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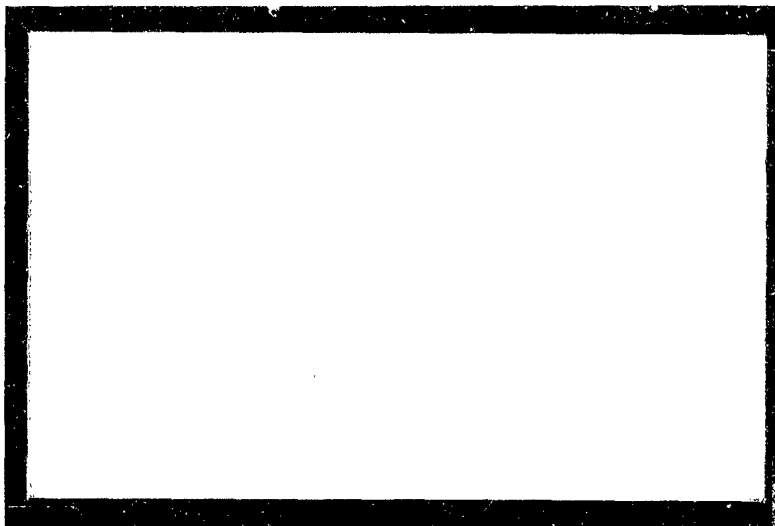
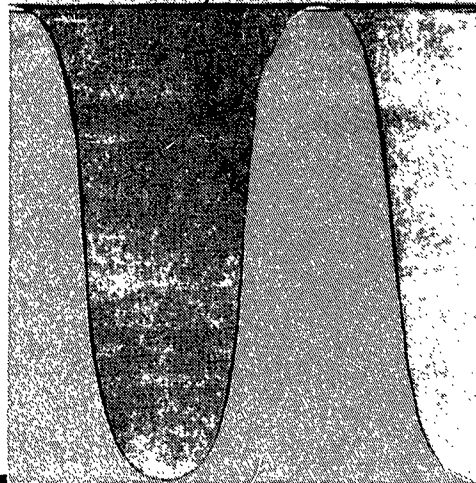
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EXTREMAL SPECTRAL FUNCTIONS
OF A SYMMETRIC OPERATOR

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ABSTRACT

It is known that the finite dimensional extensions of a symmetric operator define extremal spectral functions of the operator. Finite dimensional extensions exist, however, only for symmetric operators with equal deficiency indices. In this report it is shown that self adjoint extensions defined by the addition of maximal symmetric operators determine extremal spectral functions for a symmetric operator with unequal deficiency indices.

EXTREMAL SPECTRAL FUNCTIONS OF A SYMMETRIC OPERATOR

Richard C. Gilbert

1. Spectral functions of a symmetric operator.

Let H_1 be a symmetric operator in a Hilbert space $\mathcal{H}_1$. If H is a self adjoint operator in a Hilbert space $\mathcal{H}$ such that $\mathcal{H}_1 \subset \mathcal{H}$ and $H_1 \subset H$, then H is called a self adjoint extension of H_1 .

Suppose H is a self adjoint extension of H_1 . If $E(\lambda)$ is the spectral function of H and if P_1 is the operator in $\mathcal{H}$ of orthogonal projection on $\mathcal{H}_1$, then the operator function $E_1(\lambda) = P_1 E(\lambda)$ restricted to $\mathcal{H}_1$ is called a spectral function of H_1 . We shall say that the self adjoint extension H defines the spectral function $E_1(\lambda)$. There are in general many spectral functions, since there are in general many different self adjoint extensions. (The spectral functions of H_1 can also be characterized without going out of the space $\mathcal{H}_1$. See Achieser and Glasmann [1] and M. A. Naimark [4].) If $\mathcal{H} = \mathcal{H}_1$, then $E_1(\lambda)$ is called an orthogonal spectral function of H_1 .

The family of spectral functions of H_1 is a convex set, i. e. if $E_1^I(\lambda)$ and $E_1^{II}(\lambda)$ are spectral functions of H_1 , and if μ^I, μ^{II} are non-negative real numbers such that $\mu^I + \mu^{II} = 1$, then $\mu^I E_1^I(\lambda) + \mu^{II} E_1^{II}(\lambda)$ is also a spectral function of H_1 . A spectral function $E_1(\lambda)$ of H_1 is said to be an extremal spectral function if it is impossible to find two different spectral functions $E_1^I(\lambda), E_1^{II}(\lambda)$ and positive real numbers

μ', μ'' such that $E_1(\lambda) = \mu' E_1'(\lambda) + \mu'' E_1''(\lambda)$.

It is the purpose of this report to identify some extremal spectral functions of H_1 . Extremal spectral functions are of interest because it is often possible to construct the whole convex set from them.

denotes the end of a proof.

2. Hermitian operators

In this section we collect some information about Hermitian operators. Proofs are omitted because they are either direct verification or else are the same as for a symmetric operator. (See [1].)

Definition 1. The linear operator H in the Hilbert space $\mathcal{H}$ is Hermitian if $(Hf, g) = (f, Hg)$ for all $f, g \in \mathcal{D}(H)$. An operator H is symmetric if it is Hermitian and $\overline{\mathcal{D}(H)} = \mathcal{H}$.

Definition 2. If H is Hermitian, we define the linear manifolds $\mathcal{M}(\lambda)$ and $\mathcal{L}(\lambda)$ by the equations $\mathcal{L}(\lambda) = \mathcal{R}(A - \bar{\lambda}E)$ and $\mathcal{M}(\lambda) = \mathcal{H} \ominus \mathcal{L}(\lambda)$. $\mathcal{M}(\lambda)$ is a subspace and is called a deficiency subspace of H .

