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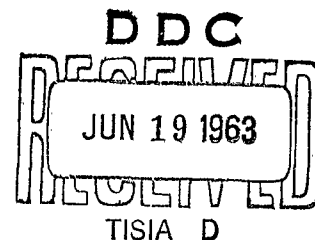
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# THE NUMERICAL SOLUTION OF THE SINGULAR INTEGRAL EQUATION FOR DIFFRACTION BY A SOFT STRIP

## 1 Introduction

Very few problems involving scattering of a wave by an obstacle can be solved exactly. The object of the present paper is to examine a method for numerical solution of the integral equation for diffraction by an obstacle for which solutions are available, namely a strip, with a view to the eventual application of the method to scattering by obstacles of more general shape.

We consider the problem of scalar two-dimensional diffraction of the incident wave

$$\phi_i = e^{ik(x \cos \Theta + y \sin \Theta)},$$

by a soft strip lying in  $y = 0$ ,  $-a \leq x \leq a$ . It is required to solve the steady-state wave equation

$$\frac{\partial^2 \phi_t}{\partial x^2} + \frac{\partial^2 \phi_t}{\partial y^2} + k^2 \phi_t = 0,$$

where  $\phi_t = \phi_i + \phi$ , is the total velocity potential consisting of the sum of the incident and scattered potentials  $\phi_i$  and  $\phi$  respectively. The scattered potential must satisfy a radiation condition at infinity. The boundary condition on the strip is

$$\begin{aligned} \phi_t &= 0, & y &= 0, \quad -a \leq x \leq a, \\ \text{or} \quad \phi &= -e^{ikx \cos \Theta}, & y &= 0, \quad -a \leq x \leq a. \end{aligned} \quad (1)$$

By superposition of simple sources of strength  $f(x)$  per unit length on the strip, the scattered potential at any point  $(x, y)$  of space is given by

$$\phi = \frac{i}{4} \int_{-a}^a f(\xi) H_0^{(1)} \{k[(x - \xi)^2 + y^2]^{\frac{1}{2}}\} d\xi. \quad (2)$$

If we let  $(x, y)$  tend to a point on the strip we obtain, on applying the boundary condition (1), an integral equation for the unknown function  $f(x)$ :

$$-\frac{i}{4} \int_{-a}^a f(\xi) H_0^{(1)}(k|x-\xi|) d\xi = e^{ikx \cos \Theta}, \quad (-a \leq x \leq a). \quad (3)$$

Of the voluminous literature on this subject we quote the exact Mathieu function solution of Morse and Rubinstein (7), the low-frequency expansions of Millar (5) using an integral equation method of Bouwkamp, and the high-frequency asymptotic expansions of Millar (4).

Low-frequency expansions have been obtained by a variational technique, using analytical methods, by De Hopp (3). An estimate of the scattering cross-section for normal incidence, using a crude constant approximation for the source-strength  $f(x)$  on the strip, has been given by Erdélyi and Papas (2). The present paper can be regarded as an extension of the method of Erdélyi and Papas using the method described in (8). The difficulty in applying a variational principle to scattering by obstacles with sharp edges is that the unknown functions have singularities at the sharp edges. If we attempt to use trial functions with the correct singularities at the edges we find that it is difficult to perform the necessary integrations. The integrations can be performed for constant trial functions, but these do not simulate the required singularities. In this paper we adopt the following compromise. To simulate the function

$$f(x) = 1/\sqrt{(a^2 - x^2)},$$

we use the approximation

$$f(x) = \begin{cases} \alpha & -p < x < p, \\ \beta & p < |x| < a, \end{cases}$$

where  $\alpha$ ,  $\beta$ ,  $p$  are determined from the variational principle. It is found by the method described later that typical values for normal incidence and low frequency are  $\beta/\alpha = 5$ ,  $p = 0.92$ .

## 2 The variational principle

Consider the variational expression

$$I(E, F) = \int_{\alpha}^{\beta} E(x) g(x) dx + \int_{\alpha}^{\beta} F(x) h(x) dx - \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} K(x, \xi) E(x) F(\xi) dx d\xi, \quad (4)$$

where  $g$ ,  $h$ ,  $K$  are known functions, and  $E$ ,  $F$  are to be determined so as to extremise  $I(E, F)$ . We set

$$E(x) = e(x) + \delta \eta(x), \quad F(x) = f(x) + \epsilon \zeta(x), \quad (5)$$

where we assume that  $e$ ,  $f$  are the functions which we wish to find, namely the functions which extremise  $I(E, F)$ . In the usual way  $\eta(x)$ ,  $\zeta(x)$  are arbitrary functions. On varying the parameters  $\delta$ ,  $\epsilon$  the functions  $E$ ,  $F$  vary round the required functions  $e$ ,  $f$ . On inserting (5) in (4) we see that

$$I(E, F) = I(e, f) + A + B,$$

where  $A$  is first order in  $\delta$  and  $\epsilon$ , and  $B$  is second order:

$$A = \epsilon \int_{\alpha}^{\beta} \zeta(x) \left\{ h(x) - \int_{\alpha}^{\beta} K(\xi, x) e(\xi) d\xi \right\} dx + \delta \int_{\alpha}^{\beta} \eta(x) \left\{ g(x) - \int_{\alpha}^{\beta} K(x, \xi) f(\xi) d\xi \right\} dx.$$

