ERRATA SHEET

RM-3632-PR, Unclassified, "Satellite Librations on an Elliptic Orbit," by H. B. Schecter

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Page 20: Change the second equation from the bottom of the page to read:

$$\sin 2\Psi_c \cong \frac{2\pi^2\sqrt{\bar{q}}}{K^2(1+\bar{q})} \quad \sin \frac{\sqrt{3}\,\pi\,H}{2K}$$

Page 21: Delete k from the denominator of the second term on the right-hand side of Eq. (45).

Page 22: Delete the second term on the right-hand side of Eq. (46).

Reports Department
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MEMORANDUM

RM-3632-PR

MAY 1963

SATELLITE LIBRATIONS ON AN ELLIPTIC ORBIT

H. B. Schechter

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PREFACE

The attitude of a satellite in orbit is materially affected by the gravitational torques which are exerted upon it. The response behavior of this vehicle to these torques will depend on both the initial orientation and angular velocity, and the orbital path followed.

Past studies, in which the distributed mass of the satellite has been approximated by a dumbbell-shaped body contained in the orbital plane, and moving along a circular orbit, have revealed the essential features of the attitude response behavior.

This Memorandum extends the analysis of the dumbbell's motion to elliptic orbits and shows to what extent the conclusions reached in the circular case are affected when orbital eccentricity is taken into account.

SUMMARY

This Memorandum discusses and analyzes the first-order effect of orbital eccentricity on the planar tumbling or oscillatory motion of a dumbbell-shaped satellite. This has been done by assuming that the angular orientation angle Ψ can be represented by a power series in eccentricity e, in which the coefficient of the e^0 term was set equal to the circular solution Ψ_c available from earlier investigations. The differential equation for the coefficient of the e^1 term is shown to be of an inhomogeneous Mathieu type, the particular solutions of which can be readily obtained if certain weak restrictions are placed on the initial magnitude of the dumbbell's angular velocity.

The analysis indicated that the orientation of the satellite in the elliptic orbit can differ substantially from the one determined for the circular orbit.

CONTENTS

SYMBOLS

A^{β}_{2r}	=	coefficients of the series expansion of the Mathieu functions
a	=	semimajor axis of the orbit
C.G.	=	center of gravity
C.M.	=	center of mass
Ce_{β}, Se_{β}	=	Mathieu functions of fractional order
e	=	eccentricity
F	=	force
F_r	=	net force exerted
K	=	quarter period of the elliptic function
k	=	modulus of the elliptic function
L	=	radial distance
ℓ	=	half length of dumbbell
M	=	distributed mass total mass mean anomaly, $n(t - t_o)$
m	=	mass
n	=	mean angular velocity in the elliptic orbit, $\sqrt{\frac{\mu}{a^3}}$
q	=	coefficient of trigonometric term in Mathieu's equation
$\overline{q}$	=	parameter defined below Eq. (43b)
r, $\overline{r}$	=	orbital radius
s	=	independent variable, $s = \alpha M$
T	=	kinetic energy
t	=	time
t_o	=	time of perigee passage
U	=	potential energy

V = velocity

W = Wronskian

z = independent variable defined by Eq. (44)

α = average value of angular velocity $d\Psi_c/dM(>3)$

β = order of the Mathieu function

μ = earth's gravitational constant, GM_E

τ = orbital period

θ = polar position angle of dumbbell's C.M.

Ψ = amplitude of oscillation

Ψ_c = amplitude of oscillation in circular orbit

$\dot{\Psi}_{c_i}$ = initial angular velocity of dumbbell

ω = absolute angular velocity of dumbbell

I. INTRODUCTION

An object of arbitrary mass distribution whose center of mass moves in a given orbit through a central inverse r^2 force field will experience a disturbing torque acting around its center of mass, which would cause its spatial orientation to vary with time. This torque vanishes when the three principal moments of inertia of the body are equal. The motion of a dumbbell-shaped object, being the simplest one to analyze mathematically while at the same time exhibiting many of the characteristics which result from the action of this imbalancing gravitational torque, has received the most attention.

An understanding of the rotational-motion characteristics of a dumbbell-shaped satellite could conceivably be of more than purely academic interest, judging by the frequent mention of such a concept in connection with a manned space station in which gravity is artificially simulated. The most recent of these proposals is General Electric Company's Pseudo Gravity Space Station concept,* which envisages two inhabited compartments hinged by means of 100-ft-long extension tubes and rotating around a centrally located nonrotating hub.

In one of the earlier papers on the subject, Klemperer and Baker,[1] drawing on the long-known stable oscillatory librational motion of the moon around its axis of rotation and being interested in the possibility of utilizing this effect for measurements of orientation on board a satellite, analyzed the librational motion of a symmetric dumbbell placed in a circular orbit. Assuming small deviations from the equilibrium position, they show that for a dumbbell of small linear dimensions (compared with the orbital radius) the dumbbell behaved essentially like an undamped simple pendulum with a natural frequency equal to $\sqrt{3}$ times the frequency of the motion in the circular orbit. They also briefly mention that for an orbit which is not precisely circular, the situation would become more complex.

Schindler,[2] confining his analysis also to circular satellite orbits but lifting the restriction on the size of the angular deviation, shows that the librational periods can range in a continuous manner anywhere from around $1/3\,\tau$ (the orbital period) on up to infinite

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*Described in Aviation Week, Vol. 78, No. 4, January 28, 1963, pp. 56-58.

values of the time, as the initial angle which the dumbbell makes with the vertical ranges from near zero to $\pi/2$ and that, therefore, an initial value of this angle exists for which the librational period just equals the orbital period.

Klemperer, in an ARS Journal note,[3] presents a first integral of the motion of a librating dumbbell and indicates that the maximum value of the amplitude of oscillation, Ψ, of the dumbbell is defined by the relation $\sin \Psi_{max} = \dot{\Psi}_o / \sqrt{3}n$, where $\dot{\Psi}_o$ is the angular velocity when $\Psi = 0$, and n is the angular velocity of the center of mass in the circular orbit. He points out, furthermore, that the motion remains oscillatory as long as $\dot{\Psi}_o < \sqrt{3}n$ but becomes one of tumbling when $\dot{\Psi}_o$ exceeds this value.

In the same issue of the ARS Journal, Baker[4] considers an orbit of small eccentricity and points out that under the restrictive assumption of small librational amplitude, the motion is governed by a forced Mathieu equation.

