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(6) SYMMETRIC VIBRATION OF THE ELASTIC SOLID CYLINDER

(15) Contract AF 49(638) - 1148

(10) by Donald S. Moseley

Approved:

F. W. Latson

F. W. Latson
Assistant Department Head

VITRO LABORATORIES
DIVISION OF VITRO CORPORATION OF AMERICA
14000 GEORGIA AVE., SILVER SPRING, MARYLAND

ABSTRACT

The exact formulation of displacements and resonant frequencies of an elastic solid cylinder ~~has been~~ WAS attempted. Progress ~~has been~~ WAS made, but the principal objective ~~has~~ WAS not ~~been~~ reached. This report traces the course of work through the early demonstration of invalidity of a formulation that was postulated prior to the contract, and through several assaults upon the problem. Novel solutions of the wave equation for radially symmetric displacements ~~have been~~ WERE discovered. A section is devoted to the recommended direction of future work.

CONTENTS

ILLUSTRATIONS

Section I
INTRODUCTION

This is the Technical Report issued upon completion of Contract AF 49(638) - 1148. It covers the period 22 March ~~through~~ 14 December 1962. This contract with Vitro Laboratories has been entitled "Elastic Solids Vibration Studies".

It has been the objective of work under this contract to advance an important area of the theory of elasticity. This area is the mathematical description of the elastic motions and resonant frequencies of isotropic solids that are bounded by traction-free surfaces. Excepting the sphere,[1] we may say that exact solutions for solids of arbitrary dimensions with these surface conditions are not yet known; other works have either presented (1) exact solutions for solids that are semi-infinite in one dimension or (2) approximate solutions for solids of arbitrary dimensions.

[1]A. E. H. Love, "A Treatise on the Mathematical Theory of Elasticity", 4th edition, Dover Publications, New York, 1944, pp 278-287.

Section II
THE ISOTROPIC SOLID CYLINDER

RESTRICTIONS

The right circular cylinder of arbitrary dimensions is the model studied during this contract. To reduce the mathematical labor, while yet demonstrating the techniques of solution, attention has been restricted to modes in which motion is symmetrical about the axis. This casts the problem in two dependent variables, one more than the sphere presents in its maximum symmetry, yet one less than presented by the rectangular parallelepiped of arbitrary dimensions. Quite arbitrarily the work has been restricted also by considering only those vibrations which are symmetrical about the midplane of the cylinder.

A BOUNDARY VALUE PROBLEM

The starting point in this contract was a set of displacement and frequency equations previously developed by the writer. Within the scope of this contract, it was the writer's initial objective to confirm these by various tests before proceeding into a computational phase in which fundamentals and overtones would be found. This was to be followed by an experimental program to verify frequencies. However, as will be seen in subsequent paragraphs, the course of the study was sharply altered when the confirmation of initial equations failed.

The displacement and frequency equations resulted from a derivation which formulates as a boundary value problem the elastic motions and dynamic stresses in the isotropic cylinder. The entire derivation is presented in Appendix A, and the displacements, resonant frequency, and components of resonant frequency are reproduced here as Eqs. (1)-(5).

Radial and axial displacements are, respectively

$$u_r = \sum_{j=-\infty}^{\infty} \left[A_{jj}\, p_j\, J_1(p_j r) \cos m_j z + A_{jj+1}\, m_{j+1}\, J_1(p_j r) \cos m_{j+1} z \right] e^{-i\omega t} \tag{1}$$

$$u_z = \sum_{j=-\infty}^{\infty} \left[A_{jj}\, m_j\, J_0(p_j r) \sin m_j z - A_{jj+1}\, p_j\, J_0(p_j r) \sin m_{j+1} z \right] e^{-i\omega t} \tag{2}$$

where A_{jj} and A_{jj+1} are arbitrary constants; p_j, m_j, m_{j+1} are propagation constants; and ω is angular frequency.

Note that the displacements are symmetrical about the axis of the cylinder, and are also symmetrical about the midplane of the cylinder. The latter property could as well have been made antisymmetric, in which case the displacements would describe the flexural modes.

Angular frequency ω in Eqs. (1) and (2) is related to propagation constants, density ρ and Lamé constants λ and μ by

$$\frac{\rho\omega^2}{\lambda + 2\mu} = (p_j^2 + m_j^2) = \frac{\mu}{\lambda + 2\mu}(p_j^2 + m_{j+1}^2) \qquad -\infty < j < \infty \tag{3}$$

Propagation constants are solved from the following pair of simultaneous, transdendental equations

$$4 p_j^2 m_j^2 F(p_{j-1} a) + (p_{j-1}^2 - m_j^2)^2 F(p_j a) = 2 p_j^2 (p_{j-1}^2 + m_j^2) \tag{4}$$

$$4 p_j^2 m_{j+1}^2 f\left(m_j \frac{h}{2}\right) + (p_j^2 - m_{j+1}^2)^2 f\left(m_{j+1} \frac{h}{2}\right) = 0 \tag{5}$$

where $$F(q) = q\frac{J_o(q)}{J_1(q)} \tag{6}$$

and $$f(q) = q \tan q \quad . \tag{7}$$

THE STAIRCASE

The meaning of Eq. (3) is illustrated by Fig. 1 in which the two lines inclined 45° to the p^2 and m^2 axes represent the locus of points of the same, constant ω^2. The inner of the two lines is associated with the dilatational components of Eqs. (1) and (2), and these are the terms in A_{jj}; the outer line is associated with shear components. For Poisson's ratio of 0.3, the dilatation and shear intercepts on either of the axes stand in ratio of 2 to 7.

The staircase effect displayed by Fig. 1 is formed by connecting all points which are predicted by Eq. (3) after selecting a specific, allowed pair of p_j, m_k, as for example based upon the point $(p_o{}^2, m_o{}^2)$.

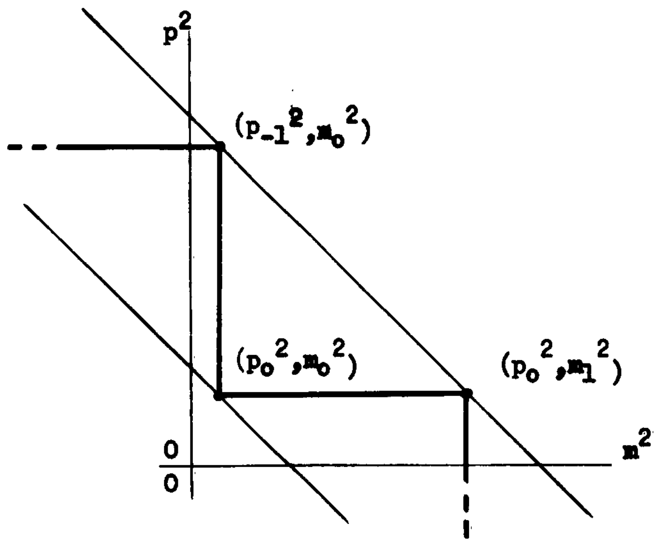

Figure 1. Equation (3) in p^2 — m^2 Space.

