable) problem of extending Lyapunov's second method to the case of conditional, i. e., non-uniform, simple and asymptotic stability. They are able to prove both "direct" and "converse" theorems and thereby establish the equivalence between the existence of suitably generalized Lyapunov functions with certain properties and the asymptotic behavior of the bounded and unbounded solutions of (1). These theorems they combine with previous ones of part I to obtain a complete characterization, in terms of Lyapunov functions, of the existence of at least one bounded solution of

$$\dot{x} + A(t)x = f(t) \tag{2}$$

with f from a certain function space. The importance of these results, as indeed of all results in Lyapunov's method, lies in the fact that they provide testable criteria in the sense that all properties of the Lyapunov functions involve solely the (homogeneous) differential equation and require therefore no knowledge of the solutions at all.

Only a bare outline of the wealth of theorems proved in this paper can be given here. Recall that, throughout, A is a continuous endomorphism of X, defined on $J = [0, \infty]$ and uniformly Bochner integrable on every finite subinterval of J; M denotes the space of such endomorphisms (or functions) for which $\int_t^{t+1} \| A(t) \| \, dt$ is bounded on J.

The authors first define the concepts of a Lyapunov function $V(x, t)$ on $X \times J$ and of a total derivative $V'(x, t)$ in such a way that the following crucial implication is true under sufficiently general assumptions: If, for a continuous function $a(x, t)$ on $X \times J$ $V' \geqq a$ holds "almost everywhere" (in a sense made precise in the paper) then for any $t \geqq t_0 \geqq 0$

$$V[x(t), t] - V[x(t_0), t_0] \geqq \int_{t_0}^{t} a[x(t), t] \, dt$$

where $x(t)$ is a solution of (1). The terms positive definite, infinitely small upper bound are defined as usual. V' is called positive almost definite if V' is positive definite save on a set H such that either its projection on J is a Lebesgue null set or, if $\dim X < \infty$, H itself is such a set; and if V has certain regularity properties. The first two main theorems can then be stated, in part, as follows.

(I) If $A \in M$ and if there exists a V with infinitely small upper bound and one of its total derivatives V' is negative almost definite then (i) the set X_0 of initial values of bounded solutions is closed and

$$(3) \qquad ||x(t)|| \leqq N||x(t_0)|| e^{-v(t-t_0)}, \quad t \geqq t_0 \geqq 0,$$

N, v being positive constants; (ii) if X_1' is a subspace such that either $x \in X_1'$ implies $V(x,0) \leqq 0$ $[V(-x,0) \leqq 0]$ or $X_1'\ X_0 = O$ and $\dim X_1' < \infty$, then $x(0) \in X_1'$ implies

$$(4) \qquad ||x(t)|| \geqq N'||x(t_0)|| e^{v'(t-t_0)}, \quad t \geqq t_0 \geqq 0,$$

N', v' being positive constants; (iii) if $x_0(0) \in X_0$, $x_1(0) \in X_1'$ then (5) $\alpha[x_0(t), x_1(t)] \geqq \alpha_0$ for $t \geqq 0$, where α is the angular distance defined in part I. The most striking feature is that V is not required to be positive definite as it is in the classical theory. There $A \in M$ need not hold; here (I) is false if $A \in' M$.

(II) If X_0, $X_1 = CX_0$ are closed, and if $x_0(t_0) \in X_0$ implies (3), $x_1(0) \in X_1$ implies (4), and $x_i(0) \in X_i$ imply (5); then there exist non-negative functions V_0, V_1 such that every total derivative of $V_0 - V_1$ is negative almost definite.

Combined with results from part I these theorems yield the following.

(III) If the hypotheses of (I) hold and if either codim $X_0 < \infty$ or $X_1 = CX_0$ is closed and $x \in X_1$ implies $V(x,0) \leqq 0$ $[V(-x,0) \leqq 0]$, then (2) has at least one bounded solution for every $f \in M$.

(IV) If $A \in M$ and X_0, $X_1 = CX_0$ are closed and if the assertion of (III) holds, then the assertion of (II) holds.

The authors point out that these results are closely related to work of Krasovskii [Mat. Sb. (N. S.) 40 (82) (1956), 57–64].

In the case of simple conditional stability the authors prove theorems of analogous type. Roughly speaking, they are able to infer from the existence of non-negative, positively homogeneous Lyapunov functions that every bounded solution satisfies $||x(t)|| \leqq N_0||x(t_0)||$, $t \geqq t_0 \geqq 0$, and for the other solutions $||x(t)|| \geqq N'||x(t_0)||$. Conversely, if this dichotomy holds, there exist such functions. By virtue of theorems of part I there follows a characterization of the bounded solutions of (2) similar to that in (III) and (IV).

Throughout the paper the authors give numerous examples to illustrate the importance of the various hypotheses in the theorems. The methods of proof are essentially classical.

138. Moore, Richard A.; Nehari, Zeev
Nonoscillation theorems for a class of non-linear differential equations. TRANS. AMER. MATH. SOC. 93:30–52, (1959)

This investigation of the solutions of (*) $y'' + p(x)y^{2n+1} = 0$, where $p(x) > 0$ is continuous in $(0, \infty)$ and n is a positive integer, falls into two parts. The first group of results depends on methods similar to those used by Atkinson, F. [Pacific J. Math. 5 (1955), 643–647] in connection with the criteria $\int^{\infty} xp(x)dx < \infty$, $\int^{\infty} x^{2n+1} p(x)dx < \infty$ for at least one or for all solutions to be non-oscillatory; here further conclusions are drawn, and the requirement (in the second case) that $p(x)$ be differentiable and monotonic is dropped. It is shown that $\int^{\infty} x^{2n+1} p(x)dx < \infty$ is equivalent to the existence of solutions such that $x^{-1}y(x) \to \text{const} > 0$ as $x \to \infty$, and that this criterion ensures the existence of "properly non-oscillatory" solutions. These have at any rate one zero, but are ultimately of one sign; the category of solutions with constant sign is viewed as too wide. Another result in this section is that if there is a non-oscillatory solution for which $\int^{\infty} x^{\nu}\{y(x)\}^{-2n} dx < \infty$, for some $\nu > -1$, then $\int^{\infty} x^{\nu+2} p(x)dx < \infty$.