Theorem 1. $\mathcal{M}(\lambda)$ has the same dimension for all λ in the same half-plane (i.e., $\text{Im } \lambda > 0$ or $\text{Im } \lambda < 0$).

Definition 3. If λ is a non-real number, let $m = \dim \mathcal{M}(\bar{\lambda})$, $n = \dim \mathcal{M}(\lambda)$. Then, (m, n) are called deficiency indices of H (with respect to λ .)

Theorem 2. If H is Hermitian and $\text{Im } \lambda \neq 0$, then

- (1) $(H - \lambda E)^{-1}$ exists and is bounded;
- (2) $U(\lambda) = (H - \bar{\lambda}E)(H - \lambda E)^{-1}$ is an isometry mapping $\mathcal{L}(\bar{\lambda})$ onto $\mathcal{L}(\lambda)$;

(3) $(U(\lambda) - E)^{-1}$ exists, and $H = (\lambda U(\lambda) - \bar{\lambda}E)(U(\lambda) - E)^{-1}$.

$U(\lambda)$ is called the Cayley transform of H .

Theorem 3. Let U be an isometric operator in $\mathcal{H}$. Suppose that $(U - E)^{-1}$ exists. Then, if $\lambda \neq 0$, there exists a Hermitian operator H such that $U = U(\lambda)$. In fact, $H = (\lambda U - \bar{\lambda}E)(U - E)^{-1}$.

Theorem 4. For fixed λ , $\lambda \neq 0$, the correspondence $H \sim U(\lambda)$ between a Hermitian operator and its Cayley transform is a one-one correspondence between the set of Hermitian operators H and the set of isometric operators U for which $(U - E)^{-1}$ exists.

Theorem 5. If $H_1 \sim U_1(\lambda)$, $H_2 \sim U_2(\lambda)$, then $H_1 \subset H_2$ if and only if $U_1(\lambda) \subset U_2(\lambda)$.

Theorem 6. H is closed $\equiv U(\lambda)$ is closed $\equiv$.

$\mathcal{L}(\lambda)$ and $\mathcal{L}(\bar{\lambda})$ are subspaces in $\mathcal{H}$.

Remark. If H is closed and $\lambda \neq 0$, then $\mathcal{H} = \mathcal{L}(\lambda) \oplus \mathcal{M}(\lambda)$.

Theorem 7. If H is a closed Hermitian operator with deficiency indices (m, n) (with respect to λ), $-H$ is a closed Hermitian operator with deficiency indices (n, m) (with respect to λ .)

Theorem 8. A subspace $\mathcal{H}_1$ reduces $H \equiv \mathcal{H}_1$ reduces $U(\lambda)$. If $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$ and H_1 is H restricted to $\mathcal{H}_1$ while $U_1(\lambda)$ is $U(\lambda)$ is restricted to $\mathcal{H}_1$, then H_1 and H_2 are Hermitian operators, $H_1 \sim U_1(\lambda)$, $H = H_1 \oplus H_2$, and $U(\lambda) = U_1(\lambda) \oplus U_2(\lambda)$.

In all that follows λ will be a fixed non-real number. Hence, we shall often write the Cayley transforms of H as U rather than $U(\lambda)$.

3. Self adjoint extensions of a symmetric operator.

The following theorem, due to M. A. Naimark [4], characterizes the self adjoint extensions of a symmetric operator.

Theorem 9. Let λ be any fixed non-real number. Let H_1 be a closed symmetric operator with deficiency indices (m_1, n_1) (with respect to λ). Then every self adjoint extension H of H_1 is obtained as follows:

(1) Let H_2 be a closed Hermitian operator in $\mathfrak{H}_2$ with deficiency indices (m_2, n_2) (with respect to λ) satisfying $m_1 + m_2 = n_1 + n_2$, $m_2 \leq n_1$.

(2) Let $H_0 = H_1 \oplus H_2$ in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. (H_0 is therefore a closed Hermitian operator with equal deficiency indices $(m_1 + m_2, n_1 + n_2)$, and if $H_i \sim U_i$, $i = 0, 1, 2$, then $U_0 = U_1 \oplus U_2$. Further, $\mathcal{M}_0(\bar{\lambda}) = \mathcal{M}_1(\bar{\lambda}) \oplus \mathcal{M}_2(\bar{\lambda})$, $\mathcal{M}_0(\lambda) = \mathcal{M}_1(\lambda) \oplus \mathcal{M}_2(\lambda)$.)

(3) Let V be an arbitrary isometric operator mapping $\mathcal{M}_0(\bar{\lambda})$ onto $\mathcal{M}_0(\lambda)$ satisfying the condition $\varphi \in \mathcal{M}_2(\bar{\lambda})$, $V\varphi \in \mathcal{M}_2(\lambda)$ implies $\varphi = 0$.

(4) Let $\mathcal{N}(H)$ be defined as all $g = f + V\varphi - \bar{\lambda}\varphi$, where $f \in \mathcal{N}(H_0)$, $\varphi \in \mathcal{M}_0(\bar{\lambda})$.

(5) If $g \in \mathcal{N}(H)$, let $Hg = H_0f + \lambda V\varphi - \bar{\lambda}\varphi$.

Then, H is a self adjoint extension in $\mathfrak{H}$ of H_1 , and every self adjoint extension of H_1 is obtained in this way. We have that $\mathcal{N}(H_2) = \mathcal{N}(H) \cap \mathfrak{H}_2$.

Definition 4. We say that H_2 and V of theorem 9 define the self adjoint extension H .

Remark 1. What has been really done in theorem 9 is that V has been used to extend the Cayley transform U_0 to a unitary operator $U = U_0 \oplus V$. The condition on V in (3) of theorem 9 allows us to show that $\mathcal{R}(U - E)$ is dense in $\mathcal{H}$ and therefore U is the Cayley transform of a self adjoint operator H . $\mathcal{A}(H)$ and H can be shown to be determined as in (4) and (5). Since $U_1 \subset U$, $H_1 \subset H$.

