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MAY 1963

A TEST OF A STATISTICAL METHOD FOR COMPUTING SELECTED INVENTORY MODEL CHARACTERISTICS BY SIMULATION

Murray A. Geisler

PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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1700 MAIN ST • SANTA MONICA • CALIFORNIA

PREFACE

This Memorandum is a product of research being conducted at RAND on the design and operation of simulation models for studying management policies and other problems that involve complex systems of random variables. Simulation techniques are being used increasingly in management studies, for instance, in examining supply and maintenance policies; and it is desirable to make the technique more rigorous.

The text is a companion piece to M. A. Geisler, The Sizes of Simulation Samples Required to Compute Certain Inventory Characteristics with Stated Precision and Confidence, The RAND Corporation, RM-3242-PR, October, 1962. Special statistical methods were used in that study to compute the sample sizes for specified inventory models. In this study, the methods were tested by applying them to particular inventory cases, and determining how well the actual precision and confidence obtained in the estimates agreed with expectation. Another related publication is R. W. Conway, Some Tactical Problems in Simulation Method, The RAND Corporation, RM-3244-PR, October, 1962.

Information compiled in this Memorandum should be of interest to Air Force statisticians and research people engaged in applying simulation techniques to management and other planning problems.

This study will be presented at the Tenth International Meeting of the Institute of Management Sciences, Tokyo, Japan, in August, 1963.

SUMMARY

RAND Memorandum 3242* presented 16 tables of sample sizes to be used for estimating certain statistical characteristics (mean shortages and overages per time period) of particular inventory models by computer simulation. The tables cover a wide range of inventory policy parameters; the inventory models used include the zero procurement lead-time case, plus the non-zero procurement cases of 2-, 5-, and 10-period lead-times with exponential demand. To compute these tables, we had to make several important assumptions that could affect the accuracy realised in simulations based on these sample sizes, compared with that expected from statistical theory. We undertook this current study to determine how well realized and expected accuracy might agree.

The test consisted of applying, to 1000 independent samples, the sampling procedure described in RM-3242, using several inventory policies and the four procurement lead-time cases mentioned. (Even though we could solve the zero lead-time case completely by mathematical analysis, we examined it to obtain an absolute test of the statistical method. In addition, the non-zero procurement lead-time cases were tested, using empirically derived standards.) In all, we tried 25 separate inventory cases, each with 1000 samples computed. For each sample size, we computed the mean number of shortages and overages per period. Then, from the 1000 samples for each inventory situation, we could compute the percentage of samples falling within the specified precision and confidence range predicted in the previous Memorandum.

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*M. A. Geisler, The Sizes of Simulation Samples Required to Compute Certain Inventory Characteristics with Stated Precision and Confidence, The RAND Corporation, RM-3242-PR, October, 1962.

We found that the actual precision and confidence obtained for each of the inventory policies and procurement lead-time cases tested did correspond closely with expectation. Thus, it was expected that 95 per cent of each of the zero procurement cases tried would be within 100 per cent of the true mean value. In the shortage calculations, the values ranged from 94.3 to 97.6 per cent, and for the overages, the range was from 95.6 to 99.9 per cent. From this test, we concluded that the statistical procedure given in RM-3242 is valid, and produces reliable estimates of mean shortages and overages per period.

CONTENTS

I. INTRODUCTION

The availability of electronic computers has made it possible to extend mathematical and statistical results beyond those attainable solely by analytical techniques. The combined use of analytical and computational methods seems quite useful and promising for a variety of purposes. First, the computational methods can feasibly be used to obtain numerical solutions to specific problems beyond the range of existing analytical results. Second, the exploration of numerical examples may provide insights concerning problem structure that could suggest further analytical extensions. Third, computers can be used to do sensitivity tests so that the relative advantages of alternative solutions can be assessed; thus, an approximate solution may not differ too much from an optimal one, but it may be much simpler to obtain. Fourth, by Monte Carlo sampling, the computers can be used to estimate variances of expected value solutions.

This Memorandum deals chiefly with the first of the purposes; and, as part of the numerical solutions, it considers the fourth purpose as well. We endeavored to compute the mean values of selected important inventory characteristics: number of shortages and number of overages per time period. This problem can be solved analytically for the (S,s) inventory policy model with a zero procurement lead-time and exponential demand; however, for the non-zero cases, analytical solutions have not been derived.

We therefore present a statistical method applicable to both zero and non-zero procurement lead-time cases, for computing the expected or mean value of shortages and overages per period. Then we apply the method to these cases. Our intent in applying this statistical procedure to the zero case, despite our ability to solve it analytically, is to check whether the method produces estimates of the true mean value which have some precision and confidence. We check this approach by applying the statistical method to a very large number of independent trials and make an estimate of mean shortages and overages each time. The frequency distribution of these estimates can then be used to evaluate the extent to which the specified precision and confidence are realized through use of the statistical method.