Schindler[5] employs a Hamiltonian formulation to study the dynamic behavior of the dumbbell in the vicinity of the two positions of equilibrium and proves in a rigorous manner not only that the vertical-dumbbell orientation is one of stable equilibrium, while the horizontal one is a position of unstable equilibrium, but also that a purely oscillatory motion around the unstable-equilibrium position is possible in an elliptic orbit, provided the initial conditions are properly chosen.

Moran[6] employs a perturbation technique to study the effect which the librational mode of motion of a dumbbell moving initially along an undisturbed circular orbit has on the motion of its center of mass. He shows that resonance conditions in the radial equation of motion can arise for certain values of the initial librational frequency and that the differential equation for the first radial correction term gives rise to a divergent oscillatory solution as time increases.

The present Memorandum describes in a qualitative and quantitative manner the first-order effects of orbital eccentricity on the planar librational motion of a dumbbell by means of a perturbation expansion in the vicinity of the nominal circular-orbit solution.

II. THE GRAVITATIONAL TORQUE

Before setting out to write down in a mechanical fashion Lagrange's equations governing the planar motion of the dumbbell satellite, it might be useful and instructive first to consider briefly one aspect of the physical origin and nature of the gravitational torque which is but seldom recognized and mentioned. Reference to Fig. 1 makes it immediately obvious that as mass m_1 of the symmetric dumbbell shown ($m_1 = m_2 = m$) is closer to the center of attraction than is mass m_2, it will experience a larger gravitational radial force than will mass m_2. The difference in magnitude and orientation of those two forces gives rise to the well-known expressions for the resultant force, which acts through the center of mass 0 and perturbs the translational motion, and to the gravitational torque around 0, which gives rise to the librational motion.

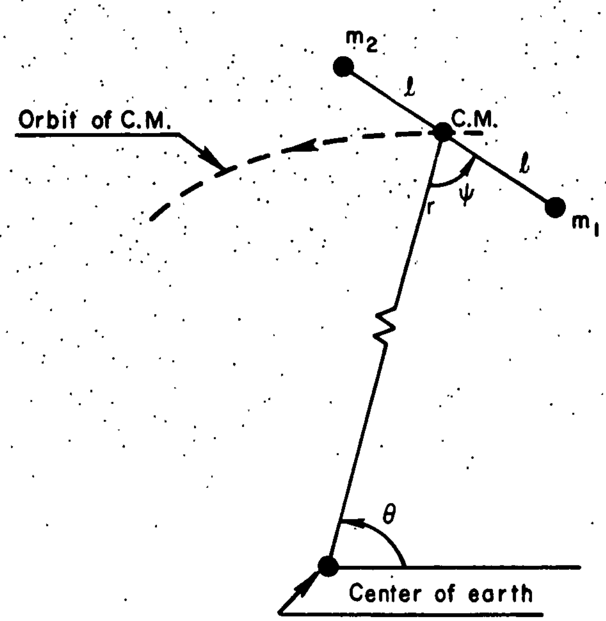

Fig. 1 — Geometry and notation

A more graphic way of interpreting the action of the gravitational torque and one which allows for a more qualitative feel for its sense and magnitude is by a visualization of the instantaneous relative position of the dumbbell's center of mass (C.M.) and center of gravity (C.G.).

For a body of arbitrary shape, these two points coincide in a constant, unidirectional force field, and for spherically symmetric bodies, also in an inverse r^2 field. In general, though, the two points are distinct.

The center of gravity of a distributed mass M is found by an application of Newton's Third Law. It is that point at which a concentrated mass equal to the total mass (M = 2m) would experience a force equal in magnitude and opposite in direction to that experienced by the earth at its center, due to the gravitational attraction exerted upon it by the mass M.

This definition is now used to locate the center of gravity of the dumbbell for two extreme cases of its orientation, $\Psi = \pi/2$ and $\Psi = 0$, as shown in Fig. 2.

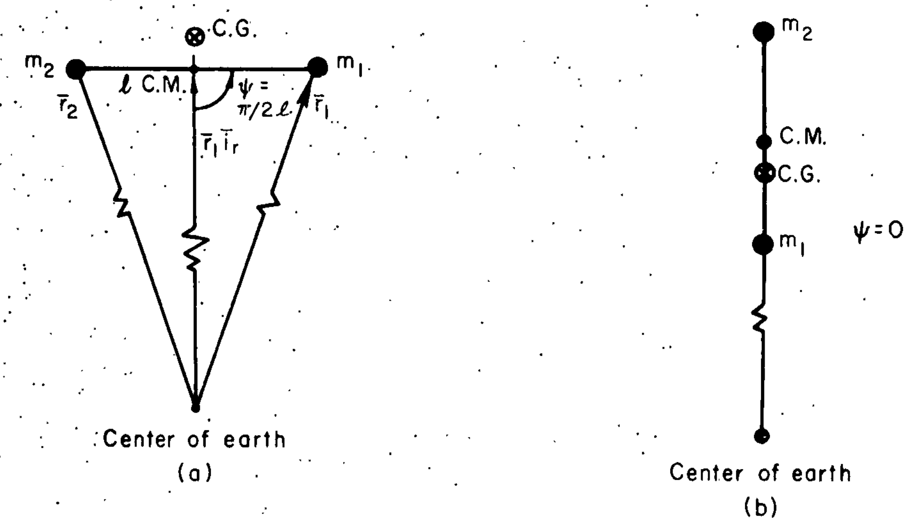

Fig. 2—Relative location of center of mass and center of gravity

Letting $\bar{F}_1$ and $\bar{F}_2$ designate the forces at m_1 and m_2, respectively, we have

$$\begin{aligned} \bar{F}_1 &= -\frac{m\mu}{r_1^3}\bar{r}_1 \\ \bar{F}_2 &= -\frac{m\mu}{r_2^3}\bar{r}_2 \end{aligned} \tag{1}$$

where $m_1 = m_2 = m$ and $\mu = GM_E$, the earth's gravitational constant. Because of symmetry, the net force exerted on the dumbbell, F_r, will be parallel to $\bar{r}$, so that

$$\begin{aligned} F_r &= \bar{F}_1 \cdot \bar{i}_r + \bar{F}_2 \cdot \bar{i}_r = -\frac{m\mu}{r_1^2}\left(\bar{i}_{r_1} \cdot \bar{i}_r\right) - \frac{m\mu}{r_2^2}\left(\bar{i}_{r_2} \cdot \bar{i}_r\right) \\ &= -\frac{2\mu m r}{\left[r^2 + \ell^2\right]^{3/2}} \end{aligned} \tag{2}$$