It is shown clearly by Fig. 1 that predicted pairs of p^2 and m^2 will consist of one positive and one negative value, or both will be positive. It is impossible for both members of the pair to be negative. Since the type solution for displacement is a Bessel-trigonometric product, a negative value makes one member of the product a hyperbolic function, recognizable as a Rayleigh-type wave.

A MATHEMATICAL DIFFICULTY

The illustrative point (p_o^2, m_o^2) is readily found from Eqs. (4) and (5). The writer assumed $p_o^2 = m_o^2 = 1$ and found $p_o a = 1.365$, $m_o h/2 = 1.0418$, and hence $h/2a = 0.764$. By Eq. (3), all values of p_j^2 and m_j^2 for $j \neq 0$ were now known. However, it was at once found that these other values did not simultaneously satisfy Eqs. (4) and (5) when the latter were written with $j \neq 0$. Therefore Eqs. (1)-(5) do not constitute a valid solution; there are too many equations for the number of unknowns.

COMPLEX PROPAGATION CONSTANTS

A complex propagation constant is admissible in Eq. (3) provided that frequency remains real. If $p_j = \alpha_j + i\beta_j$ and $m_j = \gamma_j + i\nu_j$ in Eqs. (1)-(5), the number of unknowns in the entire set of equations is doubled, and the number of equations is doubled. The mathematical impasse found with Eqs. (1)-(5) appeared to be unresolved by substitution of complex propagation constants.

A THIRD EXPRESSION FOR FREQUENCY

An alternative to the creation of more unknowns with which to validate Eqs. (3)-(6) is to terminate in a few terms the infinite series represented by Eqs. (1) and (2). It occurred to the writer that a third wave equation might lurk in the vector wave equation of elasticity in isotropic media,

$$(\lambda+2\mu)\nabla\nabla\cdot u - \mu\nabla\times\nabla\times u = \rho\frac{\partial^2}{\partial t^2} \tag{8}$$

Theorists in elasticity write of one compressional and two shear waves in any general discussion. The writer had only two.

An effort was mounted to see if Eq. (8) could possibly be converted to

$$\left[(\lambda+2\mu)\text{✡} - \rho\frac{\partial^2}{\partial t^2}\right]\left[\mu\text{✡} - \rho\frac{\partial^2}{\partial t^2}\right]^2 u = 0 \tag{9}$$

or into any 3-bracket operator. (The symbol ✡ in Eq. (9) is the vector Laplacian operator.) The result of this work was the following expression

$$\left[(\lambda+2\mu)\text{✡} - \rho\frac{\partial^2}{\partial t^2}\right]\left[\mu\text{✡} - \rho\frac{\partial^2}{\partial t^2}\right]\left[\mu\text{✡} + \rho\frac{\partial^2}{\partial t^2}\right] u = 0 \tag{10}$$

which produced a third line of constant frequency in Fig. 1. This third line made it possible to replace the infinite staircase with a closed figure involving six pairs of p^2, m^2 values. The series would then be terminated in 6 terms.

However, the third bracket would not satisfy the equation of motion from which it had come — unless $\omega^2 = 0$. Therefore the conversion of Eq. (9) into Eq. (10) had introduced multiplication by zero and was invalid. This was later shown to be the case. The direct conversion of Eq. (9) into wave equation factors can be accomplished by operating upon both sides of Eq. (9) with the factor

$$\left[\mu \nabla\nabla\cdot \; - (\lambda + 2\mu)\, \nabla\times\nabla\times \; - \rho \frac{\partial^2}{\partial t^2}\right]$$

whereupon one arrives directly at the usual

$$\left[(\lambda+2\mu) \star - \rho \frac{\partial^2}{\partial t^2}\right]\left[\mu \star - \rho \frac{\partial^2}{\partial t^2}\right] u = 0 \tag{11}$$

In performing this operation one needs the vector identities

$$\nabla\times\nabla\times\nabla\times\nabla\times A = \star^2 A - \nabla\nabla\cdot\nabla\nabla\cdot A \tag{12}$$

and

$$\nabla\nabla\cdot\nabla\nabla\cdot A = \nabla\nabla\cdot \star A = \star\, \nabla\nabla\cdot A \tag{13}$$

It has since been concluded that the second shear wave is missing from the present cylinder problem because of symmetry considerations. It would take the form of displacement in the θ direction both for a wave propagating radially and for one traveling parallel to the axis. However, u_θ is totally decoupled from u_r and u_z in the present simplified problem.

ADDITIONAL SOLUTIONS TO WAVE EQUATION

It is declared in Appendix A that the admissible functions for radial and axial displacements are

$$u_r = A\,J_1(pr)\cos mz\; e^{-i\omega t} = U_r\, e^{-i\omega t} \tag{14}$$

$$u_z = B\,J_0(pr)\sin mz\; e^{-i\omega t} = U_z\, e^{-i\omega t} \tag{15}$$

During this contract discovery has been made of other admissible solutions of the wave equation. The simplest of these, with time suppressed, are

$$U_r = C\left[mr\,J_0(pr)\cos mz + pz\,J_1(pr)\sin mz\right] \tag{16}$$

$$U_z = D\left[mr\,J_1(pr)\sin mz + pz\,J_0(pr)\cos mz\right] \tag{17}$$

and

$$U_r = E\left\{\left[(pz)^2+(mr)^2\right]J_1(pr)\cos mz + pr\,J_0(pr)\left(\cos mz - 2mz\sin mz\right)\right\} \tag{18}$$

$$U_z = F\left\{\left[(pz)^2+(mr)^2\right]J_0(pr)\sin mz - r\,J_1(pr)\left[(p^2+2m^2)\sin mz + 2p^2 mz\cos mz\right]\right\} \tag{19}$$

Substitution of Eqs. (16)-(19) in the equation of motion is illustrated in Appendix B.

It may be seen from Eqs. (14)-(19) that the general, two-displacement solution of the wave equation is

$$U_r = \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} r^{2n} z^{2q}\left\{\left[A_{2n,2q}J_1(pr) + A_{2n+1,2q}\,r\,J_0(pr)\right]\cos mz + \left[A_{2n,2q+1}J_1(pr) + A_{2n+1,2q+1}\,rJ_0(pr)\right]z\sin mz\right\} \tag{20}$$

$$U_z = \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} r^{2n} z^{2q}\left\{\left[B_{2n,2q}J_0(pr) + B_{2n+1,2q}\,rJ_1(pr)\right]\sin mz + \left[B_{2n,2q+1}J_0(pr) + B_{2n+1,2q+1}\,r\,J_1(pr)\right]z\cos mz\right\} \tag{21}$$

where Eqs. (14)-(19) are subsets of the general solution.