The condition on V also serves another purpose: It allows us to say that U (and therefore H) is not reduced by $\mathcal{H}_1$, provided $m_2 \neq 0$ or $n_2 \neq 0$.

Remark 2. We can put the operator V into correspondence with a matrix (V_{ik}) of operators such that $V_{11} : \mathcal{M}_1(\bar{\lambda}) \rightarrow \mathcal{M}_1(\lambda)$, $V_{12} : \mathcal{M}_2(\bar{\lambda}) \rightarrow \mathcal{M}_1(\lambda)$, $V_{21} : \mathcal{M}_1(\bar{\lambda}) \rightarrow \mathcal{M}_2(\lambda)$, $V_{22} : \mathcal{M}_2(\bar{\lambda}) \rightarrow \mathcal{M}_2(\lambda)$. The condition on V in (3) of theorem 9 then becomes $V_{12}\varphi = 0$ implies $\varphi = 0$. Further,

$$\mathcal{A}(H) = \{g \mid g = f_1 - \varphi_1 + V_{11}\varphi_1 + V_{12}\varphi_2 + f_2 - \varphi_2 + V_{21}\varphi_1$$

$$+ V_{22}\varphi_2, \text{ where } f_1 \in \mathcal{A}(H_1), f_2 \in \mathcal{A}(H_2), \varphi_1 \in \mathcal{M}_1(\bar{\lambda}),$$

$$\varphi_2 \in \mathcal{M}_2(\bar{\lambda})\}.$$

If $g \in \mathcal{L}(H)$,

$$Hg \cong H_1 f_1 - \bar{\lambda} \varphi_1 + \lambda (V_{11} \varphi_1 + V_{12} \varphi_2) + H_2 f_2 - \bar{\lambda} \varphi_2 + \lambda (V_{21} \varphi_1 + V_{22} \varphi_2).$$

Remark 3. If H_1 is a closed symmetric operator with deficiency indices (m_1, n_1) and if H_2 is a closed Hermitian operator with deficiency indices (m_2, n_2) such that $m_1 + m_2 = n_1 + n_2$ and $m_2 \leq n_1$, then there always exists an isometry V of $\mathcal{M}_1(\bar{\lambda}) \oplus \mathcal{M}_2(\bar{\lambda})$ onto $\mathcal{M}_1(\lambda) \oplus \mathcal{M}_2(\lambda)$ satisfying the condition that $\varphi \in \mathcal{M}_2(\bar{\lambda})$, $V\varphi \in \mathcal{M}_2(\lambda)$ implies $\varphi = 0$. For, let V map $\mathcal{M}_2(\bar{\lambda})$ isometrically onto a subspace $\bar{\mathcal{M}}_1(\lambda)$ of $\mathcal{M}_1(\lambda)$ and $\mathcal{M}_1(\bar{\lambda})$ isometrically onto $[\mathcal{M}_1(\lambda) \ominus \bar{\mathcal{M}}_1(\lambda)] \oplus \mathcal{M}_2(\lambda)$.

We now give a theorem which gives a more detailed analysis of the structure of V .

Theorem 10. Suppose that $\mathcal{M}_1(\lambda)$, $\mathcal{M}_1(\bar{\lambda})$, $\mathcal{M}_2(\lambda)$, $\mathcal{M}_2(\bar{\lambda})$ are Hilbert spaces and that V is an isometry which maps $\mathcal{M}_1(\bar{\lambda}) \oplus \mathcal{M}_2(\bar{\lambda})$ onto $\mathcal{M}_1(\lambda) \oplus \mathcal{M}_2(\lambda)$. (Note that λ has nothing to do with the theorem and is retained only as a notational convenience.) If $V = (V_{ik})$ in matrix form (note that each V_{ik} is bounded by 1), suppose that $V_{12}\varphi = 0$ implies that $\varphi = 0$. Then the following conclusions are true:

(1) If $\bar{\mathcal{M}}_1(\lambda)$ is defined by the equation $\bar{\mathcal{M}}_1(\lambda) = [V_{12}\mathcal{M}_2(\bar{\lambda})]^c$

(c indicates closure of a set) and if $\mathcal{N}_1(\lambda)$ is defined by

$\mathcal{N}_1(\lambda) = \mathcal{M}_1(\lambda) \ominus \bar{\mathcal{M}}_1(\lambda)$, then $\mathcal{N}_1(\lambda)$ is the null space of V_{12}^* . Thus,

V_{12}^* is one-one on $\bar{\mathcal{M}}_1(\bar{\lambda})$. Further, $\mathcal{M}_2(\bar{\lambda}) = [V_{12}^* \bar{\mathcal{M}}_1(\bar{\lambda})]^c$.

(2) $V^* = V^{-1}$ maps $\mathcal{N}_1(\lambda)$ onto a subspace of $\mathcal{M}_1(\bar{\lambda})$, which we denote by $\mathcal{N}_1(\bar{\lambda})$. Thus, $\mathcal{N}_1(\bar{\lambda}) = V^* \mathcal{N}_1(\lambda)$, $\mathcal{N}_1(\lambda) = V \mathcal{N}_1(\bar{\lambda})$.

(3) If $\bar{\mathcal{M}}_1(\bar{\lambda})$ is defined by the equation $\bar{\mathcal{M}}_1(\bar{\lambda}) = \mathcal{M}_1(\bar{\lambda}) \ominus \mathcal{N}_1(\bar{\lambda})$, then V maps $\bar{\mathcal{M}}_1(\bar{\lambda}) \oplus \mathcal{M}_2(\bar{\lambda})$ isometrically onto $\bar{\mathcal{M}}_1(\lambda) \oplus \mathcal{M}_2(\lambda)$.