The latter half of the paper exploits the approach of minimising the "generalized Rayleigh quotient" $J(u) = \{\int_a^b u'^2 dx\}^{n+1} / \int_a^b u^{2n+2} p(x)dx$ [cf. Z. Nehari, Trans. Amer. Math. Soc. 85 (1957), 428–445]. The minimum subject to $u(a) = 0$ is achieved by a solution of (*) such that $y(a) = y'(b) = 0$, with an analogous result for the case $u(a) = u(b) = 0$; in both cases the solution is to be positive in (a, b) but need not be unique. Making $b \to \infty$, a connection appears between the eventualities, firstly that $J(y)$ has a positive lower bound for all $b > a$, and secondly that (*) has a solution vanishing at $x = a$ and positive for $x > a$. This matter is partly clarified for the case that $\int^{\infty} x^{n+1} p(x)dx < \infty$. Finally, precise bounds are found for $J(y)$ in cases in which (*) admits explicit integration.

139. Mrówka, S.
The fixed-point theorem and its application to the theory of differential equations.
WIADOM. MAT. 2(2):292–297 (1959) (In Polish)

The existence of solutions to $y' = \varphi(t, y)$, $y(0) = 0$, where φ is continuous on $[0, 1] \times [-C, C]$, is proved by using the fact that the set of all functions satisfying a Lipschitz condition (with constant C) on $[0, 1]$, and vanishing at 0, is homeomorphic to a closed convex subset of the Hilbert cube.

140. Phakadze, A. V.; Sestakov, A. A.
On the classification of the singular points of a first order differential equation not solved for the derivative. MAT. SB. NS 49(91):3–12 (1959) (In Russian)

The authors consider the equation (1) $F(x, y, p) = 0$, where $p = y$ and F has continuous partial derivatives of the first three orders. A solution of the system $F = 0$, $F_p = 0$, $F_x + pF_y = 0$, is called a singular point of (1). Apparently, this differs from the definition given by Petrovskii [Lekcii po teorii obyknovennyh differencial'nyh uravenenii, 3d ed., GITTL, Moscow-Leningrad, 1949]. It is assumed that the singular points are isolated in the x, y, p-space. The authors first classify the singular points of the equation of first approximation – i. e., the equation using up through the 2nd order terms of the Taylor series of F – and then apply this classification to the singular points of (1). It is claimed that the qualitative behavior of the integral curves near the singular points is the same for both equations. The authors state that their analysis clarifies the Poincaré classification of singular points of an equation solved for p. {It should be noted that the Poincaré singular points have quite different properties from those analyzed in this paper. There are a number of notational errors.}

141. Sandor, Stefan
Sur l'équation différentielle matricielle de type Riccati. BULL. MATH. SOC. SCI. MATH. PHYS. R. P. ROUMAINE NS 3(51):229–249 (1959)

The Riccati-type differential equation $W' + AW + WB + WCW + D = 0$, where each letter represents an n-by-n matrix and the elements of the coefficient matrices are continuous, real functions on the entire real axis, is studied. Results obtained for this equation are related to results for the associated linear system $x' = Bx + Cy$, $y' = -Dx - Ay$, where x and y are n-vectors. For the case where A is the transpose of B, C and D are symmetric, and C is positive definite, the author investigates questions of boundedness, comparison, and extension of solutions on intervals to the right and to the left of a given point. A condition of the type introduced by Hartman [Duke Math. J. 24 (1957), 25–35] is introduced to obtain symmetry for the matrix W, and this property is used to study oscillation properties of solutions of the related linear systems.

142. Sestakov, A. A.
Existence theorems for integral and critical straight lines of a homogeneous system of n differential equations ($n \geqq 3$). USPEHI MAT. NAUK 14, 1(85):245–248 (1959) (In Russian)

Consider the system

(1) $$\dot{x}_s = X_s^{(m)}(x_1, \ldots, x_n),$$

where $s = 1, \ldots, n$; $m \geqq 3$; $X_s^{(m)}$, $s = 1, \ldots, n$, is a form of integral degree m, where m does not depend upon s. The real solutions of the system of algebraic equations

(2) $$x_1 : X_1^{(m)} = \ldots = x_n : X_n^{(m)}$$

are straight lines which are the integral straight lines of (1). If a solution of (1) tends to the origin or to infinity asymptotic to a straight line L, then the author calls L a critical line of (1).

Theorem 1: If n is odd, system (1) has at least one integral straight line. Theorem 2: If m is even, system (1) has at least one integral straight line. Theorem 3: If $(g_1, \ldots, g_n)$ is a set of direction numbers of a critical line of system (1), then $x_1 = g_1, \ldots, x_n = g_n$ is a real solution of system (2). The proof on the first theorem is based on the fact that a continuous and nonvanishing vector field defined on an even-dimensional sphere has at least one vector normal to the sphere [Alexandroff and Hopf, Topologie, Berlin, Springer 1935, p. 481]. The proof of theorem 2 depends upon Borsuk's antipodal point theorem [ibid., p. 483]. The proof of theorem 3 uses elementary methods only.

143. Sideriades, Lefteri
Méthodes topologiques et applications.
ANN. TÉLÉCOMMUN. 14:185–207 (1959)

A resumé is given of some of the things known about the nature of the singular points of systems of differential equations of dimension 2 and 3. Particular examples arising from electronics and hydraulics are discussed.

