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FINAL REPORT
COMMUNICATION SYSTEMS TECHNIQUE
BASED ON FUNDAMENTAL CONCEPTS OF
JACQUES HADAMARD

Richard G. Segers

VITRO LABORATORIES
DIVISION OF VITRO CORPORATION OF AMERICA
WEST ORANGE, NEW JERSEY

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RESEARCH AND TECHNOLOGY DIVISION
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UNITED STATES AIR FORCE
GRIFFISS AIR FORCE BASE
NEW YORK

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ABSTRACT

This report presents a variety of fundamental investigations, with common analytical foundation in the work of Jacques Hadamard, conducted for the purpose of development of advanced communications systems. Chapter I presents a functional analysis approach to the demodulation problem as formulated in a system of integral equations by D. C. Youla. Its principal content is the discussion of the engineering significance of a quadratic variable principle that underlies the Youla formulation. Chapter II concentrates on mathematical considerations with the dual purpose of completing the arguments involved in Chapter I and indicating where forward work is needed to enlarge the domain of the validity of the theory of Chapter I. Chapter III presents communication techniques derivable from incidence matrices of balanced incomplete block design configurations.

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INTRODUCTION AND SUMMARY

The research on communication system techniques reported in this document has a common basis in analytical concepts introduced by the great French mathematician, Jacques Hadamard, between the years 1893 and 1945.

Chapter I presents a functional analysis approach to a system of integral equations introduced in 1954 by D. C. Youla. A solution of Youla's equations yields a maximum likelihood estimate of continuous modulated intelligence which has been corrupted by noise. The basic result of Chapter I is the identity of Youla's equations with the first variation of a quadratic functional. In order to devote the entire first chapter to this basic result and its consequences, the existence of certain representations needed in the formulation of the quadratic functional is postulated and their actual construction is deferred to Chapter II.

Chapter II begins with the construction, under suitable restrictions, of the representations needed for Chapter I. An attempt to lighten the restrictions so as to include all cases of physical interest leads us to the conclusion, already noted by Youla, that the class of functions to be considered as candidates for solution must be widened to include distributional solutions, i. e., Dirac delta functions and their derivatives. The question of preserving integral representations for such solutions was the basic consideration in Hadamard's invention of the "finite part" of a divergent integral. This concept together with Hadamard's concept of a "properly posed" problem are introduced and shown to underlie two basic approaches of importance, namely, the representation by divergent integrals and the representation by Fourier integrals.

Chapter III presents communication systems techniques derivable from incidence matrices of balanced incomplete block design configurations. For a symmetric design the above configurations are called $\mathcal{N}, k, \lambda$ configurations and these constitute a generalization of Hadamard Matrices, i. e., for special values of $\mathcal{N}, k, \lambda$ there exists a corresponding Hadamard Matrix and conversely every Hadamard Matrix generates a $\mathcal{N}, k, \lambda$ configuration. A Hadamard Matrix is an orthogonal matrix, all of whose elements are ± 1 . For several years engineers, particularly at the Jet Propulsion Laboratory, have utilized such matrices in the problem of the optimum codes for communicating through space. In our work we regard the incidence matrix of a

ν, k, λ configuration as a generalization of orthogonal matrices; i.e., we define "almost orthogonal" matrices. These are then utilized as the basis of an "almost orthogonal multiplexing technique" which has certain attractive features for use as an anti-jamming system. To as complete an extent as time permitted we investigated questions of circuit configurations, timing, and practical implementation for ν, k, λ configurations of low order. From an engineering viewpoint it turns out that the increase in circuit complexity in going from an orthogonal system based on a Hadamard Matrix to an "almost orthogonal" system based on a ν, k, λ configuration is the additional inclusion of a set of fixed resistors at the output terminals.

The investigations described in this report were begun in April 1962 and concluded in March 1963. The work was performed at Vitro Laboratories in West Orange, New Jersey. The principal investigator was Dr. R. G. Segers, and the Rome Air Development Center Project Monitor was Mr. Alfred Kobos. The work was supported by Contract AF 30 (602)-2649.

In August, 1962, the project monitor, Mr. Alfred Kobos, brought to our attention the work of Youla which resulted in the treatments in Chapter I and Chapter II. In October, 1962, the principal investigator met with Mr. Kobos and Dr. John Lawton of the Cornell Aeronautical Laboratory in Buffalo, New York to discuss Youla's equations. Dr. Lawton's approach to Youla's equations is based on an extension of Youla's series technique for the FM case and does not overlap our approach but the discussion served to focus our attention on a functional approach and the consequent use of the concepts of Hadamard that were available. It is a pleasure to express at this time our appreciation of the technical liaison provided by the project monitor, Mr. A. Kobos, and of the stimulation of the resulting discussion with Messrs. Kobos and Lawton.

CHAPTER I

OPTIMUM ANALOG DEMODULATION THEORY

1.1 INTRODUCTION

The classical procedure for minimizing a function on a finite-dimensional space is to set its gradient equal to zero and to examine the value of the function at these points and the signature of the quadratic form whose coefficients form the Hessian matrix of the function. The corresponding procedure in variational problems is to solve the Euler equation of a variational problem and to test the solution for nature of the extrema, i.e., maximum, minimum or stationary character, by the study of the second variation. On the other hand when an equation, irrespective of its origin, can be interpreted as the Euler equation of a variational problem one can often obtain algorithms for the solution of the corresponding functional. Everything we do in this chapter, and indeed in most of the next chapter, is centered on examining a set of integral equations whose solution, or solutions, would determine the optimum receiver (in the sense of maximum likelihood) in the presence of additive noise for the case in which an analog signal $a(t)$ is transmitted by means of a carrier which is modulated by $a(t)$ in some prescribed way. The equations were introduced by Youla, (39*, Youla, page 95) in an outstanding paper in 1954. Unfortunately, in the cases of most practical interest, namely phase modulation, and frequency modulation, it has not yet been possible to solve the resulting equations. Consequently, the search for a corresponding variational problem, which has been successful, is detailed below.

The case of optimum demodulation of AM signals has been investigated and an approximate solution has been obtained (40, Thomas, et al.). A maximum likelihood estimation process for FM signals has been investigated by linearizing the integral equations of Youla (41, Lawton, page 11).

We introduce notation and assumptions compatible with Youla (39; Youla, page 90) as follows. The demodulator, or receiver, is assumed to operate on the last T seconds of data. Symbolically we have

*Notation relates to references in bibliography.

$$(1-1-1) \quad e_1(\tau) = e_2[\tau, a(\tau)] + n(\tau) \quad ; \quad t-T \leq \tau \leq t$$

in which $a(\tau)$ and $n(\tau)$, both statistical in nature, represent the intelligence and noise, respectively, and $e_2[\tau, a(\tau)]$ represents a generalized modulation scheme. For example, amplitude modulation (AM) may be designated by choosing

$$(1-1-2) \text{ AM: } e_2[\tau, a(\tau)] = E_0 [1 + \lambda a(\tau)] \sin(\omega_0 \tau + \phi)$$

whereas phase modulation (PM) is designated by choosing

$$(1-1-3) \text{ PM: } e_2[\tau, a(\tau)] = E_0 \sin[\omega_0 \tau + \phi + \lambda a(\tau)]$$

and frequency modulation (FM) is designated by choosing

$$(1-1-4) \text{ FM: } e_2[\tau, a(\tau)] = E_0 \sin\left[\omega_0 \tau + \phi + \lambda \int_{t-T}^{\tau} a(\tau) d\tau\right]$$

In these relations, E_0 is the amplitude of the unmodulated carrier whose frequency is ω_0 , ϕ is an arbitrary phase angle, and λ is a modulation index. Both the intelligence $a(\tau)$ and the noise $n(\tau)$ are assumed to be Gaussian processes with zero mean and continuous covariance functions $R_a(s, \tau)$ and $R_n(s, \tau)$.

Having received the waveform $e_1(\tau)$, defined in (1-1-1), the ideal receiver can do no more than compute the a posteriori probability density $p[a(\tau) | e_1(\tau)]$ of all possible intelligence signals $a(\tau)$ (38; Woodward; p. 62.).

The next decision, and this is a critical one, is to decide on a criterion which will produce in some well defined sense a "best" estimate of $a(\tau)$, which we designate by $a^*(\tau)$. The criterion adopted by Youla (39; Youla; p. 93) is the maximum likelihood criterion (40; Cramer; p. 498) which chooses $a^*(\tau)$ such that $p[a^*(\tau) | e_1(\tau)]$ is maximized.

We now have enough notation to write down the celebrated equation of Youla (39; Youla; p. 95) gotten by a derivation that we will discuss at a later point in this paper. The maximum likelihood estimate (or estimates) $a^*(\tau)$ is determined by the following two integral equations;

$$(1-1-5) \begin{cases} a^*(\tau) = \int_{t-T}^{\tau} \frac{\partial \rho_2[\xi, a^*(\xi)]}{\partial a} R_a(\xi, \tau) g(\xi) d\xi, & t-T \leq \tau \leq t \\ \rho_1(s) - \rho_2[s, a^*(s)] = \int_{t-T}^s R_m(\xi, s) g(\xi) d\xi, & t-T \leq \tau \leq t \end{cases}$$

In (1-1-5) we have yet to identify $g(\xi)$ in a physical context. At least to this author $g(\xi)$ has no natural physical interpretation and we will treat $g(\xi)$ as a function whose knowledge will aid us in determining $a^*(\tau)$ but not as a quantity of intrinsic interest.

Although Youla calls this a maximum likelihood criterion, it may be termed a maximum a posteriori probability criterion since the a priori probability of the modulation is used.

1.2 YOULA EQUATIONS AS EULER EQUATIONS

Since (1-1-5) is linear in $g(\xi)$ in both equations, we first solve for $g(\xi)$ and manipulate the two results. We postulate the existence of inverse kernels $R_{\perp a}(\xi, \tau)$ and $R_{\perp m}(\xi, s)$ such that

$$(1-2-1) \begin{cases} \int_{t-T}^{\tau} R_{\perp a}(\bar{\xi}, \tau) R_a(\xi, \tau) d\tau = \delta(\bar{\xi} - \xi), & t-T \leq \bar{\xi} \leq t \\ \int_{t-T}^s R_{\perp m}(\bar{\xi}, s) R_m(\xi, s) ds = \delta(\bar{\xi} - \xi), & t-T \leq \bar{\xi} \leq t \end{cases}$$

an assumption to be justified later in the discussion in Chapter II. Applying

(1-2-1) to (1-1-5) we obtain

$$(1-2-2) \quad \begin{cases} \int_{t-T}^t R_{\frac{1}{a}}(\xi, \tau) a^*(\tau) d\tau = \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} g(\xi) \\ \int_{t-T}^t R_{\frac{1}{n}}(\xi, s) \{e_1(s) - e_2[s, a^*(s)]\} ds = g(\xi) \end{cases}$$

Now multiply the first equation in (1-2-2) by $\delta a^*(\xi)$, an arbitrary function, and the second equation in (1-2-2) by

$$(1-2-3) \quad \delta a^*(\xi) \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a}$$

then integrate both equations to obtain

$$(1-2-4) \quad \begin{aligned} & \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{a}}(\xi, \tau) a^*(\tau) \delta a^*(\xi) d\xi d\tau = \\ & \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, s) \delta a^*(\xi) \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} \{e_1(s) - e_2[s, a^*(s)]\} d\xi ds \end{aligned}$$

We further postulate that both kernels are symmetric, i.e.,

$$(1-2-5) \quad \begin{cases} R_{\frac{1}{a}}(\xi, \tau) = R_{\frac{1}{a}}(\tau, \xi) \\ R_{\frac{1}{m}}(\xi, s) = R_{\frac{1}{m}}(s, \xi) \end{cases}$$

in order to symmetrize both sides of equation (1-2-4). This assumption can be by-passed by more general methods (34 ; Tricomi, page 145), which, at this early stage of investigation aren't worth the resulting complication in manipulation and notation.

We now define the quadratic functional

$$(1-2-6) \quad F[a^*(\tau)] \equiv \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{a}}(\xi, \tau) a^*(\xi) a^*(\tau) d\xi d\tau$$

and the limiting process

$$(1-2-7) \quad \delta F \equiv \lim_{\epsilon \rightarrow 0} \left\{ \frac{F[a^*(\tau) + \epsilon \delta a^*(\tau)] - F[a^*(\tau)]}{\epsilon} \right\}$$

$$= \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{a}}(\xi, \tau) \left\{ a^*(\xi) \delta a^*(\tau) + \delta a^*(\xi) a^*(\tau) \right\} d\xi d\tau$$

$$= 2 \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{a}}(\xi, \tau) a^*(\tau) \delta a^*(\xi) d\xi d\tau$$

the last step following from the symmetry of $R_{\perp}(\xi, \tau)$, i.e., from (1-2-5). We are now justified in using the notation introduced by Volterra (36 ; Volterra; page 28) as follows,

$$(1-2-8) \quad \delta F[a^*(\tau)] = \int_{t-T}^t \frac{\partial F[\xi, a^*(\tau)]}{\partial a} \delta a^*(\xi) d\xi$$

which is a generalization to a continuum of variables of the more common formula

$$(1-2-9) \quad dF = \sum_{i=1}^n \frac{\partial F}{\partial y_i} dy_i$$

for functions of n variables $F(y_1, \dots, y_n)$, and where implicitly in (1-2-8) we have defined

$$(1-2-10) \quad \frac{\partial F[\xi, a^*(\tau)]}{\partial a} = 2 \int_{t-T}^t R_{\perp}(\xi, \tau) a^*(\tau) d\tau$$

upon comparison with (1-2-7). In this notation (1-2-4) becomes

$$(1-2-11) \quad \frac{1}{2} \int_{t-T}^t \frac{\partial F[\xi, a^*(\tau)]}{\partial a} \delta a^*(\xi) d\xi =$$

$$\int_{t-T}^t \int_{t-T}^t R_{\perp}(\xi, s) \frac{\partial e_2[\xi, a^*(\tau)]}{\partial a} \{e_1(s) - e_2[s, a^*(s)]\} \delta a^*(\xi) d\xi ds$$

Now consider the functional quadratic in $e_2[\tau, a^*(\tau)]$, i.e., define

$$(1-2-12) \quad G[e_2[\tau, a^*(\tau)]] \equiv \int_{t-T}^{\tau} \int_{t-T}^t R_{\frac{1}{n}}(\xi, s) e_2[\xi, a^*(\xi)] e_2[s, a^*(s)] d\xi ds$$

and the limiting process

$$(1-2-13) \quad \delta G \equiv \lim_{\epsilon \rightarrow 0} \left\{ \frac{G[e_2[\tau, a^*(\tau) + \epsilon \delta a^*(\tau)]] - G[e_2[\tau, a^*(\tau)]]}{\epsilon} \right\}$$

$$= \int_{t-T}^{\tau} \int_{t-T}^t R_{\frac{1}{n}}(\xi, s) \left\{ \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} \delta a^*(\xi) e_2[s, a^*(s)] \right.$$

$$\left. + \frac{\partial e_2[s, a^*(s)]}{\partial a} \delta a^*(s) e_2[\xi, a^*(\xi)] \right\} d\xi ds$$

$$= 2 \int_{t-T}^{\tau} \int_{t-T}^t R_{\frac{1}{n}}(\xi, s) \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} e_2[s, a^*(s)] \delta a^*(\xi) d\xi ds$$

the last step following from the symmetry of $R_{\frac{1}{n}}(\xi, s)$, i.e., from (1-2-5). We are now justified in using the notation

$$(1-2-14) \quad \delta G[e_2[\tau, a^*(\tau)]] = \int_{t-T}^{\tau} \frac{\partial G[\xi, e_2[\tau, a^*(\tau)]]}{\partial a} \delta a^*(\xi) d\xi$$

where implicitly in (1-2-14) we have defined

$$(1-2-15) \quad \frac{\partial G[\xi, e_2[\tau, a^*(\tau)]]}{\partial a} = 2 \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) \times \\ \times \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} e_2[\tau, a^*(\tau)] d\tau$$

In this notation (1-2-11) becomes

$$(1-2-16) \quad \frac{1}{2} \int_{t-T}^t \frac{\partial F[\xi, a^*(\tau)]}{\partial a} s a^*(\xi) d\xi = \\ \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} e_2[\tau, a^*(\tau)] s a^*(\xi) d\xi d\tau \\ - \frac{1}{2} \int_{t-T}^t \frac{\partial G[\xi, e_2[\tau, a^*(\tau)]]}{\partial a} s a^*(\xi) d\xi$$