If  $I$  is stationary for variations of  $\epsilon$ ,  $\delta$  round zero, the coefficients of  $\epsilon$ ,  $\delta$  are separately zero. Since  $\zeta(x)$ ,  $\eta(x)$  are arbitrary,  $e(x)$ ,  $f(x)$  satisfy the integral equations

$$\int_{\alpha}^{\beta} K(x, \xi) f(\xi) d\xi = g(x), \quad (\alpha \leq x \leq \beta), \quad (6a)$$

$$\int_{\alpha}^{\beta} K(\xi, x) e(\xi) d\xi = h(x), \quad (\alpha \leq x \leq \beta). \quad (6b)$$

If  $e, f$  must satisfy these equations then we see from (4) that

$$I(e, f) = \int_{\alpha}^{\beta} e(x) g(x) dx = \int_{\alpha}^{\beta} h(x) f(x) dx. \quad (7)$$

Suppose that we can guess the shape of  $e(x), f(x)$  so that we can approximate  $e(x), f(x)$  by the expressions

$$E(x) = C \mathcal{E}(x), \quad F(x) = D \mathcal{F}(x),$$

where  $\mathcal{E}, \mathcal{F}$  are known functions, and  $C, D$  are constants which have to be determined. If these expressions are substituted in (4) we can determine optimum values for  $C$  and  $D$  by setting  $\frac{\partial I}{\partial C} = \frac{\partial I}{\partial D} = 0$ . This gives

$$C = \frac{(\mathcal{F}, h)'}{\Delta(\mathcal{E}, \mathcal{F})}, \quad D = \frac{(\mathcal{E}, g)}{\Delta(\mathcal{E}, \mathcal{F})},$$

$$I(E, F) = \frac{(\mathcal{E}, g)(\mathcal{F}, h)}{\Delta(\mathcal{E}, \mathcal{F})}, \quad (8)$$

where we use the notation

$$(p, q) = \int_{\alpha}^{\beta} p(x) q(x) dx,$$

$$\Delta(p, q) = \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} K(x, \xi) p(x) q(\xi) dx d\xi.$$

### 3 Application of the variational principle to diffraction by a strip

We now apply the results obtained in the last section to the problem of diffraction by a strip. By using the asymptotic expansion of the Hankel function in (2) it is readily seen that the far-field scattered by the strip is given by

$$\phi \sim A(\theta, \Theta) (8\pi kr)^{-\frac{1}{2}} e^{i(kr + \frac{1}{4}\pi)},$$

where

$$A(\theta, \Theta) = - \int_{-a}^a f(\xi) e^{-ik\xi \cos \theta} d\xi. \quad (9)$$

In these expressions we use polar coordinates  $(r, \theta)$  and write  $A(\theta, \Theta)$  to remind us that  $A$  depends on both the angle  $\theta$  at which we observe the diffracted wave, and the angle  $\Theta$  of incidence of the incident wave (since  $f(\xi)$  depends on  $\Theta$ ). As shown by several authors (for instance, de Hoop (3)) the scattering cross-section per unit width of the strip for a wave incident at angle  $\Theta$  is given by

$$t(\Theta) = - (1/4ka) \operatorname{Im} \{A(\theta, \Theta)\}. \quad (10)$$

We obtain a variational principle for  $A(\theta, \Theta)$  by first of all identifying the integral equation (3) with (6a). This indicates that  $f(\xi)$  can be taken as the same function in both equations and we can set

$$K(x, \xi) = -\frac{1}{4}i H_0^{(1)}(k|x - \xi|), \quad g(x) = e^{ikx \cos \Theta}. \quad (11a)$$

If now we identify minus the expression in (9) with (7) we see that

$$h(x) = e^{-ikx \cos \theta}, \quad (11b)$$

and from (6b) the function  $e(x)$  must satisfy the integral equation

$$-\frac{1}{4}i \int_{-a}^a e(\xi) H_0^{(1)}(k|x - \xi|) d\xi = e^{-ikx \cos \theta}. \quad (12)$$

From (7), (9) and the above identifications we see that the exact value of  $A(\theta, \Theta)$  is equal to  $-I(e, f)$  so that from (8) the variational expression for  $A(\theta, \Theta)$  is

$$A(\theta, \Theta) \approx \frac{\int_{-a}^a \mathcal{E}(x) e^{ikx \cos \Theta} dx \int_{-a}^a \mathcal{F}(x) e^{-ikx \cos \theta} dx}{\frac{1}{4}i \int_{-a}^a \int_{-a}^a H_0^{(1)}(k|x - \xi|) \mathcal{E}(x) \mathcal{F}(\xi) dx d\xi}.$$

This is the form of the variational principle used by other writers on this subject, for example Morse and Feshbach (6) and de Hoop (3).