144. Opial, Z.

Sur un problème de T. Ważewski

COLLOQ. MATH. 7:269–273 (1959/60)

According to the author, the following problem is due to T. Ważewski [New Scottish book, Prob. 29 du 14. XII. 1946]. Suppose the circle D is contained in an open set in which a continuous vector field, depending on time, is defined. This defines a system of differential equations $dx/dt = P(t, x, y)$, $dy/dt = Q(t, x, y)$ wherein P and Q are continuous. As the point (x, y) traverses D, for a fixed t, the vector $\{P(t, x, y), Q(T, x, y)\}$, attached to $(0, 0)$, describes a closed curve C_t. For each t, assume that the index of C_t with respect to $(0, 0)$ is not zero. Under these circumstances, does there exist a solution of the system of equations that remains within D for all values of t? This question is answered in the negative through an example.

145. Cherry, T. M.

The pathology of differential equations.

J. AUSTRAL. MATH. SOC. 1

This paper is an expository account (presidential address delivered in 1957) on several questions on the behavior of nonlinear differential equations in the large. Denjoy's [J. Math. Pures Appl. 11 (1932), 333–375] results on the flow on a torus are discussed, and E. Artin's [Abh. Math. Sem. Univ. Hamburg 3 (1924), 170–175] and M. Morse's [Trans. Amer. Math. Soc. 22 (1921), 84–100] theorems on transitivity of the geodesic flow on a 2-dimensional surface with negative curvature. Littlewood's [Acta. Math. 97 (1957), 267–308] investigation on van der Pol's equation with forcing terms is mentioned as well as C. L. Siegel's [Nachr. Akad. Wiss. Gottingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. 1952, 21–30] description of the solutions of an analytic system near an equilibrium by convergent series. An interesting example of a non-intergrable Hamiltonian system discussed in Section 6 seems to be new.

146. Agnew, Ralph Palmer

DIFFERENTIAL EQUATIONS. 2nd ed.

New York-Toronto-London, McGraw-Hill

Book Co., Inc., 1960. 485p.

This second edition of a well-known text is much more than a minor revision. Intended as both a text and a reference book, it gives valuable preliminary glimpses of matters encountered in advanced mathematics and science. Problems are discussed at greater length and more accurately than is usual, with a resultant difference in emphasis. A considerable part of a student's time is to be spent reading the text and answering the questions that so arise. The rules of the game are clearly stated: "We work some of the problems and read all of them." More attention than usual is given to the derivation of differential equations. This edition like the first has interesting, or humorous, side remarks. Particularly valuable are those which explain mathematical idiom or which show the relation of mathematics to physics. The problems are excellently chosen to bring out the power of mathematical analysis. Particularly for the beginning applied mathematician, analyst, or scientist this book can provide a deeper understanding than most.

There are sixteen chapters, of which the first seven are suggested for a one semester course. There is some mention of partial differential equations, but no systematic study. The following selective comments mention some of the significant features or departures from the first edition. The introductory chapter 1 contains an elegant derivation of the differential equation of a mass-spring system. Chapter 2 is devoted to applications of the fundamental theorem of calculus to differential equations. A feature is the full treatment of keplerian motion. Chapter 3 is on linear first order. Chapter 4 discusses families of curves and general and singular solutions. It is meant to be read but not dwelt upon. Chapter 5 is on first order equations. Chapter 6, 98 pp., on linear equations, contains a wealth of material in the problems. Chapter 7, on series, has some new material, for example, a long problem on the gamma function. Chapter 8 is a new and elegant treatment of numerical methods. One would hope that part of it could be included in a first course. Chapter 9 on the Laplace transform is new.

This is a well-written book which merits study by any serious student. Hopefully, it will have an influence on the authors of other books.

147. L. A. Beklemisheva

Some properties of a system of nearly canonical differential equations. VESTNIK MOSKOVSKOGO UNIVERSITETA. 4:26–36 (1960) ARS J., RUSSIAN SUPPL. 31:1614–1618, Nov 1961

148. Bogusz, W.

Application of the retract method in nonlinear engineering problems. ARCH. MECH. STOS. 12,4:437–450 (1960) (In English)

Paper concerns topological method of investigation due to Wazewski [Ann. Soc. Polon. Math. 20(1947)] of solutions of the system of ordinary differential equations

$$dx_i/dt = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots n).$$

By means of topological theorems, the behavior of solutions in the presence of singular points, or outside a singular point, can be examined. Specific applications are shown for a nonlinear dynamical system with negative elasticity, and for the differential equations for currents in an electrical circuit. In the former case, it is shown, for example, that the system $d^2x/dt^2 - k^2x + \alpha\, dx/dt + \beta\,(dx/dt)^3 = 0$ always has a bounded solution.

Left side of second equation of [2.3] should read "dx_2/dt."

149. Collatz, Lothar

Differentialgleishungen fur Ingenieure: Eine Einfuhrung. 2., neubearbeitete und erweiterte Aufl. Leitfaden der angewandten Mathematik und Mechanik, Bd. 1. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1960. 197p.

This small book gives a wealth of information on differential equations. It contains, indeed, more than can be dealt with effectively in a general course on differential equations in engineering curricula. The concise formulation is clear and precise, and it will appeal to engineers on account of its many graphic examples. The chapter titles are: Ordinary differential equations of the first order; Ordinary differential equations of higher order; Boundary value and eigenvalue problems; Special differential equations; Miscellaneous additional problems. Nonlinear problems receive more attention than is customary in books of this type.

150. J. B. Diaz

ON EXISTENCE, UNIQUENESS, AND NUMERICAL EVALUATION OF SOLUTIONS OF ORDINARY AND HYPERBOLIC DIFFERENTIAL EQUATIONS. Maryland, Univ., Inst. for Fluid Dynamics & Appl. Math. Rept. no. TN BN-216, Sep 1960. 29p. (AFOSR TN 60-1059).

Presentation of the proof for the general uniqueness theorem for the hyperbolic partial differential equation $u_{xy} \blacksquare f(x, yu)$. This theorem is proven to be an exact analog of the general uniqueness theorem for the ordinary differential equation $\frac{dy}{dx} \blacksquare f(x, y)$.