This force acts on the concentrated mass 2m located at a radial distance L. Under the assumption $\ell/r \ll 1$ we get

$$F_r = -\frac{2\mu m}{L^2} \cong -\frac{2\mu m}{r^2}\left(1 - \frac{3}{\ell}\frac{\ell^2}{r^2}\right) \tag{3}$$

from which we find that

$$L \cong r\sqrt{1 + \frac{3}{2}\frac{\ell^2}{r^2}} \cong r\left[1 + \frac{3}{4}\frac{\ell^2}{r^2}\right] = r + \Delta r \tag{4}$$

and

$$\Delta r = \frac{3}{4}\frac{\ell^2}{r} \tag{5}$$

This places the center of gravity above the center of mass. In the case of a dumbbell rotating in a counterclockwise direction, any slight increase in the angle ψ beyond $\pi/2$ would cause the center of gravity to shift to the left of the vector $\bar{r}$ and thereby give rise to an accelerating moment around the center of mass. The converse holds true during the period of time prior to the reaching of the horizontal orientation, when the torque has a retarding effect.

A repetition of the above analysis for the orientation (b) of Fig. 2 shows that the center of gravity is now located a distance $\Delta r = 3/2\ \ell^2/r$ below the center of mass. It is immediately evident that any slight deviation from the vertical position gives rise to a torque which opposes the motion, and thus labels the vertical position one of stable equilibrium.

The path described by the dumbbell's center of gravity during one complete period of rotation is shown schematically in Fig. 3 and gives an indication of the oscillatory nature of the gravitational torque.

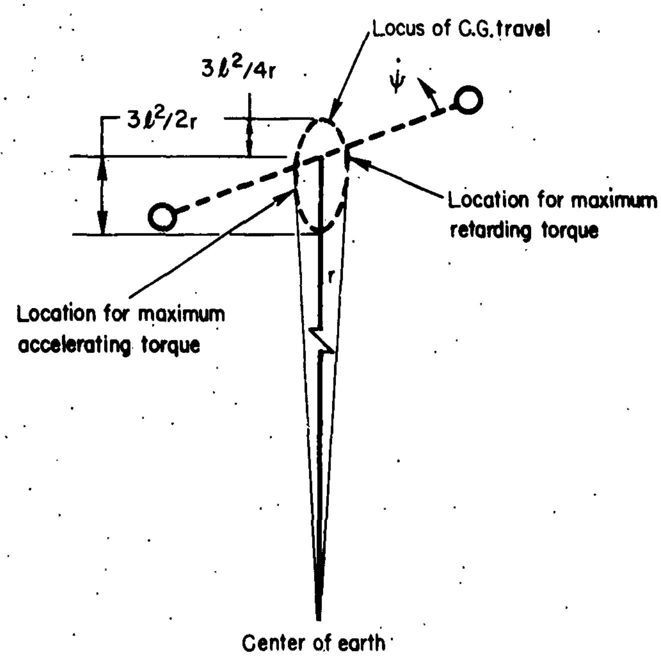

Fig. 3 — Locus of center-of-gravity travel (not drawn to scale)

III. EQUATIONS OF MOTION

The kinetic energy is

$$T = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 = \frac{1}{2} m \left(V_1^2 + V_2^2 \right) \tag{6}$$

and the potential energy is

$$U = -\frac{\mu m_1}{r_1} - \frac{\mu m_2}{r_2} \tag{7}$$

where

$$\begin{aligned}
\bar{V}_1 &= \dot{\bar{r}} + \ell\omega\bar{i}_y \\
\bar{V}_2 &= \dot{\bar{r}} - \ell\omega\bar{i}_y \\
\omega &= \dot{\theta} + \dot{\Psi} \\
\dot{\bar{r}} &= \dot{r}\bar{i}_r + r\dot{\theta}\bar{i}_\theta \\
r_1 &= \left[r^2 + \ell^2 - 2r\ell \cos \Psi\right]^{1/2} \\
r_2 &= \left[r^2 + \ell^2 + 2r\ell \cos \Psi\right]^{1/2}
\end{aligned} \tag{8}$$

Figure 4 shows the various parameters and unit vectors used in the above relations. After Eqs. (8) are substituted into Eqs. (6) and (7) and the expression for the Lagrangian T-U is obtained, one can derive the set of differential equations shown below.

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{2}\left[\frac{r - \ell \cos \Psi}{r_1^2} + \frac{r + \ell \cos \Psi}{r_2^2}\right] \tag{9}$$

$$2m\frac{d}{dt}\left[r^2\dot{\theta} + \ell^2(\dot{\theta} + \dot{\Psi})\right] = 0 \tag{10}$$

$$\frac{d}{dt}\left[\ell^2(\dot{\theta} + \dot{\psi})\right] = \frac{\mu r \ell \sin \psi}{2}\left[\frac{1}{r_2^3} - \frac{1}{r_1^3}\right] \tag{11}$$

Equation (10) indicates that the combined angular momentum of the motion is conserved.

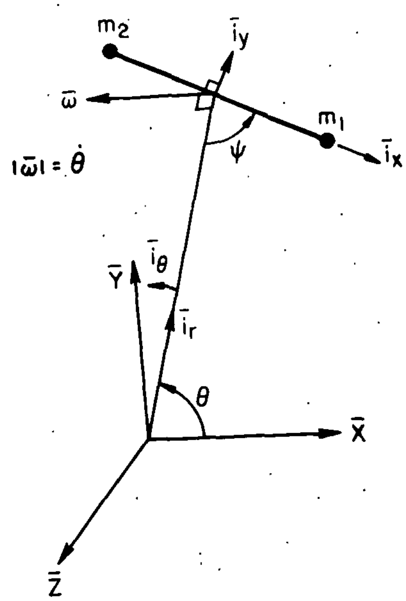

Fig. 4 — Geometry of motion

The bracket on the right-hand side of both Eq. (9) and Eq. (11) can be evaluated with the aid of the relations for r_1 and r_2 of Eq. (8). Realizing that $\ell/r \ll 1$, we can write the difference of the inverse cubes of the radii as

$$\frac{1}{r_2^3} - \frac{1}{r_1^3} = \frac{1}{r^3\left[1 + \left(\frac{\ell}{r}\right)^2 + 2\left(\frac{\ell}{r}\right)\cos\Psi\right]^{3/2}} - \frac{1}{r^3\left[1 + \left(\frac{\ell}{r}\right)^2 - 2\left(\frac{\ell}{r}\right)\cos\Psi\right]^{3/2}}$$

$$\cong \frac{1}{r^3}\left[1 - 3\left(\frac{\ell}{r}\right)\cos\Psi - 1 - 3\left(\frac{\ell}{r}\right)\cos\Psi + 0\left(\frac{\ell}{r}\right)^2 \ldots = -\frac{6\ell}{r^4}\cos\Psi\right.$$