For our purposes it is more convenient to proceed in the following way. Consider the integral equations

$$\int_{-a}^a K(|x - \xi|) f(\xi) d\xi = g(x), \quad (-a \leq x \leq a), \quad (13a)$$

$$\int_{-a}^a K(|x - \xi|) e(\xi) d\xi = h(x), \quad (-a \leq x \leq a), \quad (13b)$$

where we shall eventually make the identifications (11), with  $K(|x - \xi|) = K(x, \xi)$ . We introduce, using an obvious notation for even and odd functions,

$$f_+(\xi) = \frac{1}{2} \{f(\xi) + f(-\xi)\}, \quad g_+(x) = \frac{1}{2} \{g(x) + g(-x)\}, \quad (14a)$$

$$f_-(\xi) = \frac{1}{2} \{f(\xi) - f(-\xi)\}, \quad g_-(x) = \frac{1}{2} \{g(x) - g(-x)\}. \quad (14b)$$

On changing the sign of  $x$  in (13a) we obtain

$$\int_{-a}^a K(|x + \xi|) f(\xi) d\xi = g(-x), \quad (-a \leq x \leq a). \quad (15)$$

On adding and subtracting (13a) and (15) we readily obtain the following pair of integral equations which are together equivalent to (13a):

$$\int_0^a K_+(x, \xi) f_+(\xi) d\xi = g_+(x), \quad (16a)$$

$$\int_0^a K_-(x, \xi) f_-(\xi) d\xi = g_-(x), \quad (16b)$$

where

$$K_+(x, \xi) = K(|x - \xi|) + K(x + \xi), \quad (16c)$$

$$K_-(x, \xi) = K(|x - \xi|) - K(x + \xi). \quad (16d)$$

In an exactly similar way, equation (13b) can be replaced by the following pair, using notations analogous to (14):

$$\int_0^a K_+(x, \xi) e_+(\xi) d\xi = h_+(x) , \quad (17a)$$

$$\int_0^a K_-(x, \xi) e_-(\xi) d\xi = h_-(x). \quad (17b)$$

On comparing (16a), (17a) with (6a,b) we obtain from (4) a variational expression which we shall denote by  $I_+(E_+, F_+)$  where  $E_+$  and  $F_+$  are approximations to  $e_+$  and  $f_+$ . The exact value of this expression, from (7), (14), is:

$$\begin{aligned} I_+(e_+, f_+) &= \int_0^a e_+(x) g_+(x) dx = \int_0^a h_+(x) f_+(x) dx \\ &= \frac{1}{4} \int_0^a \{e(x) + e(-x)\} \{g(x) + g(-x)\} dx \\ &= \frac{1}{4} \int_{-a}^a \{e(x) g(x) + e(x) g(-x)\} dx. \end{aligned} \quad (18a)$$

Similarly from (16b), (17b) we can derive a variation expression which we denote by  $I_-(E_-, F_-)$  where  $E_-$ ,  $F_-$  are approximations to  $e_-$ ,  $f_-$ . The exact value of this expression is readily shown to be

$$I_-(e_-, f_-) = \frac{1}{4} \int_{-a}^a \{e(x) g(x) - e(x) g(-x)\} dx . \quad (18b)$$

On adding (18a,b), and remembering that  $A(\theta, \Theta)$  is equal to  $-I(e, f)$  whose value is given in (7) we see that

$$A(\theta, \Theta) = -2 \{I_+(e_+, f_+) + I_-(e_-, f_-)\} . \quad (19)$$

The variational expression for  $I_+$  is

$$\begin{aligned} I_+ &= \int_0^a E_+(x) g_+(x) dx + \int_0^a F_+(x) h_+(x) dx \\ &\quad - \int_0^a \int_0^a K_+(x, \xi) E_+(x) F_+(\xi) dx d\xi , \end{aligned} \quad (20a)$$

where from (11)

$$g_+(x) = \cos(kx \cos \Theta), \quad h_+(x) = \cos(kx \cos \Theta). \quad (20b)$$

From this point onwards it is convenient to set  $a = 1$ . We also substitute  $\kappa$  for  $k$  to remind us of this assumption.

We take the trial functions  $F_+$ ,  $E_+$  to be step-functions:

$$F_+(x) = \begin{cases} \alpha, & -p < x < p, \\ \beta, & p < |x| < 1, \end{cases} \quad E_+(x) = \begin{cases} \gamma, & -q < x < q, \\ \delta, & q < |x| < 1. \end{cases} \quad (21)$$

Substitution in (20a) gives

$$I_+ = \gamma D_1 + \delta D_2 + \alpha C_1 + \beta C_2 - \alpha \gamma A_{11} - \alpha \delta A_{12} - \beta \gamma A_{21} - \beta \delta A_{22} \quad (22)$$

where from (20b)

$$D_1 = \int_0^q g_+(x) dx = \frac{\sin(\kappa q \cos \Theta)}{\kappa \cos \Theta},$$

$$D_2 = \int_q^1 g_+(x) dx = \frac{\sin(\kappa \cos \Theta) - \sin(\kappa q \cos \Theta)}{\kappa \cos \Theta}.$$

Similarly

$$C_1 = \frac{\sin(\kappa p \cos \Theta)}{\kappa \cos \Theta}, \quad C_2 = \frac{\sin(\kappa \cos \Theta) - \sin(\kappa p \cos \Theta)}{\kappa \cos \Theta}.$$