151. Dolezal, Vladimir
On non-unicity of solutions of a system of differential equations. CASOPIS PEST. MAT. 85:311–337 (1960) (In Czech. Russian and English summaries)

The problems considered in this paper have their origin in a paper of Kurzweil [Czechoslovak Math. J. 8 (83) (1958), 360–388]. The main result consists in the construction of an example which shows that from the uniqueness of solutions of the vector equation $dx/dt = g(x)$ need not follow the uniqueness of solutions of the equation $x(t) = x_0 + \int_{t_0}^{t} g(x(\tau))\,d\sigma(\tau)$, where $g(x)$ is a continuous vector function and $\sigma(\tau)$ a continuous real function of bounded variation.

152. H. W. Hahnemann
Allgemeines schema für das anwenden von reihenentwicklungen mit ganzzahligen potenzen zum auflösen von differentialgleichungen. FORSCHUNG AUF DEM GEBIETE DES INGENIEURWESENS, AUSGABE A. 26(5):153–171 (1960) (In German)

Study presenting a general scheme for the solution of differential equations by way of series expansions and by application of the coefficient comparison method. This method frequently involves tedious computation, especially when series convergence in certain ranges of the argument is poor, so that a large number of terms has to be determined. To reduce computation labor, a general scheme of equations is derived for the direct determination of the desired coefficients, based on a generalized ordinary differential equation which contains most of the special cases. The labor of power-series multiplication and coefficient comparison is thus reduced to the simple task of selecting from the tables the coefficients that do not disappear. This implies that if the tables do not provide the solution for a given differential equation, this equation cannot be expanded in series of positive integer powers.

153. Harris, W. A., Jr.
Singular perturbations of two-point boundary problems for systems of ordinary differential equations. ARCH. RATIONAL MECH. ANAL. 5(3):212–225 (1960) (In English)

154. Hartman, Philip

On boundary value problems for systems of ordinary, nonlinear, second order differential equations. TRANS. AMER. MATH. SOC. 96:493–509 (1960)

The boundary-value problems considered in this paper concern systems of differential equations of the form (*) $x'' = f(t, x, x')$ for vectors $x = x(t)$. The author discusses both the non-singular case $x(0) = x_0$, $x(T) = x_T$, and the singular case in which the solution $x(t)$ satisfies $x(0) = x_0$ and is required to be either of bounded norm or to tend to zero at $t \to \infty$. In all these cases, it is shown that solutions with the requisite properties exist if the function $f(t\ x\ x')$ is subjected to suitable conditions. It is further shown that certain additional restrictions on f guarantee the uniqueness of these solutions. Finally, the preceding results are applied to prove the existence of periodic (or almost periodic) solutions of (*) if the function $f(t, x, x')$ is, for fixed x, x', periodic (or almost periodic) in t.

155. Hronec, Jur

DIFFERENTIAL EQUATIONS. I. ORDINARY DIFFERENTIAL EQUATIONS. (Diferenciálne rovnice. I. Obycajné diferenciálne rovnice). 2nd ed. Vydavatel'stvo Slovenskej Akadémie Vied, Bratislava, 1960. 447p.

A standard textbook on ordinary differential equations in the real, as well as in the complex, domain. The first two chapters are concerned with the solution of elementary types of equations of the first and second order. There follow theorems on existence, uniqueness and dependence of solutions on initial conditions and parameters. Further the linear differential equation of an arbitrary order is dealt with. The sixth chapter concerns the boundary-value problems for second-order linear equations, especially the problem of Sturm-Liouville. The following two chapters deal with the general theory of linear differential systems and their applications in mechanics. One chapter is devoted to the study of qualitative properties of the solutions of $dy/dx = P(x, y)/Q(x, y)$ in the neighborhood of a singular point. Four chapters contain the theory of lineary equations with analytic coefficients. The last two chapters were written by A. Huta. One is concerned with the Runge-Kutta method, the other with a certain method which gives us the solutions of some equations in the form of a series.

156. Ku, Y. H.; Wolf, A. A.; Dietz, J. H.
Taylor-Cauchy transforms for analysis of a class of nonlinear systems. PROC. IRE 48:912–922 (1960)

The authors consider – under mildly restrictive conditions on L, ϕ and g-nonlinear systems characterized by differential equations of the form (1) $Lx + \phi(x, \dot{x}, \ldots) = g(t)$, where x is the response function, L is a linear differential operator, and $g(t)$ is a forcing function. The Taylor-Cauchy transform of a function $x(t)$ of a real variable t is defined as $x_n = (2\pi i)^{-1} \int_C x(\lambda)\lambda^{-n-1}\, d\lambda$, where $x(\lambda)$ is an analytic continuation of $x(t)$ over the λ-plane, and C is a circle with center at the origin and enclosing all poles of $\lambda^{-n-1} x(\lambda)$. Under the assumptions made in the paper, $x(\lambda)$ is given by the series $x(\lambda) = \Sigma_{n=0}^{\infty} x_n \lambda^n$.

By applying the Taylor-Cauchy transformation to both sides of (1) and making use of formulae for the Taylor-Cauchy transform of product terms of the form $[x^{(k)}(t)]\,[x^{(l)}(t)] \ldots [x^{(m)}(t)]$, the authors obtain for the Taylor-Cauchy transform of the solution of (1) a difference equation which can be solved recursively for the x_n. This yields a power series representation for the solution of (1).

157. Kuznecov, N. N.
Problem of the disintegration of an arbitrary discontinuity for a system of quasilinear equations of the first order. DOKL. AKAD. NAUK SSSR 131:503 (1960) (In Russian)
SOVIET MATH. DOKL. 1:282–285

Hyperbolic systems of the form $u_{it} + (\varphi_i(u))_x = 0$ $(i = 1, 2)$ with some additional restrictions on φ are considered. A proof is sketched that initial-value problems of the form $u(0, x) = u^+ = \text{const}$ for $x > 0$, $u(0, x) = u^- = \text{const}$ for $x < 0$ has one and only one solution of the form $u = u(x/t)$ if only "stable" discontinuities are permitted, as is customary.