The arbitrary constants $A_{2n,2q}\cdots\cdots B_{2n+1,2q+1}$ will be determined by the constraints of the problem. From this point of view there are seven constraints, namely, the two vector wave equations, the vector equation of motion, and the four zero-stress conditions upon the flat and curved surfaces.

Considerable time during the contract period has been spent without significant progress in an effort to establish these constants. One of the results is that the constants A_{oo} and B_{oo} are independent of any other constants, which verifies the initial use of zero order terms in Eqs. (14) and (15). An insight which has been developed during this effort is that certain simplifications of the matrix of constants may be in order. Examples of these are rejection of constants with even-even subscripts, i.e., $A_{2n,2q} = B_{2n,2q} = 0$; of constants with odd-odd subscripts, i.e., $A_{2n+1,2q+1} = B_{2n+1,2q+1} = 0$; or of constants with subscript pairs in which the two indices differ by more than four or perhaps three, e.g., $A_{2n,2q} = \cdots\cdots = A_{2n+1,2q+1} = 0$ for $(2n - 2q)^2 > 16$ or $(2n - 2q)^2 > 9$, etc. No reliable guidelines to such rejection have as yet been discovered.

THE CALCULATION OF PAIRS OF (p_o^2, m_o^2)

In Appendix B are shown the three simplest forms of radial and axial displacement (through increasing n and q) after substitution in the equation of motion. Both dilatation and shear forms are presented. When these are subjected to the stress conditions, there results from the pair containing the arbitrary constant A the two transcendental Eqs. (4) and (5) as seen earlier. With some surprise it was found that Eq. (5) again emerged from the flat-surface conditions when they are applied to the pair in arbitrary constant B. Still a third time does Eq. (5) evolve from the flat-surface conditions when they are applied to the pair in arbitrary constant C.

Although the curved-surface conditions have not up to this time been applied to the second and third simplest forms of displacement, it was felt that the unfailing appearance of Eq. (5) might have significance, that it might be saying that this pair of equations, Eq. (4) and (5) when restricted to $j = 0$, are correct in predicting frequencies; and that there remains for the future only the hurdle of decoupling from the $j = 0$ case, through better choice of functions for displacement, the pairs of transcendental equations for $j \neq 0$.

Accordingly an exploratory computation of the roots of Eq. (4) and (5) for $j = 0$ has been made. The results of this effort are presented tabularly in Table I and graphically in Figs. 2 and 3. Plotted points are calculated values from which curves have been generated.

The curves in Fig. 2 are smooth whereas those of Fig. 3 suggest that roots associated with more than one mode have been confused. More computation is desirable in order to provide (1) an unscrambling of roots in Fig. 3; (2) a wider range of $(m_o/p_o)^2 < 0$ in Fig. 3; (3) and higher order roots in both Figs. 2 and 3.

TABLE I. CALCULATED ROOTS OF EQS. (3)-(5) FOR j = 0

$\frac{m_o}{p_o}$	$\frac{m_o h}{2}$	$p_o a$	$\frac{h}{2a}$	$\frac{\rho}{\lambda+2\mu}(\omega a)^2$	$\frac{\rho}{\lambda+2\mu}(\frac{\omega h}{2})^2$
31.62	1.568	.0764	.649	5.843	
10.00	1.542	.2383	.647	5.747	
4.472	1.481	.504	.658	5.323	
3.162	1.423	.673	.668	4.994	
2.00	1.313	.936	.701	4.380	
1.414	1.192	1.152	.732	3.981	
1.00	1.042	1.365	.764	3.726	
0.775	0.9204	1.509	.788	3.635	
0.707	0.880	1.552	.801	3.62	
0.447	0.665	1.749	.850	3.662	
0.316	0.524	1.858	.889	3.80	
0.100	0.1931	2.009	.924	4.08	
0.0316	0.0627	2.041	.968	4.16	
0 to ± i	No roots				
-1.183i	9.775	3.633i	2.27	5.28	27.25
-1.125i	9.16	2.73i	2.74	3.73	28.0
-1.342i	6.57	1.82i	2.79	2.65	19.15
-1.378i	6.42	1.67i	2.78	2.51	19.52
-1.449i	6.19	1.42i	3.08	2.22	20.0
-1.528i	5.97	1.163i	3.36	1.81	20.4
-1.581i	5.88	1.08i	3.44	1.75	20.7
-1.631i	4.155	0.920i	2.77	1.48	11.0
-1.732i	4.03	0.765i	3.04	1.17	10.81
-2.000i	3.78	1.98i	0.956	11.76	10.72
-2.236i	3.67	1.62i	1.012	10.5	10.78