Also

$$A_{11} = \int_0^p \int_0^q K_+(x, \xi) dx d\xi, \quad A_{12} = \int_0^p \int_q^1 K_+(x, \xi) dx d\xi,$$

$$A_{21} = \int_p^1 \int_0^q K_+(x, \xi) dx d\xi, \quad A_{22} = \int_p^1 \int_q^1 K_+(x, \xi) dx d\xi.$$

When  $K_+(x, \xi)$  has the special form given in (16c) these double integrals

can be simplified by means of the following results (cf. Noble (8)):

$$\int_a^b \int_c^d K(|x - \xi|) dx d\xi = J(|b - c|) + J(|a - d|) - J(|b - d|) - J(|a - c|),$$

$$\int_a^b \int_c^d K(x + \xi) dx d\xi = J(b + d) + J(a + c) - J(b + c) - J(a + d),$$

where

$$J(\alpha) = \int_0^\alpha (\alpha - u) K(u) du.$$

In our case  $K(u) = -\frac{1}{4i} H_0^{(1)}(\kappa u)$  and it is convenient to express all integrals in terms of

$$\begin{aligned} L(z) &= \int_0^z (z - v) H_0^{(1)}(v) dv \\ &= z \left[ \int_0^z J_0(u) du - J_1(z) \right] + i \left\{ z \left[ \int_0^z Y_0(u) du - Y_1(z) \right] - \frac{2}{\pi} \right\}. \end{aligned}$$

We have  $4i\kappa^2 J(\alpha) = L(\kappa\alpha)$ ,

and we find

$$\begin{aligned} 4i\kappa^2 A_{11} &= L\{\kappa(p + q)\} - L\{\kappa|p - q|\}, \\ 4i\kappa^2 A_{12} &= L\{\kappa(1 + p)\} - L\{\kappa(1 - p)\} + L\{\kappa|p - q|\} - L\{\kappa(p + q)\}, \\ 4i\kappa^2 A_{21} &= L\{\kappa(1 + q)\} - L\{\kappa(1 - q)\} + L\{\kappa|p - q|\} - L\{\kappa(p + q)\}, \\ 4i\kappa^2 A_{22} &= L\{2\kappa\} + L\{\kappa(p + q)\} - L\{\kappa|p - q|\} + L\{\kappa(1 - p)\} \\ &\quad - L\{\kappa(1 + p)\} + L\{\kappa(1 - q)\} - L\{\kappa(1 + q)\}. \end{aligned}$$

The optimum values of  $\alpha, \beta, \gamma, \delta$  in the variational expression (22) for  $I_+$  are determined by setting the derivatives of  $I_+$  with respect to these parameters equal to zero. This gives two sets of two simultaneous equations for  $\gamma, \delta$  and  $\alpha, \beta$ :

$$\begin{aligned} A_{11} \gamma + A_{12} \delta &= C_1, \\ A_{21} \gamma + A_{22} \delta &= C_2, \\ A_{11} \alpha + A_{21} \beta &= D_1, \\ A_{12} \alpha + A_{22} \beta &= D_2. \end{aligned}$$

If these values of  $\alpha, \beta, \gamma, \delta$  are used, equation (22) gives

$$\begin{aligned} I_+ &= \gamma D_1 + \delta D_2 = \alpha C_1 + \beta C_2 \\ &= (A_{22} C_1 D_1 - A_{12} C_2 D_1 - A_{21} C_1 D_2 + A_{11} C_2 D_2) / \Delta, \end{aligned} \quad (23)$$

where

$$\Delta = A_{11} A_{22} - A_{12} A_{21}.$$

Next consider the variational expression  $I_-$  derived from (16b), (17b):

$$I_- = \int_0^a E_-(x) g_-(x) dx + \int_0^a F_-(x) h_-(x) dx - \int_0^a \int_0^a K_-(x, \xi) E_-(x) F_-(\xi) dx d\xi, \quad (24)$$

where

$$g_-(x) = i \sin(kx \cos \Theta), \quad h_-(x) = -i \sin(kx \cos \Theta).$$

We again set  $a = 1$ , replace  $k$  by  $\kappa$ , and take the trial functions  $F_-$ ,  $E_-$  to be odd step-functions:

$$F_-(x) = \begin{cases} -\alpha, & -1 < x < -p, \\ 0, & |x| < p, \\ \alpha, & p < x < 1 \end{cases} \quad E_-(x) = \begin{cases} -\beta, & -1 < x < -q, \\ 0, & |x| < q, \\ \beta, & q < x < 1. \end{cases} \quad (25)$$

Then (24) becomes

$$I_- = \beta P + \alpha Q - \alpha \beta B \quad (26)$$

where

$$P = \frac{i \{ \cos(\kappa p \cos \Theta) - \cos(\kappa \cos \Theta) \}}{\kappa \cos \Theta}$$

$$Q = - \frac{i \{ \cos(\kappa p \cos \Theta) - \cos(\kappa \cos \Theta) \}}{\kappa \cos \Theta}$$

$$B = \int_p^1 \int_q^1 K_-(x, \xi) dx d\xi.$$

By following the procedure explained previously in connection with the  $A_{ij}$

we can show that

$$\begin{aligned} 4\kappa^2 B &= L\{\kappa(1+p)\} + L\{\kappa(1-p)\} + L\{\kappa(1+q)\} + L\{\kappa(1-q)\} \\ &\quad - L\{\kappa(p+q)\} - L\{\kappa|p-q|\} - L(2\kappa). \end{aligned}$$

