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ON HOMOMORPHIC IMAGES OF TRANSITION GRAPHS

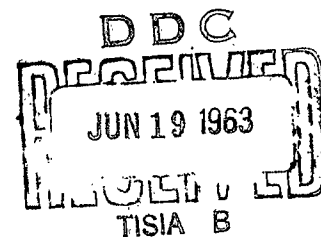
by

Michael Yoeli and Abraham Ginzburg
Technion, Israel Institute of Technology, Haifa, Israel

Technical Report No. 11

This report was prepared for the
U. S. OFFICE OF NAVAL RESEARCH, INFORMATION SYSTEMS BRANCH
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I. INTRODUCTION

The concept of transition graph has been introduced in [1] in connection with sequential machine decompositions. It has been shown there that the study of homomorphic images of transition graphs does facilitate the derivation of all admissible partitions (also called partitions having the substitution property [2, 3]) of a sequential machine.

In this report a simple method is derived for obtaining all homomorphic images of a given complete or partial transition graph.

This method consists of the successive application of elementary steps, corresponding to four types of "elementary" congruences.

Furthermore, it is shown that the number of elementary steps required to derive a given homomorphic image is constant, if the original transition graph is complete and connected.

II. BASIC CONCEPTS

A (transition) graph G is a couple (S, Γ) , where S is a finite set of vertices and Γ a map of a non-empty subset S_1 of S into S .

If $S_1 = S$, the graph is complete, otherwise partial.

The graph $G' = (S', \Gamma')$ is a homomorphic image of the graph $G = (S, \Gamma)$ if there exists a map φ of S onto S' such that:

$$\Gamma' = \psi^{-1} \Gamma \psi \quad *) .$$

We shall use the notation $G' = G\psi$.

A cycle C of length k of G is a sequence $C = (s_0, s_1, \dots, s_{k-1})$ of k different vertices of G , such that

$$s_{i-1} \Gamma = s_i, \quad (i = 1, 2, \dots, k-1) \text{ and } s_{k-1} \Gamma = s_0.$$

The vertex s of a partial G is a sink, if $s \in S - S_1$.

Every graph consists of one or more separate connected parts. Each such part contains either a single cycle or a single sink.

The equivalence relation E on S is a congruence on $G(S, \Gamma)$ if

$$s, t \in S_1 \text{ and } s \equiv t(E) \Rightarrow s\Gamma \equiv t\Gamma(E), \text{ i.e., if } \Gamma^{-1}E\Gamma \subseteq E.$$

Such a congruence determines the factor graph $G/E = (\bar{S}, \bar{\Gamma})$, where $\bar{S} = S/E$ and $\bar{\Gamma} = \psi^{-1} \Gamma \psi$, ψ denoting the natural mapping of S onto $\bar{S}$.

Evidently, G/E is a homomorphic image of G .

Conversely, if $G' = G\psi$ and E is the equivalence relation on S determined by ψ , then E is a congruence on G and G/E is isomorphic to G' .

*) $\psi^{-1} \Gamma \psi$ denotes the relational product of the relations ψ^{-1} , Γ , and ψ .

Let E_1 and E_2 be equivalences on the set S such that $E_1 \supset E_2$. We define the equivalence relation $E = E_1/E_2$ on S/E_2 in the natural way: two E_2 -classes belong to the same E -class, if and only if they are both subsets of the same E_1 -class.

Obviously, if $E = E_1/E_2$, E_1 is uniquely determined by E and E_2 (notation: $E_1 = E_2 * E$).

The following lemma is proved in [4]:

Lemma 1. Let E_1, E_2 be congruences on G such that $E_1 \supset E_2$. Then E_1/E_2 is a congruence on G/E_2 and $G/E_2/E_1/E_2 \cong G/E_1$.

It is also easily verified that if E_2 is a congruence on G , and E on G/E_2 , then $E_1 = E_2 * E$ is a congruence on G . If E is non-trivial, E_1 properly includes E_2 .

III. ELEMENTARY CONGRUENCES

The following four types of equivalence relations on S , the set of vertices of a given graph G , are easily verified to be congruences on G . They will be called elementary congruences (on G).

Type E_{α} . Let $(s^{\cap 0} = s, s^{\cap 1}, \dots, s^{\cap k-1})$ be a cycle of length k and p a prime divisor of k .