Thus, $V_{11} : \bar{\mathcal{M}}_1(\bar{\lambda}) \rightarrow \bar{\mathcal{M}}_1(\lambda)$.

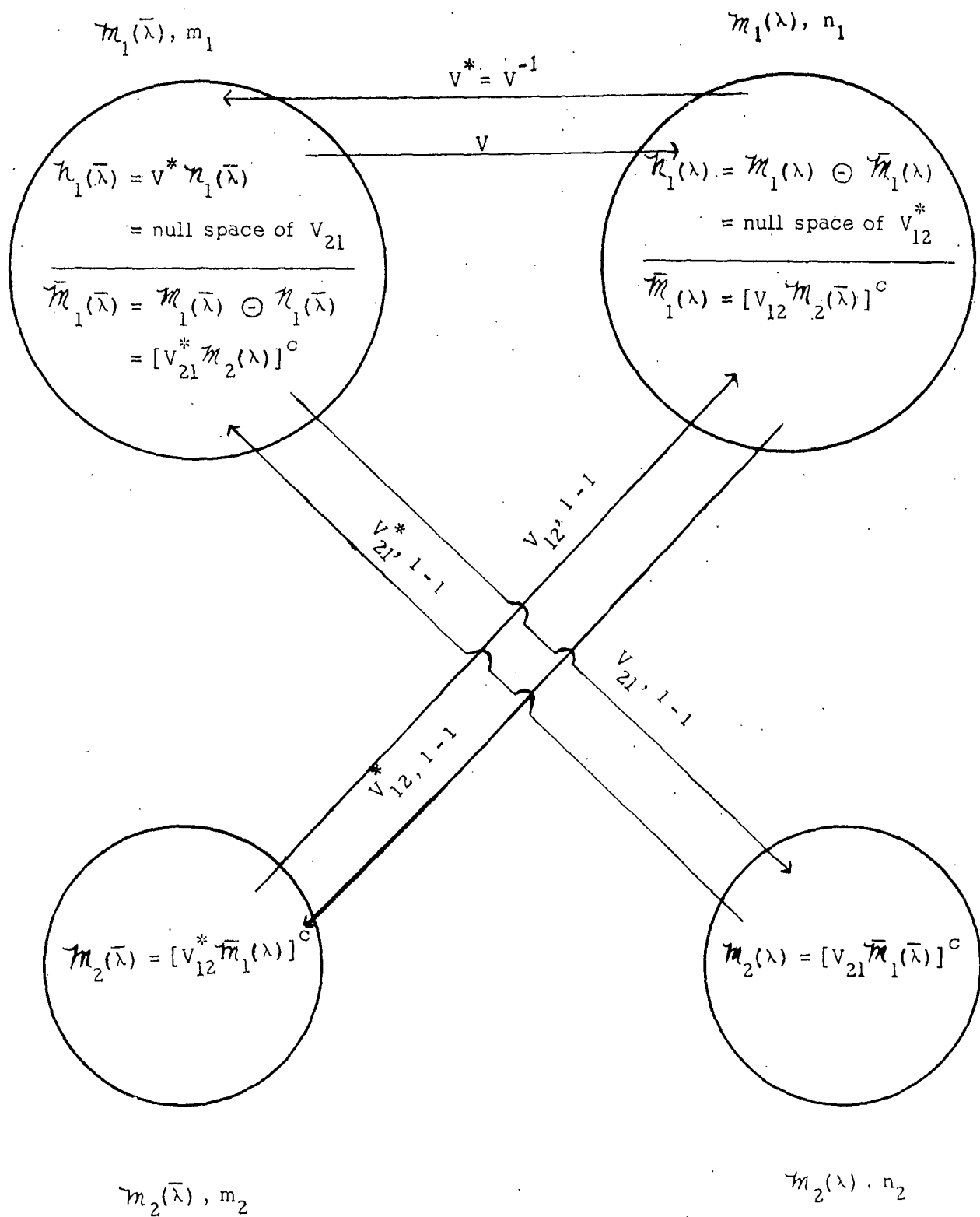
(4) V_{21} is one-one on $\bar{\mathcal{M}}_1(\bar{\lambda})$, and $\mathcal{N}_1(\bar{\lambda})$ is the null space of V_{21} .
 $\mathcal{M}_2(\lambda) = [V_{21} \bar{\mathcal{M}}_1(\bar{\lambda})]^c$.

(5) V_{21}^* is one-one on $\mathcal{M}_2(\lambda)$ and $\bar{\mathcal{M}}_1(\bar{\lambda}) = [V_{21}^* \mathcal{M}_2(\lambda)]^c$.

(6) If $m_1 = \dim \mathcal{M}_1(\bar{\lambda})$, $n_1 = \dim \mathcal{N}_1(\lambda)$, $m_2 = \dim \mathcal{M}_2(\bar{\lambda})$, $n_2 = \dim \mathcal{M}_2(\lambda)$, then $m_1 + m_2 = n_1 + n_2$, $m_2 = \dim \mathcal{M}_2(\bar{\lambda}) = \dim \bar{\mathcal{M}}_1(\lambda) \leq n_1$,
 $n_2 = \dim \mathcal{M}_2(\lambda) = \dim \bar{\mathcal{M}}_1(\bar{\lambda}) \leq m_1$.

(7) If $m_2 = n_2$, $m_1 = n_1$.

We may conveniently summarize the theorem by means of the following picture:



$$m_1 + m_2 = n_1 + n_2$$

$$m_2 \leq n_1$$

$$n_2 \leq m_1$$

Proof (1) $\mathcal{N}_1(\lambda)$ is the null space of V_{12}^* ; for, $f \in \mathcal{N}_1(\lambda) \equiv (f, V_{12}g) = 0$ for all $g \in \mathcal{M}_2(\bar{\lambda}) \equiv (V_{12}^*f, g) = 0$ for all $g \in \mathcal{M}_2(\bar{\lambda}) \equiv V_{12}^*f = 0$.