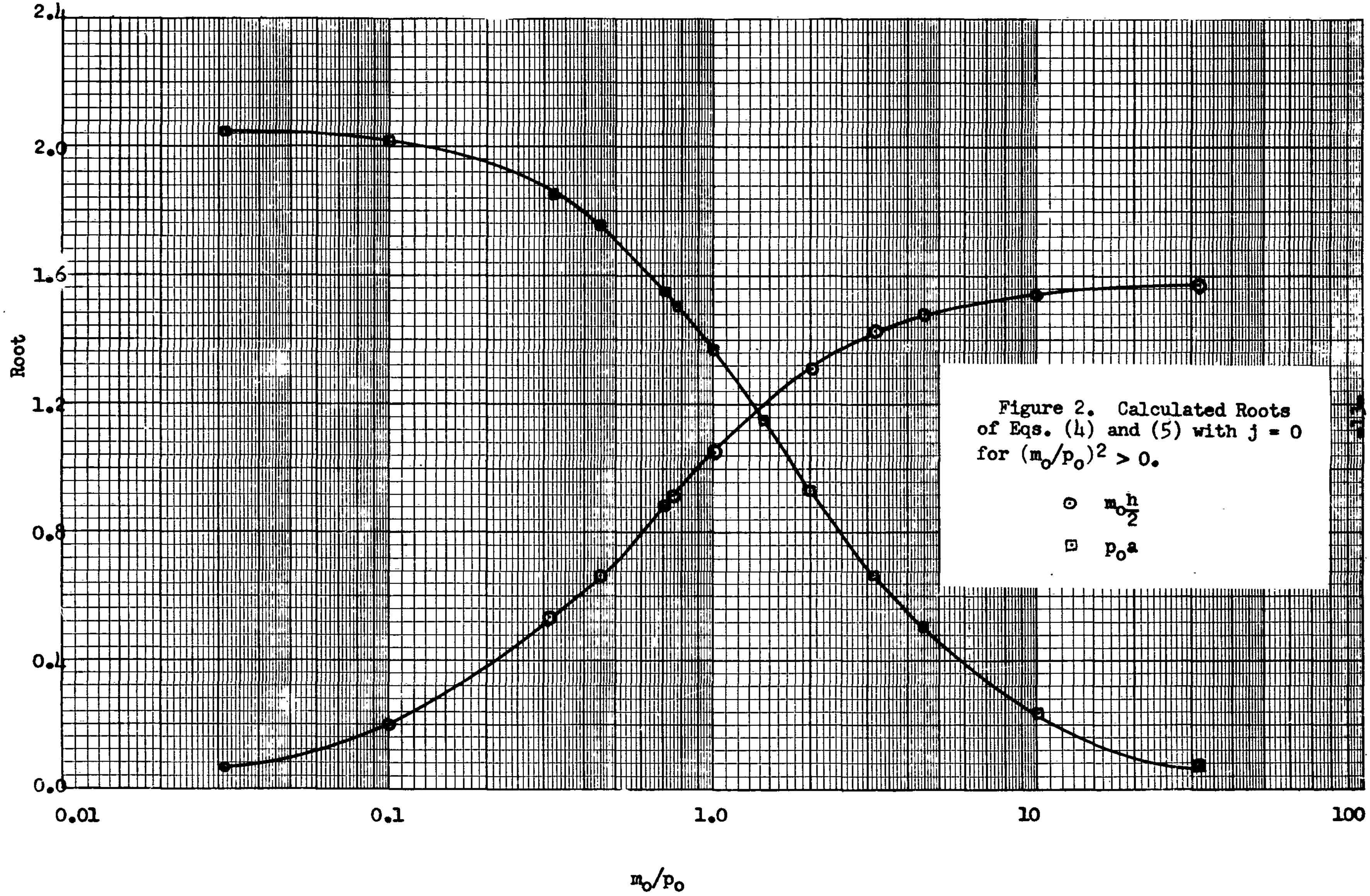

Figure 2. Calculated Roots of Eqs. (4) and (5) with $j = 0$ for $(m_o/p_o)^2 > 0$.

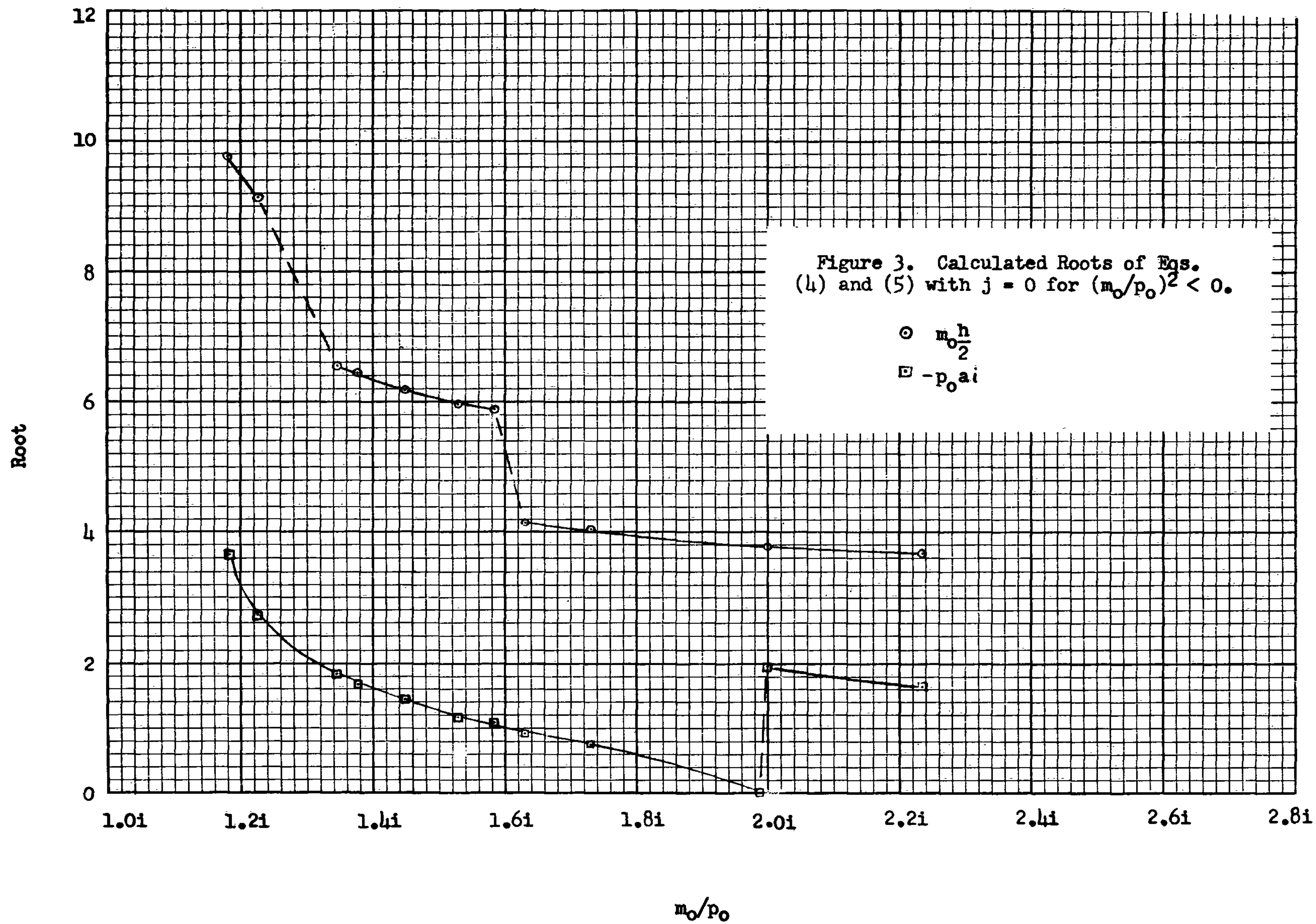

Figure 3. Calculated Roots of Eqs. (4) and (5) with j = 0 for $(m_o/p_o)^2 < 0$.