Then $s^{\cap i} \equiv s^{\cap j} (E_{\alpha}) \iff i \equiv j \pmod{\frac{k}{p}}$.

Type E_3 . Let s and t be vertices such that $s \equiv t$.

Then $s \equiv t (E_3)$.

Type E_4 . Let C_1 and C_2 be different cycles, both of length k , and let s_1, s_2 be arbitrary vertices of C_1, C_2 , respectively.

Then $s_1 \equiv s_2 (E_4)$ ($j = 0, 1, \dots, k-1$).

Type E_5 . Let s be a sink, and t an arbitrary vertex.

Then $s \equiv t (E_5)$.

We shall denote the identity congruence as Δ . Also we shall write $E_1 \supset E_2$ to indicate that E_1 includes E_2 properly.

Theorem 1. A congruence $E \neq \Delta$ on a given graph G is elementary if and only if for every congruence E' on G

$$E \supset E' \Rightarrow E' = \Delta.$$

Proof. Evidently, if E is an elementary congruence, it does not properly include any congruence $E' \neq \Delta$ on G .

Conversely, assume $E \neq \Delta$ and $E \not\supset E'$, for every congruence $E' \neq \Delta$ on G .

There exist two vertices s, t such that $s \equiv t (E)$. We shall distinguish between five cases (A-E) where A, B, C cover the alternative that s and

t belong to the same connected part of G ; $D, E, \dots$ s and t belong to different parts of G .

Case A. There exists a positive integer h such that $s \Gamma^h = t \Gamma^h$.

Then the congruence E' defined by:

$$s \Gamma^{h-1} \equiv t \Gamma^{h-1} (E')$$

is elementary of type E_β and $E \approx E' \neq \Delta$. Hence $E = E'$.

Case B. There exists a non-negative integer h such that

$s \Gamma^h = u \neq v = t \Gamma^h$ and u, v are both on the same cycle C (of length k).

Let $v = u \Gamma^m$, let d be the greatest common divisor (g. c. d.) of m and k , and p a prime number dividing $\frac{k}{d}$. We define the elementary congruence E' of type E_α by

$$u \Gamma^i \equiv u \Gamma^j (E') \iff i \equiv j \pmod{\frac{k}{p}}.$$

Then $E \approx E' \neq \Delta$. Thus $E = E'$, i. e., E is elementary.

Case C. The vertices s and t belong to the same connected part of the graph, without satisfying the conditions of cases A and B. Thus, there exist non-negative integers $h_1 \neq h_2$ such that $s \Gamma^{h_1} = t \Gamma^{h_2}$ is a sink.

Say $h_1 > h_2$. Then the congruence E' defined by $s \Gamma^{h_1} \equiv t \Gamma^{h_2} (E')$ is elementary, of type E_0 , and satisfies $E \approx E' \neq \Delta$. Hence $E = E'$.

Case D. The vertices s and t belong to two different connected

parts, containing cycles C_1 and C_2 of length k_1 and k_2 respectively.

Then there exists a non-negative integer h such that $s \cap^h = u \equiv v = t \cap^h(E)$ and $u \in C_1$, $v \in C_2$. Let d be the g. c. d. of k_1 and k_2 . If, say, $k_1 > d$ then the congruence E' given by

$$u \cap^i \equiv u \cap^j(E') \iff i \equiv j \pmod{d}$$

is properly included in E . Hence $k_1 = k_2 = d$, and the elementary congruence E'' of type E_γ merging cycles C_1 and C_2 satisfies $E \supseteq E'' \neq \Delta$.

Therefore, $E = E''$, i.e., E is elementary.

Case E. The vertices s and t belong to separate connected parts and there exists an integer $h \geq 0$, such that $s \cap^h$ (or alternatively $t \cap^h$) is a sink, whereas $t \cap^j$ ($s \cap^j$ respectively) $j < h$, is not a sink.

Then E' , defined by $s \cap^h \equiv t \cap^h(E')$ is elementary (type E_γ), and $E \supseteq E' \neq \Delta$. Hence $E = E'$.

Theorem 1 is thus proved.

Theorem 2. If G' is a homomorphic image of G , there exists a series

$$G = G_0, G_1, \dots, G_r \cong G' \quad (1)$$

such that G_{i+1} ($i = 0, 1, 2, \dots, r-1$) is a homomorphic image of G_i and the congruence on G_i determined by G_{i+1} is elementary.

Proof. Let E be the congruence on G determined by G' . If E is elementary, there is nothing to prove. Otherwise, by Theorem 1,

there must exist an elementary congruence E_1 , such that $E \supset E_1 \supset \Delta$.

$G_1 = G/E_1$ is a proper homomorphic image of G , and, by Lemma 1,

$$G/E_1/E/E_1 \cong G/E \cong G'.$$

Hence G' is a homomorphic image of G_1 . Replacing G by G_1 in the above reasoning, we obtain a graph $G_2 = G_1/E_2$ where E_2 is elementary and G' is a homomorphic image of G_2 .

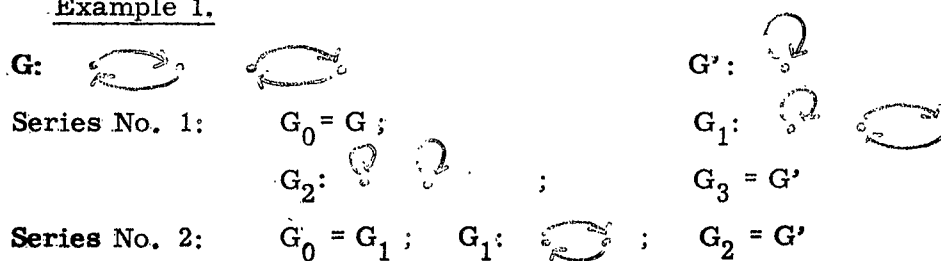
Continuing this process, which must terminate after a finite number of steps, the series (1) is obtained.

IV. JORDAN-DEDEKIND CHAIN CONDITIONS

A series of type (1) is clearly refined, i.e., no additional graph G_j can be inserted between any two elements of the series, such that the new series still satisfies the conditions of Theorem 2.

Generally, given G and G' , there exists a number of essentially different refined series from G to G' . The following two examples show that such series may even be of different lengths.

Example 1.



Example 2.

$G: \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$

$G': \bullet \rightarrow \bullet$

Series No 1: $G = G_0$,

$G_1: \bullet \rightarrow \bullet \rightarrow \bullet$, $G_2 = G'$

Series No 2: $G = G_0$,

$G_1: \bullet \rightarrow \bullet \rightarrow \bullet$,

$G_2: \bullet \rightarrow \bullet \rightarrow \bullet$, $G_3 = G'$

However, for connected, complete graphs the following holds:

Theorem 3. Let G be a complete, connected graph and G' a homomorphic image of G . Then every refined series of type (1) from G to G' is of the same length.

Proof. Let $G = G_0, G_1, \dots, G_r = G'$ be a series of the required type and

$$E_1, E_2, \dots, E_r \quad (2)$$

the corresponding congruences. It follows from the proof of Theorem 1, that each E_i is either of type E_α or E_β .

Now, let n_i be the number of vertices of G_i and k_i its cycle length.

If E_i is of type E_β , $k_i = k_{i-1}$, and $n_i = n_{i-1} - 1$.

If E_i is of type E_α , then $\frac{k_{i-1}}{k_i}$ is prime.

Let now $\frac{k_0}{k_r} = p_1^{h_1} p_2^{h_2} \dots p_m^{h_m}$ be the decomposition of $\frac{k_0}{k_r}$ into prime factors. Then the number of E_i of type E_α in (2) equals

$$h_\alpha = h_1 + h_2 + \dots + h_m.$$

Now the E_{ρ} steps have to decrease the number of vertices by $h_{\rho} = (n_0 - k_0) - (n_r - k_r)$. Thus $r = h_{\alpha} + h_{\rho}$, i.e., r is independent of the particular series chosen.

It follows from Theorem 3 that the lattice of congruences of a complete, connected graph satisfies the Jordan-Dedekind chain condition. However, this lattice is in general not modular, as shown by the following

Example 3.



The lattice of congruences of G coincides with the lattice of equivalences of the set of its vertices, which is known to be non-modular.

V. CONCLUSIONS

Considering applications to sequential machines (finite automata), we have obtained in this report a direct method to derive all admissible partitions of a single-input, complete or partial machine.

Although this method is also useful in connection with multiple-input machines, the derivation of an efficient algorithm to obtain all admissible partitions of such a machine requires further investigations.

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