We have also used a similar approach with the non-zero procurement lead-time cases, but we do not have the true value of the mean number of shortages and overages. As a substitute for these true values we have used a statistical estimate of each mean, based on several hundred observations. Therefore, the evaluation of the estimating procedure is not as rigorous as for the zero procurement lead-time case; the findings obtained, however, should give an empirical measure of the validity of the estimating procedure for the non-zero case.

The study thus tests and evaluates the statistical procedure used to estimate mean shortages and overages. It is a companion piece to an earlier RAND Memorandum*, subsequently referred to as RM-3242, which presented the method for computing such sample sizes.

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*M. A. Geisler, The Sizes of Simulation Samples Required to Compute Certain Inventory Characteristics with Stated Precision and Confidence, The RAND Corporation, RM-3242-PR, October, 1962.

II. ANALYTIC RESULTS DERIVED FOR THE ZERO PROCUREMENT LEAD-TIME CASE

We first consider an inventory model with a zero procurement lead-time governed by (S,s) policies. We assume that a particular set of values (S,s) has been selected, so that whenever the stock level x falls to or below s, positive ordering is immediately enacted to raise the level to S upon delivery. When the quantity of goods in supply x exceeds s, then no ordering is done. We allow x to assume any possible real value. A negative stock level should be interpreted as the amount owed to consumption. Thus, all demand will be ultimately satisfied, and it is therefore meaningful to refer to negative stock levels. We also assume that the density of demand $f(\xi)$ is known, so that in each time period, a demand ξ has probability $f(\xi)$ of occurring. Then, if x_n = stock level at end of period n, we have:

$$x_{n+1} = \begin{cases} x_n - \xi & \text{if} \quad s < x_n \leq S \\ \\ S - \xi & \text{if} \quad x_n \leq s \ . \end{cases}$$

For this model, if we assume $f(\xi) = \lambda e^{-\lambda\xi}$, exponential demand, we can obtain the following analytic results:

$$E(x_n) = \frac{s - \frac{1}{\lambda} + \lambda\frac{S^2}{2} - \lambda\frac{s^2}{2}}{1 + \lambda\Delta} ,$$

$$\text{Cov}\ (x_n, x_{n+1}) = \frac{\frac{\Delta^2}{12}\left[\lambda^2\Delta^2 - 2\lambda\Delta - 6\right]}{\left[1 + \lambda\Delta\right]^2} ,$$

$$\mathrm{Var}\,(x_n) = \frac{\frac{\lambda^2}{12}\Delta^4 + \frac{\lambda}{3}\Delta^3 + \Delta^2 + \frac{2}{\lambda}\Delta + \frac{1}{\lambda^2}}{\left[1 + \lambda\,\Delta\right]^2}$$

$$\rho_{x_n x_{n+1}} = \frac{\frac{\Delta^2}{12}\left[\lambda^2\Delta^2 - 2\lambda\Delta - 6\right]}{\frac{\lambda^2}{12}\Delta^4 + \frac{\lambda}{3}\Delta^3 + \Delta^2 + \frac{2}{\lambda}\Delta + \frac{1}{\lambda^2}},$$

and

$$\rho_{x_n x_{n+p}} = \rho^p_{x_n x_{n+1}}$$

where $\Delta = S - s$.

These results are derived in M. A. Geisler, "Some Statistical Properties of Selected Inventory Models," Naval Research Logistics Quarterly, June, 1962. The article also derives the following results of interest to this Memorandum. If y_n = shortages in the n-th period, we have:

$$y_n = \begin{cases} 0 & \text{if} \quad x_n \geq 0 \\ -x_n & \text{if} \quad x_n < 0. \end{cases}$$

For y_n, we have the following analytic results:

$$E(y_n) = \frac{e^{-\lambda s}}{\lambda(1 + \lambda\Delta)},$$

$$\mathrm{Cov}\ (y_n, y_{n+1}) = \frac{e^{-\lambda s}}{\lambda^2(1 + \lambda\Delta)} \left(e^{-\lambda S} - \frac{e^{-\lambda s}}{1 + \lambda\Delta}\right),$$

$$\mathrm{Var}\ (y_n) = \frac{e^{-\lambda s}}{\lambda^2(1 + \lambda\Delta)} \left(2 - \frac{e^{-\lambda s}}{1 + \lambda\Delta}\right),$$

$$\rho_{y_n y_{n+1}} = \frac{e^{-\lambda S}(1 + \lambda\Delta) - e^{-\lambda s}}{2(1 + \lambda\Delta) - e^{-\lambda s}}.$$

Since y_n is not a Markov process, we cannot infer $\rho_{y_n y_{n+p}}$, $p = 2, 3, \ldots$ from $\rho_{y_n y_{n+1}}$, as we could with x_n.