After the right-hand side of Eq. (9) is also evaluated, we come up with the following set of differential equations:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}\left[1 + \frac{3}{2}\left(\frac{\ell}{r}\right)^2 (3\cos^2\Psi - 1) + \ldots\right] \quad (12)$$

$$\frac{d}{dt}\left\{r^2\left[\dot{\theta} + \left(\frac{\ell}{r}\right)^2 (\dot{\theta} + \dot{\Psi})\right]\right\} = 0 \quad (13)$$

$$\ddot{\theta} + \ddot{\Psi} + \frac{3}{2}\frac{\mu}{r^3}\sin 2\Psi = 0 \quad (14)$$

It is observed that the dumbbell size appears only through the parameter $(\ell/r)^2$. In order to be able to express the orbital parameters as power series in $(\ell/r)^2$ it is necessary to treat $(\ell/r)^2$ as a constant. This can be done for orbits of small enough eccentricity, e, when $\ell/r \cong \ell/a$, where a is the semimajor axis of the orbit. Following Moran[6] by assuming expansions of the form

$$\theta = \sum_{n=0}^{\infty} \theta_n \left(\frac{\ell}{a}\right)^{2n}$$

$$r = \sum_{n=0}^{\infty} r_n \left(\frac{\ell}{a}\right)^{2n} \quad (15)$$

there results the zeroth set of equations

$$\ddot{r}_o - r_o\dot{\theta}_o^2 = -\frac{\mu}{r_o^2} \tag{16}$$

$$\frac{d}{dt}\left(r_o^2\dot{\theta}_o \right) = 0 \tag{17}$$

$$\ddot{\Psi} + \frac{3}{2}\frac{\mu}{r_o^3}\sin 2\Psi = -\ddot{\theta}_o \tag{18}$$

Equations (16) and (17) define of course the undisturbed orbit of the center of mass and have as their solutions the series

$$r_o = a\left[1 - e\cos M - \frac{e^2}{2}(\cos 2M - 1) - 0(e^3)\ldots\right] \tag{19}$$

$$\theta_o = M + 2e\sin M + \frac{5}{4}e^2\sin 2M + 0(e^3)\ldots \tag{20}$$

where

$M = n(t - t_o) =$ mean anomaly

$t_o =$ time of perigee passage

$n =$ mean angular velocity in the elliptic orbit $= \sqrt{\frac{\mu}{a^3}}$

Now let

$$\Psi = \Psi_c + \sum_{n=1}^{\infty} \Psi_n e^n \tag{21}$$

where Ψ_c denotes the orientation angle of the dumbbell when the center of mass describes a circular orbit. Substituting Eqs. (19), (20), and (21) into Eq. (18) and collecting like terms in eccentricity gives, to linear terms in eccentricity

$$\ddot{\Psi}_c + \frac{3}{2} n^2 \sin 2\Psi_c = 0 \tag{22}$$

$$\ddot{\Psi}_c + e\ddot{\Psi}_1 + \frac{3}{2} n^2 \left[1 + 3e \cos M\right]\left[\sin 2(\Psi_c + e\Psi_1)\right] = 2en^2 \sin M \tag{23}$$

Approximating $\sin 2(\Psi_c + e\Psi_1)$ by $(\sin 2\Psi_c + 2e\Psi_1 \cos 2\Psi_c)$ and using Eq. (22) gives the following differential equation for Ψ_1

$$\ddot{\Psi}_1 + (3n^2 \cos 2\Psi_c)\, \Psi_1 = 2n^2 \sin M - \frac{9}{2} n^2 \cos M \sin 2\Psi_c \tag{24}$$

We now replace time, t, by the mean anomaly, M, as the independent variable, which eliminates the factor n^2 in Eqs. (22) and (24). Thus

$$\Psi_c'' + \frac{3}{2} \sin 2\Psi_c = 0 \tag{25}$$

$$\Psi_1'' + (3 \cos 2\Psi_c)\, \Psi_1 = 2 \sin M - \frac{9}{2} \cos M \sin 2\Psi_c \tag{26}$$

where $(\ )'$ denotes differentiation with respect to the mean anomaly M.

IV. MOTION IN THE ELLIPTIC ORBIT

We shall now investigate the nature of the solutions of Eq. (26). These depend on the form of the solution of Eq. (25), the differential equation of the circular-orbit case. As shown by Schindler and Moran,[5,6] the solution to Eq. (25 depends on the initial value of Ψ'_{c_i} for $\Psi_c = 0$, and is given by

$$\sin \Psi_c = \left(\Psi'_{c_i} / \sqrt{3} \right) \mathrm{Sn} \sqrt{3}\, \theta_o \qquad |\Psi'_{c_i}| < \sqrt{3} \text{ (oscillation)} \qquad (27a)$$

$$\sin \Psi_c = \mathrm{Sn}\, \Psi'_{c_i} \theta_o \qquad |\Psi'_{c_i}| > \sqrt{3} \text{ (tumbling)} \qquad (27b)$$

The modulus k of the elliptic function is given by

$$k = \Psi'_{c_i} / \sqrt{3} \text{ (oscillation)} \qquad (28a)$$

$$k = \sqrt{3} / \Psi'_{c_i} \text{ (tumbling)} \qquad (28b)$$

Furthermore, for two values of the initial angular velocity, Ψ'_{c_i} = 1.66675, 1.78794, the periods of the oscillatory and tumbling modes, respectively, will equal the orbital period of motion.

A plot of Eq. (27), taken from Ref. 6, is reproduced for convenience in Fig. 5.