Because these computations were performed in the closing days of the contract, there has been scant opportunity to check them against the data of others. However, during the course of the contract the writer was invited to review a preliminary report[2] of a series of measurements on barium titanate cylinders. This report declares that the lowest frequency of vibration in fully polarized cylinders appears to obey the following relationships in which M_i and N_i are constants,

$$(fD)^2 = M_1\left[1 - M_2\left(\frac{L}{D}\right)^2\right] \qquad 0 < \left(\frac{L}{D}\right)^2 < 1 \tag{22}$$

and

$$(fL)^2 = N_1\left[1 - N_2\left(\frac{D}{L}\right)^2\right] \qquad 0 < \left(\frac{D}{L}\right)^2 < 1 \tag{23}$$

where D and L are 2a and h, respectively, in this writer's notation. Figs. 4 and 5 are displays of computed values arranged for qualitative comparison with Eqs. (22) and (23). The latter of course would be straight lines of negative slope if superposed on Figs. 4 and 5.

There is virtually no encouragement derivable from this comparison. However, in view of the limited range of h/2a covered by the computations, one cannot conclude for or against the theoretical equations, Eqs. (4) and (5), on the basis of this comparison.

[2] Work by G. K. Lucey, Jr., Diamond Ordnance Fuze Laboratory, Washington, D. C., publication pending.

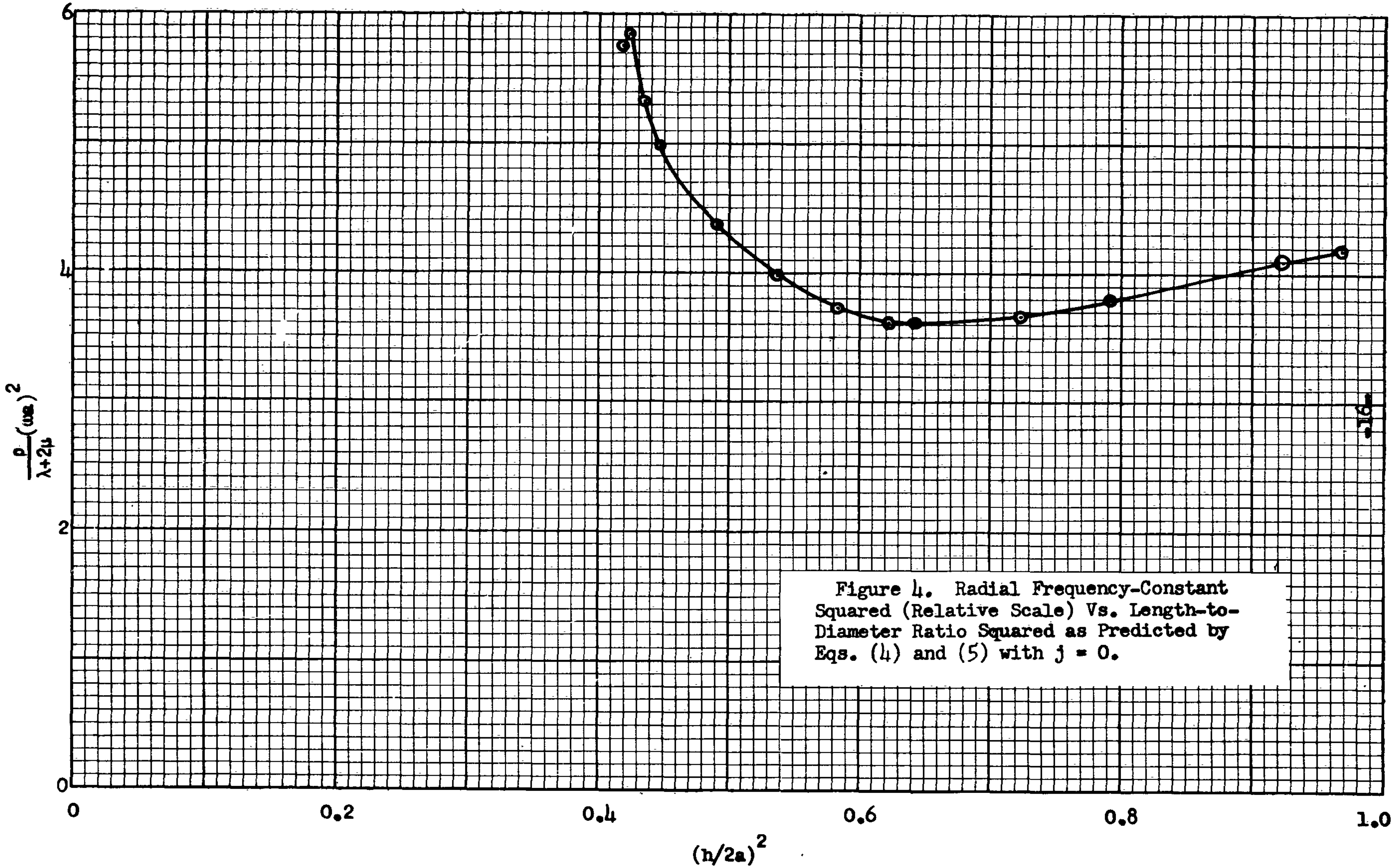

Figure 4. Radial Frequency-Constant Squared (Relative Scale) Vs. Length-to-Diameter Ratio Squared as Predicted by Eqs. (4) and (5) with j = 0.