158. Lee, Shen-ling

Topological structure of integral curves of the differential equation

$$y' = \frac{a_0x^n + a_1x^{n-1}y + \ldots + a_ny^n}{b_0x^n + b_1x^{n-1}y + \ldots + b_ny^n}$$

ACTA MATH. SINICA 10:1−21 (1960)

(In Chinese. English summary)

The author first deals with the case where the differential equation has m $(\leqq n + 1)$ solutions which are straight lines through the only critical point, the origin in xy-plane. The topological structure of the totality of integral curves in the neighborhood of the critical point is characterized and classified by those in the neighborhood of each of such straight lines. Elaborate formulas are worked out for the number of the classes when n is an arbitrary positive integer.

159. Massera, J. L.; Schäffer, J. J.

Linear differential equations and functional analysis. IV. MATH. ANN. 139:287−342 (1960)

[For part III see Ann. of Math. (2) 69 (1959), 535−574, Citation no. 137]. Let X be a Banach space, $J = [0, \infty)$, and suppose $t \to A(t)$ is a mapping of J into the Banach space of continuous linear operators of X into itself which is uniformly Bochner integrable on every bounded subinterval of J. A closed linear subspace Y of X is said to induce a dichotomy [resp. exponential dichotomy] of the solutions of (I) $x' + A(t)x = 0$ if there exist positive constants $\gamma < 1$, γ_0, N_i [and ν_i], $i = 0, 1$, such that: $x(0) \in Y$ implies, for $t \geqq t_0 \geqq 0$,

(1) $\|x(t)\| \leqq N_0 \|x(t_0)\|$ [resp. $\|x(t)\| \leqq N_0 \exp(-\nu_0(t - t_0)) \|x(t_0)\|$] ;

$x(0) \in' Y$ and $\gamma[Y, x(0)] \geqq \gamma$ imply, for $t \geqq t_0 \geqq 0$,

(2) $\|x(t)\| \geqq N_1 \|x(t_0)\|$ [resp. $\|x(t)\| \geqq N_1 \exp(\nu_1(t - t_0)) \|x(t_0)\|$] ;

$x_0(0) \in Y$, $x_1(0) \in' Y$, $x_i(0) \neq 0$, and $\gamma[Y, x_1(0)] \geqq \gamma$ imply $\gamma[x_0(t), x_1(t)] \geqq \gamma_0$ for $t \geqq 0$. Here $\gamma[y, x] = \| y/\|y\| - x/\|x\| \|$ and $\gamma[Y, x] = \inf \{\gamma[y, x] : y \in Y - \{0\}\}$ for $x \in' Y$. If $Y = \{0\}$ it is only required that $x(0) \in' Y$ imply (2).

In part I [ibid. 67 (1958), 517–573] the authors proved, among others, the following result concerning the linear manifold X_0 X of initial points $x(0)$ of bounded solutions of (I). If X_0 is closed and has a closed complement, then X_0 induces a dichotomy if and only if (II) $x' + A(t)x = f(t)$ has at least one bounded solution for every $f \in L^1$; and X_0 induces an exponential dichotomy if and only if (II) has at least one bounded solution for every $f \in L^p$, $1 < p \leqq \infty$, provided (*) $\sup_{t \in J} \int_t^{t+1} \|A(s)\| ds < \infty$.

In the present part IV the authors attack the question as to when a closed linear subspace of X induces an ordinary or exponential dichotomy in a much more general setting and from a more unified point of view. Their new results represent not only considerable extensions and refinements of their earlier ones; also they concern new types of stability properties, less local in nature than (1), (2), in that they are formulated in terms of different norms and mean values over certain "slices" of the solutions of (I). Only a bare outline of the principal results can be given here.

Let B, D be Banach spaces of mappings of J into X such that convergence in either implies convergence in the mean on every bounded subinterval of J. The authors call the pair (B, D) admissible if, for every $f \in B$, (II) has at least one solution which belongs to D. The pair (B_1, D_1) is stronger than (B_2, D_2) if B_2, D_1 are stronger than B_1, D_2, respectively. A pair (B, D) is a -pair if, roughly speaking, B and D are translation invariant [see Schaffer, Math. Ann. 137 (1959), 209–262; 138 (1959), 141–144].

In what follows, suppose that (B, D) is an admissible -pair (whence D contains non-trivial solutions). Let X_{0D} be the linear manifold of initial points $x(0)$ of solutions of (I) which belong to D, and suppose that X_{0D} is closed (but is complement need not be closed) and $X_{0D} \neq \{0\}$. Then for every solution of (I): $x(0) \in X_{0D}$ implies, for $t \geqq t_0 \geqq 0$,

$$\begin{aligned} &\int_t^{t+\Delta} \|x(s)\| ds \leqq M_0(\Delta) \int_{t_0}^{t_0+\Delta} \|x(s)\| ds, \\ (3)\qquad &\|\chi[t, t+\Delta]^x\| D \leqq M_0(\Delta) \|\chi[t_0, t_0+\Delta]^x\| D; \end{aligned}$$

and $x(0) \epsilon' X_{0D}$, $\gamma[X_{0D}, x(0)] \geqq \gamma$, imply, for $t \geqq t_0 \geqq 0$,

$$\begin{aligned} &\int_t^{t+\Delta} \|x(x)\| ds \geqq M_1(\Delta, \gamma) \int_t^{t_0+\Delta} \|x(s)\| ds, \\ (4)\qquad &\|\chi[t, t+\Delta]^x\| D \geqq M_1(\Delta, \gamma) \|\chi[t_0, t_0+\Delta]^x\| D. \end{aligned}$$

Here χ_1 is the characteristic function of I and $(-1)^i M_i$ are non-increasing with $\Delta > 0$. If, in addition, (B, D) is not weaker than (L^1, L_0^∞), then the inequalities (3), (4) hold with their right-hand sides multiplied by $e^{-\nu_0(t-t_0)}$, $e^{\nu_1(t-t_0)}$, respectively. Moreover, if D is also stronger than L^∞, then every solution of (I) which belongs to D satisfies, for $t \geqq t_0 \geqq 0$,

$$\|x(t)\| \leqq R(t_0)\|x(t_0)\| e^{-\nu(t - t_0)},$$

where $R(t_0)$ can not be taken independent of t_0 in general. Using these results, the authors then prove the following theorems on dichotomies.