SOLUTION FOR LARGE Ψ'_{c_i} (TUMBLING MOTION)

The solution of Eq. (26) with Eq. (27) taken into account is facilitated if the range of possible values of Ψ'_{c_i} is broken up into two separate regions. For large values of Ψ'_{c_i}, say $\Psi'_{c_i} \geq 3$, Fig. 5 shows that the oscillatory ripple around the constant-slope line of Ψ_c versus θ_o has nearly disappeared, so that it is reasonable to approximate sin Ψ_c

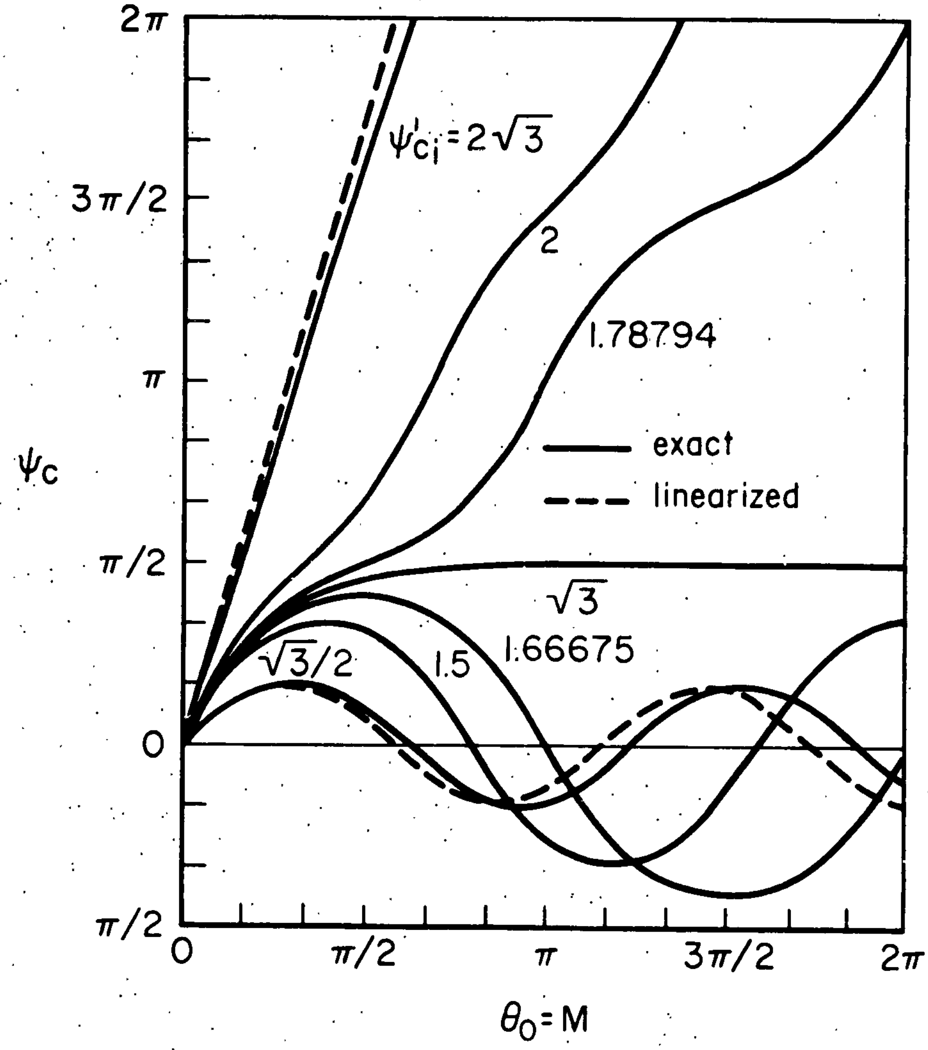

Fig. 5—Time history of rotational motion in a circular orbit (6)

For $\alpha = 3$, $q = -\dfrac{3}{2\alpha^2} = -0.1667$.

The order of the Mathieu function, β, is obtained by equating to each other the two functions v_o and $\bar{v}_o$, which are given by the expressions

$$v_o = \frac{-q/(\beta+2)^2}{1} - \frac{q^2}{(\beta+2)^2\ (\beta+4)^2} - \ldots\ 0(q^3)$$

$$v_o = -\frac{\beta^2}{q} + \frac{q}{(\beta-2)^2} - \frac{q^2}{(\beta-2)^2\ (\beta-4)^2} + \ldots\ 0(q^3) \tag{32}$$

For the present case, a value of $\beta \cong 0.118$ is found. The coefficients A^β_{2r} are solved from the recurrence relations

$$(2r+\beta)^2\ A^\beta_{2r} - q(A^\beta_{2r+2} + A^\beta_{2r-2}) = 0 \qquad (-\infty < r < \infty) \tag{33}$$

When terms beyond A^β_{-4} and A^β_4 are neglected, the other coefficients can be evaluated in terms of A^β_o. We find

$$\begin{aligned} A^\beta_{-4} &= 0.0005205\ A^\beta_o \\ A^\beta_{-2} &= -0.04706\ A^\beta_o \\ A^\beta_2 &= -0.03716\ A^\beta_o \\ A^\beta_4 &= 0.0003651\ A^\beta_o \end{aligned} \tag{34}$$

and the two solutions then become

$$\begin{aligned} \Psi_{11} = Ce_{0.118} = A^\beta_o \big[& 0.0005205 \cos 3.882s + 0.04706 \cos 1.882s \\ & + \cos 0.118s + 0.03716 \cos 2.118s + 0.0003651 \cos 4.118s + \ldots\big] \end{aligned} \tag{35}$$

by sin αM, where M is the mean anomaly, and α is the average value of the angular velocity $d\Psi_c/dM$ which is assumed to be larger than or equal to 3. For convenience of analysis α will be considered an integer. With the additional change of independent variable $s = \alpha M$, Eq. (26) becomes

$$\frac{d^2\Psi_1}{ds^2} + \left(\frac{3}{\alpha^2} \cos 2s \right) \Psi_1 = \frac{2}{\alpha^2} \sin \frac{s}{\alpha} - \frac{9}{2\alpha^2} \cos \frac{s}{\alpha} \sin 2s \qquad (29)$$

which is an inhomogeneous Mathieu equation of the form

$$\frac{d^2z}{dx^2} + \left[a - 2q \cos 2x \right] z = f(x) \qquad (30)$$

with

$$a = 0$$

and

$$q = -\frac{3}{2\alpha^2}$$

In view of the assumption $\alpha \geq 3$, all solutions to Eq. (29) are stable and oscillatory in nature and consist of Mathieu functions of fractional order. For purposes of illustration, the method of obtaining the solution for the case $\alpha = 3$ will be demonstrated below.