To further simplify (1-2-16) we define a linear functional in $e_2[\tau, a^*(\tau)]$ by

$$(1-2-17) \quad H[e_2[\tau, a^*(\tau)]] \equiv \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) e_2[\xi, a^*(\xi)] e_2[\tau, a^*(\tau)] d\xi d\tau$$

and the limiting process

$$\begin{aligned}
 (1-2-18) \quad \delta H &\equiv \lim_{\epsilon \rightarrow 0} \left\{ \frac{H[e_2[\tau, a^*(\tau) + \epsilon \delta a^*(\tau)] - H[e_2[\tau, a^*(\tau)]]}{\epsilon} \right\} \\
 &= \int_{t-T}^t \int_{t-T}^t R_{\perp}(\xi, \tau) \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} \delta a^*(\xi) e_1(\tau) d\xi d\tau
 \end{aligned}$$

We are now justified in using the notation

$$(1-2-19) \quad \delta H[e_2[\tau, a^*(\tau)]] = \int_{t-T}^t R_{\perp}(\xi, \tau) \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} e_1(\tau) d\tau$$

where implicitly in (1-2-19) we have defined

$$(1-2-20) \quad \frac{\partial H[\xi, e_2[\tau, a^*(\tau)]]}{\partial a} = \int_{t-T}^t R_{\perp}(\xi, \tau) \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} e_1(\tau) d\tau$$

In this notation (1-2-16) becomes

$$\begin{aligned}
 (1-2-21) \quad \int_{t-T}^t \frac{\partial F[\xi, a^*(\tau)]}{\partial a} \delta a^*(\xi) d\xi &= 2 \int_{t-T}^t \frac{\partial H[\xi, e_2[\tau, a^*(\tau)]]}{\partial a} \delta a^*(\xi) d\xi \\
 &\quad - \int_{t-T}^t \frac{\partial G[\xi, e_2[\tau, a^*(\tau)]]}{\partial a} \delta a^*(\xi) d\xi
 \end{aligned}$$

which is the first variation (6 ; Courant Vol. I, page 186) of the composite functional

$$(1-2-22) \quad I \equiv F[a(\tau)] + G[e_2[\tau, a(\tau)]] - 2 H[e_2[\tau, a(\tau)]]$$

at the function $a(\tau) = a^*(\tau)$, i.e., $a^*(\tau)$ makes (1-2-22) an extremum. To see this analytically we note that the first variation of (1-2-22) is, by definition,

$$(1-2-23) \quad \delta I \Big|_{a=a^*(\tau)} = \lim_{\epsilon \rightarrow 0} \left\{ \frac{I[a(\tau) + \epsilon \delta a(\tau)] - I[a(\tau)]}{\epsilon} \right\} \Big|_{a=a^*(\tau)}$$

$$= 0$$

by equation (1-2-21). Before summarizing our results as a theorem, we combine the last two functionals on the right hand side of (1-2-22). This is natural since G is quadratic and H is linear in $e_2[\tau, a(\tau)]$ so that "completing the square" produces the desired result. More explicitly we have from (1-2-22) that

$$(1-2-24) \quad I[a(\tau)] = \int_{t-T}^{t+T} \int_{t-T}^{t+T} R_{\frac{1}{a}}(\xi, \tau) a(\xi) a(\tau) d\xi d\tau$$

$$+ \int_{t-T}^{t+T} \int_{t-T}^{t+T} R_{\frac{1}{n}}(\xi, \tau) e_2[\xi, a(\xi)] e_2[\tau, a(\tau)] d\xi d\tau$$

$$- 2 \int_{t-T}^{t+T} \int_{t-T}^{t+T} R_{\frac{1}{n}}(\xi, \tau) e_2[\xi, a(\xi)] e_1(\tau) d\xi d\tau$$

so that

$$(1-2-25) \quad I[a(\tau)] = \int_{t-T}^t \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, \tau) a(\xi) a(\tau) d\xi d\tau$$

$$+ \int_{t-T}^t \int_{t-T}^t \mathcal{R}_{\frac{1}{n}}(\xi, \tau) \{e_2[\xi, a(\xi)] - e_1(\xi)\} \{e_2[\tau, a(\tau)] - e_1(\tau)\} d\xi d\tau$$

$$- \int_{t-T}^t \int_{t-T}^t \mathcal{R}_{\frac{1}{n}}(\xi, \tau) e_1(\tau) e_1(\xi) d\xi d\tau$$

where we have again used symmetry of $\mathcal{R}_{\frac{1}{n}}(\xi, \tau)$

Since

$$(1-2-26) \quad n(\tau) = e_1(\tau) - e_2[\tau, a(\tau)]$$

we can rewrite (1-2-25) as

$$\begin{aligned}
 (1-2-27) \quad I[a(\tau)] &= \int_{t-T}^t \int_{t-T}^t \mathcal{R}_{\perp a}(\xi, \tau) a(\xi) a(\tau) d\xi d\tau \\
 &+ \int_{t-T}^t \int_{t-T}^t \mathcal{R}_{\perp m}(\xi, \tau) m(\xi) m(\tau) d\xi d\tau \\
 &- \int_{t-T}^t \int_{t-T}^t \mathcal{R}_{\perp m}(\xi, \tau) e_1(\xi) e_1(\tau) d\xi d\tau
 \end{aligned}$$

In summary, the above manipulation exhibits a quadratic function, $I[a(\tau)]$ in equation (1-2-27), such that the extrema of $I[a(\tau)]$ are attained by the same $a^*(\tau)$ that solve Youla's equations, (1-1-5). In the language of Variational Calculus the Youla equations are the Euler equations (6; Courant, Vol. 1; page 184.). We summarize our results as a theorem, the discussion of whose consequences constitutes the material of the next section.

1.3 PRINCIPAL RESULT AND CONSEQUENCES

Consider the following diagram.

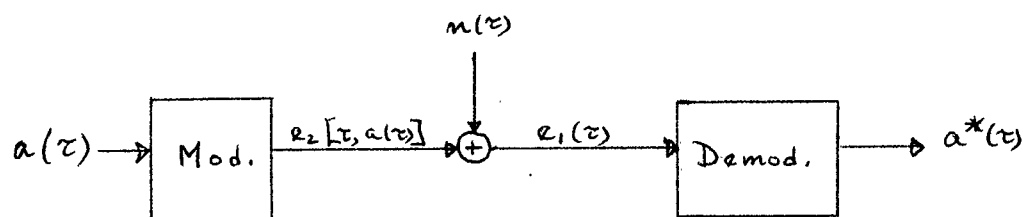


FIGURE 1 - BASIC BLOCK DIAGRAM

The problem is to design the demodulator; i.e., knowing $e_1(\tau)$ construct $a^*(\tau)$.

Theorem: The extremum, i.e., the maximum, minimum, or stationary values of the quadratic functional

$$\begin{aligned}
 (1-2-27) \quad I[a(\tau)] = & \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{a}}(\xi, \tau) a(\xi) a(\tau) d\xi d\tau \\
 & + \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) n(\xi) n(\tau) d\xi d\tau \\
 & - \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) e_1(\xi) e_1(\tau) d\xi d\tau
 \end{aligned}$$

are provided by the function (or functions) $a^*(\tau)$ for which

$$(1-2-23) \quad \delta I \Big|_{a=a^*(\tau)} = \lim_{\varepsilon \rightarrow 0} \left\{ \frac{I[a(\tau) + \varepsilon \delta a(\tau)] - I[a(\tau)]}{\varepsilon} \right\} \Big|_{a=a^*(\tau)} = 0$$

The same function (or functions) $a^*(\tau)$ satisfies the equations

$$(1-2-2) \quad \begin{cases} \int_{t-T}^t R_{\perp a}(\bar{z}, \tau) a^*(\tau) d\tau = \frac{\partial e_2[\bar{z}, a^*(\bar{z})]}{\partial a} g(\bar{z}) \\ \int_{t-T}^t R_{\perp m}(\bar{z}, s) \{e_1(s) - e_2[s, a^*(s)]\} ds = g(\bar{z}) \end{cases}$$

and, postulating the existence of inverse kernels $R_a(\bar{z}, \tau)$, $R_m(\bar{z}, s)$ such that

$$(1-2-1) \quad \begin{cases} \int_{t-T}^t R_{\perp a}(\bar{z}, \tau) R_a(\bar{z}, \tau) d\tau = \delta(\bar{z} - \bar{z}) \\ \int_{t-T}^t R_{\perp m}(\bar{z}, s) R_m(\bar{z}, s) ds = \delta(\bar{z} - \bar{z}), \text{ also satisfies} \end{cases}$$

the equations associated with the maximum likelihood estimate of Youla

(39 ; Youla; page 95).

$$(1-1-5) \left\{ \begin{array}{l} a^*(\tau) = \int_{t-T}^t \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} R_a(\xi, \tau) g(\xi) d\xi \\ e_1(s) - e_2[s, a^*(s)] = \int_{t-T}^t R_m(\xi, s) g(\xi) d\xi \end{array} \right.$$

all equations being valid for the interval $(t-T, t)$ and symmetric kernels $R(\xi, \tau) = R(\tau, \xi)$.

Proof: The manipulation involving the first three equations, (1-2-27), (1-2-23), (1-2-2), of the theorem are straightforward, though lengthy, and simply amount to reversing the steps taken in detail in Sections 1 and 2 of this chapter. The manipulation from (1-2-2) to Youla's equations (1-1-5) involves the existence of inverse kernels so as to give validity to (1-2-1) and is essentially an application of the Hilbert-Schmidt Theorem (34 ; Tricomi; page 110). In essence at this point we are excluding kernels for which the Hilbert-Schmidt Theorem is not applicable and in this precise sense the proof of the theorem is complete. Since, however, there are kernels, important in applications, for which the Hilbert-Schmidt Theorem is not applicable we will discuss the necessary additional conditions for their inclusion at a later point, namely, in Chapter II. Q.E.D.

There are many reasons why the formulation of a problem on a quadratic variational principle is important. First of all the differential equations of mathematical physics have been successfully analyzed in the light of their relation to quadratic variational principles. For linear functional equations of mathematical physics see Courant (6 ; Courant Vol. I, page 252) and for symmetrical hyperbolic systems see Courant (7 ; Courant Vol. II, page 592). Secondly, the work of Synge (31 ; Synge) has pointed out the advantage of a geometrical interpretation of a quadratic functional in which the "distance" of the approximations to the exact solution can be deduced. Since usually the engineering problem is not to attain the exact solution but to ascertain that a structure that can be constructed is not far from the best possible the approach of Synge could turn out to be as significant for communication theory as it presently is for elasticity. Thirdly, there are direct numerical techniques for establishing upper and lower bounds

on functionals. A definitive modern treatment is provided by a translation of the Russian "All-Union Conference on Functional Analysis and its Applications" (1 ; American Math. Soc. Trans.).

Since variational principles have provided some of the greatest generalizations in all of physical and biological science it is appropriate to present a short history of the attempts of early natural philosophers to discover a minimizing principle in nature. The first such discovery was the "principle of least action", where action is to be understood as the mean value of the difference between the kinetic and potential energies of a physical system averaged over some fixed interval of time, which principle originated with P. L. M. Maupertuis (1698-1759). Note that our functional $I[a(\tau)]$ in the equation (1-2-17) contains the difference of two integrals with the same kernel $R_{\mu}(\xi, \tau)$ and consequently the mathematical form for an analogous principle is available for speculation. The general statement of Maupertuis was made in an attempt to extend the theorem of P. Fermat (1601-65) that a ray of light, when traveling in a homogeneous medium, will pass from one point to another either directly or by reflection by the shortest path and in the shortest time. Having found the quantity that tends to a minimum, Maupertuis regarded the principle as all-inclusive: the laws of movement derived from it would apply to all natural phenomena. The astonishing degree to which he was correct in his prophecy can be seen not only in modern physics but in biology as well. For Maupertuis' principle is none other than Claude Bernard's principle of the maintenance of the internal environment, Walter B. Cannon's principle of homeostasis, or Le Chatelier's law of chemical equilibrium: "In a system in equilibrium, when one of the factors which determine the equilibrium is made to vary, the system reacts in such a way as to oppose the variation of the factor, and partially to annul it." For a fascinating account of the consequences of the above principles see Lotka's classic work on mathematical biology (17 ; Lotka; page 152).

The history of the calculus of variations, in the sense of obtaining necessary and sufficient conditions for obtaining extrema of integrals, is founded mainly on the work of Euler (1707-83) and Lagrange (1736-1813) with Lagrange's work extended by Hamilton (1805-65) in his classical papers on dynamics.

1.4 THRESHOLD PHENOMENA IN A VARIATIONAL SETTING

One of the dominant ideas in modern engineering is to employ "threshold" types of communication systems. In essence, if the modulated signal has sufficient power relative to the noise, a virtually perfect transmission is possible (27 ; Shannon; page 9). This "all or nothing" principle is natural to quadratic functionals, indeed it appears in the prototype of all problems in elastic stability, namely, the problem of the buckling of a long slender elastic column or rod under compressive forces at its ends. This problem was treated some two hundred years ago by Euler. The physical occurrence is quite easy to explain and understand. Obviously, if the column is long and slender, the straight unbent state of it will not remain stable if the end compression is made too large: that is clear to everyone's physical sense. What is interesting to us about the problem is its mathematical description in terms, to be described below, of the difference of two positive quadratic forms and the resulting close correspondence of the functional defined by this problem with $I[u(x)]$.

More explicitly, if the rod or column is of length L and if its lateral displacement is denoted by $u(x)$, $0 \leq x \leq L$, then the potential energy $\mathcal{U}$ is, apart from material constants, given by (6 ; Courant Vol. I; page 273).

$$(1-4-1) \quad \mathcal{U}[u(x)] = \int_0^L [u''(x)]^2 dx - P \int_0^L [u'(x)]^2 dx$$

The first integral is the energy of bending, the second integral the energy of elongation, so, since we are compressing, the negative sign appears.

Lemma: For sufficiently small values of P the minimum of $\mathcal{U}[u(x)]$, with the boundary conditions $u(0) = u(L) = 0$, has the value zero. Moreover, the minimum is attained only by $u(x) \equiv 0$.

Proof: For example, this is true for $P < L^{-1}$. Since

$$(1-4-2) \quad \int_0^L u'(x) dx = u(L) - u(0) = 0$$

it follows that there exists at least one point x_0 at which

$$(1-4-3) \quad u'(x_0) = 0 ;$$

then we have that

$$(1-4-4) \quad u'(x) = \int_{x_0}^x u''(x) dx$$

and consequently

$$(1-4-5) \quad [u'(x)]^2 \leq \left[\int_{x_0}^x u''(x) dx \right]^2 \leq \int_{x_0}^x [u''(x)]^2 dx \leq \int_0^L [u''(x)]^2 dx$$

Since (1-4-5) is true for $0 \leq x \leq L$ it follows that

$$(1-4-6) \quad \int_0^L [u'(x)]^2 dx \leq L \max_x [u'(x)]^2 \leq L \int_0^L [u''(x)]^2 dx$$

so that the functional $\mathcal{U}[u(x)]$ defined in (1-4-1) satisfies

$$\begin{aligned} (1-4-7) \quad \mathcal{U}[u(x)] \Big|_{P < L^{-1}} &= \int_0^L [u''(x)]^2 dx - P \int_0^L [u'(x)]^2 dx \Big|_{P < L^{-1}} \\ &= \frac{1}{L} \left[L \int_0^L [u''(x)]^2 dx - \int_0^L [u'(x)]^2 dx \right] \\ &\quad + \left(\frac{1}{L} - P \right) \int_0^L [u'(x)]^2 dx \Big|_{P < L^{-1}} \geq 0 \end{aligned}$$

Moreover, equality in (1-4-7) is attained only for $u(x) \equiv 0$. This is clear from the last term in (1-4-7). Q. E. D.