The term "overage", refers to the positive amount of stock left at the end of the period before ordering. If v_n = overage in n-th period, then:

$$v_n = \begin{cases} x_n, & \text{if} \quad x_n > 0 \\ 0, & \text{if} \quad x_n \leq 0. \end{cases}$$

For v_n, we have the following results: **true for each n:**

$$E(v_n) = \frac{\lambda s + e^{-\lambda s} - 1}{\lambda(1 + \lambda\Delta)} + \frac{\lambda}{2(1 + \lambda\Delta)} (S^2 - s^2),$$

$$\mathrm{Var}\ (v_n) = \frac{\lambda^2 s^2 - 2\lambda s + 2 - 2e^{-\lambda s}}{\lambda^2(1 + \lambda\Delta)} + \frac{\lambda(S^3 - s^3)}{3(1 + \lambda\Delta)}$$

$$- \left[\frac{\lambda s + e^{-\lambda s} - 1}{\lambda(1 + \lambda\Delta)} + \frac{\lambda}{2(1 + \lambda\Delta)} (S^2 - s^2)\right]^2,$$

$$\text{Cov}\ (v_n, v_{n+1}) = \left(\frac{\lambda S + e^{-\lambda S} - 1}{1 + \lambda\Delta}\right)\left(\frac{\lambda s + e^{-\lambda s} - 1}{\lambda^2}\right)$$

$$+ \frac{\lambda}{1 + \lambda\Delta}\left(\frac{S^3 - s^3}{3}\right) - \frac{1}{1 + \lambda\Delta}\left(\frac{S^2 - s^2}{2}\right)$$

$$- \frac{1}{1 + \lambda\Delta}\left(\frac{Se^{-\lambda S}}{\lambda} + \frac{e^{-\lambda S}}{\lambda^2} - \frac{se^{-\lambda s}}{\lambda} - \frac{e^{-\lambda s}}{\lambda^2}\right)$$

$$- \left[\frac{\lambda s + e^{-\lambda s} - 1}{\lambda(1 + \lambda\Delta)} + \frac{\lambda}{2(1 + \lambda\Delta)}\,(S^2 - s^2)\right]^2 .$$

Finally,

$$\rho_{v_n v_{n+1}} = \frac{\text{Cov}(v_n, v_{n+1})}{\text{Var}(v_n)} \ .$$

The expressions given above for Cov (v_n, v_{n+1}) and $\text{Var}(v_n)$ can then be substituted in $\rho_{v_n v_{n+1}}$ to get an explicit solution for $\rho_{v_n v_{n+1}}$ in terms of λ, s, and Δ. Here, too, v_n is not a Markov process so that we cannot infer the behavior of $\rho_{v_n v_{n+p}}$, $p = 2, 3\ldots$ from $\rho_{v_n v_{n+1}}$.

To summarize, the analytic derivation of results for the zero procurement lead-time case with exponential demand is complete for x_n in that we can compute exactly the mean, variance, covariances for <u>all</u> time lags, and the corresponding correlation coefficients. For y_n and v_n, however, we can compute analytically only the mean and variance, and the covariance and correlation coefficient for only <u>one</u> period lag.

III. ANALYTIC RESULTS DERIVED FOR THE NON-ZERO PROCUREMENT LEAD-TIME CASE

Analytically, we can say much less about the properties of the non-zero procurement lead-time case. We have been able to do the following. If we let z_n = sum of on-order plus on-hand stock in n-th period, before ordering, we then can show that z_n for the non-zero procurement lead-time case is equivalent to x_n for the procurement lead-time case. We can thus compute exactly, for z_n, its mean, variance, covariance for all time lags, and the corresponding correlation coefficients. Also, the expressions are identical with those given above for x_n, replacing x_n by z_n.

We can say further that if:

x_n = on-hand stock level at end of period n (which can assume any real number value),

t = procurement lead-time, measured in number of time periods from order to delivery,

then the following relation holds:

$$x_{n+t-1} = z_n - \xi_n - \dots - \xi_{n+t-2} ,$$

where ξ_n = demand in n-th period.

If we then specialize this relation to the case of $t = 2$, we obtain $x_{n+1} = z_n - \xi_n$. For this 2-period procurement lead-time case, we can derive the following results analytically:

$$E(x_n) = E(z_n) - \frac{1}{\lambda} ,$$

$$\mathrm{Var}\ (x_n) = \mathrm{Var}\ (z_n) + \frac{1}{\lambda^2}\ ,$$

$$\mathrm{Cov}\ (x_n, x_{n+1}) = \mathrm{Cov}\ (z_n, z_{n+1}) + \frac{\lambda^2 S^2 - \lambda S - \lambda^2 s^2 + 3\lambda s - 3}{\lambda^2(1 + \lambda\Delta)}\ ,$$

$$\rho_{x_n x_{n+1}} = \frac{\mathrm{Cov}\ (z_n, z_{n+1}) + \dfrac{\lambda^2 S^2 - \lambda S - \lambda^2 s^2 + 3\lambda s - 3}{\lambda^2(1 + \lambda\Delta)}}{\mathrm{Var}\ (z_n) + \dfrac{1}{\lambda^2}}$$

These results are meager indeed, so we therefore explored the possibility of using computing techniques to augment and extend them. In the following sections, we present the statistical technique and the results obtained with it.