$\mathcal{M}_2(\bar{\lambda}) = [V_{12}^* \mathcal{N}_1(\lambda)]^c$; for, suppose $g \in \mathcal{M}_2(\bar{\lambda})$, $g \perp V_{12}^* \mathcal{N}_1(\lambda)$. Then, $(g, V_{12}^*f) = 0$ for all $f \in \mathcal{N}_1(\lambda)$, or $(V_{12}g, f) = 0$ for all $f \in \mathcal{N}_1(\lambda)$. Therefore, $V_{12}g = 0$, and $g = 0$ since V_{12} is one-one.

(2) $V^* = V^{-1}$ maps $\mathcal{N}_1(\lambda)$ onto a subspace of $\mathcal{M}_1(\bar{\lambda})$; for

$$V^* = \begin{pmatrix} V_{11}^* & V_{21}^* \\ V_{12}^* & V_{22}^* \end{pmatrix}$$

Hence, $V^* \mathcal{N}_1(\lambda) = V_{11}^* \mathcal{N}_1(\lambda) \subset \mathcal{M}_1(\bar{\lambda})$.

(3) Clear, since $\mathcal{N}_1(\lambda) = V \mathcal{N}_1(\bar{\lambda})$.

(4) V_{21} is one-one on $\mathcal{M}_1(\bar{\lambda})$; for, suppose $f \in \mathcal{M}_1(\bar{\lambda})$, $V_{21}f = 0$.

Then, $Vf = V_{11}f + V_{21}f = V_{11}f \in \mathcal{M}_1(\lambda)$. Let $g = V_{11}f = Vf$, so that $f = V^*g = V_{11}^*g + V_{12}^*g$. Then, since $f \in \mathcal{M}_1(\bar{\lambda})$, $V_{11}^*g \in \mathcal{M}_1(\bar{\lambda})$, $V_{12}^*g \in \mathcal{M}_2(\bar{\lambda})$, we have that $V_{12}^*g = 0$. Since $g \in \mathcal{M}_1(\lambda)$, by (1) $g = 0$. Thus, $f = V^*g = 0$.

$\mathcal{N}_1(\bar{\lambda})$ is the null space of V_{21} ; for, $\mathcal{N}_1(\lambda) = V \mathcal{N}_1(\bar{\lambda})$ and thus $V_{21}f = 0$ for all $f \in \mathcal{N}_1(\bar{\lambda})$. On the other hand, V_{21} is one-one on $\mathcal{M}_1(\bar{\lambda})$.

$V_{21}^* \mathcal{M}_2(\lambda) \subset \bar{\mathcal{M}}_1(\bar{\lambda})$; for, if $f \in \mathcal{N}_1(\bar{\lambda})$, $(f, V_{21}^* g) = (V_{21} f, g) = 0$, since $\mathcal{N}_1(\bar{\lambda})$ is the null space of V_{21} . Thus, $\mathcal{N}_1(\bar{\lambda}) \perp V_{21}^* \mathcal{M}_2(\lambda)$ and therefore $V_{21}^* \mathcal{M}_2(\lambda) \subset \bar{\mathcal{M}}_1(\bar{\lambda})$.

$\mathcal{M}_2(\lambda) = [V_{21}^* \bar{\mathcal{M}}_1(\bar{\lambda})]^C$; for, suppose $g \in \mathcal{M}_2(\lambda)$ and $g \perp V_{21}^* \bar{\mathcal{M}}_1(\bar{\lambda})$. Therefore, $0 = (V_{21} f, g) = (f, V_{21}^* g)$ for all $f \in \bar{\mathcal{M}}_1(\bar{\lambda})$. Since $V_{21}^* \mathcal{M}_2(\lambda) \subset \bar{\mathcal{M}}_1(\bar{\lambda})$, $V_{21}^* g = 0$. Thus, $V^* g = V_{22}^* g \in \mathcal{M}_2(\bar{\lambda})$. Let $f = V^* g$. Then, $g = Vf = V_{12} f + V_{22} f$, where $g \in \mathcal{M}_2(\lambda)$, $V_{12} f \in \bar{\mathcal{M}}_1(\lambda)$, $V_{22} f \in \mathcal{M}_2(\lambda)$. Therefore, $V_{12} f = 0$ and $f = 0$. Whence, $g = Vf = 0$.

(5) V_{21}^* is one-one on $\mathcal{M}_2(\lambda)$; for, suppose $V_{21}^* f = 0$. Then, $0 = (V_{21}^* f, g) = (f, V_{21} g)$ for all $g \in \bar{\mathcal{M}}_1(\bar{\lambda})$. Therefore, $f \perp V_{21}^* \bar{\mathcal{M}}_1(\bar{\lambda})$ and $f = 0$ by (4).

$\bar{\mathcal{M}}_1(\bar{\lambda}) = [V_{21}^* \mathcal{M}_2(\lambda)]^C$; for, suppose $f \in \bar{\mathcal{M}}_1(\bar{\lambda})$, $f \perp V_{21}^* \mathcal{M}_2(\lambda)$. Then, $0 = (f, V_{21}^* g) = (V_{21} f, g)$ for all $g \in \mathcal{M}_2(\lambda)$. Therefore $V_{21} f = 0$, and $f = 0$ since V_{21} is one-one on $\bar{\mathcal{M}}_1(\bar{\lambda})$.

(6) $m_1 + m_2 = n_1 + n_2$ follows from the fact that V maps $\mathcal{M}_1(\bar{\lambda}) \oplus \mathcal{M}_2(\bar{\lambda})$ isometrically onto $\mathcal{M}_1(\lambda) \oplus \mathcal{M}_2(\lambda)$.