On setting the derivatives of  $I_-$  with respect to  $\alpha, \beta$  equal to zero we find the optimum values of these parameters, and substitution in (26) gives

$$I_- = \beta P = \alpha Q = PQ/B. \quad (27)$$

#### 4 Numerical results for the scattering cross-section

The numerical results reported below are confined to the scattering cross-section per unit width of the strip,  $t(\Theta)$ , defined in (10) above. We then require only  $A(\Theta, \Theta)$  so that we can set  $\theta = \Theta, p = q$ , in the expressions in the last section. On using (10) and (19), remembering that we now set  $a = 1, k = \kappa$  in (10), we have

$$t(\Theta) = \frac{1}{2\kappa} \text{Im} (I_+ + I_-) = \text{Re}(J_+ + J_-), \text{ say,}$$

where we can show from (23), (27) that

$$\begin{aligned} J_+ &= -\frac{1}{2}(i/\kappa) I_+ \\ &= \frac{2\{L(2\kappa)\sin^2(\kappa p \cos \Theta) + 2M(\kappa, p)\sin(\kappa p \cos \Theta)\sin(\kappa \cos \Theta) + L(2\kappa p)\sin^2(\kappa \cos \Theta)\}}{\kappa \cos^2 \Theta [L(2\kappa) L(2\kappa p) - M^2(\kappa, p)]} \\ &= \frac{2}{\kappa \cos^2 \Theta L(2\kappa)} \left\{ \sin^2(\kappa \cos \Theta) + \frac{[L(2\kappa)\sin(\kappa p \cos \Theta) - M(\kappa, p)\sin(\kappa \cos \Theta)]^2}{L(2\kappa) L(2\kappa p) - M^2(\kappa, p)} \right\}, \end{aligned}$$

where we have separated out the dominant part of  $J_+$ , and

$$M(\kappa, p) = L\{\kappa(1 - p)\} - L\{\kappa(1 + p)\}. \text{ Also}$$

$$\begin{aligned} J_- &= -\frac{1}{2}(i/\kappa) I_- \\ &= \frac{2\{\cos(\kappa p \cos \Theta) - \cos(\kappa \cos \Theta)\}}{\kappa \cos^2 \Theta [2\{L\{\kappa(1+p)\} + L\{\kappa(1-p)\}\} - L(2\kappa) - L(2\kappa p)]}. \end{aligned}$$

The optimum value of  $p$ , which is so far undetermined, is found empirically by using the variational property of the two above expressions. Thus for a given value of  $\kappa$  and  $\Theta$ , the quantity  $J_+$  is evaluated for various  $p$ , and the required value of  $p$  is that for which  $J_+$  is stationary. A difficulty arises since  $J_+$  is complex and the optimum value of  $p$  may not be the

same for the real and imaginary parts. Fortunately it turns out that the optimum values of  $p$  are not essentially different when  $\kappa$  is not too large, as illustrated in a numerical example below. The optimum value of  $p$  need not of course be the same for  $J_+$  and  $J_-$  since these are obtained from independent integral equations.

The above calculations to determine the optimum value of  $p$  were carried out for various angles of incidence  $\Theta = 0(10)90$  degrees and  $\kappa = 0.4(0.4)2.8$ . The results were similar for all angles of incidence and figures for a typical example, namely  $\Theta = 50^\circ$ , are given in Tables 1-4. For small  $\kappa$  the optimum value of  $p$  for both the real and imaginary parts of  $J_+$  was approximately 0.92 as shown in Tables 1 and 3. The optimum value increases slightly as  $\kappa$  increases to 2.0, and then increases fairly rapidly to unity as  $\kappa$  increases from 2.0 to 3.0. The optimum value of  $p$  for the imaginary part of  $J_+$  increases slightly more rapidly than the optimum value for the real part but no significant error is involved in the assumption that  $p$  is a constant equal to 0.92 for all  $\kappa$  between 0 and 2.0. For small  $\kappa$  the optimum  $p$  for both the real and imaginary parts of  $J_-$  was about 0.68. The optimum value for the imaginary part increases, and for the real part it decreases, as  $p$  increases, but the estimates of the real and imaginary parts of  $J_-$  are not seriously in error if we assume  $p$  constant equal to 0.68 for all  $\kappa$  between 0 and 2.0.