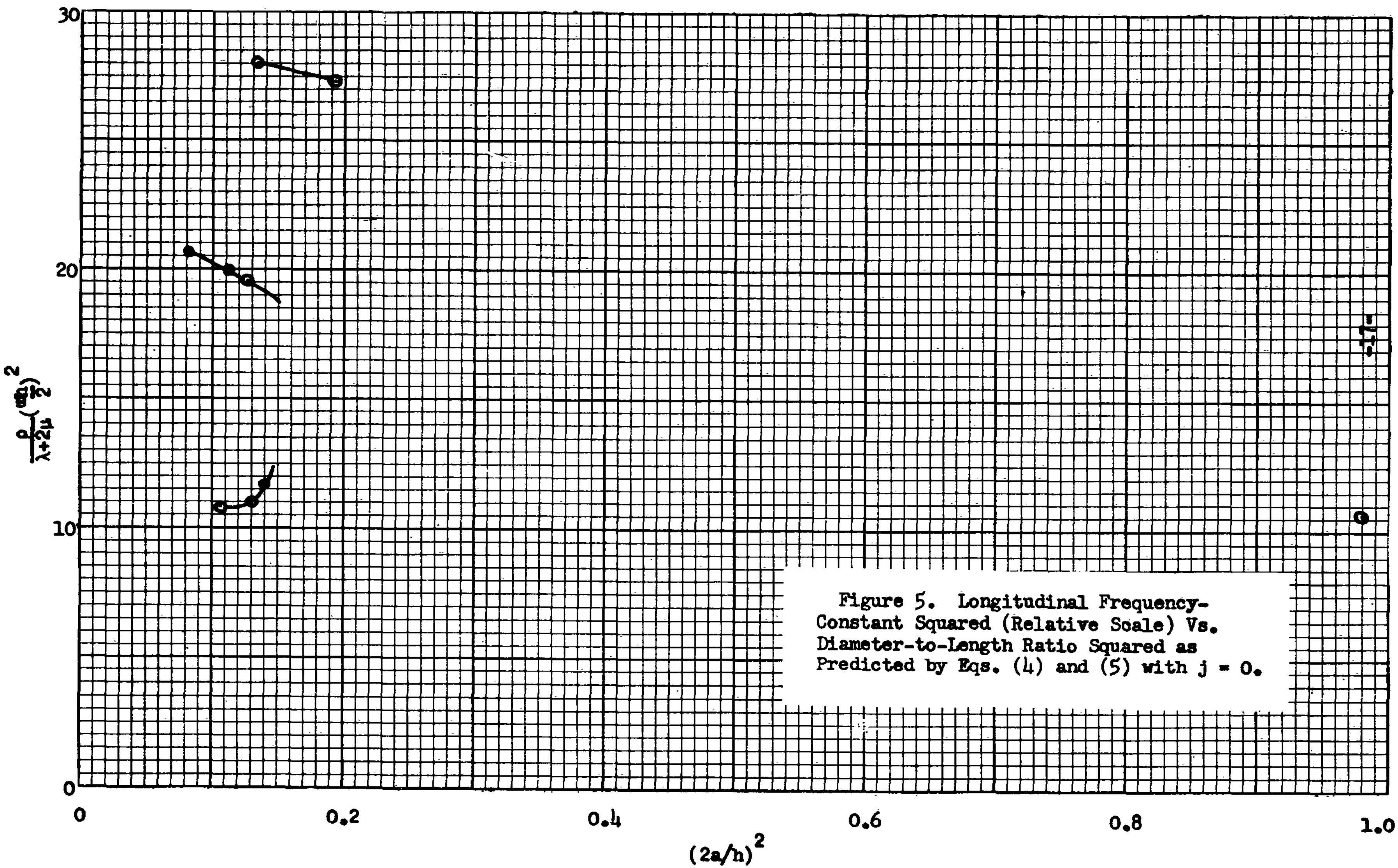

Figure 5. Longitudinal Frequency-Constant Squared (Relative Scale) Vs. Diameter-to-Length Ratio Squared as Predicted by Eqs. (4) and (5) with j = 0.

Section IV
DIRECTION OF FUTURE WORK

Three areas within the search for other solutions have been designated as incompletely tested. These are (1) the assumption that p_j and m_j are complex, (2) the discovery of additional solutions to the wave equation, and (3) a more comprehensive evaluation of Eqs. (4) and (5) for $j = 0$.

It is not possible to say which has the most promise. Further testing of (1) could probably be completed in the least time and would therefore be recommended for priority. It would be the choice of the writer to put second priority upon a part of (3), namely the evaluation of fundamentals sufficient to cover the full range of h/2a, before undertaking (2). The evaluation of overtones would be designated last, it being contingent upon earlier results.

The further approach to complex propagation constants would be illustrated by the following frequency and displacement equations, wherein the asterisk represents complex conjugate.

$$\frac{\rho\omega^2}{\lambda+2\mu} = p_j^2 + m_j^2 = p_j^{*2} + m_j^{*2} = \frac{\mu}{\lambda+2\mu}\left(p_j^2 + m_{j+1}^2\right) = \frac{\mu}{\lambda+2\mu}\left(p_j^{*2} + m_{j+1}^{*2}\right) \tag{24}$$

$$u_\lambda = \sum_{j=-\infty}^{\infty}\Big\{J_1(p_j\lambda)\left[A_{jj}\cos m_j z + A_{jj+1}\cos m_{j+1} z\right] + J_1(p_j^*\lambda)\left[C_{jj}\cos m_j^* z + C_{jj+1}\cos m_{j+1}^* z\right]\Big\}e^{-i\omega t} \tag{25}$$

$$u_z = \sum_{j=-\infty}^{\infty}\Big\{J_0(p_j\lambda)\left[B_{jj}\sin m_j z + B_{jj+1}\sin m_{j+1} z\right] + J_0(p_j^*\lambda)\left[D_{jj}\sin m_j^* z + D_{jj+1}\sin m_{j+1}^* z\right]\Big\}e^{-i\omega t} \tag{26}$$

It is felt that there are sufficient unknowns in Eqs. (24)-(26) to equal the number of equations if the arbitrary constants C and D are not complex conjugates of A and B respectively.

The approach to testing additional solutions of the wave equation for efficacy in solving the cylinder problem is not completely planned at this time. A first step would be to assemble three, say, pairs of displacement equations that would be next in increasing order of complexity beyond Eqs. (14)-(19). Since these six pairs are independent solutions, there are initially as many arbitrary constants as there are pairs chosen. The constraints of the problem would then be applied, hoping that the conditions could be satisfied and the problem solved.

If not, a second step is the search for a generating function by which the six pairs and all possible pairs can be predicted. The sum of all possible pairs would then be tried as a solution.

If the second step fails, a third step is a renewed assault upon the manipulation of the general equations which are Eqs. (20) and (21). For this there are no definite guidelines known at this time.

Section VI
ACKNOWLEDGEMENT

I count it a privilege to have been supported in the effort described in this report, and I hope that the material rendered here will be helpful to future work, whether performed by myself or others.

Donald S. Moseley

APPENDIX A

Natural Modes of an Isotropic Cylinder
A Mathematical Derivation

Let cylindrical coordinates be chosen such that the origin is at the center of the cylinder. The coordinates of the bounding surfaces are then given by $(r, \frac{h}{2})$, $(r, -\frac{h}{2})$ and (a, z).

Let the modes be restricted to those symmetric in r and z.

Notation, with the exception of the imaginary i, is that of Mason in "Piezoelectric Crystals and Their Application to Ultrasonics" (D. Van Nostrand Company, Inc., New York, 1950) p. 488 ff.

Within the isotropic solid the mechanical displacements must satisfy the equation

$$(\lambda + 2\mu)\nabla\nabla\cdot u - \mu\nabla \times \nabla \times u = \rho\, \partial^2 u/\partial t^2$$

and all stresses must be zero on the bounding surfaces.

Mathematically the problem is stated in Equations 1-8

(1) $(\lambda + 2\mu)\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r u_r + \mu \frac{\partial^2 u_r}{\partial z^2} - \rho \frac{\partial^2 u_r}{\partial t^2} + (\lambda+\mu)\frac{\partial^2 u_z}{\partial r \partial z} = 0$

(2) $(\lambda + 2\mu)\frac{\partial^2 u_z}{\partial z^2} + \mu \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u_z}{\partial r} - \rho \frac{\partial^2 u_z}{\partial t^2} + (\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u_r}{\partial z} = 0$

(3) u_r is bounded at $r=0, z=0$ for all t

(4) u_z is " " " " " " "

(5) $T_{rr} = 0$ at $r = a$

(6) $T_{rz} = 0$ at $r = a$

(7) $T_{zz} = 0$ at $z = \pm\frac{h}{2}$

(8) $T_{rz} = 0$ at $z = \pm\frac{h}{2}$

For convenience the following are given.

(9) $T_{rr} = (\lambda+2\mu)\frac{\partial u_r}{\partial r} + \lambda \frac{u_r}{r} + \lambda \frac{\partial u_z}{\partial z}$

(10) $T_{zz} = \lambda \frac{\partial u_r}{\partial r} + \lambda \frac{u_r}{r} + (\lambda+2\mu)\frac{\partial u_z}{\partial z}$

(11) $T_{rz} = \mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$

Equations (1) and (2) may be transformed to separate u_r from u_z:

$$(12)\quad \left[(\lambda+2\mu)\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2}\right) - \rho\frac{\partial^2}{\partial t^2}\right]$$
$$\left[\mu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2}\right) - \rho\frac{\partial^2}{\partial t^2}\right]u_r = 0$$

$$(13)\quad \left[(\lambda+2\mu)\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) - \rho\frac{\partial^2}{\partial t^2}\right]\cdot$$
$$\left[\mu\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) - \rho\frac{\partial^2}{\partial t^2}\right]u_z = 0$$

Solutions which satisfy (12) and (13) are

$$(14)\quad u_r = \begin{Bmatrix} J_1(pr) \\ N_1(pr) \end{Bmatrix}\begin{Bmatrix} \cos mz \\ \sin mz \end{Bmatrix} e^{-i\omega t}$$

$$(15)\quad u_z = \begin{Bmatrix} J_0(pr) \\ N_0(pr) \end{Bmatrix}\begin{Bmatrix} \sin mz \\ \cos mz \end{Bmatrix} e^{-i\omega t}$$

Of these the Neumann functions are rejected by (3), (4). For symmetry about $z=0$, take

$$(16)\quad u_r = A\, J_1(pr)\cos mz\, e^{-i\omega t}$$
$$(17)\quad u_z = B\, J_0(pr)\sin mz\, e^{-i\omega t}$$

Putting (16) and (17) in (12) and (13), one finds two expressions for frequency, namely

$$(18)\quad \rho\omega^2 = (\lambda+2\mu)(p^2+m^2)$$

$$(19)\quad \rho\omega^2 = \mu\,(p^2+m^2)$$

Substituting in (1) and (2) the solutions of frequency given by (18), and by (19)

(20) $\frac{B}{A} = \frac{m}{p}$, $\frac{B}{A} = -\frac{p}{m}$, respectively.

Then one solution of the differential equations (1), (2), (12), (13) can be

$$(21) \quad u_r = J_1(pr)\left[A_1 p \cos m_1 z + A_2 m_2 \cos m_2 z\right] e^{-i\omega t}$$

$$(22) \quad u_z = J_0(pr)\left[A_1 m_1 \sin m_1 z - A_2 p \sin m_2 z\right] e^{-i\omega t}$$

Another solution can be

$$(23) \quad u_r = \cos mz \left[A_3 p_1 J_1(p_1 r) + A_4 m J_1(p_2 r)\right] e^{-i\omega t}$$

$$(24) \quad u_z = \sin mz \left[A_3 m J_0(p_1 r) - A_4 p_2 J_0(p_2 r)\right] e^{-i\omega t}$$

When one proceeds to form the stress relations using (21) and (22), or (23) and (24), one realizes the apparent futility of satisfying flat-surface conditions for all r and peripheral-surface conditions for all z.

The only answer is for each p to be linked with two m's and vice versa.

Therefore, take the following system of equations expressing the frequency:

(25) $$\frac{\rho\omega^2}{\lambda+2\mu} = \dots p_{-1}^2 + m_{-1}^2 = \frac{\mu}{\lambda+2\mu}\left(p_{-1}^2 + m_0^2\right) =$$

$$p_0^2 + m_0^2 = \frac{\mu}{\lambda+2\mu}\left(p_0^2 + m_1^2\right) = \dots =$$

Then displacements may be written as

(26) $$u_r = \sum_{-\infty}^{+\infty}\left(A_{jj}\, p_j\, J_1(p_j r)\cos m_j z + A_{j\,j+1}\, m_{j+1}\, J_1(p_j r)\cos m_{j+1} z\right)e^{-i\omega t}$$

(27) $$u_z = \sum_{-\infty}^{+\infty}\left(A_{jj}\, m_j\, J_0(p_j r)\sin m_j z - A_{j\,j+1}\, p_j\, J_0(p_j r)\sin m_{j+1} z\right)e^{-i\omega t}$$

Application of conditions (5) and (6) leads to

(28) $$4 p_j^2 m_j^2\, F(p_{j-1} a) + (p_{j-1}^2 - m_j^2)^2 F(p_j a) = 2\, p_j^2 (p_{j-1}^2 + m_j^2)$$

Application of conditions (7) and (8) leads to

(29) $$\frac{f\left(m_{j+1}\frac{h}{2}\right)}{f\left(m_j\frac{h}{2}\right)} = -\frac{4 p_j^2 m_{j+1}^2}{(p_j^2 - m_{j+1}^2)^2}$$

In the above, $F(q) \equiv \dfrac{q\, J_0(q)}{J_1(q)}$ and $f(q) \equiv q \tan q$.

Frequency equation is Eq. (25), wherein the propagative constants p_0 and m_0 are defined through simultaneous equations (28) and (29).

Supplement to Appendix A

Equations (28) and (29) arise from direct substitution of (26) and (27) into (9), (10) and (11), after which conditions (5-8) are applied.

Putting (26) and (27) into (9), one obtains

$$T_{rr} = \sum_{-\infty}^{+\infty} \Big[A_{jj} \Big\{ \big[(\lambda+2\mu) p_j^2 + \lambda m_j^2 \big] J_0(p_j r) - 2\mu p_j \frac{J_1(p_j r)}{r} \Big\} \cos m_j z + A_{jj+1}\, 2\mu\, m_{j+1} \Big\{ p_j J_0(p_j r) - \frac{J_1(p_j r)}{r} \Big\} \cos m_{j+1} z \Big] e^{-i\omega t} \qquad (9\text{-}1)$$

Condition (5) requires that $T_{rr} = 0$ at $r = a$ for all z. We note that the A_{j-1j} and A_{jj} terms have $(\cos m_j z) e^{-i\omega t}$ as a common factor, and therefore the coefficient of this factor can be equated to zero. The result is a set of equations of which the following is representative:

$$A_{j-1j} \Big\{ 2 m_j \Big[p_{j-1} J_0(p_{j-1} a) - \frac{J_1(p_{j-1} a)}{a} \Big] \Big\} + A_{jj} \Big\{ (p_{j-1}^2 - m_j^2) J_0(p_j a) - 2 p_j \frac{J_1(p_j a)}{a} \Big\} = 0 \qquad (5\text{-}1)$$

Note that $(\lambda+2\mu) p_j^2 + \lambda m_j^2$ has been replaced with its equivalent, $\mu(p_{j-1}^2 - m_j^2)$, from Eq. (25).