If (B, D) is stronger than $({}_\tau L, L^1, L^\infty)$ for some $\tau \geqq 0$, where ${}_\tau L^1$ denotes the subspace $\{\chi(\tau, \infty)u : u \in L^1\}$, then X_{0D} induces a dichotomy. Conversely, if a closed linear subspace induces a dichotomy then (L^1, L^∞) is admissible. If (B, D) is not stronger than $({}_\tau L^1, L^\infty)$ then X_{0D} need not induce a dichotomy unless some other condition such as (*) is imposed, in which case X_{0D} always induces a dichotomy.

If (B, D) is not weaker than (L^1, L_0^∞) and if there exists a subspace Y which induces a dichotomy, then X_{0D} induces an exponential dichotomy. This is a necessary condition as well provided (*) holds.

As in the previous parts, the authors again give numerous examples to illustrate the interplay between the various hypotheses and to show that in almost all cases the results obtained are best possible. They also point out that, given a -pair (B, D), there does not in general exist an equation (II) with respect to which it is admissible.

160. Nehari, Zeev

On a class of nonlinear second-order differential equations. TRANS. AMER. MATH. SOC. 95:101–123 (1960)

This treatment of $y'' + yF(y^2, x) = 0$ is closely parallel to the paper of Moore and Nehari [see citation no. 138] on the special case $F(t, x) = t^n p(x)$; however for the variational approach a distinct functional is used. $F(t, x)$ is assumed continuous for $t \geqq 0$, $x > 0$ and positive for $t >$, $x > 0$, and such that $t^{-\epsilon} F(t, x)$ is increasing for fixed $x > 0$ and some $\epsilon > 0$. The existence of bounded non-oscillatory solutions is shown to be equivalent to $\int^\infty xF(c, x)dx < \infty$, for some positive c. In generalization of the criterion $\int^\infty x^{2n+1}p(x)dx < \infty$, one has $\int^\infty tF(ct^2, t)dt < \infty$ either for some $c > 0$, or again for all $c > 0$; the former ensure the existence of properly non-oscillatory solutions, the latter that they are all either bounded or of the asymptotic form βx, and that if in addition $F(\eta, x)$ is decreasing in x, then all solutions are non-oscillatory.

For the variational aspect, the author considers $\lambda(a, b) = \min \int_a^b [y'^2 - G(y^2, x)] dx$, where $G(\eta, x) = \int_0^\eta F(t, x) dt$, and y is subject to $y(a) = 0$, $y(x) \in D^1[a, b]$, and is normalized by $\int_a^b y'^2 dx = \int_a^b y^2 F(y^2, x) dx$. The minimum is achieved by a solution for which $y(a) = y'(b) = 0$, $y(x) > 0$ in $(a, b]$. Since $\lambda(a, b)$ is decreasing in b, one may define $\lambda(a) = \lim \lambda(a, b)$ as $b \to \infty$. Noteworthy results are that if there is a solution vanishing for $x = a$ and positive thereafter, then $\lambda(a) > 0$, and that this in turn is equivalent to $x \int_x^\infty F(mt, t) dt \leqq M$ for some m and M. On the other hand, this does not ensure the existence of properly non-oscillatory solutions. For the latter it is sufficient that $\int^\infty x F(\beta x, x) dx < \infty$ for all positive β; with this condition there is for every a a bounded solution whose zero is at $x = a$. The work closes with two results concerning solutions $y(x)$ for which $\liminf_{x \to \infty} x^{-1/2} y$ has a positive upper or lower bound.

161. Shintani, Hisayoshi

On the behavior of paths of the analytic two-dimensional autonomous system in a neighborhood of an isolated critical point. J. SCI. HIROSHIMA UNIV. SER. A 23:171–194 (1959); 24:173–187 (1960)

Nell'intorno dell'origine $(0, 0)$ si considerano le traiettorie del sistema $dx/dt = P_m(x, y) + 0(r^m)$, $dy/dt = Q_n(x, y) + 0(r^n)$ con i secondi membri analitici, P_m e Q_n polinomi omogenei dei gradi rispettivi m ed n, e con $r = (x^2 + y^2)^{1/2}$. Lo studio, molto dettagliato, e fatto seguendo il metodo di Frommer ed utilizzando un teorema di K. A. Keil e conduce ad una classificazione delle traiettorie tendenti all'origine in una direzione determinata e quindi ad una descrizione delle possibili configurazioni del piano fase in un intorno dell'origine stessa. Il procedimento rappresenta una semplificazione di quelli precedentemente usati da altri autori in quanto richiede l'applicazione, una sola volta, di una semplice trasformazione di coordinate.

162. G. Temple

LECTURES ON TOPICS IN NONLINEAR DIFFERENTIAL EQUATIONS. U. S., Navy Dept., David Taylor Model Basin, Appl. Math. Lab., Res. & Devel. Rept. no 1415. Mar 1960. 49p.

Presentation of a rigorous, analytical theory of Lighthill's technique for solving nonlinear differential equations with an "irregular" perturbation. The theory is developed from a number of simple examples and given a rigorous form by means of the theory of "dominant functions." A search for a "superposition" principle for ordinary non-linear differential equations is also undertaken, and the class of equations for which such a principle exists is determined.

163. Weinstein, Alexander

On a singular differential operator. ANN. MAT. PURA APPL. 49(4):359–365 (1960)

For operators of the form

$$D_k(u) = \frac{d^2u}{dt^2} + \frac{k}{t}\frac{du}{dt}$$

certain recursion formulas had previously been established by the author [Proc. Symposia Appl. Math., Vol. V, pp. 137–147, McGraw-Hill, New York, 1954; Ann. Mat. Pura Appl. (4)43 (1957), 325–340]. Let $X(u)$ be a linear operator and consider solutions of $D_k(u) = X(u)$. The recursion formulas are extended to cover more general operators and, further, an application is given which generalizes a classical formula for confluent hypergeometric functions. Finally the author establishes the principle of the Kelvin transformation for equations of the form

$$\frac{\partial^2 \nu}{\partial r^2} + \frac{k}{r}\frac{\partial \nu}{\partial r} = \frac{1}{r^2}\Phi(\nu),$$

where Φ is linear, independent of r and vanishes for $\nu \equiv 0$. This contains the result of Huber [Proc. Conference on Differential Equations, pp. 147–155, Univ. of Maryland Bookstore, College Park, Md., 1956] as well as the standard theorem for harmonic functions.

165. Yosida, Kosaku

LECTURES ON DIFFERENTIAL AND INTEGRAL EQUATIONS. New York-London, Pure and Applied Mathematics, Vol. X. Interscience Publishers, 1960. 220p.

We have here a modern introductory monograph on the theory of ordinary differential equations, together with those aspects of integral equations which are required for the former. There are five main chapters, whose headings are self explanatory. These are: 1. The initial value problem for ordinary differential equations; 2. The boundary value problem for linear differential equations; 3. Fredholm integral equations; 4. Volterra integral equations; and 5. The general expansion theorem (Weyl-Stone-Titchmarsh-Kodaira's theorem). The book closes with some comments on non-linear integral equations, and an appendix which contains several theorems of a more advanced nature from the theory of functions of a complex variable.

It is the author's attention to take us through the basic development of the initial value and boundary value problems (regular as well as singular) of ordinary differential equations, and a very readable account is given. The book can be read with profit by a student who has a thorough grounding in the elements of the real and complex calculus minus Lebesgue integration. Of special value are the many examples which are given to illustrate the theory as well as to fulfill the author's desire to let the reader know why one proceeds in a particular direction or why a theory fails and modifications are necessary.

165. Campbell, E. S., Buehler, R., Hirschfelder, J. O. and Hughes, D.
Numerical construction of Taylor series approximations for a set of simultaneous first order differential equations. ASSN. COMPUTING MACHINERY, J., 8:374–383 (1961)

USAF-Navy-supported presentation of specific computational procedures for the construction of series solutions to a set of simultaneous first-order ordinary differential equations subject to (1) a restriction to moderately general functional forms, and (2) two sets of boundary conditions. Procedures are developed for series construction about a certain type of singular point which arises in important applications. Series construction about an ordinary point is reduced to its operational essentials.

166. Chester, C. R. and Keller, J. B.
Asymptotic solutions of systems of linear ordinary differential equations with discontinuous coefficients. J. MATH. & MECH. 10:557–567, Jul 1961

USAF-supported analysis of the vector solution $u(x, k)$ of the system of N first-order linear ordinary differential equations $du/dk = kA(x, k)u$, where some derivative of A is discontinuous, and consequently only a finite number of terms in the expansion of u are known. The solution to the problem of finding additional terms when A and its derivatives have finite jump discontinuities is demonstrated, and the next term beyond those previously known is explicitly found. This term is discontinuous, whereas the preceding terms are all continuous. The significance of this result in various applications – e. g., in the one-dimensional propagation of time-harmonic waves in an inhomogeneous medium – is noted.

167. Hukuhara, Masuo; Kimura, Tosihusa; Matuda, Tizuko
EQUATIONS DIFFERENTIELLES ORDINAIRES DU PREMIER ORDRE DANS LE CHAMP COMPLEXE. The Mathematical Society of Japan, Tokyo, Publications of the Mathematical Society of Japan, 7. 1961. 155p.

The principal object of this monograph is to present an analysis of the behavior of the solutions of a differential equation $y' = R(x, y)$, in which R is a rational function of the complex variables x, y, with special emphasis on solutions in the neighborhood of a singular point. For much of the theory, R is required to be rational only in $y : R = P(x, y)/Q(x, y)$, where P and Q are polynomials in y with coefficients which are analytic functions of x in neighborhood of a point x_0. The results described are partly classical (including in particular various theorems of Malmquist) and partly recent achievements of the authors, especially M. Hukuhara.

A preliminary chapter includes existence theorems and basic ideas concerning singular points of differential equations in the complex domain. The second chapter presents a detailed analysis of an equation of form (1) $x^{\sigma+1}y' = P(x, y)/Q(x, y)$. An involved analysis of the algebraic character of P and Q leads to introduction of six canonical forms and methods of reduction to these forms. In the following chapter the equations in canonical form are integrated formally with the aid of certain power series; the formal solutions are shown to represent the actual solutions asymptotically and, under certain conditions, to be convergent representations.

A fourth chapter presents a detailed study of an equation possessing a solution having an essential singularity in a neighborhood of which there are no movable critical points. In particular, an extension of the Picard theorem is established. The final chapter considers the behavior of solutions when x is restricted to real values. An equation of form $dy/dt = R(y)$ is first considered, where $R(y)$ is rational; here one is dealing with a special case of a dynamical system whose phase space is the complex y-plane. The results obtained are used to obtain conclusions for the more general equation (1) for x close to 0.

The problems considered are difficult for the most part because of their complexity, rather than their depth. A great variety of special cases must be considered, and much ingenuity is required to keep track of all relationships. The authors have made an impressive contribution in organizing the theory. The reviewer believes that further efforts towards putting the theory in elegant form are justified, and that success in such efforts would be of much value in creating interest in a field in which many problems remain open.