IV. DESCRIPTION OF STATISTICAL TECHNIQUE*

We would like to compute estimates of the mean number of shortages and overages occurring in an inventory model operating under (S,s) policies. We consider both the zero and the non-zero procurement lag cases. We present the zero lag case to provide an absolute assessment of the statistical method which we are compelled to use to obtain the properties of the non-zero lag cases.

ZERO PROCUREMENT LEAD-TIME MODEL

We are interested in calculating the sample sizes needed to assure that the sample estimates of the mean shortages and overages (per time period) differ in absolute value from the corresponding true value no more than K_1 and K_2, respectively, with 95-per-cent confidence. Thus, if $\bar{y}$ is the sample estimate of the mean shortages per time period and Y is the true value, then we want to find that minimum value of n_y, the sample size, such that

$$\Pr\left\{|\bar{y} - Y| > K_1\right\} \leq 0.05 ,$$

where

$$\bar{y} = \frac{\sum_{n=1}^{n_y} y_n}{n_y} .$$

*Much of the material of this section has been drawn from the companion piece, RM-3242, to provide that necessary description which will help the reader to follow the test procedure and results presented later in this Memorandum.

Similarly, if $\bar{v}$ is the sample estimate of the mean overages per time period, and V is the true value, then we want to find that minimum value of n_v, the sample size, such that

$$Pr\left\{|\bar{v} - V| > K_2\right\} \leq 0.05 ,$$

where

$$\bar{v} = \frac{\sum_{n=1}^{n_v} v_n}{n_v} .$$

Subsequently, where $\hat{Y}$ and $\hat{V}$ are estimates of the true values Y and V, we let $K_1 = \hat{Y}$ and $K_2 = \hat{V}$. The manner of estimating $\hat{Y}$ and $\hat{V}$ is described in Sec. V. Thus, we seek those minimum sample sizes for estimating $\bar{y}$ and $\bar{v}$ such that there will be 95-per-cent confidence that $\bar{y}$ and $\bar{v}$ will differ from their corresponding true values by no more than approximately the true value in absolute amount. This is the same as requiring the sample sizes to differ no more than approximately 100 per cent from the correspondingly true value, with 95-per-cent confidence.

If we assume $\bar{y}$ and $\bar{v}$ have approximately normal distributions, a reasonable assumption in view of the Central Limit Theorem, then we can use Tschebycheff's inequality, and choose n_y and n_v such that

$$(1.96)\sigma_{\bar{y}} \leq \hat{Y} ,$$

$$(1.96)\sigma_{\bar{v}} \leq \hat{V} ,$$

for attaining a 95-per-cent confidence level.

We can now calculate $\sigma^2_{\bar{y}}$ and $\sigma^2_{\bar{v}}$ using the following formulas:

$$\sigma^2_{\bar{y}} = \frac{\sigma^2_y}{n_y} \left\{1 + 2 \sum_{p=1}^{m} (1 - \frac{p}{m+1}\, \rho_{p,y})\right\},$$

$$\sigma^2_{\bar{v}} = \frac{\sigma^2_v}{n_v} \left\{1 + 2 \sum_{p=1}^{m} (1 - \frac{p}{m+1}\, \rho_{p,v})\right\},$$

where

σ^2_y = variance of shortages (per period);

σ^2_v = variance of overages (per period);

$\rho_{p,y}$ = p-th order lag correlation for shortages;

$\rho_{p,v}$ = p-th order lag correlation for overages; and

m = maximum lag for which correlations are computed; so that

$p = 1, 2, \ldots m$.

These expressions are modifications of the more usual forms for calculating the variance of the mean of the autocorrelated series, as given by Moran.* They allow for the necessity of truncating the calculation of autocorrelations; and by weighting more heavily the autocorrelations for the shorter lags, they attempt to reduce the likelihood of computing negative values for $\sigma^2_{\bar{y}}$ and $\sigma^2_{\bar{v}}$.