$\dim \mathcal{M}_2(\bar{\lambda}) = \dim \bar{\mathcal{M}}_1(\lambda)$; for, let $\{\varphi_\alpha\}$ be a complete orthonormal system in $\mathcal{M}_2(\bar{\lambda})$. Then $\{V_{12} \varphi_\alpha\}$ is a fundamental set in $\bar{\mathcal{M}}_1(\lambda)$. (See Nagy [3] for definitions.) Therefore $\dim \mathcal{M}_2(\bar{\lambda}) = P\{\varphi_\alpha\} = P\{V_{12} \varphi_\alpha\} \geq \dim \bar{\mathcal{M}}_1(\lambda)$, where P stands for cardinality. Similarly, using V_{12}^* , $\dim \bar{\mathcal{M}}_1(\lambda) \geq \dim \mathcal{M}_2(\bar{\lambda})$. Thus, $m_2 = \dim \mathcal{M}_2(\bar{\lambda}) = \dim \bar{\mathcal{M}}_1(\lambda) \leq n_1$.

Similarly, using V_{21} and V_{21}^* , $n_2 = \dim \mathcal{M}_2(\lambda) = \dim \overline{\mathcal{M}}_1(\bar{\lambda}) \leq m_1$.

(7) Clear from (6). #

Definition 5. A self adjoint extension H in $\mathcal{H}$ of a symmetric operator H_1 in $\mathcal{H}_1$ is said to be minimal if H is not reduced by $\mathcal{H} \ominus \mathcal{H}_1$ nor by any of its subspaces different from zero.

Theorem 11. (M.A. Naimark [4]) For each self adjoint extension H in $\mathcal{H}$ of a symmetric operator H_1 in $\mathcal{H}_1$ there exists a minimal self adjoint extension H_0 in $\mathcal{H}_0$ such that

$$(1) \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H};$$

$$(2) H_1 \subset H_0 \subset H;$$

$$(3) H_0 \text{ and } H \text{ define the same spectral function of } H_1.$$

Theorem 12. Suppose that H_1 is a closed symmetric operator and that H_2 and V define a self adjoint extension H of H_1 . Let H_0 be a self adjoint extension of H_1 having the properties that $\mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}$ and $H_1 \subset H_0 \subset H$. Then the following statements are true:

(1) If we write $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_3$, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_4 = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$, $\mathcal{H}_2 = \mathcal{H}_3 \oplus \mathcal{H}_4$, then H is reduced by $\mathcal{H}_4$ and $H = H_0 \oplus H_4$, where H_4 is a self adjoint operator in $\mathcal{H}_4$.

$$(2) \mathcal{H}_4 \subset \mathcal{L}_2(\bar{\lambda}) \cap \mathcal{L}_2(\lambda), m_2(\bar{\lambda}) \subset \mathcal{H}_3, m_2(\lambda) \subset \mathcal{H}_3.$$

(3) H_2 is reduced by $\mathfrak{h}_4$ and $H_2 = H_3 \oplus H_4$, where H_3 is a closed Hermitian operator in $\mathfrak{h}_3$ with the same deficiency subspaces $\mathcal{M}_2(\bar{\lambda}), \mathcal{M}_2(\lambda)$ as H_2 .

(4) H_0 is defined by H_3 and V .

(5) H and H_0 define the same spectral function of H_1 .

Proof. Since $H_1 \subset H_0 \subset H$, we have by theorem 5 that $U_1 \subset U_0 \subset U$. Since $U_0 : \mathfrak{h}_0 \xrightarrow{\text{isom. onto}} \mathfrak{h}_0$ and $U : \mathfrak{h} \xrightarrow{\text{isom. onto}} \mathfrak{h}$, we must have that $U : \mathfrak{h}_4 \xrightarrow{\text{isom. onto}} \mathfrak{h}_4$. Thus $\mathfrak{h}_4$ reduces U , and by theorem 8, $U = U_0 \oplus U_4$, $H = H_0 \oplus H_4$, where H_4 is a self adjoint operator in $\mathfrak{h}_4$ with Cayley transform U_4 . This proves (1).

We claim now that $\mathfrak{h}_4 \subset \mathcal{L}_2(\bar{\lambda})$. Let $f \in \mathfrak{h}_4 \subset \mathfrak{h}_2$. Since $\mathfrak{h}_2 = \mathcal{M}_2(\bar{\lambda}) \oplus \mathcal{L}_2(\bar{\lambda})$, $f = f' + f''$, where $f' \in \mathcal{M}_2(\bar{\lambda})$, $f'' \in \mathcal{L}_2(\bar{\lambda})$. Hence, $Uf = Uf' + Uf'' = Vf' + U_2f'' = V_{12}f' + V_{22}f' + U_2f''$, where $Uf \in \mathfrak{h}_4 \subset \mathfrak{h}_2$, $U_2f'' \in \mathcal{L}_2(\lambda) \subset \mathfrak{h}_2$, $V_{12}f' \in \mathcal{M}_1(\lambda) \subset \mathfrak{h}_1$, $V_{22}f' \in \mathcal{M}_2(\lambda) \subset \mathfrak{h}_2$. Thus, $V_{12}f' = 0$ and therefore $f' = 0$ by theorem 9.

We have, then, that $f = f'' \in \mathcal{L}_2(\bar{\lambda})$ and hence $\mathfrak{h}_4 \subset \mathcal{L}_2(\bar{\lambda})$.