Taking as self evident the principle that the minimum of the potential energy $\mathcal{U}[u(x)]$ yields the equilibrium states $u(x)$ the physical meaning of the lemma is that for sufficiently small P the unbent state of the rod, $u(x) \equiv 0$, is the only equilibrium state. Since it is clear physically that bent states $u(x) \neq 0$ exist, and mathematically for any such admiss-

ible states $u(x) \neq 0$ we can choose

$$(1-4-8) \quad \mathcal{I} > \frac{\int_0^L [u''(x)]^2 dx}{\int_0^L [u'(x)]^2 dx},$$

we see the possibility that $\mathcal{V}[u(x)] < 0$. From the lemma and (1-4-8) it is now clear that there exists a critical value of $\mathcal{P}$, call it $\mathcal{P}_0$, which causes buckling, i.e., the straight state ceases to be the only possible equilibrium state and a bifurcation of the solution of the equilibrium problem takes place; the bent state is then the stable state of equilibrium. This is truly a bifurcation phenomenon in the sense meant by Poincaré (34; Tricomi; page 161); that is, $u(x) = u(x, \mathcal{P})$, a function of both the independent variable x and of the load parameter $\mathcal{P}$ such that when $\mathcal{P}$ attains the critical value $\mathcal{P}_0$ new types of solutions of the equilibrium problem appear in the neighborhood of the unbent state which co-exist for the same values of the physical parameters. It is also clear that $\mathcal{P}_0$ is the minimum of the right hand side of (1-4-8) for $u(x) \neq 0$. A goodly number of the readers of this report will be familiar with the totality of results in the present case (30; Stoker; page 251), i.e., that there are infinitely many critical values and correspondingly many different buckled shapes or modes of the column.

What can the above discussion add to our understanding of the optimum demodulation problem? First of all we have never decided what type of extrema, i.e., maximum, minimum or stationary point, of $\mathcal{I}[u(x)]$ is appropriate. The above discussion yields an approach that indicates a minimum is appropriate. To see this consider the functional $\mathcal{I}[u(x)]$, rewritten here for convenience as

$$\begin{aligned}
 (1-4-9) \quad I[a(\tau)] &= \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{a}}(\xi, \tau) a(\xi) a(\tau) d\xi d\tau \\
 &+ \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) n(\xi) n(\tau) d\xi d\tau \\
 &- \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) e_1(\xi) e_1(\tau) d\xi d\tau
 \end{aligned}$$

As with $\mathcal{U}[u(x)]$ we are interested in the reference or zero state $a(\tau) \equiv 0$ for which we expect the estimate $a^*(\tau) \equiv 0$. Since

$$(1-1-1) \quad e_1(\tau) = e_2[\tau, a(\tau)] + n(\tau)$$

we get from (1-4-9) that

$$\begin{aligned}
 (1-4-10) \quad I[0] &= 2 \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) n(\xi) e_2[\tau, 0] d\xi d\tau \\
 &- \int_{t-T}^t \int_{t-T}^t R_{\frac{1}{n}}(\xi, \tau) e_2[\xi, 0] e_2[\tau, 0] d\xi d\tau
 \end{aligned}$$

It is also clear that for any reasonable form of modulation, say the AM, PM, and FM cases, we have characterized by (1-1-2), (1-1-3), and (1-1-4), that the modulation is proportional to E_0 , the amplitude of the unmodulated carrier. Consequently from the last integral in (1-4-10) we can factor out E_0^2 which will play the role of $R > 0$ in our previous reasoning. We need one more general fact about the integrals in (1-4-9) before proceeding on our particular tack. We want to establish conditions such that each integral in (1-4-9) is non-negative. Fortunately such conditions are known, and have been known for a long time.

Theorem (Hilbert). A necessary and sufficient condition for the quadratic form $J(\phi, \phi)$, defined by

$$(1-4-11) \quad J(\phi, \phi) = \iint K(x, y) \phi(x) \phi(y) dx dy,$$

to be non-negative in L_2 - space, i.e., for the inequality.

$$(1-4-12) \quad J(\phi, \phi) \geq 0, \quad \phi \in L_2$$

to hold, is that all the eigenvalues of $K(x, y)$ be positive, i.e.,

$$(1-4-13) \quad 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

Proof: See Tricomi (34; Tricomi; page 124). Q. E. D. If in equation (1-4-12) we have the inequality sign whenever ϕ is non-zero over a set of finite measure the kernel $K(x, y)$ is called positive-definite.

We now postulate that the kernels $R_{\alpha}(\xi, \tau)$ and $R_{\mu}(\xi, \tau)$ are positive definite. The plausibility of the assumption is, of course, evident from (1-2-1) which defines the kernels in question as "inverses" of the covariance functions $R_{\alpha}(\xi, \tau)$ and $R_{\mu}(\xi, \tau)$ which have only positive eigenvalues and moreover are positive definite. Precise conditions for the validity of the postulate will, as in our basic theorem, be discussed in Chapter II.

We now return to our reasoning with (1-4-10). To get a definite point of reference for $I[a(\tau)]$ we not only put $a(\tau) \equiv 0$ but turn the carrier power down to zero, i.e., from (1-4-10) we have that

$$(1-4-14) \quad I[a(\tau)] \Big|_{\substack{a(\tau) \equiv 0 \\ E_0 = 0}} = 0$$

What happens as we increase E_0 from zero to quite a large value in (1-4-10)? The first integral containing $\mu(\xi)$ depends linearly on $e_2[\tau, 0]$ and hence on E_0 ; while the second integral, whose contribution we know to be positive by itself and negative when preceded by a negative sign as in (1-4-10), depends on the product of $e_2[\tau, 0]$ and $e_2[\xi, 0]$ and hence on E_0^2 . For a fixed but arbitrary sample of noise $\mu(\xi)$ it is clear that the last term in (1-4-10) will for sufficiently large E_0 make $I[a(\tau)] \Big|_{a(\tau) \equiv 0} < 0$.

For from (1-4-10) it is only necessary to choose

$$(1-4-15) \quad E_0 > \frac{\left| 2 \int_{t-T}^t \int_{t-T}^t R_{\perp}(\xi, \tau) \mu(\xi) \hat{e}_2[\tau, 0] d\xi d\tau \right|}{\int_{t-T}^t \int_{t-T}^t R_{\perp}(\xi, \tau) \hat{e}_2[\xi, 0] \hat{e}_2[\tau, 0] d\xi d\tau}$$

where we have put

$$(1-4-16) \quad \hat{e}_2[\tau, a(\tau)] = E_0^{-1} e_2[\tau, a(\tau)]$$

i.e., the modulation $\hat{e}_2[\tau, a(\tau)]$ is normalized so as to make the role of E_0 more apparent. There is a close correspondence between (1-4-15) and (1-4-8) since the fulfillment of each implies the respective functionals achieve negative values. The reason for assuming the kernel $R_{\perp}(\xi, \tau)$ is negative definite is now clear from (1-4-15) since otherwise the possibility of the denominator vanishing in (1-4-15) exists. Actually this assumption is not that critical since if it were not fulfilled for some $R_{\perp}(\xi, \tau)$ of interest it would merely be necessary to choose the modulation $e_2[\tau, a(\tau)]$ so as to avoid the null function of $R_{\perp}(\xi, \tau)$. It is also of interest to comment on the numerator of (1-4-15). Since we have no way of knowing the sign of $\mu(\xi)$ the only reasonable course in assigning E_0 , which would be fixed in application, is the conservative inclusion of absolute value signs in the numerator of (1-4-15).

It is now clear how to handle the general case $a(\tau) \neq 0$. Referring now to (1-4-9) instead of (1-4-10) we note that the contribution of the first integral is positive and hence more power, i.e., a larger value of E_0 , is required to make $I[a(\tau)] < 0$. Essentially the same mechanism comes into play as with (1-4-9). The last integral involving $e_1 = e_2 + \mu$ is capable of forcing $I[a(\tau)] < 0$ by using a sufficiently large value of E_0 . We summarize our results as follows.

Theorem: For fixed, but arbitrary, samples $a(\tau)$ and $n(\tau)$ of the intelligence and noise in the interval $t-T \leq \tau$, $\tau \leq t$, and for positive definite kernels $R_a(\tau, \tau)$ and $R_n(\tau, \tau)$ there exists a sufficiently large value of E_0 , the amplitude of the unmodulated carrier, to make $I[a(\tau)] < -M$ for any arbitrary magnitude $M > 0$.

Proof: The term in $I[a(\tau)]$ which is quadratic in E_0 and is inherently negative has been demonstrated by our previous remarks to dominate the value of $I[a(\tau)]$. Q. E. D.

Equating more power, i.e., larger E_0 with better transmission, which by the above theorem is equivalent to minimizing $I[a(\tau)]$, we draw the inference that for fixed power, i.e., fixed E_0 , the choice $a(\tau) = a^*(\tau)$, which by our basic theorem gives an extremum to $I[a(\tau)]$, must correspond to a minimum of $I[a(\tau)]$.

We now show the consistency of the above conclusion with the basic postulates on which Youla has based his theory. Following Youla (39 ; Youla; page 93) the theorem of conditional probabilities is that

$$(1-4-17) \quad p(a|e_1) = \frac{p(a) p(e_1|a)}{p(e_1)}$$

where the quantities in (1-4-17) are defined by

$$(1-4-18) \quad \begin{aligned} p(a) &= \text{likelihood of } a(\tau) \\ p(e_1) &= \text{likelihood of } e_1(\tau) \\ p(e_1|a) &= \text{conditional likelihood of } e_1(\tau) \text{ given } a(\tau) \text{ at the transmitter} \\ p(a|e_1) &= \text{a posteriori likelihood of } a(\tau) \text{ given } e_1(\tau) \text{ at the receiver.} \end{aligned}$$

The procedure, called the maximum likelihood estimate, adopted by Youla is to utilize the knowledge of $e_1(\tau)$ to maximize $p(a|e_1)$ by choice of $a(\tau)$. Youla then reasons (39 ; Youla; page 94) that since $e_1 - e_2 = n$, $p(e_1|a) = \text{likelihood that } e_1 - e_2[\tau, a(\tau)] = n(\tau)$,

noise, so that (1-4-17) is equivalent to

$$(1-4-19) \quad p(a|e_1) = \frac{p(a) p(m)}{p(e_1)}$$

and we desire to maximize $p(a|e_1)$.

Lemma: The extrema of the functional $I[a(\tau)]$ are attained by the same function $a^*(\tau)$ (or functions) that yield extrema of the functionals $\exp\{I[a(\tau)]\}$ and $\exp\{-I[a(\tau)]\}$ since both mappings are monotone.

Choosing the negative exponential in the above lemma we obtain from (1-2-27) that

$$(1-4-20) \quad \exp\{-I[a(\tau)]\} = \frac{\exp\left\{-\int_{t-T}^t \int_{t-T}^t R_{\frac{1}{2}}(z, \tau) a(z) a(\tau) dz d\tau\right\} \times \exp\left\{-\int_{t-T}^t \int_{t-T}^t R_{\frac{1}{2}}(z, \tau) m(z) m(\tau) dz d\tau\right\}}{\exp\left\{-\int_{t-T}^t \int_{t-T}^t R_{\frac{1}{2}}(z, \tau) e_1(z) e_1(\tau) dz d\tau\right\}}$$

Compare (1-4-20) and (1-4-19) term by term noting the correspondence of our double integrals with the quadratic forms in equations (8), (9), and (12) of Youla (39; Youla; page 94). Since our double integrals are sometimes called "quadratic forms with an infinite number of variables" (34; Tricomi; p. 118) the correspondence is quite close, except of course for a normalizing factor to make probability interpretations possible. Hence the right hand sides of (1-4-20) and (1-4-19) suggest the reasonableness of using our functionals in place of the series technique of Youla. The correspondence of the left hand sides requires special comment. More explicitly we have that

$$(1-4-21) \quad \exp\{-I[a(\tau)]\} \longleftrightarrow p(a|e_1)$$

where by $\longleftrightarrow$ we mean identical except for a normalizing constant factor. First of all the minimization of $I[a(\tau)]$, which by our basic theorem and

previous reasoning in this section is accomplished by the same $a^*(\tau)$ that satisfy Youla's equations (1-1-5), yields a maximization of (1-4-21) which is precisely the maximum likelihood criterion. Secondly, the fact that $e_1(\tau)$ is a known function, as indicated by the notation $p(a|e_1)$, played an essential role in the manipulations leading us to $I[a(\tau)]$. More explicitly, even though

$$(1-1-1) \quad e_1(\tau) = e_2[\tau, a(\tau)] + n(\tau),$$

and consequently depended on $a(\tau)$, we did not allow for any variation in $e_1(\tau)$ when we formed the expression $a(\tau) + \varepsilon \delta a(\tau)$. This was correct, of course, only because $e_1(\tau)$ was explicit data, known at the receiver, and our search for a critical $a^*(\tau)$ by variation over all admissible $a(\tau)$ had to be compatible with the fixed nature of $e_1(\tau)$. What we are saying, of course, is that the correspondence (1-4-21) is very satisfying in the number of checks it yields on the correctness of the theory of this chapter.

1.5 SOME SPECULATIVE CONSIDERATIONS

We now proceed with one of the author's favorite ways of turning an intractable problem into an at least approachable problem which, hopefully, bears resemblance to the original. First of all we introduce a geometry, which will again require the assumption that any relevant kernels like $R_{\alpha\beta}(\xi, \tau)$ and $R_{\alpha\beta}(\xi, \tau)$ be positive definite so that a metric can be defined. As in the previous section the discussion of the conditions for which the assumption is valid will be deferred until Chapter II. Having a geometry we then employ the trick of assuming that the solutions of interest have certain simple features. For example, in what follows we construct a triangle and there are certain heuristic reasons for asserting the triangle is a right triangle. The resulting simplification in analysis is considerable so we proceed with optimism. Finally a simple structure emerges, but we must change the setting of the problem somewhat in order to obtain a consistent structure. The reader who is scornful of such tinkering or who feels no need of geometrical motivation may proceed at once to Chapter II without any loss in logical continuity.

In order to associate a geometrical picture with $I[a(\tau)]$ we introduce a vector notation.

$$(1-5-1) \quad \vec{a} = a(\tau), \quad \vec{n} = n(\tau), \quad \vec{e}_1 = e_1(\tau), \quad \vec{e}_2 = e_2[\tau, a(\tau)]$$

and the inner products appropriate to the kernels $R_{\frac{1}{a}}(\xi, \tau)$ and $R_{\frac{1}{m}}(\xi, \tau)$, i.e., put

$$(1-5-2) \quad \begin{cases} (\vec{x}, \vec{y})_{\frac{1}{a}} = \int_{t-T}^{t+T} \int_{t-T}^{t+T} R_{\frac{1}{a}}(\xi, \tau) x(\xi) y(\tau) d\xi d\tau \\ (\vec{x}, \vec{y})_{\frac{1}{m}} = \int_{t-T}^{t+T} \int_{t-T}^{t+T} R_{\frac{1}{m}}(\xi, \tau) x(\xi) y(\tau) d\xi d\tau \end{cases}$$

For either inner product in (1-5-2) we introduce the notation

$$(1-5-3) \quad \|\vec{x}\| = \sqrt{(\vec{x}, \vec{x})}$$

which is the distance from $\vec{x}$ to the null vector $\vec{0}$ at the origin of the coordinate system. Then from (1-2-27) we obtain in the above notation,

$$(1-5-4) \quad I[a(\tau)] = \|\vec{a}\|_{\frac{1}{a}}^2 + \|\vec{m}\|_{\frac{1}{m}}^2 - \|\vec{e}_1\|_{\frac{1}{m}}^2$$

Since the data received, $a(\tau)$, is an additive combination of the modulated signal, $e_2[\tau, a(\tau)]$, and the noise, $m(\tau)$, we can write

$$(1-5-5) \quad \vec{e}_1 = \vec{e}_2 + \vec{m}$$

Inserting (1-5-5) in (1-5-4) we obtain another expression for $I[a(\tau)]$, namely,

$$(1-5-6) \quad I[a(\tau)] = \|\vec{a}\|_{\frac{1}{a}}^2 + \|\vec{e}_2\|_{\frac{1}{m}}^2 - 2(\vec{e}_2, \vec{m})_{\frac{1}{m}}$$

Note that (1-5-6) is identical to (1-2-22) which is the form of our functional before "completing the square", accomplished by equations (1-2-22) to (1-2-27).