NUMERICAL-EXAMPLE SOLUTION

Employing the notation of Ref. 7, the two linearly independent solutions to the homogeneous part of the equation, Ψ_{11} and Ψ_{12}, are the functions $Ce_\beta(s, -q)$ and $Se_\beta(s, -q)$ which are given by the series

$$\begin{aligned} \Psi_{11} &= Ce_\beta(s, -q) = \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}^\beta \cos (2r + \beta)s \\ \Psi_{12} &= Se_\beta(s, -q) = \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}^\beta \sin (2r + \beta)s \end{aligned} \qquad (31)$$

$$\Psi_{12} = Se_{0.118} = A_o^{\beta}\Big[-0.0005205 \sin 3.882s - 0.04706 \sin 1.882s$$

$$+ \sin 0.118s + 0.03716 \sin 2.118s + 0.0003651 \sin 4.118s + \ldots\Big] \quad (36)$$

The complementary solution to Eq. (29) is then

$$\Psi_1 = c_1 Ce_{0.118} + c_2 Se_{0.118} \quad (37)$$

The solutions are nonperiodic but are bounded and tend neither to 0 or ∞ as $s \to \infty$.

If we now define the set of fundamental solutions $\overline{\Psi}_{11}$, $\overline{\Psi}_{12}$ such that

$$\begin{aligned} \overline{\Psi}_{11}(0) &= \overline{\Psi}'_{12}(0) = 1 \\ \overline{\Psi}'_{11}(0) &= \overline{\Psi}_{12}(0) = 0 \end{aligned} \quad (38)$$

the complete solution is given by

$$\Psi_1 = C_1(s)\,\Psi_{11}(s) + C_2(s)\,\Psi_{12}(s) + \Psi_1(0)\,\overline{\Psi}_{11}(s) + \Psi_1'(0)\,\overline{\Psi}_{12}(s) \quad (39)$$

where

$$C_1(s) = -\int_o^s \frac{h(\xi)\,\Psi_{12}(\xi)}{W\big[\Psi_{11}(\xi),\,\Psi_{12}(\xi)\big]}\,d\xi$$

$$C_2(s) = \int_o^s \frac{h(\xi)\,\Psi_{11}(\xi)}{W\big[\Psi_{11}(\xi),\,\Psi_{12}(\xi)\big]}\,d\xi$$

$$h(\xi) = \frac{2}{\alpha^2} \sin \frac{\xi}{\alpha} - \frac{9}{2\alpha^2} \cos \frac{\xi}{\alpha} \sin 2\xi$$

$$W\left[\Psi_{11}(\xi), \Psi_{12}(\xi)\right] = \text{Wronskian of } \Psi_{11} \text{ and } \Psi_{12}$$

Since all the initial conditions on the motion are assumed to have been satisfied by the circular-orbit solution, Ψ_c, we have $\Psi_1(0) = \Psi_1'(0) = 0$, and the complementary portion of the solution consequently vanishes.

The Wronskian W will be a constant because Eq. (29) has no first derivative of Ψ_1, and can therefore be pulled out from the integral sign. Solution (39) can then be written

$$W(0)\Psi_1 = Se_{0.118}(s) \int_0^s h(\xi)\, Ce_{0.118}(\xi) d\xi - Ce_{0.118}(s) \int_0^s h(\xi)\, Se_{0.118}(\xi) d\xi \tag{40}$$

When all the operations are carried out we finally end up with the solution

$$\begin{aligned} \Psi_1 = {} & 1.00028 - 1.0565 \sin 0.3333s + 0.10892 \sin 1.6667s \\ & - 0.01276 \sin 2.3333s - 0.00124 \sin 4.3333s \\ & - 0.00112 \sin 3.6667s + \ldots. \end{aligned} \tag{41}$$

The result is plotted in Fig. 6, which shows the behavior of Ψ_c, Ψ_1, and Ψ_1' as a function of M for one orbital period of the dumbbell.

The magnitude of the perturbation caused by an orbital eccentricity of e = 0.1 is shown by the dashed curve. It is apparent that orientation errors as large as 80 deg could occur during the first orbital period.