If we now substitute for $\sigma^2_{\bar{y}}$ and $\sigma^2_{\bar{v}}$ in the above inequalities, and solve for the minimum values of n_y and n_v that satisfy these inequalities,

*P. A. P. Moran, The Theory of Storage, John Wiley and Sons, New York, 1959.

we obtain:

$$\min(n_y) = \left[\frac{(1.96)^2 \sigma_y^2 \left\{ 1 + 2 \sum_{p=1}^{m} (1 - \frac{p}{m+1}) \rho_{p,y} \right\}}{\gamma^2} \right] + 1;$$

and

$$\min(n_v) = \left[\frac{(1.96)^2 \sigma_v^2 \left\{ 1 + 2 \sum_{p=1}^{m} (1 - \frac{p}{m+1}) \rho_{p,v} \right\}}{\gamma^2} \right] + 1,$$

where [x] = greatest integer in x.

As stated above, we can compute σ_y^2 and σ_v^2 analytically and have therefore used their analytical values in computing min (n_y) and min (n_v). We cannot, however, compute $\rho_{p,y}$ and $\rho_{p,v}$ analytically. Instead, we have to estimate them with Monte Carlo techniques, thereby deriving $r_{p,y}$ as an estimate of $\rho_{p,y}$ and $r_{p,v}$ as an estimate of $\rho_{p,v}$. The technique by which the calculations of $r_{p,y}$ and $r_{p,v}$ were done in a particular study and the resulting average sample sizes with their standard deviations are presented in Tables 1 and 2 of RM-3242.

NON-ZERO PROCUREMENT LEAD-TIME MODEL

The approach taken to compute the sample sizes for this model is the same as that used for the zero procurement lead-time model. The main problem in using the formulas for min (n_y) and min (n_v) is that we cannot derive the exact values σ_y^2 and σ_v^2 analytically. Instead, we have to estimate σ_y^2 by S_y^2, and σ_v^2 by S_v^2. The procedure for doing this in computing sample sizes is also described in RM-3242. This procedure was applied to compute sample sizes for the non-zero procurement cases of 2-, 5-, and 10-period lead-times. The relevant average sample sizes with their standard deviations are given in RM-3242: Tables

5-6 for the 2-period procurement lead-time, Tables 9-10 for the 5-period procurement lead-time, and Tables 13-14 for the 10-period procurement lead-time.

V. TEST OF STATISTICAL TECHNIQUE

To make the sample size calculations, it was necessary, in effect, to create a statistical model of the sampling process. This was primarily represented by the formulas specified for computing min (n_y) and min (n_v). To apply these formulas we had to make three decisions. First, we had to choose an appropriate value of p, the truncation point; and second, we had to select large enough Monte Carlo samples to ensure reasonably valid estimations of the covariances and variances used in the formulas. Third, although theoretically in the case of exponential demand distributions the sample size computations are independent of the value of λ used, we wished to use one value of λ in the exponential demand distribution, and have it apply to the range of values used for s and Δ. We chose the value of $\lambda = 1$. Also, we were assuming that the normal distribution assumption could be used in computing confidence intervals for Y and V, and that this assumption would also hold for computing values for min (n_y) and min (n_v) that would result in the specified precision and confidence levels. Thus, a number of fairly significant but subtle assumptions and decisions were required to establish the basis for utilizing the formulas given above for min (n_y) and min (n_v).

TEST OF ZERO PROCUREMENT LEAD-TIME CASE

To evaluate the statistical technique, we then reversed the procedure used in RM-3242 to compute min (n_y) and min (n_v). We used the limiting density for x = stock level, given by:

$$\varphi(x) = \begin{cases} \dfrac{\lambda}{1 + \lambda\Delta} & \text{if} \quad s < x \le \Delta + s \\ \\ \dfrac{\lambda e^{-\lambda(s-x)}}{1 + \lambda\Delta} & \text{if} \quad x \le s. \end{cases}$$

For specific values of λ, Δ, and s, $\varphi(x)$ is defined. Using this density function, a value of x was selected randomly, and set equal to x_0. Then, using the value of $\lambda = 1$, a specified number k_y of random samplings of ξ_n was made using the demand density function $f(\xi) = e^{-\xi}$, so that $\xi_0, \ldots \xi_{k_y-1}$ were obtained. The sample size k_y was selected so that it would be equal to the mean sample size as given in Table 1 of RM-3242, for the specified s and Δ.

Then, using the transition equations:

$$x_{n+1} = \begin{cases} x_n - \xi_n & \text{if} \quad s < x_n \le \Delta + s \\ \\ S - \xi_n & \text{if} \quad x_n \le s, \end{cases}$$

we calculated the random variables $x_1 \ldots x_{k_y}$, thus providing a random sequence of k_y values for x_n from which we calculated

$$y_n = \begin{cases} 0 & \text{if} \quad x_n \ge 0 \\ \\ -x_n & \text{if} \quad x_n < 0. \end{cases}$$

Then we computed

$$\bar{y} = \frac{\sum_{n=1}^{k_y} y_n}{k_y} ,$$

which gave the mean number of shortages experienced in that particular sample.