Putting (26) and (27) into (11), one obtains

$$(11\text{-}1)\quad T_{rz} = \sum_{-\infty}^{+\infty} \mu \Big[2A_{jj}\, m_j\, p_j\, J_1(p_j r) \sin m_j z \; - \; A_{jj+1}(m_{j+1}^2 - p_j^2)\, J_1(p_j r) \sin m_{j+1} z \Big] e^{-i\omega t}$$

Condition (6) requires that $T_{rz} = 0$ at $r = a$ for all z. We note that the A_{j-1j} and A_{jj} terms have $(\sin m_j z)e^{-i\omega t}$ as a common factor, and therefore the coefficient of this factor can be equated to zero. The result is a second set of equations in A_{j-1j} and A_{jj} of which the following is representative:

$$(6\text{-}1)\qquad A_{j-1j}(m_j^2 - p_{j-1}^2)\, J_1(p_{j-1} a) + 2A_{jj}\, m_j\, p_j\, J_1(p_j a) = 0$$

Elimination of A_{j-1j} and A_{jj} from equations (5-1) and (6-1) leads directly to

$$(5\text{-}6)\qquad 4 m_j^2 p_j J_1(p_j a)\Big[p_{j-1} J_0(p_{j-1} a) - \frac{J_1(p_{j-1} a)}{a} \Big] = (m_j^2 - p_{j-1}^2) J_1(p_{j-1} a)\Big[(p_{j-1}^2 - m_j^2) J_0(p_j a) - 2 p_j \frac{J_1(p_j a)}{a} \Big],$$

which is easily rearranged to be equation (28).

Putting (26) and (27) into (10), one obtains

$$(10\text{-}1)\quad T_{zz} = \sum_{-\infty}^{+\infty} \Big\{ A_{jj}\big[\lambda p_j^2 + (\lambda + 2\mu) m_j^2\big] \cos m_j z \; - \; A_{jj+1}\, 2\mu\, p_j\, m_{j+1} \cos m_{j+1} z \Big\} J_0(p_j r)\, e^{-i\omega t}$$

Condition (7) requires that $T_{zz} = 0$ at $z = \pm \frac{h}{2}$ for all r. Imposing this condition upon equation (10-1) produces a set of equations in A_{jj} and A_{jj+1} of which the following is representative:

(7-1) $$A_{jj}(m_{j+1}^2 - p_j^2)\cos m_j \frac{h}{2} - A_{jj+1}\, 2\, p_j m_{j+1} \cos m_{j+1}\frac{h}{2} = 0$$

Note that $\lambda p_j^2 + (\lambda + 2\mu) m_j^2$ has been replaced by its equivalent, $\mu(m_{j+1}^2 - p_j^2)$, from equation (25).

Condition (8) requires that $T_{rz} = 0$ at $z = \pm \frac{h}{2}$ for all r. Imposing this condition upon equation (11-1) produces a second set of equations in A_{jj} and A_{jj+1} of which the following is representative:

(8-1) $$A_{jj}\, 2\, m_j p_j \sin m_j \frac{h}{2} + A_{jj+1}(m_{j+1}^2 - p_j^2)\sin m_{j+1}\frac{h}{2} = 0$$

Elimination of A_{jj} and A_{jj+1} from equations (7-1) and (8-1) leads directly to

$$4 p_j^2 m_j m_{j+1} \sin m_j \frac{h}{2} \cos m_{j+1}\frac{h}{2} + (p_j^2 - m_{j+1}^2)^2 \sin m_{j+1}\frac{h}{2} \cos m_j \frac{h}{2} = 0,$$

which is easily rearranged to be equation (29).

APPENDIX B

Additional Solutions

To formulate displacements which satisfy the equations of motion, Eqs. (1) and (2) from Appendix A, one must use combinations of Eqs. (14)--(19) from the text.

Let U_{rd} and U_{zd} be the radial and axial components of dilatational displacement, and let U_{rs} and U_{zs} be the radial and axial components of shear displacement, respectively, with time suppressed. In order of increasing degree in r and z, the displacements are*

$$U_{rd} = p A_d J_1 \cos$$

and

$$U_{zd} = m A_d J_0 \sin \quad ,$$

$$U_{rs} = m A_s J_1 \cos$$

and

$$U_{zs} = -p A_s J_0 \sin \quad ;$$

$$U_{rd} = p B_d [m(mr J_0 \cos + pz J_1 \sin) + p J_1 \cos]$$

and

$$U_{zd} = -m B_d [m(mr J_1 \sin + pz J_0 \cos)] \quad ,$$

$$U_{rs} = B_s [pm(mr J_0 \cos + pz J_1 \sin) - (p^2 + 2m^2) J_1 \cos]$$

and

$$U_{zs} = B_s [p^2(mr J_1 \sin + pz J_0 \cos)] \; ;$$

* Arguments pr and mz omitted for brevity.

$$U_{rd} = p^2 C_d \Big\{ m^2 \left[(pz)^2 + (mr)^2 \right] J_1 \cos - m^2 p r J_0 (\cos + 2mz \sin) - 2p^2 J_1 (\cos + mz \sin) \Big\}$$

and

$$U_{zd} = m^3 C_d \Big\{ p \left[(pz)^2 + (mr)^2 \right] J_0 \sin - r J_1 \left[(p^2 + 2m^2) \sin + 2p^2 m z \cos \right] \Big\},$$

$$U_{rs} = C_s \Big\{ mp \left[(pz)^2 + (mr)^2 \right] J_1 \cos + + m r J_0 \left[(3p^2 + 4m^2) \cos - 2p^2 m z \sin \right] + pz J_1 \sin \Big\}$$

and

$$U_{zs} = -C_s \Big\{ p^2 \left[(pz)^2 + (mr)^2 \right] J_0 \sin - pr J_1 \left[(p^2 + 2m^2) \sin + 2p^2 m z \cos \right] 2(3p^2 + 4m^2) J_0 \sin \Big\} .$$