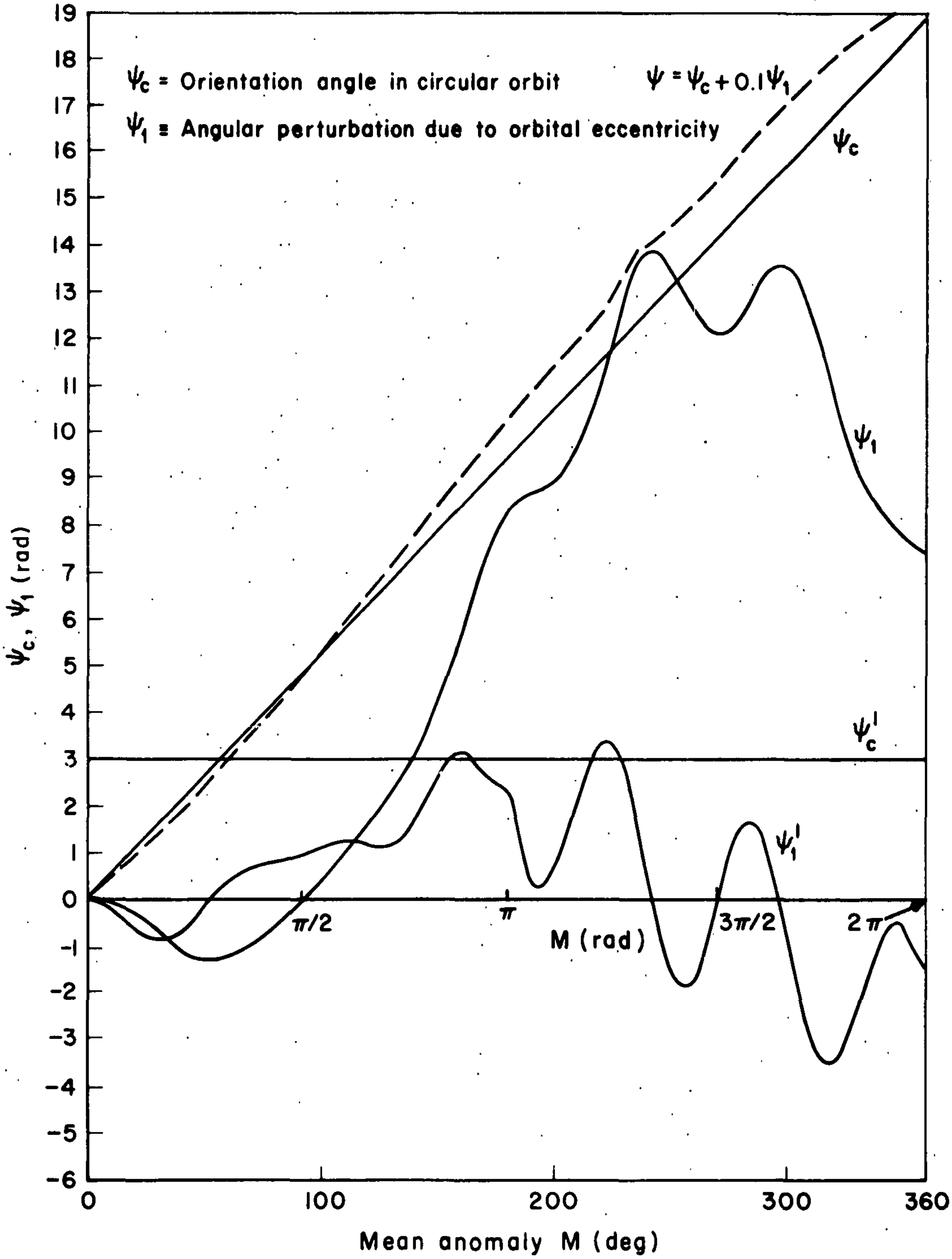

Fig. 6 — Solution for the tumbling case with $\alpha = 3$

SOLUTION FOR SMALL Ψ'_{c_i} (OSCILLATORY MOTION)

As Ψ'_{c_i} is reduced from a value of about 3 and approaches $\sqrt{3}$, the accuracy of the approximation $\Psi_c \approx \Psi'_{c_i} \cdot M$ becomes progressively worse, and the above solution ceases to describe the motion.

Neglecting the region around the condition $\Psi'_{c_i} = \sqrt{3}$, which is only of limited interest, we consider now the form of the solution of Eq. (26) for $\Psi'_{c_i} < \sqrt{3}$.

The functions $\cos 2\Psi_c$ and $\sin 2\Psi_c$ appearing in the differential Eq. (26) can be expressed as functions of M by means of Eq. (27a).

$$\begin{aligned} \cos 2\Psi_c &= 1 - 2\sin^2\Psi_c = 1 - 2k^2 \mathrm{sn}^2 \sqrt{3}\, M \\ \sin 2\Psi_c &= 2\sin\Psi_c \cos\Psi_c = 2k\, \mathrm{sn}\sqrt{3}\, M\, \mathrm{dn}\sqrt{3}\, M = -2k \frac{d}{d(\sqrt{3}\, M)} \mathrm{cn}\sqrt{3}\, M \end{aligned} \tag{42}$$

In order to bring Eq. (26) into a more tractable form, the functions $\mathrm{sn}\sqrt{3}\, M$ and $\mathrm{cn}\sqrt{3}\, M$ are expanded in Fourier series.

$$\begin{aligned} \frac{kK}{2\pi} \mathrm{sn}\sqrt{3}\, M &= \frac{\sqrt{\bar{q}}}{1 - \bar{q}} \sin\frac{\sqrt{3}\,\pi M}{2K} + \frac{\sqrt{\bar{q}^3}}{1 - \bar{q}^3} \sin\frac{3\sqrt{3}\,\pi M}{2K} + \frac{\sqrt{\bar{q}^5}}{1 - \bar{q}^5} \sin\frac{5\sqrt{3}\,\pi M}{2K} + \ldots \\ &= \sum_{r=0}^{\infty} \frac{\bar{q}^{r+1/2}}{1 - \bar{q}^{2r+1}} \sin\frac{(2r+1)\sqrt{3}\,\pi M}{2K} \end{aligned} \tag{43a}$$

$$\frac{kK}{2\pi} \mathrm{cn}\sqrt{3}\, M = \sum_{r=0}^{\infty} \frac{\bar{q}^{r+1/2}}{1 + \bar{q}^{2r+1}} \cos\frac{(2r+1)\sqrt{3}\,\pi M}{2K} \tag{43b}$$

$$\bar{q} = e^{-\pi K'/K}$$

$$K = K(k) = \int_0^{\pi/2} \frac{d\phi}{1 - k^2 \sin^2\phi}$$

$$K' = K(k')$$

$$k'^2 = 1 - k^2$$

For sufficiently small values of $\bar{q}$, the retention of the leading term in each of Eqs. (43a) and (43b) might give sufficient accuracy. A rough idea of the size of q is obtained by taking for instance the case $\Psi'_{c_i} = 1.66$ as a rough indication of an upper bound on the angular velocity. This corresponds to equality in orbital and librational periods. For this case

$$k = \frac{1.66}{\sqrt{3}} = 0.958 \qquad K(k) = 2.8$$

$$k' = 0.2868 \qquad K(k') \cong 1.6$$

$$\bar{q} = e^{-\pi\frac{1.6}{2.8}} = 0.165$$

As Ψ'_{c_i} decreases, k will decrease and k′ will increase, so that K will be reduced and K′ increased; this will lead to a decrease in the size of $\bar{q}$. Thus, for

$$\Psi'_{c_i} < 1.66$$

$$\bar{q} < 0.165$$

Placing an upper limit of 1.66 on Ψ'_{c_i}, it appears that the retention of only one term in the Fourier expansions would be satisfactory in the present case. Under this approximation terms which are of magnitude $\bar{q}$ have been neglected in comparison with unity. Thus

$$\cos 2\Psi_c \cong 1 - P^2 \sin^2 \frac{\sqrt{3}\,\pi M}{2K} = 1 - \frac{P^2}{2} + \frac{P^2}{2} \cos \frac{\sqrt{3}\,\pi M}{K}$$

$$\sin 2\Psi_c \cong \frac{2\pi^2 \sqrt{\bar{q}}}{kK^2(1 + \bar{q})} \sin \frac{\sqrt{3}\,\pi M}{2K}$$

$$P^2 = \frac{8\pi^2 \bar{q} \Psi'^2_{c_i}}{3k^2K^2(1 - \bar{q})^2} = \frac{8\pi^2 \bar{q}}{K^2(1 - \bar{q})^2}$$

(Note that $k = \Psi'_{c_i} / \sqrt{3}$ by definition.)