Since $\mathfrak{h}_4 \subset \mathcal{L}_2(\bar{\lambda})$ while $U : \mathfrak{h}_4 \xrightarrow{\text{isom. onto}} \mathfrak{h}_4$ and $U : \mathcal{L}_2(\bar{\lambda}) \xrightarrow{\text{isom. onto}} \mathcal{L}_2(\lambda)$, it follows that $\mathfrak{h}_4 \subset \mathcal{L}_2(\lambda)$. By the preceding paragraph and what we have just proved, $\mathfrak{h}_4 \subset \mathcal{L}_2(\bar{\lambda}) \cap \mathcal{L}_2(\lambda)$. This proves (2).

Since $U_2 = U$ on $\mathcal{L}_2(\bar{\lambda})$, $U_2 : \mathcal{L}_2(\bar{\lambda}) \xrightarrow{\text{isom. onto}} \mathcal{L}_2(\lambda)$ and $U_2 : \mathfrak{h}_4 \xrightarrow{\text{isom. onto}} \mathfrak{h}_4$. Thus, $\mathfrak{h}_4$ reduces U_2 and $U_2 = U_3 \oplus U_4$, where $U_3 : \mathcal{L}_2(\bar{\lambda}) \ominus \mathfrak{h}_4 \xrightarrow{\text{isom. onto}} \mathcal{L}_2(\lambda) \ominus \mathfrak{h}_4$. We note that $\mathfrak{h}_3 = \mathcal{M}_2(\bar{\lambda}) \oplus [\mathcal{L}_2(\bar{\lambda}) \ominus \mathfrak{h}_4] = \mathcal{M}_2(\lambda) \oplus [\mathcal{L}_2(\lambda) \ominus \mathfrak{h}_4]$. Hence, by theorem 8, $H_2 = H_3 \oplus H_4$ where H_3 is a closed Hermitian operator in $\mathfrak{h}_3$ with Cayley transform U_3 and deficiency subspaces $\mathcal{M}_2(\bar{\lambda})$, $\mathcal{M}_2(\lambda)$. This proves (3).

By theorem 9, H_3 and V define a self adjoint extension H'_0 of H_1 in $\mathfrak{h}_0 = \mathfrak{h}_1 \oplus \mathfrak{h}_3$. The Cayley transform U'_0 of H'_0 is given by $U_1 = U$ on $\mathcal{L}_1(\bar{\lambda})$, $V = U$ on $\mathcal{M}_1(\bar{\lambda}) \oplus \mathcal{M}_2(\bar{\lambda})$ and $U_3 = U$ on $\mathcal{L}_2(\bar{\lambda}) \ominus \mathfrak{h}_4$. Since $\mathfrak{h}_1 = \mathcal{L}_1(\bar{\lambda}) \oplus \mathcal{M}_1(\bar{\lambda})$ and $\mathfrak{h}_3 = \mathcal{M}_2(\bar{\lambda}) \oplus [\mathcal{L}_2(\bar{\lambda}) \ominus \mathfrak{h}_4]$, $U'_0 = U$ on $\mathfrak{h}_1 \oplus \mathfrak{h}_3 = \mathfrak{h}_0$. But since $U_0 \subset U$, $U_0 = U$ on $\mathfrak{h}_1 \oplus \mathfrak{h}_3$. Thus, $U_0 = U'_0$ and $H_0 = H'_0$, so that H_0 is defined by H_3 and V . This proves (4).

As we have shown, $H = H_0 \oplus H_4$. Thus, $E(\lambda) = E_0(\lambda) \oplus E_4(\lambda)$ and therefore $E(\lambda)f = E_0(\lambda)f$ for all $f \in \mathfrak{h}_1$. If P is the operator of orthogonal projection of $\mathfrak{h}$ onto $\mathfrak{h}_1$ and if P_0 is the operator of orthogonal projection of $\mathfrak{h}_0$ onto $\mathfrak{h}_1$, $PE(\lambda)f = PE_0(\lambda)f = P_0E_0(\lambda)f$ for all $f \in \mathfrak{h}_1$, so that H and H_0 define the same spectral function of H . This proves (5). #

Theorem 13. Let H be a minimal self adjoint extension of the closed symmetric operator H_1 . Suppose that for any bounded self adjoint operator A in $\mathcal{H}$ with matrix representation

$$A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix},$$

the property that A commutes with H implies that $B \equiv 0$. Here E is the identity in $\mathcal{H}_1$, $B: \mathcal{H}_2 \rightarrow \mathcal{H}_1$, $C: \mathcal{H}_2 \rightarrow \mathcal{H}_2$, C is self adjoint. Then, H defines an extremal spectral function of H_1 .

Proof. M. A. Naimark [5] has shown that the spectral function $E_1(\lambda)$ of H_1 defined by a minimal self adjoint extension H of H_1 is extremal if and only if every bounded self adjoint operator in $\mathcal{H}$ which commutes with H and satisfies the condition $(Af, g) = (f, g)$ for all $f, g \in \mathcal{H}$ is reduced by $\mathcal{H}_1$. The operator A defined in the theorem has the general form of every bounded self adjoint operator which satisfies the condition $(Af, g) = (f, g)$ for all $f, g \in \mathcal{H}$. Further, $B \equiv 0$ means that $\mathcal{H}_1$ reduces A . Thus, if the property that A commutes with H implies that $B \equiv 0$, we know that $E_1(\lambda)$ is extremal by the theorem of M. A. Naimark.

Remark. If A commutes with H , then A commutes with the Cayley transform U of H . If we write U in matrix form, $U \sim (U_{jk})$,

where $U_{jk} : \mathfrak{h}_k \rightarrow \mathfrak{h}_j$, $j, k = 1, 2$, then the hypothesis that A commutes with H implies the validity of the following equations:

$$BU_{21} = U_{12}B^*,$$

$$U_{12} + BU_{22} = U_{11}B + U_{12}C,$$

$$B^*U_{11} + CU_{21} = U_{21} + U_{22}B^*,$$

$$B^*U_{12} + CU_{22} = U_{21}B + U_{22}C.$$

4. Extremal spectral functions of symmetric operators with equal deficiency indices.

In this section we shall deduce implications of the hypothesis $\mathcal{N}(H_2) = \{0\}$. Among these implications is the fact that the spectral function is extremal.