We repeated the process to compute the mean number of overages, $\bar{v}$. From Table 2, we obtained the requisite mean sample size k_v. If $k_y \geq k_v$, we used the first k_v random samplings of ξ_n: $\xi_0, \ldots \xi_{k_v-1}$, taken from $\xi_0, \ldots \xi_{k_y-1}$. If $k_v > k_y$, however, we then added random samplings $\xi_{k_y}, \ldots \xi_{k_v-1}$ to the earlier series $\xi_0, \ldots \xi_{k_y-1}$, using the same starting value x_0 drawn from $\varphi(x)$. We could thus obtain $x_1, \ldots x_{k_v}$, from which v_n could be computed by means of:

$$v_n = \begin{cases} x_n & \text{if} \quad x_n \geq 0 \\ 0 & \text{if} \quad x_n < 0. \end{cases}$$

This gave $v_1, \ldots v_{k_v}$, from which we computed

$$\bar{v} = \frac{\sum_{n=1}^{k_v} v_n}{k_v} .$$

And this, in turn, gave the mean number of overages, $\bar{v}$, in that particular sample.

To test the precision and confidence such a statistical procedure would give, we repeated the above procedure 1000 times, selecting x_0 from $\varphi(x)$ and a sample of ξ_n values for each of these 1000 random samples. We thus obtained 1000 values of $\bar{y}$ and $\bar{v}$ for the specified sample sizes of k_y and k_v, respectively. In RM-3242, k_y and k_v were computed to produce sample estimates of mean shortages and overages differing no more than approximately 100 per cent from the corresponding true mean values, with 95-per-cent confidence. Further, for the zero procurement lag case, we know the true values of the mean shortages and overages, which we designate by Y and V respectively.* We could then compute:

$$\frac{\bar{y} - Y}{Y} \text{ and } \frac{\bar{v} - V}{V}$$

for each of the 1000 samples. If for a given sample,

$$\left|\frac{\bar{y} - Y}{Y}\right| \leq 1,$$

then we had an estimate which satisfied the precision condition of being within 100 per cent of the true value. We could then count the number of samples out of the 1000 independent ones, say N_y, satisfying this condition. Then,

$$\frac{N_y}{1000} \times 100 \text{ per cent}$$

gave the confidence realized with the estimating procedure. This same procedure was followed with the mean number of overages,

$$\frac{N_v}{1000} \times 100 \text{ per cent}$$

and the corresponding confidence values computed.

This test was done for 7 pairs of values $(s\lambda, \Delta\lambda)$ that seemed to encompass interesting inventory policy cases. The results of this test appear in Table 1 in which we show two interesting statistics: first, the confidence levels actually achieved (with 95-per-cent confidence expected), and, second, the comparison of the true mean values for shortages and overages with the estimate obtained from averaging the sample means over the 1000 samples computed. To be more precise,

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*These values of Y and V are given in RM-3242, Tables 3 and 4 respectively.

if $\bar{y}_i$ and $\bar{v}_i$ are the mean shortages and mean overages estimated from the i-th sample, where i = 1, ... 1000, then we computed

$$\frac{\sum_{i=1}^{1000} \bar{y}_i}{1000} \quad \text{and} \quad \frac{\sum_{i=1}^{1000} \bar{v}_i}{1000} ,$$

Table 1

EVALUATION OF STATISTICAL ESTIMATING PROCEDURES
Percentage Confidence Attained, and Comparison of True and Estimated Mean Shortages and Overages: Zero Procurement Lead-Time Case

Inventory Policies ($s\lambda$, $\Delta\lambda$)	Per cent Confidence Attained		Mean Shortages		Mean Overages		Sample Sizes Used	
	Short[a]	Ovg[b]	Est[c]	True	Est	True	Short	Ovg
0.01, 0.1	97.6	95.6	1.05	0.90	0.006	0.006	5	36
0.1, 0.1	96.7	95.7	0.96	0.82	0.02	0.02	6	19
0.1, 5.0	95.0	98.9	0.15	0.15	2.9	2.2	32	2
1.0, 0.01	95.3	99.1	0.37	0.36	0.45	0.37	16	5
1.0, 1.0	94.3	99.8	0.18	0.18	1.38	0.93	30	2
1.0, 5.0	97.1	99.9	0.06	0.06	5.8	3.0	115	1
3.0, 0.01	97.2	99.9	0.05	0.05	4.1	2.1	197	1

[a] Shortages
[b] Overages
[c] Estimated

and these values are the estimated mean shortages and estimated mean overages given in Table 1.