Some typical results for  $t(\Theta)$  are plotted in Figures 1 and 2, where the optimum values of  $p$  for the real parts of  $J_+$  and  $J_-$  have been used. In Figure 1 our results for  $t(\Theta)$  are plotted against  $\kappa$  for  $\kappa = 0.4$  to 2.0, and  $\Theta = 0, 30, 60^\circ$ . Also in Figure 1, for the sake of comparison, are the exact

TABLE 1

Values of  $\text{Re}[J_+]$  for  $\Theta = 50^\circ$ ,  $\kappa = 0.4$  to  $2.8$ ,  $p = 0.88$  to  $0.96$

| $\kappa \backslash p$ | 0.88   | 0.90   | 0.92   | 0.94   | 0.96   |
|-----------------------|--------|--------|--------|--------|--------|
| 0.4                   | 1.1278 | 1.1284 | 1.1286 | 1.1281 | 1.1265 |
| 0.8                   | 0.8824 | 0.8829 | 0.8831 | 0.8828 | 0.8817 |
| 1.2                   | 0.7362 | 0.7365 | 0.7367 | 0.7367 | 0.7362 |
| 1.6                   | 0.6094 | 0.6095 | 0.6097 | 0.6098 | 0.6097 |
| 2.0                   | 0.4933 | 0.4932 | 0.4930 | 0.4930 | 0.4930 |
| 2.4                   | 0.3932 | 0.3924 | 0.3916 | 0.3907 | 0.3898 |
| 2.8                   | 0.3161 | 0.3143 | 0.3121 | 0.3094 | 0.3060 |

TABLE 2

Values of  $\text{Re}[J_-]$  for  $\Theta = 50^\circ$ ,  $\kappa = 0.4$  to  $2.8$ ,  $p = 0.62$  to  $0.70$

| $\kappa \backslash p$ | 0.62    | 0.64    | 0.66    | 0.68    | 0.70    |
|-----------------------|---------|---------|---------|---------|---------|
| 0.4                   | 0.00559 | 0.00561 | 0.00562 | 0.00563 | 0.00562 |
| 0.8                   | 0.0350  | 0.0351  | 0.0352  | 0.0352  | 0.0351  |
| 1.2                   | 0.0909  | 0.0911  | 0.0912  | 0.0912  | 0.0910  |
| 1.6                   | 0.1656  | 0.1659  | 0.1659  | 0.1657  | 0.1652  |
| 2.0                   | 0.2486  | 0.2488  | 0.2485  | 0.2479  | 0.2468  |
| 2.4                   | 0.3292  | 0.3289  | 0.3279  | 0.3262  | 0.3238  |
| 2.8                   | 0.3970  | 0.3950  | 0.3920  | 0.3877  | 0.3820  |

TABLE 3

Values of  $\text{Im}[J_+]$  for  $\Theta = 50^\circ$ ,  $\kappa = 0.4$  to  $2.0$ ,  $p = 0.88$  to  $0.96$

| $\kappa \backslash p$ | 0.88    | 0.90    | 0.92    | 0.94    | 0.96    |
|-----------------------|---------|---------|---------|---------|---------|
| 0.4                   | -1.2512 | -1.2513 | -1.2513 | -1.2512 | -1.2509 |
| 0.8                   | -0.5909 | -0.5906 | -0.5905 | -0.5905 | -0.5907 |
| 1.2                   | -0.3078 | -0.3074 | -0.3071 | -0.3070 | -0.3073 |
| 1.6                   | -0.1541 | -0.1536 | -0.1532 | -0.1529 | -0.1529 |
| 2.0                   | -0.0741 | -0.0737 | -0.0733 | -0.0728 | -0.0724 |

TABLE 4

Values of  $\text{Im}[J_-]$  for  $\Theta = 50^\circ$ ,  $\kappa = 0.4$  to  $2.0$ ,  $p = 0.62$  to  $0.70$

| $\kappa \backslash p$ | 0.62    | 0.64    | 0.66    | 0.68    | 0.70    |
|-----------------------|---------|---------|---------|---------|---------|
| 0.4                   | -0.1080 | -0.1082 | -0.1083 | -0.1084 | -0.1084 |
| 0.8                   | -0.1918 | -0.1922 | -0.1924 | -0.1926 | -0.1925 |
| 1.2                   | -0.2486 | -0.2492 | -0.2496 | -0.2499 | -0.2500 |
| 1.6                   | -0.2779 | -0.2788 | -0.2796 | -0.2803 | -0.2808 |
| 2.0                   | -0.2814 | -0.2830 | -0.2846 | -0.2862 | -0.2878 |

solutions of Morse and Rubinstein (7) for  $\Theta = 0^\circ$ , and the curves of Millar (4) and (5). It should be noted that Millar's curves consist of a low-frequency approximation extending up to  $\kappa \approx 1$ , and a high-frequency approximation extending down to  $\kappa \approx 1.25$ . These two approximations do not match exactly. In Figure 2 our results for  $t(\Theta)$  are plotted against  $\kappa$  going from  $\kappa = 0.4$  to 4.0 for normal incidence ( $\Theta = 90^\circ$ ). Also included are the exact solutions of Morse and Rubinstein, Millar's solutions for normal incidence, and the solution of Erdélyi and Papas (2). Our results agree well with the exact solution of Morse and Rubinstein for  $0.4 < \kappa < 3$  in Figure 2 and  $0.4 < \kappa < 2$  in Figure 1. Our results are also in fair agreement with those of Millar, the agreement being worst at grazing incidence where in any case Millar's low- and high-frequency approximations do not join smoothly in the region  $1 < \kappa < 1.25$ .