Theorem 14. Let H be a self adjoint extension of the closed symmetric operator H_1 . Suppose that H is defined by H_2 and V . Then the following statements are equivalent:

- (1) $\mathcal{N}(H_2) = \{0\}$.
- (2) $m_2(\bar{\lambda}) = m_2(\lambda) = \mathfrak{h}_2$.
- (3) $\mathcal{N}(H) \cap \mathfrak{h}_2 = \{0\}$.

Further, $\mathcal{N}(H_2) = \{0\}$ implies that

- (i) $m_1 = n_1$, i.e., the deficiency indices of H_1 are equal;
- (ii) H is minimal.

Proof. That (1) implies (2) is clear from the definition of

$m_2(\bar{\lambda})$ and $m_2(\lambda)$. Suppose, on the other hand, that $m_2(\bar{\lambda}) = m_2(\lambda) = \frac{1}{2}$.

Then, $\mathcal{R}(H_2 - \lambda E) = \mathcal{R}(H_2 - \bar{\lambda} E) = \{0\}$. If $f \in \mathcal{N}(H_2)$, $H_2 f - \lambda f = 0$

and $H_2 f - \bar{\lambda} f = 0$. Hence, $(\lambda - \bar{\lambda})f = 0$, and therefore $f = 0$. Thus,

$\mathcal{N}(H_2) = \{0\}$, and we have proved that (2) implies (1).

By theorem 9, $\mathcal{N}(H_2) = \mathcal{N}(H) \cap \frac{1}{2}$, so that (1) and (3) are clearly equivalent.

Suppose, now, that $\mathcal{N}(H_2) = \{0\}$. Then $m_2(\bar{\lambda}) = m_2(\lambda)$ and $m_2 = n_2$. By theorem 10, (7), $m_1 = n_1$. This proves (i). Since

$\mathcal{N}(H_2) = \{0\}$ implies that $m_2(\bar{\lambda}) = m_2(\lambda) = \frac{1}{2}$ and therefore that

$\mathcal{L}_2(\bar{\lambda}) = \mathcal{L}_2(\lambda) = \{0\}$, it follows from theorems 11 and 12 that H is minimal.

This proves (ii). #

Theorem 15. Let H_1 be a closed symmetric operator. Let H be a self adjoint extension of H_1 defined by H_2 and V . If $\mathcal{N}(H_2) = \{0\}$, then the spectral function $E_1(\lambda)$ of H_1 defined by H is extremal.

Proof. By theorem 14, H is a minimal self adjoint extension of H_1 .

Suppose the operator

$$A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix}$$

commutes with H . By theorem 13, we need only show that it follows that $B \equiv 0$.

By the remark to theorem 13, we have that $BU_{21} = U_{12}B^*$ and that $U_{21}^*B^* = BU_{12}^*$, where $U \sim (U_{jk})$ is the Cayley transform of H . We know, further, that $U = V$ on $\mathcal{M}_1(\bar{\lambda}) \oplus \mathcal{M}_2(\bar{\lambda})$ and that $U^* = U^{-1} = V^{-1} = V^*$ on $\mathcal{M}_1(\lambda) \oplus \mathcal{M}_2(\lambda)$. Using the fact that $\mathcal{M}_2(\bar{\lambda}) = \mathcal{M}_2(\lambda) = \mathfrak{H}_2$, we obtain, then, that $BV_{12}^*\bar{\mathcal{M}}_1(\lambda) = BU_{12}^*\bar{\mathcal{M}}_1(\lambda) = U_{21}^*B^*\bar{\mathcal{M}}_1(\lambda) \subset U_{21}^*\mathfrak{H}_2 = U_{21}^*\mathcal{M}_2(\lambda) = V_{21}^*\mathcal{M}_2(\lambda) \subset \mathcal{M}_1(\bar{\lambda})$. Since $V_{12}^*\bar{\mathcal{M}}_1(\lambda)$ is dense in $\mathcal{M}_2(\bar{\lambda}) = \mathfrak{H}_2$ by theorem 10, and since B is bounded, it follows that $B\mathfrak{H}_2 \subset \mathcal{M}_1(\bar{\lambda})$.

Similarly, $BV_{21}\bar{\mathcal{M}}_1(\bar{\lambda}) = BU_{21}\bar{\mathcal{M}}_1(\bar{\lambda}) = U_{12}B^*\bar{\mathcal{M}}_1(\bar{\lambda}) \subset U_{12}\mathfrak{H}_2 = U_{12}\mathcal{M}_2(\bar{\lambda}) = V_{12}\mathcal{M}_2(\bar{\lambda}) \subset \mathcal{M}_1(\lambda)$, and therefore $B\mathfrak{H}_2 \subset \mathcal{M}_1(\lambda)$.

Thus, $B\mathfrak{H}_2 \subset \mathcal{M}_1(\bar{\lambda}) \cap \mathcal{M}_1(\lambda)$. But $\mathcal{M}_1(\bar{\lambda}) \cap \mathcal{M}_1(\lambda) = \{0\}$, because $\mathcal{M}_1(\bar{\lambda})$ and $\mathcal{M}_1(\lambda)$ are the deficiency subspaces of a symmetric operator. Hence $B \equiv 0$ on $\mathfrak{H}_2$. #

Definition 6. Let H be a self adjoint extension of the closed symmetric operator H_1 . H is called a finite dimensional self adjoint extension of H_1 if $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$ is finite dimensional.

Remark 1. