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EXTINCTION PROBABILITIES FOR AGE AND POSITION-DEPENDENT BRANCHING PROCESSES

Howard E. Conner

MRC Technical Summary Report #390
March 1963

UNITED STATES ARMY
MATHEMATICS RESEARCH CENTER

MATHEMATICS RESEARCH CENTER, UNITED STATES ARMY
THE UNIVERSITY OF WISCONSIN

Contract No.: DA-11-022-ORD-2059

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ABSTRACT

In this paper we study a model for the population transition probabilities for a branching process composed of particles diffusing in a finite interval. The model is in general non-Markovian since we assume the branching transformation probabilities for a particle depend on its age and position. The process is described by the random number $N_t(x)$ of particles in the interval I at time t that are generated by a single particle initially at the point x in I. By considering $N_t(x)$ as a regenerative process with respect to the random age and position of the initial particle when it is transformed, we develop a functional equation for the generating function for $N_t(x)$. This functional equation is the basis for our study of the population probabilities $P(N_t(x) = n)$, $n = 0, 1, 2, \ldots$, as function of x in I and t in $[0, \infty)$.

Our principal result concerns the extension to our model of the fundamental result on the probability for ultimate extinction in a Galton-Watson branching process, [3, Chapter XII]. Letting $f_0(x)$ be the probability for ultimate extinction in a process generated by a particle initially at x, we give a necessary and sufficient condition for $f_0(x) = 1$. We also give a characterization of $f_0(x)$, x in I, as the minimal positive solution of a functional equation which is asymptotically related to the basic equation.

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Introduction

The process is composed of particles diffusing in a bounded open interval I_0, and it is generated by one particle initially at x in I_0. The diffusion is represented by the conditional probability density function (p.d.f) $p(x, y; s)$, where $p(x, y; s)\,dy$ is the probability for a particle at x at time t to be at y at time $t + s$, assuming it has not been previously to the boundary Γ of I_0. Therefore, we assume

1.1 (a) $p(x, y; s) = 0$ for x in Γ, y in I and $0 \le s < \infty$ and

(b) $p(x, y; s)$ is non-negative and continuous for x, y in I and $0 \le s < \infty$.

The boundary Γ is an absorbing barrier for which $a(x, t)$ is the probability distribution function for a particle initially at x to be absorbed at Γ within the time t and $a(x) = \lim_{t\to\infty} a(x, t)$ is the probability for the ultimate absorption at Γ for a particle at x. Consequently, we assume

Sponsored by the Mathematics Research Center, U. S. Army, Madison, Wisconsin, under Contract No.: DA-11-022-ORD-2059.

1.2 (a) $a(x, t)$ is non-negative and continuous for x in I and $0 \leq t < \infty$ with $a(x, t) = 1$ for x in Γ and $0 \leq t < \infty$,

(b) for each x in I, $a(x, t)$ is nondecreasing as t increases and

(c) $a(x) = \lim_{t \to \infty} a(x, t)$ is continuous for x in I with $a(x) = 1$ for x in Γ and not identically 1 for x in I_0.

Each particle has an independent random life span with the p.d.f. $g(s)$ where $g(s)\,ds$ is the probability for a particle formed at time T to end its life at time $T + s$; and so, we assume

1.3 $g(s)$ is non-negative and continuous for $0 \leq s < \infty$.

A particle whose life has ended at the age T and at the point x in I_0 is replaced by a random number k of particles with probability $b_k(x, T)$ and the expected number of particles with which it is replaced is $\sum_{k=1}^{\infty} kb_k(x, t)$. Therefore, we assume

1.4 (a) $b_k(x, s)$ is a non-negative continuous function for x in I_0 and $0 \leq s < \infty$,

(b) $\sum_{k=0}^{\infty} b_k(x, s) = 1$ for x in I_0, $0 \leq s < \infty$.

(c) $\mu(x, s) = \sum_{k=1}^{\infty} kb_k(x, s)$ is positive, bounded and continuous

for x in I_0 and $0 \leq s < \infty$.

At any given time t a particle initially at x in I_0

1.5 (a) has been absorbed at Γ with probability $a(x, t)$ or

(b) has ended its life in I_0 with probability $\int_I \int_0^t p(x, y; s) g(s) \, ds dy$ or

(c) is diffusing in I_0 with probability $b(x, t)$.

Ultimately a particle is either absorbed at Γ or transformed into new particles at some interior point x. Consequently, we assume

1.6 $b(x, t)$ is non-negative and continuous with $\lim b(x, t) = 0$ as $t \to \infty$ for x in I_0 and $0 \leq t < \infty$.

Since the events described in 1.5 are mutually exclusive and exhaustive, we assume

1.7 (a) $1 = a(x, t) + b(x, t) + \int_I \int_0^t p(x, y; s) g(s) \, ds dy$ and

$$\text{(b)}\quad 1 = a(x) + \int_I \int_0^\infty p(x, y; s) g(s)\, ds\, dy$$

for x in I and $0 \leq t < \infty$, where the integral in (b) is assumed to converge uniformly for x in I.

We define the generating function for the transition probabilities

$$f_k(x, t) = P(N_t(x) = k),\ k = 0, 1, 2, \ldots, \text{ by}$$

1.8 $$f(x, t; z) = \sum_{k=0}^{\infty} f_k(x, t) z^k \ ;\ f(x, t; 1) = 1,$$

defined for x in I, $0 \leq t$ and $|z| \leq 1$. If the initial particle is replaced at the age s by k particles at the point y in I_0, the generating function for the population size at $t > s$ is then $f^k(y, t - s; z)$. Since the probability for the initial particle to be replaced at an age s with k particles at y is given by $b_k(y, s) p(x, y; s) g(s)\, ds\, dy$, this and the previous discussion suggests $f(x, t; z)$ satisfies the functional equation,

1.9 $$f(x, t; z) = a(x, t) + zb(x, t) + \int_I \int_0^t h[y, s; f(y, t-s; z)] p(x, y; s) g(s)\, ds\, dy$$

for x in I, $0 \leq t < \infty$ and $|z| \leq 1$, where

1.10 $$h(x, t; z) = \sum_{k=0}^{\infty} b_k(x, t) z^k \ ;\ h(x, t; 1) = 1$$

Having formally derived 1.9, it is necessary to establish the existence of a unique solution $f(x, t; z)$ bounded by 1 which is a generating function in z for each x and t.

Before doing this, we introduce

1.11 $$K(x, y) = \int_0^\infty \mu(y, s)\, p(x, y; s)\, g(s)\, ds,$$

the expected number of particles replacing a particle initially at x which ends its life at y. We infer from the conditions of nonnegativity, continuity and uniform integrability that $K(x, y)$ is nonnegative and continuous for x, y in I. Therefore $K(x, y)$ defines a linear integral operator K

1.12 $$Kf(x) = \int_I K(x, y) f(y)\, dy \ , \ x \text{ in } I$$

(a) transforming the class of functions continuous on I into the subclass of functions vanishing on Γ and

(b) transforming the convex cone of nonnegative functions on I into itself.

Consequently, the operator K has special spectral properties similar to those for a matrix operator with nonnegative elements. Of these we use the existence of a positive characteristic number of minimum modulus to form a necessary and sufficient condition for ultimate extinction for $N_t(x)$ with probability 1.

A number of mathematical models for branching processes are covered in the papers of T. E. Harris, [4]. Limiting theorems for specific models with transformation probabilities depending on age or position are also given in these papers and in papers by R. Bellman and T. E. Harris, [1], N. Levinson, [6] and the author, [2]. The problem of ultimate extinction for a time-dependent branching process is given in papers by B. A. Sevast'janov, [7, 8].

§2. Existence of generating function.

The existence of a solution to 1.9 is given by the

Theorem 2.1. Suppose $a(x,t)$, $b(x,t)$, $p(x,y;t)$, $g(t)$, $h(x,t;z)$ and $\mu(x,t)$ satisfy the conditions given in 1.1-1.4, 1.6, 1.7 and 1.10. With these conditions, there exists a unique solution $f(x,t;z)$ to 1.9 defined for x in I, $0 \le t < \infty$ and $|z| \le 1$ and satisfying:

(a) $f(x,t;z)$ is continuous on its domain,

(b) $|f(x,t;z)| \le 1$ with $f(x,t;1) \equiv 1$ on its domain and

(c) for each x in I and $t \ge 0$, $f(x,t;z)$ is analytic on $|z| < 1$ and therefore has a representation

$$f(x,t;z) = \sum_{k=0}^{\infty} f_k(x,t)\, z^k$$

where each $f_k(x,t)$ is continuous and nonnegative for x in I and $0 \le t$.

Proof. We first list some properties of $h(x, t; z)$ which follow directly from the condition 1.4 and 1.10 and an application of the mean value theorem:

2.1 (i) $0 < h(x, t; \tau_1) < h(x, t; \tau_2)$,

(ii) $0 \leq |h(x, t; z)| \leq h(x, t; |z|)$,

(iii) $0 \leq |\frac{\partial}{\partial z} h(x, t; z)| \leq \mu(x, t)$ and

(iv) $|h(x, t; z) - h(x, t; w)| \leq \mu(x, t)|z - w|$

for x in I, $0 \leq t$, $|z|$ and $|w| \leq 1$ and $0 < \tau_1 < \tau_2 \leq 1$.

We define the sequence $\{f_n(x, t; z)\}$ with $f_0(x, t; z) \equiv 0$ by

2.2 $$f_{n+1}(x,t;z) = a(x,t) + zb(x,t) + \int_I \int_0^t h[y, s; f_n(y, t-s, z)]\, p(x, y; s)\, g(s)\, ds\, dy$$

for x in I, $0 \leq t$ and $|z| \leq 1$.

An induction argument using properties 2.1 and 1.7 implies for each n, $n = 0, 1, 2, \ldots$,

2.3 $$|f_n(x, t; z)| \leq 1 \text{ and } f_n(x, t; 1) \equiv 1$$

for x in I, $0 \leq t$ and $|z| \leq 1$.

A second induction argument using the continuity properties of the given functions and the analyticity of $h(x, t; z)$ in z shows each $f_n(x, t; z)$ is continuous for x in I, $t \geq 0$ and $|z| \leq 1$ and for each x in I and $t \geq 0$ it is analytic for $|z| < 1$. Another induction argument using the nonnegativity of the given functions shows the derivatives $\left(\frac{\partial}{\partial z}\right)^k f_n(x, t; 0)$ are continuous and nonnegative for $k = 0, 1, 2, \ldots$. Therefore we assert each $f_n(x, t; z)$ has the representation for x in I, $0 \leq t$ and $|z| < 1$ given by

2.4 $$f_n(x, t; z) = \sum_{k=0}^{\infty} f_{n,k}(x, t) z^k \text{ with}$$

$$f_{n,k}(x, t) \text{ continuous and nonnegative, } k = 0, 1, 2, \ldots .$$

We use 2.1(iv) and 2.2 to make the following estimate, for general n

2.5 $$|f_{n+1}(x,t;z) - f_n(x,t;z)| \leq \int_0^t \int_I |f_n(y,t-s;z) - f_{n-1}(y,t-s;z)| \mu(y,s) p(x,y;s) g(s) \, ds \, dy$$

on x in I, $0 \leq t$ and $|z| \leq 1$. The substitution of 2.3 with $n = 1$ into the integrand of 2.5 with $n = 1$ gives

$$|f_2(x, t; z) - f_1(x, t; z)| \leq \int_0^t \int_I \mu(y,s) p(x, y; s) g(s) \, ds \, dy.$$

Setting $\Delta(T) = \max\{\int_I \mu(y, s) p(x, y, s) g(s) \, dy \mid x \text{ in } I, 0 \leq s \leq T\}$ for $0 < T$, we have $|f_2(x, t; z) - f_1(x, t; z)| \leq t\Delta(T)$; and so, by induction, we have for $n = 0, 1, 2, \ldots,$

2.6 $$|f_{n+1}(x, t; z) - f_n(x, t; z)| \leq t^n \Delta^n(T)/n!$$

on x in I, $0 \leq t \leq T$ and $|z| \leq 1$.

We infer from 2.6 that the $f_n(x, t; z)$ converge as $n \to \infty$ to a limit $f(x, t; z)$ and that this convergence is uniform for any $T > 0$ on x in $I, 0 \leq t \leq T$ and $|z| \leq 1$. Therefore using the Moore-Weierstrass theorems, we know

(a) $f(x, t; z)$ is continuous for x in I, $t \leq 0$ and $|z| \leq 1$,

(b) $f(x, t; z)$ is analytic in $|z| < 1$ for each x in I, $0 \leq t$ and

by letting $n \to \infty$ in 2.2,

(c) $f(x, t; z)$ is a solution to 1.9 for x in I, $0 \leq t$ and $|z| \leq 1$.

The properties $|f(x, t; z)| \leq 1$ and $f(x, t; 1) \equiv 1$ are consequences of the convergence and 2.3. We use the Cauchy formula to write for each x in I and $0 \leq t$,

2.7 $$f_{n,k}(x, t) = \frac{1}{2\pi}\int_0^{2\pi} f_n(x, t, e^{i\theta})e^{-ik\theta}d\theta.$$

Since $f_n(x, t; z)$ converges uniformly on x in I, $0 \leq t \leq T$ and $|z| \leq 1$ for each $T > 0$, we know the right side of 2.7 converges uniformly on the same sets to

$f_k(x, t) = \frac{1}{k!}\left(\frac{\partial}{\partial z}\right)^k f(x, t; 0)$. Therefore, for any $T > 0$ each $f_k(x, t)$ is the uniform limit on x in I, $0 \leq t \leq T$ and $|z| \leq 1$ of the continuous nonnegative functions $f_{n,k}(x, t)$ given in 2.4; and so, we know each $f_k(x, t)$ is continuous and nonnegative for x in I and $0 \leq t$.

To complete the proof we must show the solution $f(x, t; z)$ is unique in the class of functions continuous and bounded in magnitude by 1 on x in I, $0 \leq t$ and $|z| \leq 1$. Suppose $g(x, t; z)$ is a continuous function with $|g(x, t; z)| \leq 1$ on x in I, $0 \leq t$ and $|z| \leq 1$ satisfying 1.9 and consider $r(x, t; z) = |f(x, t; z) - g(x, t; z)|$. Since $|g(x, t; z)| \leq 1$ we can use 1.9 and 2.1(iv) to make the estimate

2.8 $$r(x, t, z) \leq \int_0^t \int_I r(y, t-s, z)\mu(y, s)\, p(x, y; s)\, g(s)\, dy ds.$$

Letting $R(T, \lambda, z) = \max\{r(x, t; z)e^{-\lambda t} \mid x \text{ in } I;\ 0 \leq t \leq T\}$ for T and $\lambda \geq 0$ and $|z| \leq 1$, we have $R(T, \lambda, z) < \infty$ and

$$R(T, \lambda, z) \leq R(T, \lambda, z) \int_0^\infty \int_I e^{-\lambda s} \mu(y, s) p(x, y; s)\, g(s)\, dy ds.$$

We know the integral exists, is finite, 1.4 and 1.7, and is a decreasing function of λ; so, we choose λ_0 so that the value of the integral is less than $\frac{1}{2}$. This shows $0 \leq R(T, \lambda_0, z) \leq 0$ for each $T > 0$ and $|z| \leq 1$ and consequently $g(x, t; z) = f(x, t; z)$. Therefore $f(x, t; z)$ is a unique continuous solution to 1.9 satisfying $|f(x, t; z)| \leq 1$, completing the proof.

Similar functional equations can be derived and solved for the multivariate generating functions

$$F(x, t_1, \dots, t_n; z_1, \dots, z_n) = \sum_{k_1, \dots, k_n} P(N_{t_1}(x) = k_1; \dots ; N_{t_n}(x) = k_n) z_1^{k_1} \dots z_n^{k_n}$$

satisfying the consistancy condition that for every set of positive integers $(i_1, \dots, i_m)$, $1 \leq m \leq n$,

$$F(x, t_1, \dots, t_n; z_1, \dots, z_n = F(x, t_1, \dots, t_m; z_{i_1}, \dots, z_{i_m})$$

where $z_i = 1$ if i is not in $(i_1, \dots, i_m)$.

§ 3. Probability for eventual extinction.

Having established the existence of a unique continuous solution $f(x, t; z)$ to 1.9 on x in I, $0 \leq t$ and $|z| < 1$ satisfying

3.1 $$f(x, t; z) = \sum_{k=0}^{\infty} f_k(x, t) z^k \quad ; \quad \sum_{k=0}^{\infty} f_k(x, t) = 1$$

where each $f_k(x, t)$ is nonnegative and continuous, we formally identify the $f_k(x, t)$ as the population transition functions for the population size $N_t(x)$ for a system generated by a particle initially at x. In particular setting $z = 0$ in 1.9, we have

3.2 $$f_0(x,t) = a(x,t) + \int_I \int_0^t h[y,s;f_0(y,t-s)]\,p(x,y;s)\,g(s)\,dyds \,.$$

where $f_0(x,t)$ formally gives the Prob. $\{N_t(x) = 0 \mid N_0(x) = 1\}$. Since $f_0(x,t)$ is nonnegative, bounded above by 1 and continuous on x in I and $0 \leq t$, Theorem 2.1, we know from the uniqueness argument following 2.8 that $f_0(x,t)$ is a unique continuous solution to 3.2 satisfying $|f_0(x,t)| \leq 1$. We will use this uniqueness property to show $f_0(x,t)$ is an increasing function of t for each x in I, a property which is suggested by the probability context.

We define a sequence of functions for x in I and $0 \leq t$ by $g_0(x,t) = 0$ and $g_{n+1}(x,t) = Tg_n(x,t)$ where T is a non-linear operator defined by

3.3 $$Tg(x,t) = a(x,t) + \int_I \{ \int_0^t h[y,s;g(y,t-s)]\,p(x,y;s)\,g(s)\,ds\}dy,$$

for a continuous function $g(x,t)$ on x in I and $0 \leq t$.

An induction argument shows each $g_n(x,t)$ is a nonnegative continuous function bounded above by 1 on x in I and $0 \leq t$, and it is a monotone increasing function of t for each x in I. The last statement follows directly from the nonnegativity of the given functions and the nondecreasing behavior of $a(x,t)$ as a function of t, 1.2. With the same estimates as used in Theorem 2.1 we have $|g_1(x,t) - g_0(x,t)| \leq 1$ and

$$|g_{n+1}(x, t) - g_n(x, t)| \leq \int_I \int_0^t |g_n(y, t-s) - g_{n-1}(y, t-s)| \mu(y, s) p(x, y; s) g(s) \, dyds .$$

Therefore we have by induction on n; for each $T > 0$, $|g_{n+1}(x, t) - g_n(x, t)| \leq t^n \Delta^n(T)/n!$ for x in I and $0 \leq t \leq T$ where $\Delta(T)$ is defined in 2.6. Using the uniqueness of $f_0(x, t)$ as a continuous bounded solution to 3.2, this shows $f_0(x, t)$ is for each $T > 0$ the uniform limit as $n \to \infty$ for x in I and $0 \leq t \leq T$ of the sequence $g_n(x, t)$. Consequently, $f_0(x, t)$ is a monotone increasing function of t for each x in I. Summarizing we have

3.4 $f_0(x, t)$ is a nonnegative continuous unique solution to 3.2 bounded in magnitude by 1 and for each x in I, it is a monotone increasing function of t.

In particular 3.4 implies $\lim_{t \to \infty} f_0(x, t) = f_0(x)$ exists for each x in I. The probability context suggests $f_0(x)$ is the probability for eventual extinction in a process generated by a particle initially at x. If we can show $f_0(x, t)$ is continuous in x uniformly in t then we know $f_0(x)$ is continuous in x and therefore by the Moore-Osgood theorem on iterated limits

3.5 $$\lim_{t \to \infty} f_0(x, t) = f_0(x) \leq 1$$

uniformly for x in I. To show $f_0(x, t)$ is continuous in x uniformly in t, we make the estimate

$$|f_0(x_1,t) - f_0(x_2,t)| \le |a(x_1,t) - a(x_2,t)| + \int_0^t \int_I h[y,s;f_0(y,t-s)]\,|p(x_1,y;s) - p(x_2,y;s)|\,g(s)\,dyds$$

$$\le |a(x_1,t) - a(x_2,t)| + \int_0^\infty \int_I |p(x_1,y;s) - p(x_2,y;s)|\,g(s)\,dyds\,.$$

The uniform integrability condition 1.7 gives the existence of a positive $\gamma(\epsilon)$, $0 < \epsilon$, such that the 2nd term is $\le \frac{\epsilon}{2}$ if $|x_1 - x_2| < \gamma(\frac{\epsilon}{2})$. Since we have assumed $\lim_{t\to\infty} a(x,t) = a(x)$ uniformly in x, 1.3, there exists a positive $\delta(\epsilon)$, $0 < \epsilon$, such that the first term is $\le \frac{\epsilon}{2}$ if $|x_1 - x_2| < \delta(\frac{\epsilon}{2})$. This completes the argument and establishes 3.5.

Letting $t \to \infty$ in 3.2 and using 1.7 and 3.5 shows $f_0(x)$ is a continuous solution to

3.6 $$f_0(x) = a(x) + \int_I \{ \int_0^\infty h[y,s;f_0(y)]\,p(x,y;s)\,g(s)\,ds\}\,dy$$

with $0 \le f_0(x) \le 1$ for x in I. Since the function $\mathbf{1}(x)$, $\mathbf{1}(x) = 1$ for x in I, is a solution to 3.6 by 1.7, we have the problem of finding a necessary and sufficient condition for 3.6 to have a nonnegative continuous solution different from $\mathbf{1}(x)$.

We define the nonlinear Urysohn operator U on the class of nonnegative continuous functions bounded in magnitude by $\mathbf{1}$ on I, 3.7 $$Uf(x) = a(x) + \int_I \{ \int_0^\infty h[y, s; f(y)] p(x), y; s) g(s) ds \} dy.$$

The proof of the next Theorem is based on the relationship between U and the nonlinear operator T defined in 3.3 and the spectral properties of the derivative of U at $\mathbf{1}(x) \equiv 1$, which is defined by

$$Kf(x) = \int_I \{ \int_0^\infty \frac{\partial}{\partial \tau} h(y, s; 1) p(x, y; s) g(s) ds \} f(y) dy = \int_I K(x, y) f(y) dy,$$

where $K(x, y)$ is the expected number of particles replacing at y a particle initially at x. The transformation properties of K are given in 1.12, and,as stated there, K has some special spectral properties.

The spectral properties we want can be most easily obtained by approximating $K(x, y)$ and the continuous functions on which it operates by step functions, forming a matrix operator with nonnegative elements to which the Perron-Frobenius' theory can be applied. A more general theory is developed in the monograph by M. G. Krein and M. A. Rutman, [5]. We now list for convenience those properties for which we have use. Specifically, assuming $K(x, y)$ is nonnegative, not identically zero and continuous for x, y in I, the following statements are valid.

3.8 The kernel $K(x, y)$ has a positive characteristic number λ_0 such that for any other characteristic number λ, $\lambda_0 \leq |\lambda|$, [5; Chapter 6].

A sufficient condition for $\lambda_0 < 1$ is the existence of a nonnegative function $f(x)$ such that

$$3.9 \qquad f(x) < \int_I K(x, y) f(y)\, dy$$

for x in I, [5; Chapter 6].

The next property is a result of the asymptotic behavior of the n^{th} iterate $K^{(n)}$ of the operator K when λ_0 is less than 1. If $\lambda_0 < 1$, then for any nontrivial, nonnegative continuous function $f(x)$ there is some point x_0 in I such that

$$3.10 \qquad f(x_0) \leq \int_I K(x_0, y) f(y)\, dy \,.$$

We eliminate those processes having no absorption by assuming a positive probability for a particle initially at x to be eventually absorbed at Γ or in I_0,

$$3.11 \qquad 0 < a(x) + \int_I \{ \int_0^\infty b_0(y, s)\, p(x, y; s)\, g(s)\, ds \}\, dy$$

for x in I.

We can now state the

Theorem 3.1. Suppose the condition 3.11 is satisfied in addition to the conditions listed in Theorem 2.1. A necessary and sufficient condition for 3.6 to have a positive continuous solution $g(x)$ different from the function $1(x) \equiv 1$ is

3.12 $$\lambda_0 < 1,$$

where λ_0 is the unique characteristic number determined by 3.8.

We apply the result of Theorem 3.1 to the probability $f_0(x)$ for eventual extinction defined in 3.5 and obtain the

Theorem 3.2. Assuming the conditions in Theorem 3.1,

(a) if $\lambda_0 \geq 1$, $f_0(x) = 1$ for x in I and

(b) if $\lambda_0 < 1$, $f_0(x)$ is the minimum positive solution $g(x)$ to 3.6.

Proof for Theorem 3.1. We list some properties for the operator U which are direct results of the strict monotonicity of $h(y, s; \tau)$ as a function of τ on $0 \leq \tau \leq 1$ and the normalization 1.7. Letting $J = \{x \text{ in } I | a(x) = 1\}$ and $I - J = \{x \text{ in } I | a(x) < 1\}$, we then have the following for any nonnegative continuous function $f(x)$ which is bounded above by 1 and not identically 1, 3.13 $$Uf(x) = 1 \text{ for } x \text{ in } J \text{ and}$$

$$Uf(x) < 1 \text{ for } x \text{ in } I - J.$$

Suppose $g(x)$ is a solution to 3.6, $g(x) = Ug(x)$. Then 3.13 implies $1 - g(x)$ is positive for x in $I - J$. Since $1(x) = U1(x)$, 1.7, 1.11 and 2.1 imply

3.14 $$0 < 1 - g(x) = \int_I \{ \int_0^\infty [1 - h[y, s; g(y)]] p(x, y; s) g(s) ds \} dy$$

$$< \int_I [1 - g(y)] K(x, y) dy$$

for x in $I - J$. Therefore by 3.9, the characteristic number λ_0 for $K(x, y)$, determined by 3.8, satisfies $\lambda_0 < 1$, proving the necessity of the condition.

To prove the sufficiency, we define $g_0(x) = 0$ and

3.15 $$g_{n+1}(x) = Ug_n(x)$$

for x in I. We have $g_n(x) = 1$ for x in J, 3.13. Using the strict montonicity of $h(y, s; \tau)$ in τ on $0 \leq \tau \leq 1$, we have

3.16 $$Uf(x) < Ug(x)$$

for x in $I - J$ when $f(x) < g(x)$ on $I - J$. Assumption 3.11 and property 3.13 imply $0 < g_1(x) < 1$ on $I - J$; and so, 3.16 implies $g_1(x) = Ug_0(x) < Ug_1(x) = g_2(x)$ on $I - J$. By induction on n, we have for $n = 1, 2, 3, \ldots$

3.17 $$0 < g_n(x) < g_{n+1}(x) < 1$$

for x in $I - J$.

The inequality

$$|g_n(x_1) - g_n(x_2)| \leq |a(x_1) - a(x_2)| + \int_I |K(x_1, y) - K(x_2, y)| \, dy$$

and the continuity of $K(x, y)$ show $g_n(x)$ is continuous in x uniformly in n. This and 3.17 imply the existence of

3.18 $$\lim_{n \to \infty} g_n(x) = g(x)$$

uniformly for x in I. Therefore letting $n \to \infty$ in 3.15 shows $g(x)$ is a continuous solution to 3.6 satisfying $0 < g(x) \leq 1$ for x in $I - J$ and $g(x) = 1$ for x in J.

We now show $g(x)$ is different from $1(x) = 1$ when $\lambda_0 < 1$. For this purpose we introduce the collection of kernels defined by

$$K(x, y; \tau) = \int_0^\infty \frac{\partial}{\partial \tau} h(y, s; \tau)\, p(x, y; s)\, g(s)\, ds$$

for x, y in I and $0 < \tau \leq 1$; so that $K(x, y; 1) = K(x, y)$, defined in 1.11. We use the assumptions in 1.1, 1.3, 1.4 and 1.7 to state $K(x, y; \tau)$ is nonnegative and continuous for x, y in I and $0 < \tau \leq 1$. Since $\mu(y, s)$ is positive on its domain, $K(x, y; \tau)$ and the kernel

$$P(x, y) = \int_0^\infty p(x, y; s)\, g(s)\, ds$$

vanish together.

Each kernel $K(x, y; \tau)$ has a positive characteristic number $\lambda_0(\tau)$ determined by 3.8. Using the continuity properties of $K(x, y; \tau)$ in x, y and τ and the Fredholm theory for integral operators with continuous kernels, we can show that the Fredholm determinant $d(\lambda; \tau)$ for $K(x, y; \tau)$, where λ is the spectral parameter, is a continuous function for $|\lambda| < \infty$ and $0 < \tau \leq 1$ and an entire function of λ for fixed τ. This is sufficient to assert $\lambda_0(\tau)$ is a continuous function on $0 < \tau \leq 1$. This and the assumption $\lambda_0 = \lambda_0(1) < 1$ imply the existence of a $\tau_1 < 1$ such that $\lambda_0(\tau) < 1$ for $\tau_1 \leq \tau \leq 1$. We now develop a contradiction from the assumption $g(x) = 1(x)$. If this be valid, the uniform convergence of $g_n(x)$ to $g(x)$, 3.18, implies the existence of a positive integer $N(\tau_1)$ such that $g_n(x)$ is uniformly close to $\mathbf{1}(x)$, $g_n(x) \geq \tau_1 + \frac{1-\tau_1}{2}$, for $n \geq N(\tau_1)$. Therefore using the mean value theorem and the strict monotonicity of $\frac{\partial}{\partial\tau} h(y, s; \tau)$ as a function of τ, we have

$$3.19 \qquad (1 - g_{N+1}(x)) > \int_I (1 - g_N(y))\, K(x, y; \tau_1)\, dy$$

for x in $I - J$. Assuming $\lambda_0(\tau_1) < 1$, property 3.10 gives the existence of some x_0 in I such that

$$\int_I (1 - g_N(y))\, K(x_0, y; \tau_1)\, dy \geq 1 - g_N(x_0) .$$

Since $K(x, y; \tau_1)$ and $P(x, y)$ vanish together, we know x_0 is in $I - J$. Therefore this and 3.19 imply

$$1 - g_{N+1}(x_0) > 1 - g_N(x_0) \quad \text{or} \quad g_{N+1}(x_0) < g_N(x_0),$$

contradicting 3.17. Consequently, $g(x)$ is different from $\mathbf{1}(x)$ when $\lambda_0 < 1$, proving the sufficiency of the condition.

The operators U and T have a structural relation which is utilized for the proof of Theorem 3.2.

Proof for Theorem 3.2. If $\lambda_0 \geq 1$, the results of Theorem 3.1 assert $f_0(x) = 1$, x in I, proving (a). To prove (b), we recall the probability for eventual extinction is given by $f_0(x) = \lim f_0(x, t)$ as $t \to \infty$, 3.5. We have also shown $f(x, t) = \lim Tg_n(x, t)$ as $n \to \infty$ with the operator T and the functions $g_n(x, t)$ defined in 3.3. The operator T has a monotone property similar to that for the operator U, 3.16. It is

3.20 $$Tf(x, t) < Tg(x, t)$$

for x in $I - J$ and $0 < t$ if $f(x, t) < g(x, t)$ on the same set. Since $g_0(x, t) \equiv 0$ and $a(x, t) \leq a(x)$, 1.2, we have

$$g_1(x, t) = Tg_0(x, t) \leq Ug_0(x) = g_1(x)$$

for x in $I - J$ and $0 \leq t < \infty$, and so, an induction argument using 3.20 gives

$$g_{n+1}(x, t) = Tg_n(x, t) \leq Ug_n(x) = g_{n+1}(x)$$

on the same set, where the $g_n(x, t)$ are given by 3.3 and the $g_n(x)$ by 3.15. Therefore we have

3.21 $$f_0(x,t) = \lim_{n\to\infty} g_n(x,t) \leq \lim_{n\to\infty} g_n(x) = g(x) \quad \text{and}$$

$$f_0(x) = \lim_{t\to\infty} f_0(x,t) \leq g(x)$$

for x in I where $g(x)$ is defined in 3.18.

We next show that $g(x)$ is the minimal positive solution to 3.6. Suppose $h(x)$ is a continuous solution to 3.6 satisfying $0 \leq h(x) \leq 1$ for x in I. Using the monotone property 3.16 for U, we have

$$g_1(x) = Ug_0(x) \leq Uh(x) = h(x)$$

for x in I; and so using induction, for each n

$$g_{n+1}(x) = Ug_n(x) \leq Uh(x) = h(x)$$

for x in I. This result shows $\lim_{n\to\infty} = g_n(x) = g(x) \leq h(x)$. Therefore 3.21 implies $f_0(x) = g(x)$ on I, completing the characterization of $f_0(x)$ as the minimal positive solution to 3.6.

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