The solution of Erdélyi and Papas for normal incidence given in Figure 2 is obtained from the variational principle which is used in this paper but a trial function is assumed which is simply constant over the whole strip. The trial function of Erdélyi and Papas is obtained by taking

$$p = q = 1, \quad F_+(x) = E_+(x) = \alpha, \quad (-1 < x < 1)$$

in equation (21) above. We have already mentioned that the optimum value of  $p$  for the even solution tended to unity as  $\kappa$  increases from 2.0 to 3.0, so that the graph of our results in Figure 2 tends to the solution of Erdélyi and Papas for  $\kappa > 3.0$ . Unfortunately the asymptotic behaviour of the Erdélyi and Papas solution does not agree with that of the exact solution as  $\kappa$  tends to infinity. However our results show a considerable

Figure 1, 10

Scattering cross-section  $t(\Theta)$  as a function of angle of incidence  $\Theta = 0(30)60^\circ$   
and  $k = ka$ .

— Present Theory.  
--- Millar.  
--- Morse and Rubinstein.

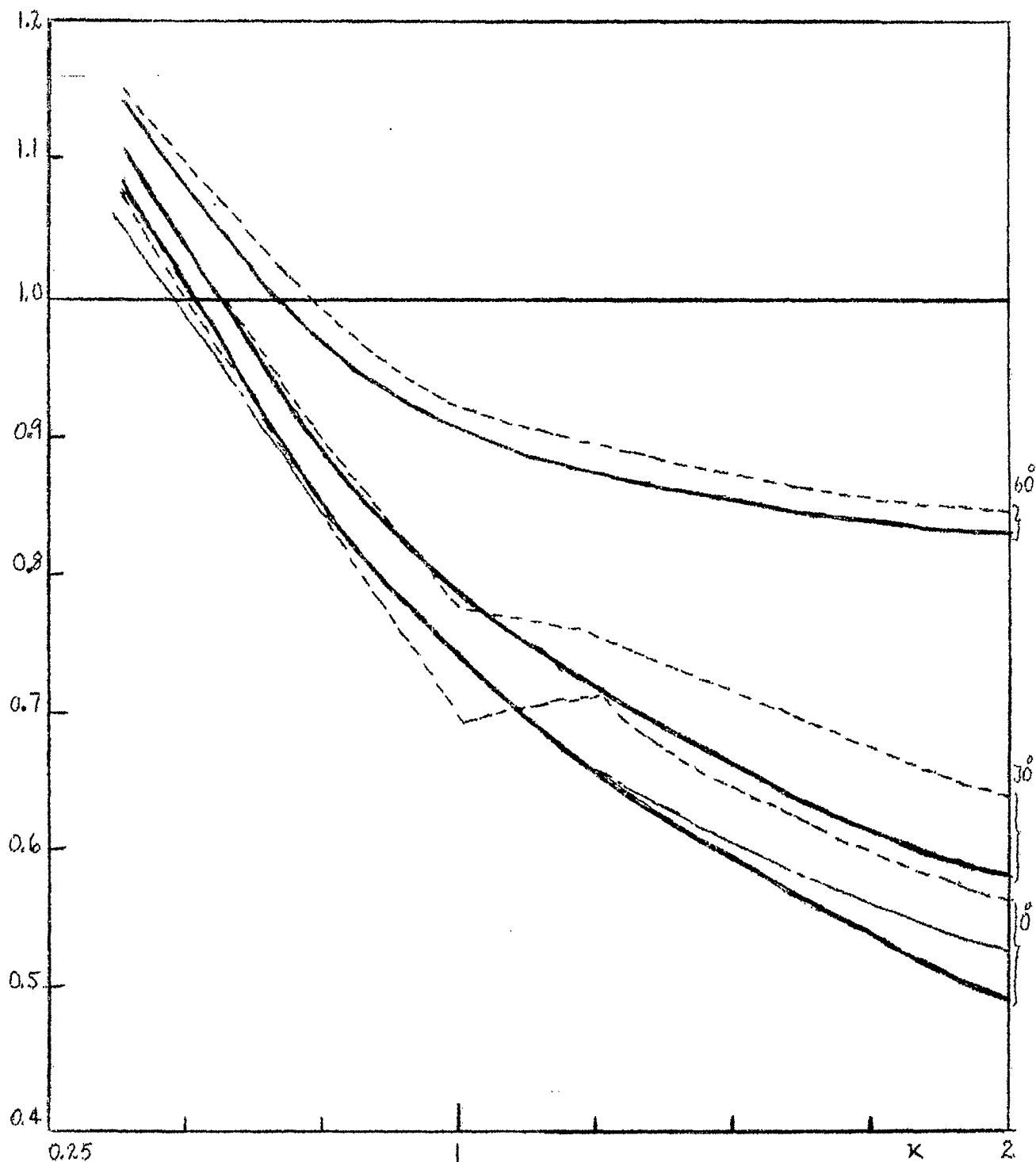
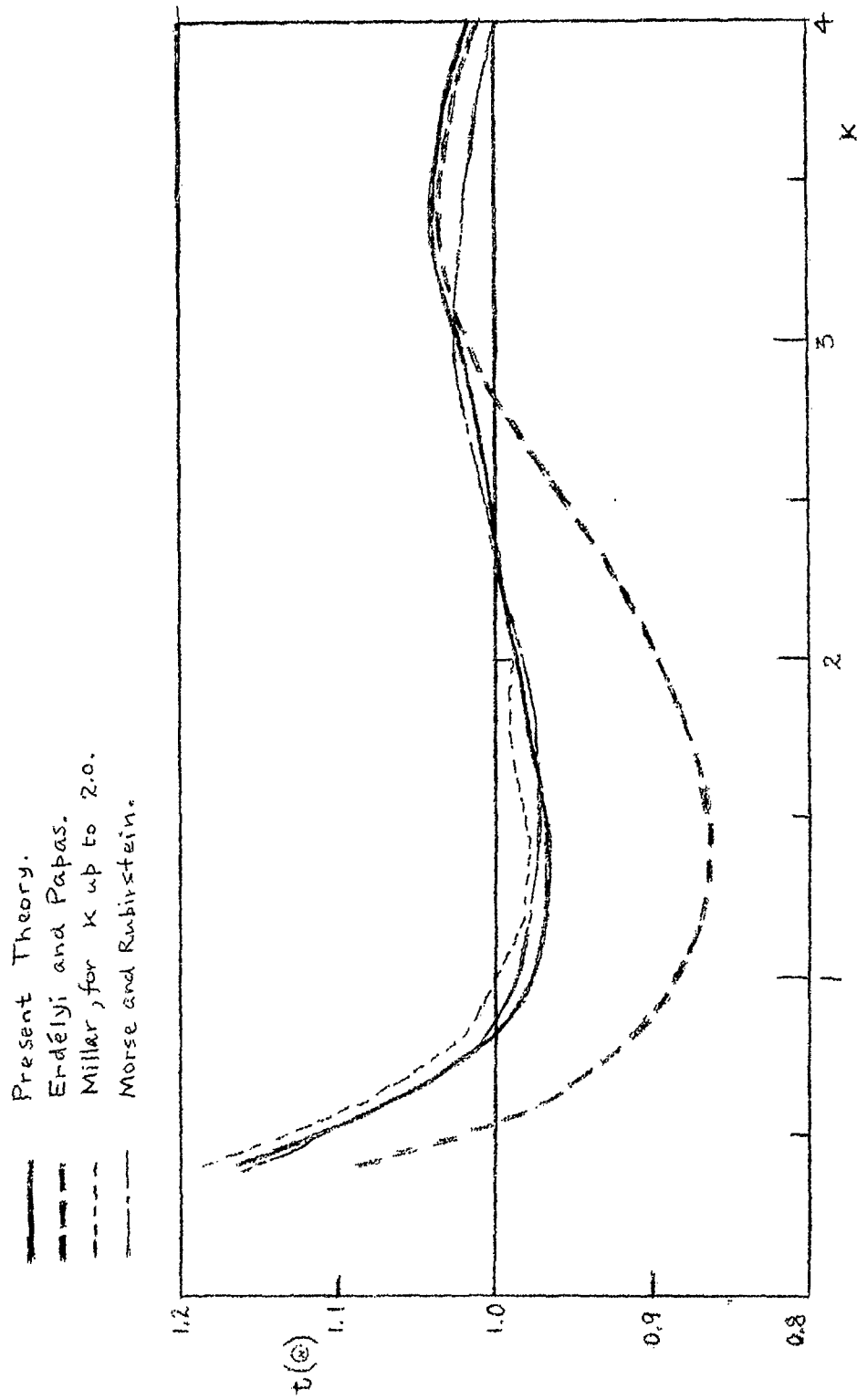