Having before us the simplest possible representations for $I[a(\tau)]$, namely, (1-5-4) and (1-5-6) we ask what choice of $a(\tau) = a^*(\tau)$ can lead to an extremum of $I[a(\tau)]$? The last term in (1-5-6) yields a clue to a contribution to $I[a(\tau)]$ that we cannot control by choice of $a(\tau) = a^*(\tau)$. Recalling that our knowledge of $\vec{m} = m(\tau)$, the noise being added to our modulated signal, is embodied in the kernel $R_m(s, \tau)$, which is the covariance function of the noise, assumed to be a continuous second-order process with mean zero (39; Youla; page 92), it follows that the numerical sign of $m(\tau)$ in the term $(\vec{e}_2, \vec{m})_{\vec{m}}$ cannot be known to us. This is a basic point. Having assumed that the processes of interest have mean zero and having characterized our problem by covariance functions, which are expected values of the product of two processes, we have no possibility of knowing the numerical sign of the noise contribution in the term $(\vec{e}_2, \vec{m})_{\vec{m}}$. Note that all other terms in (1-5-4) and (1-5-6) are quadratic and hence insensitive to the numerical sign of $\vec{a}$, $\vec{m}$, $\vec{e}_2$, or $\vec{e}_1$. Since the term $(\vec{e}_2, \vec{m})_{\vec{m}}$ cannot be controlled relative to yielding an extremum of $I[a(\tau)]$, it is of interest to investigate the consequences of assuming that $(\vec{e}_2, \vec{m})_{\vec{m}} = 0$. Even more to the point is the question of the consistency of the condition $(\vec{e}_2, \vec{m})_{\vec{m}} = 0$ with the solution (or solutions) of Youla's equations.

Before plunging into the particular question posed by the last paragraph some general remarks are in order to achieve the proper perspective. In the theory of ordinary maxima and minima the existence of a solution is ensured by the fundamental theorem of Weierstrass (6; Courant; page 57). In contrast, the characteristic difficulty of the calculus of variations (and of formulations like Youla's which our Theorem shows is equivalent to a question in the calculus of variations) is that problems which can be meaningfully formulated may not have solutions. In essence it is not in general possible to choose the domain of admissible functions as a "compact set" in which a principle of points of accumulation is valid. This is best illustrated by the following geometric example (6; Courant, Vol. I; page 173). Two points on the x-axis are to be connected by the shortest possible line of continuous curvature which is perpendicular to the x-axis at the end points. This problem has no solution. For, the length of such a line is always greater than that of the straight line connecting the two points, but it may approximate this length as closely as desired. Thus there exists a greatest lower bound but no minimum for admissible curves.

Consequently, in the calculus of variations the existence of an extremum in a particular problem cannot be taken for granted. A special existence proof is needed for the solution of each problem or class of problems. Such existence proofs have not been published relative to the equations of Youla (1-1-5). However, an indication of how such proofs can be accomplished is contained in the work of Krasnosel'skii and Rutickii (16 ; Krasnosel'skii; Chapter IV) on convex functions and Orlicz spaces (i.e. normal spaces of which the L -spaces are a special case). Since the Youla equations are expressible as the gradient of the functional $I[a(r)]$ and a number of theorems (16 ; Krasnosel'skii; page 214) on the existence of characteristic functions for such a gradient, called a "potential operator", are known, it may be possible to establish the existence of solutions of the Youla equations by means of the available theorems. Such an effort in the future is contemplated by the author as part of a continuing interest in this area of research.

The point of the above discussion is that there may or may not exist solutions of Youla's equations or equivalently extrema of the functional $I[a(r)]$ since the existence of a bound does not imply the existence of functions attaining the bound. Moreover the additional condition, $(\vec{e}_2, \vec{n})_{\perp} = 0$, that we have reasoned from (1-3-6) as desirable, may not be attainable. Regardless of these possible negative results we now investigate the consequences of assuming that

$$(1-5-7) \quad (\vec{e}_2, \vec{n})_{\perp} = 0$$

Combining (1-5-5) and (1-5-7) in a geometric construction, using the metric $\| \cdot \|_{\perp}$ as the basis of length, we obtain the following figure.

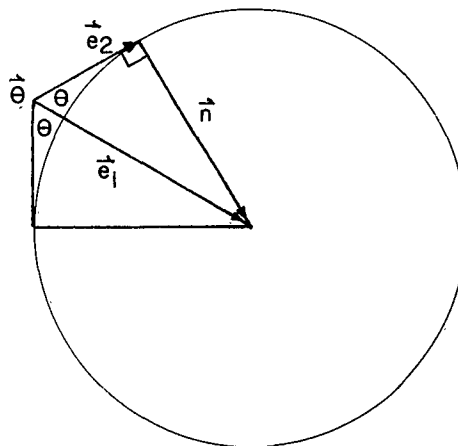


FIGURE 2 - GEOMETRY OF RECEPTION PROCESS

To construct the above figure, called a hypercircle by Synge (31; Synge; page 87), the received signal $\vec{e}_1$ is surrounded by a circle of radius $\|\vec{n}\|_{\perp}$, the magnitude of $\|\vec{e}_1\|_{\perp}$ being assumed larger than the assigned radius so that the origin of coordinates $\vec{e}$ lies outside the circle. From the origin $\vec{e}$ we draw the tangents to the circle. Of the two possible constructions we arbitrarily choose the one in the above figure. The essence of the construction is embodied in the single condition (1-5-7). From the figure we see two interpretations of this condition. First of all, the modulation $\vec{e}_1$ has been positioned orthogonal to the noise $\vec{n}$ which is what we have been postulating all along. A second, and fresh point of view, is that for the assigned value of $\|\vec{n}\|_{\perp}$ the angle Θ which separates $\vec{e}_1$ and $\vec{e}_2$ has been maximized. We are now in a position to draw a surprising conclusion concerning Θ .

Theorem: The condition of orthogonality, (1-5-7) $(\vec{e}_2, \vec{n})_{\perp} = 0$, suffices to determine the angle Θ which separates $\vec{e}_1$ and $\vec{e}_2$. More explicitly $\Theta = 60^\circ$.

Proof: Put (1-5-7) into (1-5-6) to obtain (1-5-8)

$$(1-5-8) \quad \mathcal{I}[a(\tau)] = \|\vec{a}\|_{\perp}^2 - \|\vec{e}_2\|_{\perp}^2$$

We now retrace our steps to obtain the equation satisfied by

$$(1-5-9) \quad \delta \mathcal{I}[a(\tau)] \Big|_{a(\tau) = a^*(\tau)} = 0$$

From (1-5-9) we obtain

$$(1-5-10) \quad \begin{aligned} \delta \mathcal{I} &= 2 \int_{t-T}^t \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(z, \tau) a^*(\tau) \delta a^*(z) d\tau dz \\ &\quad - 2 \int_{t-T}^t \int_{t-T}^t \mathcal{R}_{\frac{1}{n}}(z, s) \frac{\partial \mathcal{R}_2[z, a^*(z)]}{\partial a} a^*(z) \delta a^*(z) dz ds \\ &= 0 \end{aligned}$$

Since $\delta a^*(\xi)$ is arbitrary we obtain from (1-5-10) that

$$(1-5-11) \quad \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, \tau) a^*(\tau) d\tau = \frac{\partial \mathcal{L}_2[\xi, a^*(\xi)]}{\partial a} \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, s) \mathcal{L}_2[s, a^*(s)] ds$$

Comparing (1-5-11) with

$$(1-2-2) \quad \begin{cases} \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, \tau) a^*(\tau) d\tau = \frac{\partial \mathcal{L}_2[\xi, a^*(\xi)]}{\partial a} g(\xi) \\ \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, s) \{ \mathcal{L}_1(s) - \mathcal{L}_2[s, a^*(s)] \} ds = g(\xi) \end{cases}$$

we obtain two expressions for $g(\xi)$, namely

$$(1-5-12) \quad \begin{cases} g(\xi) = \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, s) \mathcal{L}_2[s, a^*(s)] ds \\ g(\xi) = \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, s) \{ \mathcal{L}_1(s) - \mathcal{L}_2[s, a^*(s)] \} ds \end{cases}$$

From (1-5-12) we have

$$(1-5-13) \quad 2 \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, s) \mathcal{L}_2[s, a^*(s)] ds = \int_{t-T}^t \mathcal{R}_{\frac{1}{a}}(\xi, s) \mathcal{L}_1(s) ds$$

Multiply both sides of (1-5-13) by $e_2[\xi, a^*(\xi)]$ and integrate to obtain

$$(1-5-14) \quad \|\vec{e}_2\|_{\perp_m}^2 \equiv (\vec{e}_2, \vec{e}_2)_{\perp_m} = \frac{1}{2} (\vec{e}_1, \vec{e}_2)_{\perp_m}$$

Multiply both sides of (1-5-13) by $e_1(\xi)$ and integrate to obtain

$$(1-5-15) \quad 2(\vec{e}_1, \vec{e}_2)_{\perp_m} = (\vec{e}_1, \vec{e}_1)_{\perp_m} = \|\vec{e}_1\|_{\perp_m}^2$$

From (1-5-14) and (1-5-15) we conclude that

$$(1-5-16) \quad \|\vec{e}_2\|_{\perp_m}^2 = \frac{1}{4} \|\vec{e}_1\|_{\perp_m}^2$$

and since $\|\cdot\|$ is inherently positive that

$$(1-5-17) \quad \|\vec{e}_2\|_{\perp_m} = \frac{1}{2} \|\vec{e}_1\|_{\perp_m}$$

There is an elementary theorem in Euclidean geometry, well known to every school boy, that states "In a right triangle where the short side is one half the hypotenuse the angles are 30° , 60° , 90° ." Hence, in Figure 2, $\theta = 60^\circ$ and the sides of the triangle are in the proportions.

$$(1-5-18) \quad \|\vec{e}_1\|_{\perp_m} : \|\vec{e}_2\|_{\perp_m} : \|\vec{m}\|_{\perp_m} = 1 : \frac{1}{2} : \frac{\sqrt{3}}{2}$$

Q. E. D.

Now that we have arrived at a definitive ratio of metrics in (1-5-18) it is natural to attempt to "pack" the space, whose metric is determined by

$\|\cdot\|_{\perp_m}$, with triangles of the form of Figure 2, whose sides are compatible with (1-5-18). Except for the special case given below the following fundamental difficulty is encountered. The geometry of the space whose metric is determined by $\|\cdot\|_{\perp_m}$ is the geometry of Hilbert Space. This is easily seen by comparing our construction of quadratic functionals with the treatment of reciprocal quadratic variational problems as given by Courant (6; Courant, Vol. I; page 252). Since Hilbert Space is infinite-dimensional, i.e., for any integer m one can find in Hilbert Space m linearly independent vectors, the constraint

in (1-5-18) and consequently the 30° , 60° , 90° angle relation is not sufficient to pin down the construction of $\vec{e}_1$, $\vec{e}_2$ and $\vec{m}$ unless we postulate some further condition. Again following our philosophy of seeking the simplest solution compatible with the conditions of our problem we now restrict our attention to a structure that will produce a plane in the Hilbert Space of interest. See Figure 3.

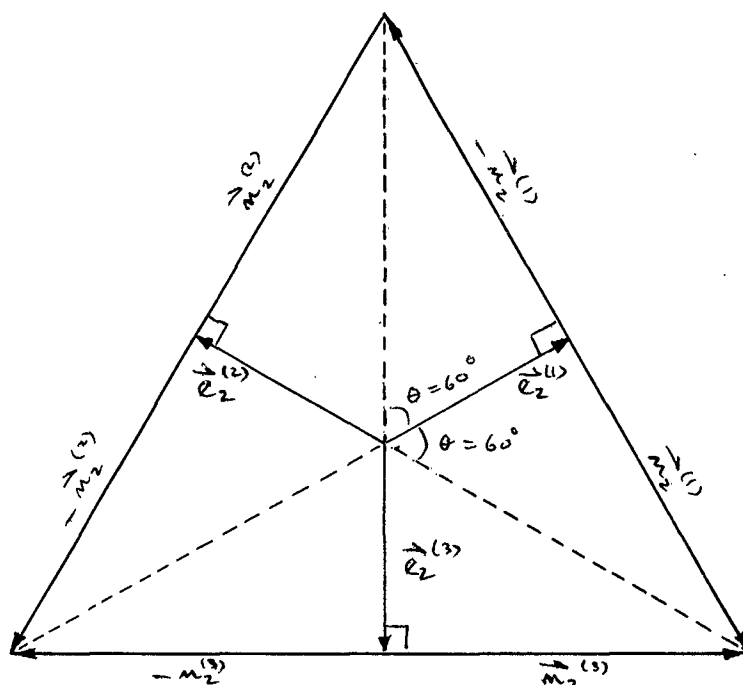


FIGURE 3 - GEOMETRY OF RECEPTION IN A PLANE

What we have done is restrict our possible inputs $a(\tau)$ to three distinct functions $a^{(1)}(\tau)$, $a^{(2)}(\tau)$, and $a^{(3)}(\tau)$ and postulated that the resulting modulated signals $e_2^{(1)}[\tau, a^{(1)}(\tau)]$, $e_2^{(2)}[\tau, a^{(2)}(\tau)]$ and $e_2^{(3)}[\tau, a^{(3)}(\tau)]$ make angles of 120° with each other. Each modulated signal is of the same magnitude and, subject to the planar restriction, is separated as much as possible from the other modulated signals. From the point of view of coding we would designate the modulated signals as a maximal code (37 ; Wolfowitz; page 78) with the decoding system consisting of the partitioning of the plane by the dotted lines, each of which is the negative extension of one of the modulated signals making an angle of 60° with the other two modulated signals. To decode, or demodulate, a received vector $\vec{e}_1$, the

partitioning of the plane assigns the nearest modulated noiseless signal, i.e., either $\vec{e}_1^{(1)}$, $\vec{e}_2^{(2)}$, or $\vec{e}_3^{(3)}$ from which the original $a(\tau)$ is obtained as one of the functions $\hat{a}^{(1)}(\tau)$, $\hat{a}^{(2)}(\tau)$, or $\hat{a}^{(3)}(\tau)$. If $\vec{e}_1$ happens to coincide with one of the dotted lines then there are two modulated signals at the same distance, either of which may be used without increasing the probability of error.

In summary of the construction we note that our search for a simple geometry has required consideration of a discrete set of signals $a(\tau)$ so that the modulations might all lie in an easily visualized plane. Moreover, the desired condition of orthogonality (1-5-7) and the consequent angle $\theta = 60^\circ$ are only satisfied by the $\vec{e}_1$ which fall on the lines which partition the plane into pie shaped wedges. From a heuristic point of view one could reason that only the vectors which partition the plane need be identified in an optimal fashion and consequently in this sense the problem has been satisfactorily handled. From a logical point of view much remains to be done to give substance to the above approach.

CHAPTER II

BASIC CONCEPTS OF HADAMARD AND THEIR APPLICATION

2.1 INTRODUCTION

The subject matter of this chapter has application to specific questions raised in treating Youla's integral equations (1-2-27) and to the general question of integral equations of the first kind (34 ; Tricomi; page 143) with symmetric kernels. We proceed from the specific to the general. First of all, in proving our principal result concerning Youla's equations (1-2-27), i.e., the identity of (1-2-27) with the first variation (1-2-23) of a quadratic functional (1-2-27), we assumed the existence of kernels $\mathcal{R}_\Delta(z, \tau)$ and $\mathcal{R}_\Delta(\bar{z}, \tau)$ defined by (1-2-1). The validation of this assumption is absolutely essential to the basic ideas in Section I and we shall correspondingly spend much effort on this topic in what follows. A second but less critical assumption, since it can be by-passed to some extent (see the discussion following (1-4-16)), is the postulate that $\mathcal{R}_\Delta(z, \tau)$ and $\mathcal{R}_\Delta(\bar{z}, \tau)$ are positive definite.

(For real noise $n(t)$ and information $a(t)$, the kernels are symmetric and thus so would be the inverse kernels.)

Our organization is as follows. Since the construction of the kernels $\mathcal{R}_\Delta(z, \tau)$ and $\mathcal{R}_\Delta(\bar{z}, \tau)$ follow the same pattern, we choose one of them, in particular $\mathcal{R}_\Delta(z, \tau)$, and discuss it in complete detail. This will validate the assumption of Section I and show us that our theory is not vacuous. We then observe the desirability of alternate conditions for validation. The point is that our conditions for validation, though sufficient, may be awkward when applied to a particular problem. This point of view for integral equations of the first kind has been put forward by Shinbrot (28 ; Shinbrot; page 3) and when our kernels are difference kernels, corresponding to stationary statistics, Shinbrot (28 ; Shinbrot; page 4) has an alternate approach based on the use of Fourier transforms. Shinbrot's work was done in 1960. It turns out that the concept of a "properly posed" problem (10 ; Hadamard; page 33), a concept of the first importance for partial differential equations, was introduced by Hadamard in 1917. In 1938 Petrovskii, motivated by the above mentioned work of Hadamard, used a Fourier transform to investigate the proper posing (the phrase "correct setting" is more common in the Russian literature) of the Cauchy problem for one or several partial differential equations when the initial functions are assumed bounded (22 ; Petrovskii). We observe that both Shinbrot and Petrovskii obtain important and similar results when the Fourier transforms increase, in the neighborhood of infinity, no faster than some power of the independent

variable. Moreover, in 1957 Gel'fond and Silov generalized the work of Petrovskii by constructing the Fourier transform for functions which increase with arbitrary rapidity (9; Gel'fond). It is entirely possible that a continuation of the work of Shinbrot in analogy to that of Gel'fond and Silov would produce results of importance for integral equations of the first kind.

At least as important are two other pertinent concepts of Hadamard, the notions of finite part (p.f.) and logarithmic part (p.l.) of divergent integrals, which are basic to the "method of singularities" developed in the years from 1904 to 1932 as an alternative approach to the use of Fourier methods. Rather than describe these concepts in general forms we now proceed to the exposition we have outlined, working in the new concepts as specific questions require their introduction.

The results in Chapter I depend on showing the equivalence of the two integral equations introduced by Youla, namely,

$$(1-1-5) \quad \begin{cases} a^*(\tau) = \int_{t-T}^t \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} R_a(\xi, \tau) g(\xi) d\xi; t-T \leq \tau \leq t \\ e_1(s) - e_2[s, a^*(s)] = \int_{t-T}^t R_m(\xi, s) g(\xi) d\xi; t-T \leq \tau \leq t \end{cases}$$

and the equations

$$(1-2-2) \quad \begin{cases} \int_{t-T}^t R_{\perp}(\xi, \tau) a^*(\tau) d\tau = \frac{\partial e_2[\xi, a^*(\xi)]}{\partial a} g(\xi); t-T \leq \xi \leq t \\ \int_{t-T}^t R_{\perp}(\xi, s) \{e_1(s) - e_2[s, a^*(s)]\} ds = g(\xi); t-T \leq \xi \leq t \end{cases}$$

Since we are given the covariance functions $R_a(\xi, \tau)$ and $R_m(\xi, s)$ we first show that (1-1-5) implies (1-2-2) by construction of $R_{\perp}(\xi, \tau)$ and $R_{\perp}(\xi, s)$. It is then easily seen how to reverse the arguments to show that (1-2-2) implies (1-1-5). Since the construction of the kernels $R_{\perp}(\xi, \tau)$ and $R_{\perp}(\xi, s)$ follow the same pattern we choose to discuss in detail only one of them, in particular $R_{\perp}(\xi, s)$, since the notation in its equation is a bit simpler. Since the covariance function

$R_m(\xi, s)$ is continuous, positive definite, and symmetric (39 ; Youla, page 100), it can by Mercer's Theorem be expanded into the absolutely and uniformly convergent series

$$(2-1-1) \quad R_m(\xi, s) = \sum_{n=1}^{\infty} \frac{\psi_n(\xi) \psi_n(s)}{\mu_n} ; \quad t-T \leq \xi, s \leq t$$

where the $\psi_n(\xi)$ and μ_n are the eigen functions and eigenvalues of the kernel $R_m(\xi, s)$, i.e., they satisfy

$$(2-1-2) \quad \psi_n(s) = \mu_n \int_{t-T}^t R_m(\xi, s) \psi_n(\xi) d\xi$$

In addition, the $\psi_n(\xi)$ are orthonormal, i.e.,

$$(2-1-3) \quad \int_{t-T}^t \psi_n(\xi) \psi_j(\xi) d\xi = \delta_{nj}$$

and because of the positive definite nature of $R_m(\xi, s)$ the set $\{\psi_n(\xi)\}$ is complete (34, Tricomi; page 124).

The general analysis utilizing (2-1-1) with an infinite number of terms is going to be lengthy. To obtain insight into the construction and to verify in complete detail the results of Chapter I in an at least restricted setting we first restrict (2-1-1) to a finite number N of terms obtaining what is called a degenerate or Pincherle-Goursat kernel (34 ; Tricomi, page 55). With this restriction we now assert that the kernel $R_m(\xi, s)$ is given by

$$(2-1-4) \quad R_m(\xi, s) = \sum_{n=1}^N \mu_n \psi_n(\xi) \psi_n(s)$$

To prove this we need only insert (2-1-4) in the left hand side of the second equation in (1-2-2) and show that the right hand side results. We obtain

$$(2-1-5) \quad \int_{t-T}^t R_{\perp n}(\xi, s) \{ r_1(s) - r_2[s, a^*(s)] \} ds =$$

$$\int_{t-T}^t g(\bar{\xi}) d\bar{\xi} \int_{t-T}^t R_{\perp n}(\xi, s) R_n(\bar{\xi}, s) ds =$$

$$\sum_{n=1}^N \left[\int_{t-T}^t g(\bar{\xi}) \psi_n(\bar{\xi}) d\bar{\xi} \right] \psi_n(\xi) = g(\xi)$$

the critical steps in the manipulation following from

$$(2-1-6) \quad \int_{t-T}^t R_{\perp n}(\xi, s) R_n(\bar{\xi}, s) ds =$$

$$\int_{t-T}^t \sum_{n=1}^N \mu_n \psi_n(\xi) \psi_n(s) \sum_{j=1}^N \frac{\psi_j(\bar{\xi}) \psi_j(s)}{\mu_j} ds =$$

$$\sum_{n=1}^N \psi_n(\xi) \psi_n(\bar{\xi}) = K_I(\xi, \bar{\xi})$$

where $K_I(\xi, \bar{\xi})$ is the "unit kernel" with the property

$$(2-1-7) \quad \int_{t-T}^t K_I(\xi, \bar{\xi}) g(\bar{\xi}) d\bar{\xi} = g(\xi)$$

which is just the last equation in (2-1-5). Note that the manipulations from (2-1-4) to (2-1-7) are critically dependent on the finite N assumption which implicitly carries with it the completeness of the set of functions $\{\psi_\lambda(\xi)\}$, $\lambda = 1, \dots, N$. The removal of this assumption will introduce into the manipulations convergence difficulties relative to the infinite series to be considered and consequent difficulty in interchange of integration and summation in reasoning as in (2-1-5). The class of functions to be considered must then be widened from the continuous functions of the finite dimensional case to the L_2 functions and even beyond in some cases to distributional solutions which are also called generalized functions. Youla was not unaware of these considerations. See in particular his Appendix A(39; Youla; page 100). Before plunging into these matters we summarize our results.

Theorem. Restricting the positive definite kernels $R_a(\xi, \tau)$ and $R_m(\xi, s)$ to a finite number N of eigen functions and eigenvalues, the so-called degenerate or Pincherle-Goursat kernels, it is possible to define inverse or reciprocal positive definite kernels $R_a^{-1}(\xi, \tau)$ and $R_m^{-1}(\xi, s)$ so that the set of equations (1-1-5) is equivalent to the set (1-2-2). Consequently the assumptions in Chapter I that invoked later demonstrations in Chapter II are validated under the above mentioned restriction to N dimensional space.

Proof: Given the positive definite expansions

$$(2-1-8) \quad R_a(\xi, \tau) = \sum_{\lambda=1}^N \frac{\phi_\lambda(\xi) \phi_\lambda(\tau)}{\lambda_\lambda},$$

$$(2-1-1) \quad R_m(\xi, s) = \sum_{\lambda=1}^N \frac{\psi_\lambda(\xi) \psi_\lambda(s)}{\mu_\lambda},$$

define the inverses

$$(2-1-9) \quad R_a^{-1}(\xi, \tau) = \sum_{\lambda=1}^N \lambda_\lambda \phi_\lambda(\xi) \phi_\lambda(\tau)$$

$$(2-1-10) \quad R_m^{-1}(\xi, s) = \sum_{\lambda=1}^N \mu_\lambda \psi_\lambda(\xi) \psi_\lambda(s)$$

which are clearly positive definite. By direct substitution in (1-1-5) and (1-2-2) and manipulations analogous to (2-1-4) to (2-1-7) the equivalence of (1-1-5) and (1-2-2) is clear. Moreover the symmetry implicit in the construction of $R_a^{-1}(\xi, \tau)$ and $R_m^{-1}(\xi, s)$ shows that the postulate (1-2-5)

is actually not a postulate but a consequence and hence non-symmetric kernels play no part in the theory. Q. E. D.

To investigate the general case we streamline our notation and consider the Fredholm integral equation of the first kind.

$$(2-1-11) \quad \phi(x) = \int_{-1}^1 K(x,y) f(y) dy, \quad -1 < x < 1$$

where we assume the kernel $K(x,y)$ is continuous, symmetric, and positive definite and consequently possesses a complete orthonormal set of eigenfunctions $\{\phi_i(x)\}$ and associated set of eigenvalues $\{\lambda_i\}$, $i=1,2,\dots$. Results applicable to Youla's equations will be collected at the end of this section and exhibited in the next. To further simplify manipulations we introduce the notation

$$(2-1-12) \quad \int_{-1}^1 f(y) g(y) dy = (\vec{f}, \vec{g})$$

$$\int_{-1}^1 K(x,y) f(y) dy = K \vec{f}$$

so that the eigenfunctions and eigenvalues satisfy

$$(2-1-13) \quad \begin{cases} (\vec{\phi}_i, \vec{\phi}_j) = \delta_{ij} \\ K \vec{\phi}_i = \frac{1}{\lambda_i} \vec{\phi}_i \end{cases}$$

Note that capital letters denote linear operations while small letters with an arrow above denote functions and small letters without the arrow are scalars. This is the usual notation in studying integral equations when one thinks of a function as a vector in an infinite dimensional space. Even without this attitude, the simplification in expression is worth the identification as the following illustrates. We rewrite (2-1-11) as

$$(2-1-14) \quad \vec{\phi} = K \vec{f}$$

Our objectives in this section are centered on the question of the possibility of writing the solution of (2-1-14) as

$$(2-1-15) \quad \vec{f} = K^{-1} \vec{\phi}$$

The class of functions in which we find $\vec{f}$ clearly depends on the nature of both κ^{-1} and $\vec{\Phi}$. From (2-1-4) we can immediately obtain the Fourier components of $\vec{f}$, relative to the complete orthonormal set of functions $\{\vec{\Phi}_i\}$, by taking the inner product of (2-1-14) with $\vec{\Phi}_i$ to obtain

$$(2-1-16) \quad (\vec{\Phi}, \vec{\Phi}_i) = (\kappa \vec{f}, \vec{\Phi}_i) = (\vec{f}, \kappa \vec{\Phi}_i) \\ = (\vec{f}, \frac{1}{\lambda_i} \vec{\Phi}_i) = \frac{1}{\lambda_i} (\vec{f}, \vec{\Phi}_i),$$

where we have used the symmetry of $\kappa(x, y)$ to provide the critical manipulation in (2-1-16), i.e.,

$$(2-1-17) \quad (\kappa \vec{f}, \vec{\Phi}_i) = (\vec{f}, \kappa \vec{\Phi}_i)$$

From (2-1-16) we have the desired result

$$(2-1-18) \quad (\vec{f}, \vec{\Phi}_i) = \lambda_i (\vec{\Phi}, \vec{\Phi}_i)$$

We now invoke the Riesz-Fisher Theorem (6 ; Courant, page 110) in the formulation due to Riesz.

Theorem (Riesz 1907): If $\{\vec{\Phi}_i(x)\}$ is an arbitrarily given complete orthonormal system of functions and if $a_1, a_2, \dots$ are arbitrary real numbers for which $\sum_{i=1}^{\infty} a_i^2$ converges, then there exists a summable function $f(x)$ with summable square for which $a_i = (\vec{f}, \vec{\Phi}_i)$.

By virtue of the above theorem there are only two possibilities; either (I) There is a unique (neglecting functions which vanish almost everywhere) function $\vec{f}$ satisfying (2-1-14) which can be calculated as the limit in the mean

$$(2-1-19) \quad \vec{f} = \lim_{N \rightarrow \infty} \sum_{i=1}^N \lambda_i (\vec{\Phi}, \vec{\Phi}_i) \vec{\Phi}_i$$

where

$$(2-1-20) \quad \sum_{i=1}^{\infty} \lambda_i^2 (\vec{\Phi}, \vec{\Phi}_i)^2 < \infty$$

so that the solution $\vec{f}$ is in L_2 ; or

(II) The infinite series diverges, i.e.,

$$(2-1-21) \quad \sum_{\lambda=1}^{\infty} \lambda^2 (\vec{\phi}, \vec{\phi}_{\lambda})^2 = \infty$$

and (2-1-14) has no solution of class L_2 .

To appreciate how stringent a condition (2-1-20) imposes on $\vec{\phi}$ we now show that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. This follows from the fact that

$$(2-1-22) \quad K(x, y) = \sum_{\lambda=1}^{\infty} \frac{\phi_{\lambda}(x) \phi_{\lambda}(y)}{\lambda^2}$$

where by Mercer's Theorem (6 ; Courant Vol. I; page 138) the convergence is uniform as a consequence of the continuous, symmetric, positive definite nature of $K(x, y)$. From (2-1-22) we find

$$(2-1-23) \quad \iint_{-1}^1 [K(x, y)]^2 dx dy = \sum_{\lambda=1}^{\infty} \frac{1}{\lambda^2} < 4 \max_{-1 \leq x, y \leq 1} K^2(x, y) < \infty$$

so that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ in order that the series converge. Since the simultaneous satisfaction of (2-1-14) and (2-1-15) implies

$$(2-1-24) \quad K K^{-1} = K^{-1} K = I,$$

the identity, it is clear that the eigenvalues and eigenfunctions of K^{-1} follow from the calculation

$$(2-1-25) \quad \frac{1}{\lambda_n} K^{-1} \vec{\phi}_n = K^{-1} K \vec{\phi}_n = I \vec{\phi}_n = \vec{\phi}_n$$

so that

$$(2-1-26) \quad K^{-1} \vec{\phi}_n = \lambda_n \vec{\phi}_n, \quad n = 1, 2, \dots,$$

i.e., the inverse kernel has the same eigenfunctions but inverse eigenvalues. The impossibility of the simultaneous satisfaction of

$$(2-1-27) \quad \sum_{\lambda=1}^{\infty} \frac{1}{\lambda^2} < \infty, \quad \sum_{\lambda=1}^{\infty} \lambda^2 < \infty$$

which would follow if we tried to express $K^{-1}(x, y)$ in analogy to (2-1-22) shows us the necessity of adopting a fresh approach in representing K^{-1} .

What we have hinted at in the above observations, in particular in noting that

$\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$ and that

$$(2-1-18) \quad (\vec{f}, \vec{\phi}_\lambda) = \lambda_\lambda (\vec{\phi}, \vec{\phi}_\lambda), \quad \lambda = 1, 2, \dots;$$

$$(2-1-26) \quad K^{-1} \vec{\phi}_\lambda = \lambda_\lambda \vec{\phi}_\lambda, \quad \lambda = 1, 2, \dots,$$

is that both the solution $\vec{f}$ and the operator K^{-1} may be unbounded in general and require expression through the concept of distributions or "ideal functions". The name "distributions" indicates that ideal functions, such as Dirac's delta-function and its derivatives, may be interpreted by mass distributions, dipole distributions, etc., concentrated in points, or along lines, or on surfaces, etc. An excellent introduction to ideal functions or distributions is available in Courant (7; Courant Vol. II; pages 766-798) but in the following pages we choose to give meaning to the phrase, "Either the kernel is an ordinary function or it must be expressed as a distribution, that is, obtained from continuous functions by differentiation processes" (7; Courant Vol. II; page 727). On the same page in a footnote Courant continues "Such a representation for a single equation of second order is the subject of Hadamard's famous theory."

What we attempt to do in the following is to give an explicit example of how Hadamard, seeking an integral representation of the Cauchy problem for the wave equation (to be defined) had to generalize the meaning of integral to achieve the representation. In the words of Courant, "Hadamard's invention of the finite part (of a divergent integral) may be regarded as an important motivation for the modern theory of distribution" (7; Courant Vol. II, page 743). In view of our interest in integral representations in Section I the possible importance of the above concepts to our theory should be evident to the reader.

The following formula are in the notation employed by Bureau in a paper published in 1955 (5; Bureau, page 154). We give only enough of the problem to indicate the form of the difficulty faced by Hadamard and the final form of solution achieved. Detailed study of Bureau to appreciate the constructions, by no means trivial, is recommended to the reader. That the notion of a finite part of a divergent integral seems at first sight somewhat strange was commented on in "An Essay on the Psychology of Invention in the Mathematical Field" where J. Hadamard himself wrote: "Certainly, considering it in itself, it looks typically like 'thinking aside'. But in fact, for a long while my mind refused to conceive that idea until positively compelled to. I was

led to it step by step --- I could not avoid it any more than the prisoner in Poe's tale The Pit and the Pendulum could avoid the hole at the center of his cell". (12 ; Hadamard; page 110).