From Table 1, we see that the percentage of confidence actually attained in the test came reasonably close to the desired 95-per-cent value, the results for the shortages ranging from 94.3 to 97.6 per cent, and for the overages ranging from 95.6 to 99.9 per cent. Of the 7 values, 6 exceeded 95 per cent in the case of the shortages; all 7 exceeded that figure for the case of overages. A partial explanation for the tendency of the confidence level to exceed 95 per cent is that in selecting min (n_y) and min (n_v) from Tables 1 and 2 respectively of RM-3242, the values given there were rounded upward in all instances. Thus, if n_v was 35.4 in Table 2, then 36 was used. The effect of rounding would be greatest with the smaller sample sizes, and may help explain the relatively higher confidence values found for the overage cases when samples of 2 and 1 were used.

Relatively good agreement was also found between the true mean shortage values Y and those estimated from the 1000 sample values of $\bar{y}$. We note that the agreement was better, the larger the sample size used to estimate $\bar{y}$. A similar finding occurred in comparing the true value of mean overages V and the mean overage value estimated from the 1000 sample values of $\bar{v}$.

Thus, from the data contained in Table 1, we would infer that the statistical procedure tends to do what it was devised to accomplish: it produces estimates of the mean value of shortages and overages within 100 per cent of the true value, with 95-per-cent confidence.

TEST OF NON-ZERO PROCUREMENT LEAD-TIME CASES

A procedure similar to that described for the zero procurement lead-time case was followed for the non-zero procurement lag models of 2, 5, and 10 time periods. As mentioned in Sec. III, if z_n stands for the on-hand plus on-order stock in the n-th period (before ordering), then z_n has the same statistical properties in the non-zero procurement lag case as x_n in the zero lag case. Thus, for the exponential distribution, $f(\xi) = \lambda e^{-\lambda \xi}$:

$$\varphi(z) = \begin{cases} \dfrac{\lambda}{1+\lambda\Delta} & \text{if} \quad s < z \le s + \Delta \\[2ex] \dfrac{\lambda e^{-\lambda(s-z)}}{1+\lambda\Delta} & \text{if} \quad z \le s. \end{cases}$$

In addition, if x_n = the amount of stock on hand at start of period n, including stock delivered in period n, and

ξ_n = demand in n-th period, and

t = procurement lead-time in number of time periods from order to delivery, then, as stated in Sec. III:

$$x_{n+t-1} = z_n - \xi_n - \dots - \xi_{n+t-2}, \text{ for } t \ge 2.$$

We thus could select z_0 from $\varphi(z)$, then select the appropriate series of ξ_n values, and derive the series: $y_1, \dots y_{k_y}$ for shortages and $v_1, \dots v_{k_v}$ for overages, paralleling the procedure described in this section for the zero lag case. We thus computed

$$\bar{y} = \frac{\sum_{n=1}^{k_y} y_n}{k_y} \quad \text{and} \quad \bar{v} = \frac{\sum_{n=1}^{k_v} v_n}{k_v}$$

for the particular sample of values $z_0, \xi_1, \dots \xi_{k-1}$ where $k = k_y$ or $k = k_v$ depending on which is larger. We then repeated this sampling and computing process 1000 times, obtaining 1000 independent values each of $\bar{y}$ and of $\bar{v}$. In the non-zero lag cases, we did not know the true values of Y and V, the mean values of shortages and mean overages per period, respectively. Instead, we used approximate values designated by $\tilde{Y}$ and $\tilde{V}$, which were computed from a time series of 500 values of

y_n and v_n. The values of $\tilde{Y}$ and $\tilde{V}$ used are given in RM-3242: Tables 7 and 8 for the 2-period lag case, Tables 11 and 12 for the 5-period lag case, and Tables 15 and 16 for the 10-period lag case. With this information we could compute

$$\frac{\bar{y} - \tilde{Y}}{\tilde{Y}} \quad \text{and} \quad \frac{\bar{v} - \tilde{V}}{\tilde{V}}$$

for each of the 1000 samples, and establish whether

$$\left|\frac{\bar{y} - \tilde{Y}}{\tilde{Y}}\right| \leq 1 \quad \text{and} \quad \left|\frac{\bar{v} - \tilde{V}}{\tilde{V}}\right| \leq 1$$

in each sample. We could then compute

$$\frac{N_y}{1000} \quad \text{and} \quad \frac{N_v}{1000}$$

which gave the percentage of samples for

$$\left|\frac{\bar{y} - \tilde{Y}}{\tilde{Y}}\right| \leq 1 \quad \text{and} \quad \left|\frac{\bar{v} - \tilde{V}}{\tilde{V}}\right| \leq 1 ,$$

respectively. These two percentages thus gave an indication of the confidence, within 100 per cent of the approximate true values, that the statistical method would yield in its estimates of mean shortages and overages. We also computed the average of the $\bar{y}$ and of the $\bar{v}$ values for the 1000 samples to determine how well these averages agreed with the corresponding approximate true value $\tilde{Y}$ and $\tilde{V}$.