168. G. A. Kamenskii
Dvukhtochechnala kraevaia zadacha dlia nelineinogo differentsial'nogo uravneniia vtorogo poriadka I teoremy o promezhutochnykh znacheniiakh. USSR, AN, DOKLADY. 139:541−543, 21 Jul 1961 (In Russian)

Analysis of a two-point boundary problem for a second-order, nonlinear differential equation, and development of intermediate-value theorems.

169. Kukles, I. S. ; Suyarsaev, A. M.
Frommer's generalized method. DOKL. AKAD. NAUK SSSR 136:29−32 (1961) (In Russian). SOVIET MATH. DOKL. 2:20−23

Consider the differential equation

(1) $$dy/dx = \sum_{i=0}^{m} \alpha_i y^{m-i} \Big/ \sum_{j=0}^{n} \beta_j y^{n-j},$$

where α_0 and β_0 are constants $(\alpha_0^2 + \beta_0^2 \neq 0)$, the functions $\alpha_i(x)$ $(i = 1, \ldots, m)$ and $\beta_j(x)$ $(j = 1, \ldots, n)$ are differentiable for small $x > 0$, conserve their signs and vanish with x. The problem is to examine whether there exist characteristics of equation (1), entering the origin from the right; and, if they exist, whether the set of such characteristics is finite or infinite. Also the problem of estimating the order of smallness of the solutions $y(x)$ of (1), vanishing at the origin, is discussed. If $\alpha_i(x)$ and $\beta_j(x)$ are analytic, the above problems are solved by Frommer's method [Math. Ann. 99 (1928), 222−272; cf. also I. S. Kikles, same Dokl. 117 (1957), 367−370]. In the present more general case use is made of a more general method, of which Frommer's method is a particular case. For the details of the method and results the reader is referred to the paper itself.

170. Ku, Y. H.
Theory of nonlinear control. J. FRANKLIN INST. 2(271):108–144, Feb 1961

Paper is a review of the recent advances in analytical methods for studying nonlinear control problems. Part I is an introduction discussing some of the well-known nonlinear differential equations arising in the study of several scientific areas. Attention is then directed to a brief discussion of feedback control, linear and nonlinear. Part II describes three stages in physical systems study as identified by the author. The first is idealization of actual systems as linear and the development of linear analysis. The second stage involves actual analysis of nonlinearities by such methods as perturbation, iteration, and phase-space. The third treats of performance improvement and the concept of nonlinear control synthesis.

Part III reviews a number of linearization techniques. The describing function method developed by Kochenburger, Klotter, and others and the techniques of quasi-linearization, equivalent linearization, and piecewise linearization are discussed. Part IV covers the general nature of phase-space analysis. Application of simultaneous phase-plane equations to nonlinear control systems is considered.

Part V deals with new analytic techniques and transforms. Methods covered include a reversion method of Pipes based on an algebraic procedure used in reverting power series, a number series transform method and a complex convolution method. Part VI presents a discussion of certain new concepts and theorems. Stability theory and new stability criteria are considered. Sampled-data systems and systems with random inputs are also discussed.

Reviewer believes this paper to be an excellent starting point for an investigator wishing to obtain a quick survey of the state of the art. The bibliography is very good and will prove quite helpful to anyone wishing to pursue certain topics in depth.

171. Lees, Milton
A boundary value problem for nonlinear ordinary differential equations. J. MATH. & MECH. 10:423–430, May 1961.

USAF-supported development of a theorem for two-point boundary value problems. Proof of the theorem is obtained by the method of finite difference.

172. Leipholz, H.

A foundation of the application of the Laplace transform to nonlinear differential equations

ZAMM 41(5):208–214, May 1961 (In German)

The differential equation $D(x) = f(x, \dot{x}) + z(t)$ under the initial conditions $x^{(i)}(0) = c_i$, $1 \le i \le n - 1$ is considered. D is an n^{th}-order $(n \ge 2)$ linear differential operator with constant coefficients. f is assumed to involve no linear functions of its arguments. Despite nonlinearity, Laplace transforms combined with a successive approximation technique are used after suitable restrictions on z and f are imposed. The applicability of the Laplace transform is proved. Then the convergence of successive approximations is shown and followed by a discussion of error estimates. The basic iteration formula is

$$x_{k+1}(t) = \mathcal{L}^{-1}\left\{\frac{M_{ax}(s) + \mathcal{L}\{(x_k, \dot{x}_k)\} + Z(s)}{D(s)}\right\}$$

where $M_{ax}(s)$ is the polynomial corresponding to the initial conditions, $Z(s) = \mathcal{L}(z(t))$, $D(s)$ is the polynomial corresponding to the operator D.

173. Sugai, Iwao

A class of solved Riccati's equations. ELEC. COMMUN. 37(1):56–60 (1961)

Presentation of exact solutions for a certain class of Ricatti equations, in explicit and analytical forms. The class considered has two arbitrary variable coefficients in specified products.

174. Sugai, Iwao

Exact solutions for ordinary nonlinear differential equations. ELEC. COMMUN. 37(1):47–55 (1961)

Presentation of an elementary approach to the exact solution of some nonlinear differential equations. The three equations considered are Riccati's equation, a second-order equation for a plasma problem, and a second-order equation obtained from an extension of the Sturm-Liouville differential equation.

175. Vejvoda, O.
On perturbed nonlinear boundary value problems. CZECH. MATH. J. 11(3):323–364 (1961)

Analysis of the existence of a solution to a perturbed nonlinear boundary problem. In particular, the autonomous case (where f and g do not depend upon t explicitly) is studied in detail, and a class of these problems, which show the same anomalies as those with periodic boundary conditions, is singled out.

176. Whitbeck, R. F.
Phase plane analysis. INFO. & CONTROL 4:30–47, Mar 1961

Study of the behavior of a second-order nonlinear differential equation of the form $\ddot{x} + f(x)\dot{x} + g(x)x = 0$ and the two changes of the variables necessary to place it in a form suitable for the application of Lienard's construction.

AUTHOR INDEX

SUBJECT INDEX