Figure 2.  
Scattering cross-section  $t(\infty)$  as a function of  $k=ka$ , for normal incidence.



improvement over those of Erdélyi and Papas for  $\kappa$  less than about 3.0.

The trial functions used in this paper are quite crude in that they are step-functions with, effectively, only one step. In spite of this the results are remarkably good. It is also noteworthy that empirically it appears that no great loss of accuracy is involved in computing the scattering cross-section if it is assumed that for all  $\theta$  we have  $p = q = 0.92$  for the even function (21) and  $p = q = 0.68$  for the odd function (25), in the range  $0 < \kappa < 2.5$ . It is the chief conclusion of this paper that good results can be obtained with these simple trial functions. This encourages the hope that it may be possible to apply the same method to calculate the scattering by objects of more complicated shapes in the awkward wavelength region where the wavelength is comparable with the size of the obstacle.

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| <p>THE ROYAL COLLEGE OF<br/>SCIENCE AND TECHNOLOGY,<br/>GLASGOW, SCOTLAND<br/>DEPARTMENT OF MATHEMATICS<br/>31 July 1962</p> <p>THE NUMERICAL SOLUTION OF THE SINGULAR INTEGRAL<br/>EQUATION FOR DIFFRACTION BY A SOFT STRIP</p> <p>ABSTRACT: It is difficult to obtain practical solutions for the scattering of waves by obstacles when the dimensions of the obstacle are comparable with the wavelength. In this paper it is shown that numerical solution of the integral equation for diffraction by a two-dimensional strip gives accurate results in this awkward range of wavelengths. A variational principle is used, with a comparatively simple step-function approximation to the unknown function in the integral equation. The results obtained justify further investigations to see whether the method can deal with diffraction by objects of more complicated shape.</p> | <p>EOAR 61-49<br/>TR (Part II)<br/>MECHANICS</p> |
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