The results of these calculations for selected inventory policies are given in Table 2 for the 2-period lead-time case, in Table 3 for the 5-period lead-time case, and in Table 4 for the 10-period lead-time case.

The results found in each of the non-zero lead-time cases seem to agree well with those for the zero lead-time case. That is, the confidence actually attained with the computed sample size is not far from the expected value of 95 per cent, with most of the confidence values exceeding that figure. The highest confidence values tend to

Table 2

EVALUATION OF STATISTICAL ESTIMATING PROCEDURES
Percentage Confidence Attained, and Comparison of "Approximately True" and Estimated Mean Shortages and Overages: 2-Period Procurement Lead-Time Case

Inventory Policies ($s\lambda$, $\Delta\lambda$)	Per Cent Confidence Attained		Mean Shortages		Mean Overages		Sample Sizes Used	
	Short[a]	Ovg[b]	Est[c]	Approx True[d]	Est	Approx True	Short	Ovg
0.01, 0.01	97.4	--[e]	2.0	2.0	--	--	5	--
0.01, 4.0	97.3	98.3	0.7	0.6	1.1	1.0	20	7
1.0, 0.01	97.2	99.9	1.2	1.1	0.1	0.2	13	32
1.0, 0.1	98.5	--	1.2	1.0	--	--	14	--
2.0, 0.01	98.9	99.9	0.7	0.5	0.7	0.5	29	12
2.0, 3.0	--	99.9	--	--	2.0	2.1	--	3
3.0, 10.0	--	99.9	--	--	7.0	6.4	--	2

[a]Shortages
[b]Overages
[c]Estimated
[d]Approximately true
[e]Not calculated

be found with the smallest sample sizes; this occurrence is believed to be attributable to the author's practice of rounding the average sample sizes in RM-3242 upward to the next integer value. Also, the estimated mean shortages and mean overages computed from the 1000 samples of specified size seem to agree quite well with the approximately true mean shortages and overages, especially for the larger specified sample sizes.

Table 3

EVALUATION OF STATISTICAL ESTIMATING PROCEDURES
Percentage Confidence attained, and Comparison of "Approximately True" and Estimated Mean Shortages and Overages: 5-Period Procurement Lead-Time Case

Inventory Policies (sλ, Δλ)	Per cent Confidence Attained		Mean Shortages		Mean Overages		Sample Sizes Used	
	Short[a]	Ovg[b]	Est[c]	Approx True[d]	Est	Approx True	Short	Ovg
0.01, 0.01	98.3	--[e]	4.9	5.0	--	--	5	--
0.01, 3.0	96.2	--	2.6	3.2	--	--	11	--
0.01, 50.0	--	99.9			20.0	20.9	--	6
1.0, 2.0	96.6	--	2.8	2.8	--	--	13	--
2.0, 0.01	98.0	--	3.0	3.0			13	--
2.0, 2.0	97.7	--	1.9	1.9	--	--	22	--
4.0, 3.0	--	99.4	--	--	1.4	1.5	--	20
4.0, 5.0	--	99.9	--	--	3.0	2.3	--	11
5.0, 4.0	--	99.8	--	--	2.9	2.7	--	10
10.0, 0.01	--	99.9	--	--	4.0	5.0	--	5

[a]Shortages
[b]Overages
[c]Estimated
[d]Approximately true
[e]Not Calculated

Table 4

EVALUATION OF STATISTICAL ESTIMATING PROCEDURES
Percentage Confidence attained, and Comparison of "Approximately True" and Estimated Mean Shortages and Overages: 10-Period Procurement Lead-Time Case

Inventory Policies (sλ,Δλ)	Per cent Confidence Attained		Mean Shortages		Mean Overages		Sample Sizes Used	
	Short[a]	Ovg[b]	Est[c]	Approx True[d]	Est	Approx True	Short	Ovg
0.01, 0.01	99.7	--	10.0	10.0	--	--	5	--
0.01, 5.0	96.0	89.8	5.2	7.1	16.5	16.7	9	9
1.0, 3.0	98.6	--	7.2	6.9	--	--	10	--
3.0, 4.0	98.5	--	4.7	4.6	--	--	18	--
4.0, 0.01	98.3	--	5.9	6.0	--	--	13	--
4.0, 50.0	--	93.4	--	--	19.6	20.0	--	5
5.0, 50.0	--	99.9	--	--	22.2	25.6	--	6
10.0, 10.0	--	99.9	--	--	8.3	10.4	--	11
50.0, 10.0	--	99.9	--	--	45.3	45.4	--	1

[a]Shortages
[b]Overages
[c]Estimated
[d]Approximately true
[e]Not Calculated

Thus, on the basis of this test, we can further state that the estimating procedure presented in RM-3242 for computing sample sizes of the (S,s)-type inventory models seems to be valid, in that it does provide the confidence desired in the estimates of mean shortages and mean overages.