We consider real quantities $x_1, \dots, x_p, t, u$ and write as abbreviated notations (5 ; Bureau; page 154)

$$(2-1-28) \begin{cases} x = (x_1, \dots, x_p), (p \geq 1); y = (y_1, \dots, y_p); \\ u(x; t) = u(x_1, \dots, x_p; t); \quad \rho^2 = |x - y|^2, (\rho > 0) \\ \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_p^2} \end{cases}$$

We consider the Cauchy problem for the wave equation, i.e., consider

$$(2-1-29) \begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u \\ u(x; 0) = 0, \quad u_t(x; 0) = g(x) \end{cases}$$

where $g(x)$ is a regular function and we put $u(x; 0) = 0$ to obtain simple formulas. The case for $u(x; 0) \neq 0$ can easily be solved once the above problem is resolved (5 ; Bureau; page 155). We shall say that a function $g(x)$ is regular if it is continuous together with its derivatives up to a certain order s ; the order will vary according to the nature of the problem. This class of functions occurs over and over again in the work of Hadamard and we will meet it later in discussing his concept of "properly posed" problems. The solution of (2-1-29) is given by (5 ; Bureau; page 157)

$$(2-1-30) \quad u(x; t) = \frac{1}{(p-2)!} \frac{\partial^{p-2}}{\partial t^{p-2}} \int_0^t (t^2 - \alpha^2)^{\frac{p-3}{2}} \alpha \overline{g(\alpha)} d\alpha$$

where

$$(2-1-31) \quad \overline{g(\rho)} = \frac{1}{\Omega \rho} \int_{\Omega_{x, \rho}} g(x) d\Omega_{x, \rho}$$

is the mean value of the regular function $g(x)$ on the hypersphere $\Omega_{x, \rho}$ of center x and radius ρ in the p -dimensional Euclidean space. The temptation is great to rid (2-1-30) of the derivatives by differentiating the integral but since

$$\frac{p-3}{2} - (p-2) = \frac{1-p}{2} \leq 0,$$

the resulting terms in $(t^2 - r^2)$ would be raised to negative powers and consequently diverge for $\alpha = t$, i.e., at the end of the interval. Yet this is the key to the problem. In order to obtain an "integral" from (2-1-30) one must generalize the meaning of integral to cover the case where infinities occur. Hadamard conceived the idea of discarding the "infinite part" of the integral and saving the "finite part" (p.f.) and moreover was able to find a consistent setting for his concepts. Little wonder that he felt drawn to the "pit"! He thus achieved one of the very basic advances in mathematics, the extension of a given set of mathematical objects $\mathcal{S}$ by additional new "ideal elements" not defined as entities in the original set $\mathcal{S}$ (here not defined because of divergence), and not defined descriptively but defined merely by relationships such that in the extended set $\overline{\mathcal{S}}$ the original rules for basic operations are preserved.

The final form of Hadamard's results is sensitive to the even or odd nature of p , which is tied up to the study of propagation phenomena and Huyghens' Principle valid only for p odd and greater than one (7; Courant Vol. II, page 764), so that separate new concepts, finite part (p.f.) and logarithmic part (p.l.) of divergent integrals enter the following:

For p even, (2-1-30) can be replaced by

$$(2-1-32) \quad u(x; t) = A_p \text{ p.f. } \int_{|x-y| \leq t} g(y) (t^2 - r^2)^{\frac{1-p}{2}} dy$$

where

$$(2-1-33) \quad A_p = (-1)^{\frac{p-2}{2}} 2^{-1} \pi^{-\frac{p+1}{2}} \Gamma\left(\frac{p-1}{2}\right)$$

For p odd, (2-1-30) can be replaced by

$$(2-1-34) \quad u(x; t) = B_p \text{ p.l. } \int_{|x-y| \leq t} g(y) (t^2 - r^2)^{\frac{1-p}{2}} dy$$

where

$$(2-1-35) \quad B_p = (-1)^{\frac{p-1}{2}} 2^{-1} \pi^{\frac{1-p}{2}} \Gamma\left(\frac{p-1}{2}\right)$$

Since $(t^2 - r^2)^{\frac{1-p}{2}}$ is a solution of $u_{tt} - \Delta u = 0$ the reader should note the resemblance of (2-1-32) and (2-1-34) to the usual convolution formulas used in communication theory. For precise definitions of finite part (p.f.) and logarithmic part (p.l.) of the divergent integrals see Bureau (5; Bureau; page 146). The number of derivatives s of $g(x)$

that enter (2-1-32) and (2-1-34) are then clearly seen and it follows that the solution $u(x,t)$ depends in a continuous fashion on $g(x)$ and derivatives of $g(x)$ up to order s .

The nature of the above solution contrasts sharply with the behavior of the solution of the Cauchy problem for Laplace's equation, i.e., consider

$$(2-1-36) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = g_1(x); u_t(x,0) = g_2(x) \end{cases}$$

In his Yale lectures (10; Hadamard; page 33) Hadamard made the following observation concerning the solution of (2-1-36). It is possible to modify the solution of (2-1-36) arbitrarily much and arbitrarily near the line $t=0$ by changing the values of $g_1(x)$ and $g_2(x)$ and of all their derivatives at $t=0$ as little as we please. To do this one adds to the given solution

$$(2-1-37) \quad v(x,t) = e^{-\sqrt{m}t} \sin mx \sinh mt$$

where m is a positive constant. The function v , which is a solution of the differential equation in (2-1-36), and an arbitrary number of its partial derivatives with respect to t and x are arbitrarily small at $t=0$, if m is chosen sufficiently large, but for $t > \epsilon > 0$ the function $v(x,t)$ assumes arbitrarily large values dependent on the choice of m .

Hadamard then draws a distinction between "properly posed" problems, of which (2-1-29) is an example, and "improperly posed" problems, of which (2-1-36) is an example. More precisely we say that a Cauchy initial value problem is properly posed if there exists a number s such that for initial functions $g_s(x)$ that are regular of order s , i.e., continuous together with their derivatives up to order s , then there exists exactly one solution $u(x,t)$ and $u(x,t)$ depends continuously on the $g_s(x)$ and on their derivatives of order $\leq s$.

In 1938, I. G. Petrovskii (22; Petrovskii) motivated by Hadamard's ideas analyzed by Fourier methods the following system of linear partial differential equations

$$(2-1-38) \quad \frac{\partial u(x,t)}{\partial t} = T\left(\frac{1}{\partial x}, t\right) u(x,t)$$

where $u(x,t) = \{u_1(x,t), \dots, u_n(x,t)\}$ is the desired vector

function with m components, depending on the time t and also on the spatial point $x = (x_1, \dots, x_N)$. P is a matrix with m rows and m columns, whose elements are polynomials in the differential operators with respect to the variables x_j , $j = 1, \dots, N$ of various orders and with coefficients which depend continuously upon the time. The initial conditions are given by

$$(2-1-39) \quad u(x, 0) = u_0(x)$$

where $u_0(x)$ is a given vector function. Applying a Fourier transform, denoted by $v(\alpha, t) = \widehat{u(x, t)}$ to (2-1-38) we obtain the system of ordinary differential equations,

$$(2-1-40) \quad \frac{dv(\alpha, t)}{dt} = P(\alpha, t) v(\alpha, t),$$

where the matrix $P(\alpha, t)$ is obtained by the correspondence

$$(2-1-41) \quad \frac{1}{2\pi i} \frac{\partial}{\partial x_j} \longleftrightarrow \alpha_j, \quad j = 1, \dots, N$$

Petrovskii obtains a necessary and sufficient condition that (2-1-38), (2-1-39) be properly posed. (Actually Petrovskii's setting is slightly more general, the precise concept he investigated being that of "uniformly properly posed" problems where the initial instant is not necessarily $t=0$ but $0 \leq t \leq T$, but we shall not introduce the additional notation here (22; Petrovskii; page 3.) The distinguished condition, called Condition A in the literature, expresses a restriction on a fundamental set of solutions of (2-1-40) relative to their rate of growth with respect to α_M , where

$$(2-1-42) \quad \alpha_M = \max_i |\alpha_i|, \quad i = 1, \dots, N,$$

as $\alpha_M \rightarrow \infty$. In essence Condition A states that no solution of (2-1-40) grows faster than a fixed power of α_M as $\alpha_M \rightarrow \infty$. The point of introducing fundamental solutions is that the above condition need be checked only for m independent solutions of (2-1-40). The number s in the definition of properly posed can be related to the maximum power of α_M in the verification of Condition A, when the condition is fulfilled. This fact is our connecting link with the work of Shinbrot on integral equations of the first kind and ultimately with Youla's equations!

In 1960 Shinbrot (25; Shinbrot) studied the integral equation (2-1-11) with the additional restriction

$$(2-1-43) \quad k(x, y) = k(|x-y|),$$

i.e., that $k(x, y)$ is a difference kernel. The equation (2-1-11) then becomes

$$(2-1-44) \quad \phi(x) = \int_{-1}^1 k(|x-y|) f(y) dy, \quad -1 < x < 1$$

Using the same notation as with Petrovskii but noting that now x and α are one dimensional we denote the Fourier transform of $k(|x|)$ by $\bar{K}(\alpha)$. In order to state his results Shinbrot makes three assumptions (25; Shinbrot, page 4)

(i) The derivative

$$\frac{\partial}{\partial \alpha} \log \bar{K}(\alpha) = \frac{\bar{K}'(\alpha)}{\bar{K}(\alpha)}$$

is continuous for real α and belongs to h_2 .

(ii) The derivative

$$\frac{\partial}{\partial \alpha} \arg \bar{K}(\alpha) = O\left(\frac{1}{\alpha}\right)$$

as $|\alpha| \rightarrow \infty$ along the real axis.

(iii) The quantities

$$\lambda = \lim_{\alpha \rightarrow \infty} \sup \frac{\log \left| \frac{1}{\bar{K}(\alpha)} \right|}{\log \alpha}$$

$$\mu = \lim_{\alpha \rightarrow \infty} \inf \frac{\log \left| \frac{1}{\bar{K}(\alpha)} \right|}{\log \alpha}$$

are both finite.

Note that for $k(|x-y|)$ a covariance function it follows that $\bar{K}(\alpha)$ is real so (ii) is trivially satisfied. The assumption (i) is familiar from consideration of network synthesis ideas and says essentially that an acceptable $\bar{K}(\alpha)$ does not change too rapidly. Recalling Condition A from the

work of Petrovskii, "No solution of (2-1-40) grows faster than a fixed power, say $\mathcal{P}$, of α_M as $\alpha \rightarrow \infty$ " we see that assigning $\overline{K}(\alpha)$ the rate of growth from Condition A yields in (iii)

$$(2-1-45) \quad \lambda = \mu = \lim_{\alpha \rightarrow \infty} \frac{\log |\frac{1}{\alpha^{\mathcal{P}}}|}{\log \alpha} = -\mathcal{P}$$

We are not of course asserting a direct connection but merely that the rate of growth of Petrovskii's solutions yield finite, acceptable limits in (iii).

At this point we quote pertinent sentences from Shinbrot concerning this purposes and his main result which we summarize as a Theorem (28; Shinbrot; page 5). "In most applications, one is interested in solutions that belong to some $\mathcal{L}$ -class. In the case of the problem of a filter with a finite memory, however, the solution itself represents an operator that is to be applied to the signal plus the noise to obtain the best linear approximation to the signal itself. Consequently the solutions of interest do not necessarily have to belong to any $\mathcal{L}$ -class; indeed they need not be functions at all but may be distributions in the sense of Laurent Schwarz (e.g., $\mathcal{S}$ -functions and their derivatives)".

Theorem: (Shinbrot) If $[\lambda/2]$ is used to denote the integral part of $\lambda/2$, the solution of (2-1-44) consists of an "ordinary" function belonging to some $\mathcal{L}$ -class plus a finite sum of derivatives of the $\mathcal{S}$ -function, the highest derivative that can occur being $[\lambda/2]-1$.

Remark: Of the many possible approaches to the $\mathcal{S}$ -function and its derivatives currently available Shinbrot depends on the treatment of Lighthill which is especially suited to Fourier methods (7; Lighthill; p. 767).

Example: For $K(|x|) = e^{-|x|}$, $\lambda = 2$ so a $\mathcal{S}(x)$ can occur in the solution of (2-1-44).

2.2 APPLICATION TO YOULA'S EQUATIONS

For finite dimensional kernels the theory is complete as indicated in the first theorem in the last section. The general situation when one attempts to produce kernels $\mathcal{R}_\pm(\xi, \tau)$ and $\mathcal{R}_\pm^*(\xi, \tau)$ to show the equivalence of Youla's equations (1-1-5) and the equations (1-2-2), which are solved for $\mathcal{G}(\xi)$ in order to eliminate $\mathcal{G}(\xi)$ and produce the functional

$I[a(v)]$, is that distributional solutions are appropriate. This is especially clear from (2-1-18) and (2-1-26) which indicate in general notation that both the function ϕ and the operator K^{-1} associated with (2-1-15) may be unbounded. That integral operators are bounded and differential operators are unbounded is well known. The question then arises, must differential operators be admitted to the theory and, if so, to what extent?

At this point we attempted to illustrate two concepts of J. Hadamard that are of great potential for further progress in analyzing Youla's equation. The first was the concept of finite or logarithmic part of a divergent integral, the divergency occurring when one attempted to eliminate the derivatives appearing in (2-1-30), the solution of (2-1-29). The use of divergent integrals in the systematic fashion initiated by Hadamard and carried on today by Bureau is an approach almost unknown in communication theory. One of the main virtues of the approach is the directness with which it answers the question raised above concerning the number of derivatives that must appear as a result of the generalized meaning of the integral. See (2-1-32) to (2-1-35) and the discussion that followed.

The second and related concept was the properly posed nature of a problem which implied through subsequent work of Patrovskii an approach through Fourier integrals that again emphasizes how many derivatives must enter the solution of a Cauchy problem. The similarity between this work and that of Shinbrot which is applicable for the specialization to difference kernels, hence stationary statistics, seems especially strong in light of equation (2-1-45). Finally the generalization of Petrovskii's work by Gel'fand and Silov (9; Gel'fand) may provide tools to generalize Shinbrot and ultimately Youla's work.

CHAPTER III

COMMUNICATION SYSTEMS BASED ON HADAMARD MATRICES

3.1 INTRODUCTION

In 1909 Hadamard's Theorem on the maximum value of a determinant with bounded elements was utilized to prove the convergence of the celebrated Fredholm Formulae (34; Tricomi; page 66) which explicitly represent the resolvent kernel of an integral equation of the second kind with a bounded kernel. Having spent the first two chapters reasoning with integral equations of the first kind it is pertinent to remark that when the kernel in an equation of the first kind is a composite of an ordinary function and a distribution or ideal function then the equation of the first kind includes within its scope equations of the second kind and integro-differential equations. The central idea of Fredholm's method was a heuristic construction for a bounded kernel

$K(x, y)$ that would mimic the finite dimensional construction applicable for a degenerate or Pincherle-Goursat kernel $K_n(x, y)$. Rather than attempt to justify all aspects of the passage to the limit as $n \rightarrow \infty$, which would correspond to the approach of the last chapter, Fredholm simply wrote down the infinite series forms made plausible by finite dimensional arguments and directly verified their convergence by means of a majorant series, the convergence of the majorant series following from Hadamard's Theorem. We now give an explicit statement of the theorem (34; Tricomi; page 223).

Theorem (Hadamard) If D is a determinant of the n -th order of the matrix $\{a_{rs}\}$, ($r, s = 1, \dots, n$), then

$$(3-1-1) \quad D^2 \leq \sum_{s=1}^n a_{1s}^2 \sum_{s=1}^n a_{2s}^2 \dots \sum_{s=1}^n a_{ns}^2$$

$$= \prod_{r=1}^n \|\vec{a}_r\|^2$$

where

$$(3-1-2) \quad \begin{cases} \vec{a}_r = (a_{r1}, a_{r2}, \dots, a_{rn}) \\ \|\vec{a}_r\|^2 = \sum_{s=1}^n a_{rs}^2 \end{cases}$$

Moreover, equality in (3-1-1) is attained only when the row vectors $\vec{a}_r$ are

mutually orthogonal, i.e.,

$$(3-1-3) \quad (\vec{a}_n, \vec{a}_p) \equiv \sum_{s=1}^m a_{ns} a_{ps} = 0, \quad n \neq p$$

Geometrically (3-1-3) implies that the volume of the polyhedron formed from m vectors of given length in m dimensional space is greatest if the factors are mutually orthogonal. Provided that

$$(3-1-4) \quad |a_{ps}| \leq B$$

it follows from (3-1-1) that

$$(3-1-5) \quad D^2 \leq m^m B^m$$

Modern communication theory assumes a normalization of B , namely $B=1$, so that the following definition emerges⁽²⁾; Peterson, page 79). A Hadamard matrix H is an orthogonal $n \times n$ matrix whose elements are the real numbers $+1$ and -1 . The orthogonality is in the sense of (3-1-3).

Our purpose in this section is to present some communication system techniques that are derivable from a mathematical concept that reduces to a Hadamard matrix in certain special cases and that constitutes a generalized Hadamard matrix in the remaining cases. The sense of the generalization is relative to the concept of orthogonality. Clearly a Hadamard matrix H satisfies the equation

$$(3-1-6) \quad H H^T = n I$$

where T means transpose and I is the identity matrix.

Any matrix satisfies the equation

$$(3-1-7) \quad A A^T = k I + R$$

where k is a scalar and R is a "remainder" matrix. We associate the rank r of R with the degree of orthogonality of A . If $r=0$, i.e., R vanishes, then A is an orthogonal matrix. If $r=1$, we say that A is "almost orthogonal". The cases $r=2, 3, \dots, n$ correspond to lesser degrees of orthogonality and will not be distinguished by name. The "almost orthogonal" matrices consti-

tute the basis of a multiplexing technique which has certain attractive features for use as an anti-jamming communication system. The basic philosophy of our approach is that orthogonal matrices are of too simple a structure to analyze and hence jam, but that almost orthogonal structures are sufficiently simple to construct and sufficiently variable to make jamming difficult. The multiplexing approach will occupy our attention in section 3.2.

Having outlined our purpose we now show the naturalness of our approach by putting the Hadamard matrix H in perspective with modern coding theory.

It is important in the theory of symmetric binary codes (29; Slepian) to determine the maximum number $A(n, d)$ of different n -place binary sequences that can be constructed such that the Hamming distance (14; Hamming) between any two of them is greater than or equal to a preassigned positive integer d . A set of n n -place binary sequences such that the distance between any two is greater than or equal to d is called an n -place code of length n and minimum distance d . Denote such a code by $M(n, d; n)$ (23; Plotkin).

Since n mutually orthonormal vectors in an n -dimensional space have, in the usual Euclidean metric, the maximum possible separation one suspects that, mapping the real numbers $+1$ and -1 into the binary notation of coding theory, a Hadamard matrix can be associated with a code $M(n, d; n)$. That such is indeed the case is the content of the following theorem (21; Peterson; page 79).

Theorem: If there exists an $n \times n$ Hadamard matrix, there exists a binary code $M(n, d; n)$ where $n = 2^k$, $d = n/2$.

Proof: Let H be a Hadamard matrix. The code is constructed as follows: Form a set of $2n$ vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n; -\vec{a}_1, -\vec{a}_2, \dots, -\vec{a}_n$ where the $\vec{a}_1, \dots, \vec{a}_n$ are the rows of H . Then in each of these change the 1 's to 0 's and the -1 's to 1 's. This gives a set of $n = 2n$ vectors of n binary symbols each. Since corresponding components of $\vec{a}_i$ and $-\vec{a}_i$ are different, the Hamming distance between $\vec{a}_i$ and $-\vec{a}_i$ is n . Since $\pm \vec{a}_i$ and $\pm \vec{a}_j$ are orthogonal, equation (3-1-3), if $i \neq j$, they must match in half the positions and differ in the other half, and thus the corresponding binary vectors are at distance $n/2$. Q. E. D.

For example, it is easy to construct a Hadamard matrix H_n for $n = 2^k$ where k is a positive integer. The resulting code is the same as a first order Reed-Muller code (24; Reed) or a MacDonald code (19; MacDonald).

That codes generated by Hadamard matrices are maximal in the sense that no other code with the same minimum distance and the same number of places has greater length was demonstrated by Bose and Shrikhande: "This note establishes a connection between Hadamard matrices H_{4t} and the maximal binary codes $M(4t, 2t; 8t)$, $M(4t-1, 2t; 4t)$ and $M(4t-2, 2t; 2t)$ in two symbols 0 and 1, where by $M(n, d; m)$, we mean a set of m n -place sequences with 0 and 1 such that the Hamming distance between any two sequences is greater than or equal to d " (4; Bose; page 183).

In order to state more explicitly the "connection" in the above quotation, we now introduce the concept of a symmetric balanced incomplete block design, the so-called ν, k, λ problem, a concept sufficiently general to include the Hadamard matrices as a special case.

Foregoing the symmetric case for the moment, we define a balanced incomplete block design as an arrangement of ν objects into b sets called blocks such that every object occurs exactly r times and every pair of objects occurs exactly λ times in a block. The subject of the design of experiments, in which the above concept has played a central role, was built up largely by two men, R. A. Fisher and F. Yates. The subject of the enumeration of designs was dealt with by Bose in 1939 whose principal tools were the use of finite geometries and symmetrically repeated differences (3; Bose). Viewing the design of experiments as an attempt to extract information against a background of interference we see the naturalness of Bose producing papers in 1939 and 1959 using virtually the same tools on what at first sight seem distinct disciplines, namely design of experiments and code construction.

For a symmetric design, $b = \nu$ and $k = r$, and the arrangement is called a ν, k, λ configuration. Irrespective of its connection with statistical considerations, as outlined above, the ν, k, λ problem has played a distinguished role in combinatorial mathematics (25; Ryser; page 456).

The ν, k, λ Problem. Let ν elements $x_1, \dots, x_\nu$ be arranged into ν sets $X_1, \dots, X_\nu$ such that every set contains exactly k distinct elements and such that every pair of sets has exactly λ elements in common ($0 < \lambda < k < \nu$). Such a configuration is called a ν, k, λ configuration. Let $a_{ij} = 1$ if x_j is in set X_i and $a_{ij} = 0$ otherwise. The matrix $A = [a_{ij}]$ of order ν is called the incidence matrix of the ν, k, λ configuration. Let ν, k, λ be integers such that $0 < \lambda < k < \nu$. Then a ν, k, λ configuration exists if and only if there exists a $(0, 1)$ matrix A of order ν such that

$$(3-1-8) \quad A A^T = B$$

where A^T denotes the transpose of A and where B has k in the main diagonal and λ in all other positions. Clearly, the k on the main diagonal of B is associated with k ones in each row of A , and λ elsewhere is associated with the overlap of λ ones in each distinct pair of rows of A . Since each element occurs equally frequently with a specified one within a set, we have that $k(k-1) = \lambda(v-1)$, so that

$$(3-1-9) \quad \lambda = \frac{k(k-1)}{v-1}.$$

The central problem in the study of these configurations is the determination of the precise range of values of v, k, λ for which configurations exist. Our need for a variety of configurations is dependent on specific requirements of the multiplexing scheme we have in mind, so we defer the general question of construction until later.

A specialization of v, k, λ to $v = 4t-1$, $k = 2t-1$, $\lambda = t-1$ leads us back to a Hadamard matrix of order $4t$ (33; Todd). More precisely, the existence of such a v, k, λ configuration is equivalent to the existence of a Hadamard matrix of order $4t$. An easy construction to exhibit the equivalence is to start with a Hadamard matrix of order $4t$ in normal form, i.e., by permutations normalize the Hadamard matrix to have only ± 1 's in the first row and column, delete the first row and column to form a matrix of order $v = 4t-1$, and change the -1 's to 0 's. Concerning 1 's there are clearly $k = 2t-1$ in each row and since the orthogonality of H_{4t} required the coincidence of 1 's in t columns the coincidences have been reduced to $\lambda = t-1$. Hence the resulting matrix meets the requirement (3-1-8) with $v = 4t-1$, $k = 2t-1$, $\lambda = t-1$ and the reasoning can clearly be reversed so the equivalence is complete.

With the above concepts in mind we return to a definitive statement of Bose and Shrikhande in the following theorem (4; Bose; page 186).

Theorem: The following statements are equivalent:

- (a) $A(4t, 2t) = 8t$ i.e., there exists a code $M_1(4t, 2t; 8t)$
- (b) $A(4t-1, 2t) = 4t$ i.e., there exists a code $M_2(4t-1, 2t; 4t)$
- (c) There exists a v, k, λ configuration with the following parameters,

$$v = 4t-1, k = 2t-1, \lambda = t-1$$

- (d) There exists a Hadamard matrix H_{4t} .

We now complete the cycle of reasoning of Bose and Shrikhande by stating a result of Plotkin (23; Plotkin).

Theorem: For any positive integer t

- (i) $A(4t, 2t) \leq 8t$
- (ii) $A(4t-1, 2t) \leq 4t$
- (iii) $A(4t-2, 2t) \leq 2t$

Combining (a), (b), (c), (d), (i), (ii) the maximal nature of the codes $M_1(4t, 2t; 8t)$ and $M_2(4t-1, 2t; 4t)$ is established.

The results of our discussion so far can now be summarized in the following figure.

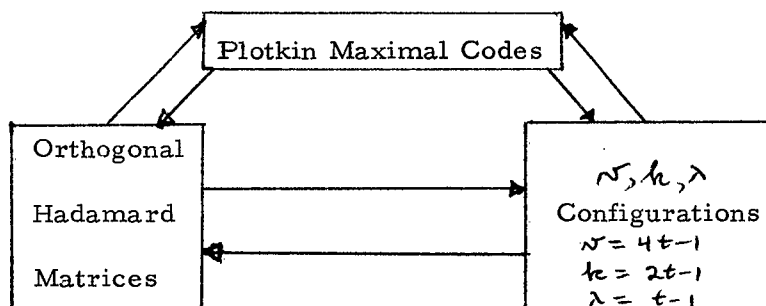


FIGURE 4

EQUIVALENCE THEOREM OF BOSE SHRIKHANDE

The most notable feature of the research summarized in Figure 4 is the use of a special case of the ν, k, λ configurations. We set ourselves the task of obtaining interesting consequences of the general ν, k, λ configurations characterized by (3-1-8) and (3-1-9).

We first rewrite (3-1-8) as

$$(3-1-10) \quad A A^T = (k - \lambda) I + \lambda J$$

where I denotes the identity matrix, J denotes a matrix of all 1's and

$$k-\lambda > 0 \quad \text{since} \quad 0 < \lambda < k < n.$$

Now observe that

$$(3-1-11) \quad A \mathcal{S} = k \mathcal{S}$$

since A has k 1's in each row, so that each column of $\mathcal{S}$ is an eigenfunction of A with eigenvalue k . Note that $\mathcal{S}$ is of rank 1 so that A is "almost" orthogonal in the sense of the discussion following (3-1-7). Putting (3-1-11) in (3-1-10) we get

$$(3-1-12) \quad A A^T = (k-\lambda) I + \frac{\lambda}{k} A \mathcal{S}$$

so that

$$(3-1-13) \quad A \left(A^T - \frac{\lambda}{k} \mathcal{S} \right) = (k-\lambda) I$$

which yields

$$(3-1-14) \quad A^{-1} = \frac{k A^T - \lambda \mathcal{S}}{k(k-\lambda)}$$

In addition, if A is symmetric, (3-1-14) becomes

$$(3-1-15) \quad A^{-1} = \frac{k A - \lambda \mathcal{S}}{k(k-\lambda)}, \quad A^T = A$$

If in (3-1-15) it is possible to choose $k=2\lambda$, then using (3-1-9) we obtain

$$(3-1-16) \quad A^{-1} \Big|_{\substack{k=2\lambda \\ n=4\lambda-1}} = \frac{2A - \mathcal{S}}{2\lambda}, \quad A^T = A$$

Since A consists of 1's and 0's, $2A$ consists of 2's and 0's and $2A - \mathcal{S}$ (remembering that $\mathcal{S}$ has 1's in every position) differs from A only in that the 0's in A are replaced by -1's in $2A - \mathcal{S}$. For example, a n, k, λ incidence matrix A_1 for $n=7, k=4, \lambda=2$ is

$$(3-1-17) \quad A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad A_1^T = A_1$$

whose inverse is

$$(3-1-18) A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

It is a curious fact that if we border (3-1-18) with a column of -1's and a row of -1's that we obtain a Hadamard matrix H_8 where the usual roles of -1 and +1 have been interchanged in that -1's appear exclusively in the first row and column.

The main point we want to emphasize about the choice in (3-1-16) that simplifies A^{-1} so remarkably is that Ryser (25; Ryser, page 457) has shown, and we quote, "Another important specialization of values of n, k, λ is $n = 4t-1, k = 2t, \lambda = t$. Such a n, k, λ configuration is equivalent to a Hadamard matrix of order $4t$ ". We summarize Ryser's result in Figure 5.

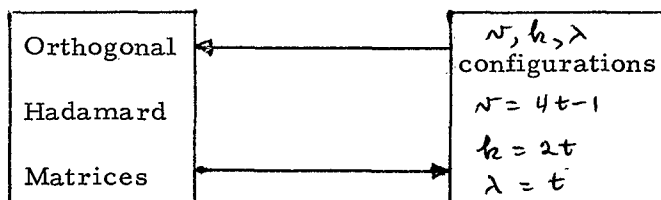


FIGURE 5
EQUIVALENCE THEOREM OF RYSER

Comparing Figure 4 and Figure 5 one might suspect the existence of Plotkin Maximal Codes distinct from those obtainable by the Bose, Shrikhande scheme. That such is not the case, that indeed one could hook the configuration in Figure 4 and Figure 5 together by equivalence arrows is shown by the following reasoning. Assume the existence of a n, k, λ configuration with an incidence matrix A where $n = 4t-1, k = 2t, \lambda = t$. Then the matrix $I - A$ is a matrix of zeros and ones which we assert is an inci-

dence matrix of a n', k', λ' configuration where $n' = 4t-1$, $k' = 2t-1$, $\lambda' = t-1$. To see this we calculate, using (3-1-10), that

$$\begin{aligned}
 (3-1-19) \quad (S-A)(S-A)^T &= [S^2 - AS - SA^T] + AA^T \\
 &= [nS - kS - kS] + (k-\lambda)I + \lambda S \\
 &= (k-\lambda)I + \{n - 2k - \lambda\}S \\
 &= (k'-\lambda')I + \lambda'S
 \end{aligned}$$

so that

$$(3-1-20) \quad \begin{cases} \lambda' = n - 2k - \lambda = t-1 \\ k' = 2t-1 \end{cases} \text{ since } k' - \lambda' = k - \lambda = t$$

which satisfies the equivalence, since, of course, $n' = 4t-1$. In essence the configurations of Bose, Shrikhande and Ryser are related merely by interchanging zeros and ones.

3.2 A MULTIPLEXING TECHNIQUE

The simplicity of the equation (3-1-15), exhibiting the inverse matrix A^{-1} , suggests to us a multiplexing technique that has sufficient arbitrariness in its structure to consider its use for an anti-jamming system. Fixing n , which specifies the number of channels to be multiplexed in a given application, one looks for appropriate solutions of (3-1-9), i.e., permissible values of λ, k . Assume there are m such pairs, call them $(\lambda_1, k_1), \dots, (\lambda_m, k_m)$. THEN WE ASSERT THAT THERE EXIST m MODES OF OPERATION OF A MULTIPLEXING SYSTEM SUCH THAT AN ENEMY'S PICKING UP A TRANSMITTED SIGNAL IN ONE MODE, AND THEN JAMMING IT, CAN BE AVOIDED BY SWITCHING TO ANOTHER MODE, THE SWITCHING PROCEDURE BEING SIMPLY IMPLEMENTED. MOREOVER, FOR A DISTINGUISHED SET OF n 'S THERE EXISTS A SIMPLEST MODE WHICH COULD BE USED UNTIL JAMMING COMMENCES, AT WHICH TIME SWITCHING TO A MORE COMPLICATED MODE OCCURS.

The simplest mode corresponds to the simplest form that A^{-1} can take, namely, that specified by (3-1-16). For the physical realization we have in mind see Figure 6.

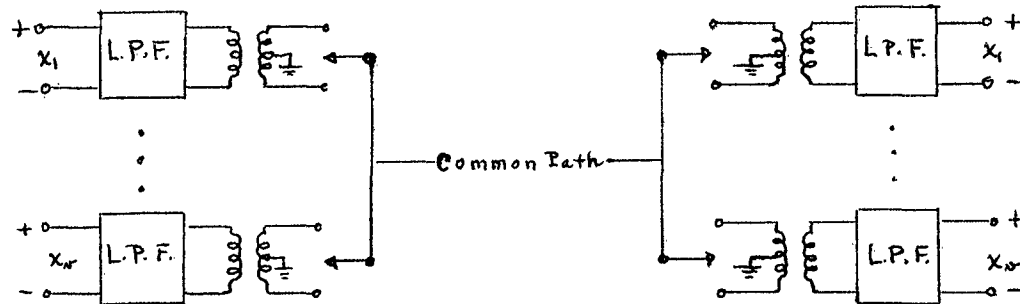


FIGURE 6

ANTI-JAMMING (MULTIPLEXING) REALIZATION FOR $k = 2\lambda$.

The indicated switches act synchronously and serve as gating functions. The low pass filters (L.P.F.) eliminate the side bands associated with gating. In addition they are assumed to be sufficiently narrow band that the filtered version of any of the signals to be transmitted, say x_i , will not change during λ sequential settings of the gating functions. This, of course, relates the necessary switching rate to the allowed band width of the signals $x_1, \dots, x_n$ to be transmitted.

The proposed operation of the switches is as follows. Instead of transmitting $x_1, \dots, x_n$ sequentially on the common path, we transmit the "composite" sequence $y_1, \dots, y_n$ where

$$(3-2-1) \quad \vec{y} = A \vec{x}$$

and A is an incidence matrix of a λ, k, λ configuration. Both $\vec{x}$ and $\vec{y}$ are column vectors of length λ . The input gating functions, which correspond to the rows of the A matrix, are either 1 or 0. Note that the arrangement of grounds in Figure 6 allows the gating functions to be 1, -1, or 0. Thinking of the A_i matrix exhibited in (3-1-17) for the case

$N=7, k=4, \lambda=2$, we see for example, that

$$(3-2-2) \quad y_1 = x_1 + x_3 + x_5 + x_7$$

Hence an enemy, who requires that he understand how we are transmitting in order to efficiently jam our transmission, must obtain A^{-1} and apply it to $\vec{y}$. Even if he does this we rely on our ability to find other k, λ corresponding to a fixed N to switch "modes" of transmission.

Regarding the 2λ in (3-1-16) as an inessential gain factor, we set the output gating functions, which corresponds to the rows of the $2\lambda A^{-1}$ matrix, to either +1 or -1. Again thinking of the A_1 matrix exhibited in (3-1-17) we see, for example, that from (3-1-18) it follows that the output of the first channel on the right hand side of Figure 6 is given by

$$(3-2-3) \quad 4x_1 = y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7$$

For every 1, 0 setting of the input gating functions there is synchronously a 1, -1 setting of the output gating functions for the case $k=2\lambda$, our simplest mode of operation.

For $k \neq 2\lambda$ we vary the scheme in Figure 6 only in that we now use two switches in the output gating functions. To see that this suffices examine (3-1-15) rewritten as

$$(3-2-4) \quad (k-\lambda) A^{-1} = A - \frac{\lambda}{k} S, \quad A^T = A$$

In (3-2-4) we see that $(k-\lambda) A^{-1}$ can be realized as a composite of A , i.e., output gating functions varying precisely in step with the input gating functions, and a fixed set of switches, i.e., invariant in time, attached to the negative terminal of each output channel with an attenuator of magnitude λ/k in each. See Figure 7.

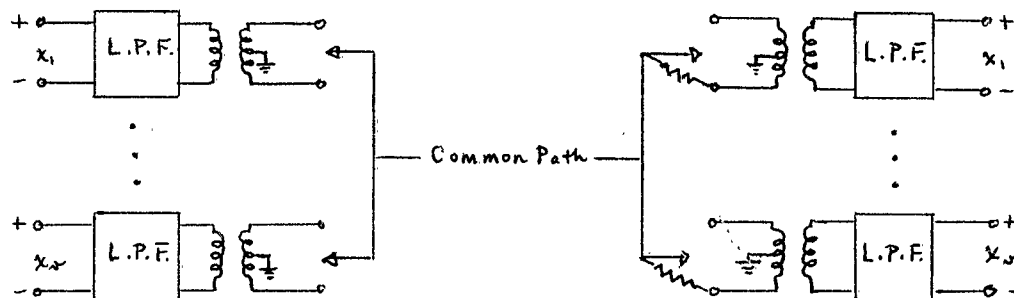


FIGURE 7

ANTI-JAMMING (MULTIPLEXING) SCHEME FOR $k \neq 2\lambda$.
(Note Additional Switches Characterized by γ_k Are Required)

We could, of course, not make any distinction and always use Figure 7, or equivalently (3-2-4) if this proves more convenient in engineering practice.

To illustrate the above scheme we write down a r, k, λ incidence matrix A_2 for $r=7, k=3, \lambda=1$, i.e.,

$$(3-2-5) \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

so that corresponding to (3-2-1) we now have

$$(3-2-6) \quad y_1 = x_2 + x_4 + x_6$$

and corresponding to (3-2-4) we now have

$$(3-2-7) \quad 2x_1 = y_2 + y_4 + y_6 - \frac{1}{3}(y_1 + \dots + y_7)$$

at the output of the first channel on the right hand side of Figure 7 .

We now make some comments on the relation between the process depicted by Figure 7 and an orthogonal multiplexing technique based directly on an associated Hadamard matrix H_{4t} , whenever the N, t, λ parameters are such that an H_{4t} matrix exists in the sense of either Figure 4 or Figure 5 . For example corresponding to the A_2 in (3-2-5) there clearly corresponds the Hadamard matrix $H_{4t} = H_8$, specified by

$$(3-2-8) \quad H_8 = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix}$$

which has the property

$$(3-2-9) \quad H_8^2 = 8 I ; \quad H_8^{-1} = \frac{1}{8} H_8$$

so that H_8 is essentially, except for a "gain" factor, its own inverse. Hence if in Figure 7 we omit the fixed set of switches and allow the output gating functions to vary precisely in step with the input gating functions, both sets of switches taking the values ± 1 in accordance with the rows of H_8 , we have an orthogonal multiplexing scheme. This orthogonal scheme was jointly studied by M. E. Hines^(15; Hines) and the author^(26; Segers) in connection with time division multiplexing transmission, without companding, the mix of "loud" and "soft" talkers accomplished by H_8 providing improved, i.e., steadier signals for transmission. By the same token the effect of the mixing technique on interference is to spread noise power concentrated in time in the common path over a large range of time at the output terminals. The essential generalization of the scheme of Figure 7 over that of Hines and Segers (the engineering was first conceived by Hines and the study of the matrices involved was then the work of Segers) is the inclusion of the fixed set of switches supplementing the synchronously varying ones of the orthogonal scheme. The essential mathematical structure underlying the possibility of such a scheme is the "almost orthogonal" character of A embodied

in the fact that $\mathcal{S}$ is of rank one. The possibility of the utility of the structure of Figure 4 for anti-jamming is clearly dependent on the number of $\mathcal{N}, k, \lambda$ configurations, for a given $\mathcal{N}$, that we can concoct.

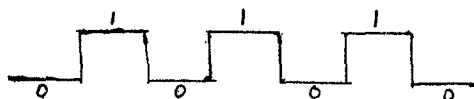
The search for $\mathcal{N}, k, \lambda$ configurations has become almost a career in itself to many mathematicians and engineers. We content ourselves with providing references to literature that has helped us rather than attempt a short, and consequently inadequate, survey. In a 1962 book edited by Todd (32; Todd; page 518) Marshall Hall, Jr. contributes a chapter on "Discrete Variable Problems" and includes a section on "Systematic Search for Block Designs". The work of Baumert, Easterling, Golomb, and Viterbi at Jet Propulsion Laboratory has been outstanding and a good summary up to 1961 is provided by (2; Baumert) while recent contributions have appeared in almost every issue of the Research Summaries of the Jet Propulsion Laboratory. The work of the group at Sylvania is extensive but just in the process of being published, except for a 1961 report, a goodly number of whose ten chapters were contributed by Turyn (35; Turyn). Recent issues of the Proceedings of the American Mathematical Society have much relevant information on new techniques and results.

There is a further relation of interest between the incidence matrix A of a $\mathcal{N}, k, \lambda$ configuration and a Hadamard matrix H_n . Assuming for convenience that $A = A^T$ then by (3-1-8) $A^2 = B$, where B has k in the main diagonal and λ in all other positions. We show below that B , which differs only by a constant factor from a "correlation matrix" whose eigenfunctions and eigenvalues were studied by Max (20; Max; page 114), has only two eigenvalues. The larger eigenvalue has associated with it an eigenmanifold of dimension one and the smaller eigenvalue has associated with it an eigenmanifold containing all other vectors perpendicular to the first eigenmanifold. Consequently, the uniqueness of the first eigenvector makes it of great interest as a candidate for timing purposes. Quite clearly our assumption of synchronously operated switches at input and output, in the face of natural noise and jamming, is a critical one. Indeed, since the ability of spread-spectrum systems to reject noise and jamming when synchronized is inherently superior to their ability when synchronization is being established, it is not an exaggeration to say that a major problem, if not the major problem, is to establish initial synchronization.

Before plunging into mathematical detail on the ideas outlined in the last section we first simplify the implementation in Figure 7 so as to put the essential question in better perspective. For instance, using the matrix A_2 in (3-2-5) as a model, the switch on channel 1 goes through the sequence of

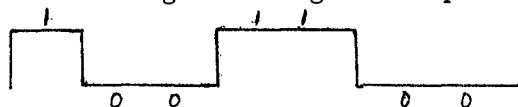
values

(3-2-10)



and the switch on channel 2 goes through the sequence

(3-2-11)



This is hard to implement; instead we assert the possibility of using combinations of resistors, say, unit resistors connecting x_2, x_4, x_6 to switch position s_1 ; unit resistors connecting x_1, x_4, x_5 to switch position s_2 ; etc.; finally, unit resistors connecting x_3, x_5, x_6 to switch position s_7 ; and the usual commutator switch to gather all the above sequentially from $s_1, \dots, s_7$.

Since we also need fixed connections to negative terminals on the output in Figure 7 and we might want to reverse the process, i.e., put fixed connections to negative terminals on the input, the most general structure of interest is gotten by replacing the "inner core" of Figure 7, i.e., the structure of Figure 8.

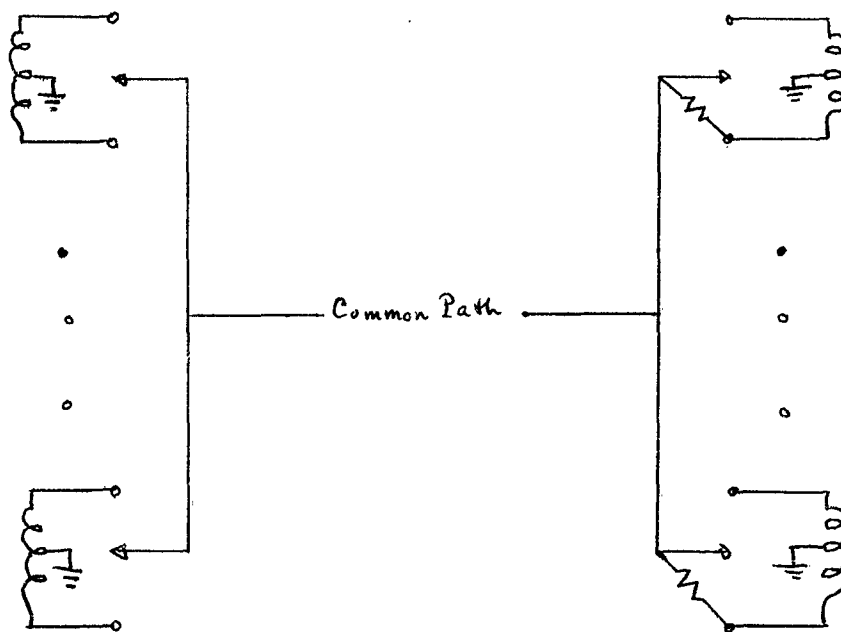


FIGURE 8
"INNER CORE" of FIGURE 7

We replace the inner core in Figure 8 by the structure Figure 9.

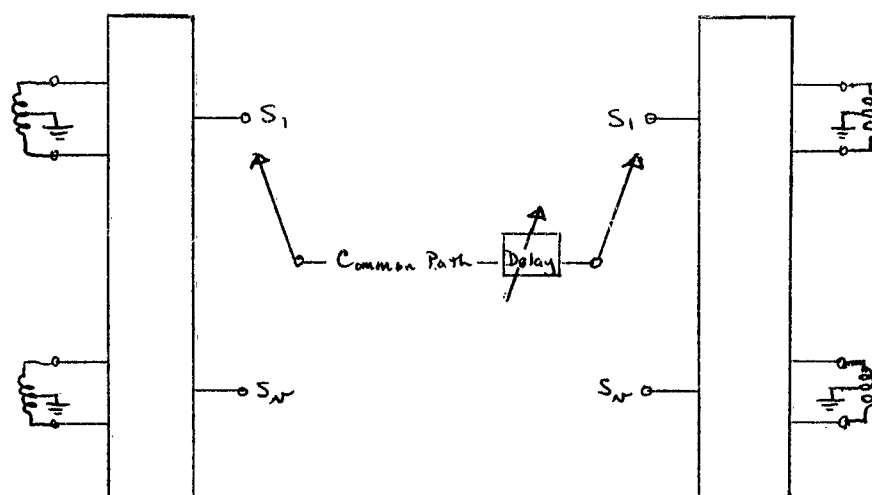


FIGURE 9

ALTERNATE CIRCUITRY WITH COMMUTATORS,
RESISTANCE MATRICES AND VARIABLE DELAY

The transmission properties of Figure 7 and Figure 9 are identical. We have merely replaced the complicated switching pattern by interconnection matrices plus simple switches and indicated a variable delay in the common path to emphasize the problem of putting the information in the correct output channel. We now return to mathematical considerations concerning the matrix B .

Theorem: The matrix B of order $N=4t$, in (3-1-8), is specified by

$$(3-2-10) \quad A^2 = B = (k - \lambda) I + \lambda J; \quad A = A^T$$

This matrix has only two distinct eigenvalues, $k - \lambda$ and $k + (n-1)\lambda$. The eigenvector associated with the larger eigenvalue is the column of H_{4t} consisting entirely of 1's. The $n-1$ remaining columns of H_{4t} are eigenvectors associated with $1 - \frac{\lambda}{k}$.

Proof: Clearly we have that

$$(3-2-11) \quad \begin{bmatrix} k & & \lambda \\ & \ddots & \\ \lambda & & k \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \{k + (n-1)\lambda\} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = k^2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

so half of the statement is established. Now consider

$$(3-2-12) \quad B \hat{x} = (k - \lambda) \hat{x}$$

i.e., try to find eigenvectors $\hat{x}$ associated with $k - \lambda$. Using (3-2-10) in (3-2-12) we obtain

$$(3-2-13) \quad (k - \lambda) \hat{x} + \lambda S \hat{x} = (k - \lambda) \hat{x}$$

and since $\lambda \neq 0$, (3-2-13) yields

$$(3-2-14) \quad S \hat{x} = 0$$

Since the rank of S is one, (3-2-14) reduced to the single equation

$$(3-2-15) \quad x_1 + \dots + x_n = 0 \quad ; \quad n = 4t$$

which is satisfied by the $n-1$ columns of $H_n = H_{4t}$ that are not all 1's. Clearly, there are no more than n eigenvectors. Q. E. D.

The ratio of eigenvalues

$$(3-2-16) \quad R = \frac{k + (n-1)\lambda}{k - \lambda} = \frac{k^2}{k - \lambda} \leq k$$

where we have used (3-1-9) to simplify the numerator, is a measure of how much "magnification" the matrix B produces on the distinguished vector of all 1's relative to the other vectors.

The essential problem that remains is to relate the eigenvalues and eigenfunctions of A , the square root of B , to those of B . Here we have an

ambiguity in taking the square root of eigenvalues. Granting that this can be resolved there remains the problem of how precisely in the mechanism of transmission to reserve the timing signal so that the information process will not initiate it, or noise initiate it, to give fake timing information. The solution of these problems requires further study.

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