Making the additional transformation of independent variables

$$z = \frac{\sqrt{3}\,\pi M}{2K} \tag{44}$$

we obtain Eq. (26) in its final form

$$\frac{d^2\Psi_1}{dz^2} + \left[a - 2q \cos 2z\right] \Psi_1 = \frac{8K^2}{3\pi^2} \sin \frac{2K}{\sqrt{3}\,\pi} z - \frac{6\sqrt{\bar{q}}}{k(1+\bar{q})} \left[\sin\left(\frac{2K}{\sqrt{3}\,\pi} + 1\right) z + \sin\left(1 - \frac{2K}{\sqrt{3}\,\pi}\right) z\right] \tag{45}$$

where

$$a = \frac{4K^2}{\pi^2} - 16 \frac{\bar{q}}{(1-\bar{q})^2}$$

$$q = -8 \frac{\bar{q}}{(1-\bar{q})^2}$$

As in the last case, here too the motion of libration is governed by an inhomogeneous Mathieu equation. The bounded oscillatory form of the forcing function suggests immediately that for finite values of Ψ'_{c_i} the nature of the forced solution of Eq. (45) depends on the stability or instability exhibited by the solution to the homogeneous Mathieu equation. This question is easily resolved with the aid of Fig. 7, which presents in the plane of (a) versus (q) the stable and unstable regions of the solution to Mathieu's equation. Due to symmetry with respect to the a axis, the right side of the Mathieu plane has been used.

The curves Ce_o, Se_1, and Ce_1 denote respectively the loci of zeroth and first-order periodic solutions to Mathieu's equations and represent the transition curves between the stable and unstable regions shown.

The directed dashed curve indicates the trace of the (a,q) point of Eq. (45) as the initial value of angular velocity Ψ'_{c_i} tends toward zero. Except for a narrow region around $a = 1$, $q = 0$, the homogeneous solution is at all times oscillatory and divergent as z increases, making the complete solution unstable. That segment of the dashed curve which appears to follow closely along the Se_1 curve has associated with it divergent, though periodic, homogeneous solutions.

Although the scale of Fig. 7 does not make this apparent, a simplified analysis shows that the dashed curve approaches the point $a = 1$, $q = 0$ with a slope $da/d|q| = -2$. Since the Se_1 curve exhibits at this point a slope of -1, it appears that the dashed curve penetrates into the stable region of the plane as Ψ'_{c_i} assumes vanishingly small values.

In the limit, as

$$\Psi'_{c_i} \to 0$$

$$K \to \pi/2$$

$$a \to 1$$

$$\frac{k}{\sqrt{q}} \to 4$$

Equation (45) is reduced to the form

$$\frac{d^2\Psi_1}{dz^2} + \Psi_1 = \frac{2}{3} \sin \frac{1}{\sqrt{3}} z - \frac{3}{2}\left[\sin\left(1 + \frac{1}{\sqrt{3}}\right) z + \sin\left(1 - \frac{1}{\sqrt{3}}\right) z\right] \tag{46}$$

The complementary solution of Eq. (46) will be bounded and periodic, but the forced solution, though bounded too, will not exhibit any periodicity. The form of the arguments of the trigonometric terms in the forcing function precludes the occurrence of resonance conditions.

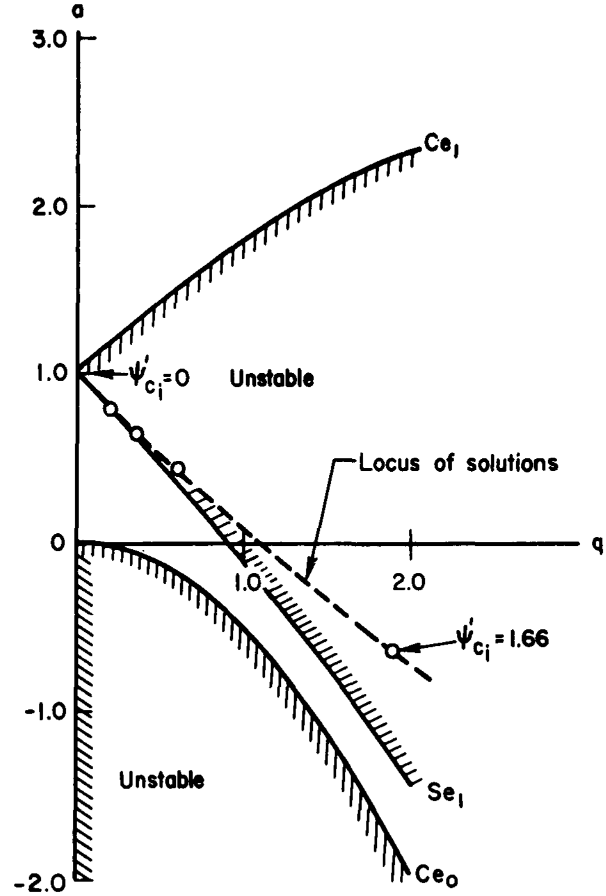

Fig.7—Effect of initial angular velocity ψ'_{c_i} on the location of the solution point in the a, q plane

VI. CONCLUSION

The planar rotational motion of a dumbbell-shaped satellite whose center of mass moves along an undisturbed elliptic orbit of slight eccentricity has been analyzed. It has been shown that the correction terms due to eccentricity, which must be superimposed on the angular-position time behavior of the same satellite when moving along a circular orbit, are always bounded and oscillatory for all values of initial angular velocity in excess of roughly 3 times the orbital angular velocity. Depending on the magnitude of the eccentricity, these perturbations could be quite appreciable.

When the initial motion is one of pendulous oscillations, the first-order effect of eccentricity is to introduce a divergent oscillatory term into the time behavior of the first perturbation term and thus to cause the orientation angle in the elliptic orbit to differ significantly from that assumed in the circular orbit. Because of possible phase differences in the time behavior of the circular orientation angle Ψ_c and the perturbed correction term Ψ_1, it is still possible for the complete solution not to exhibit an actual boundless increase in amplitude as long as the restrictions imposed by linearity are not violated.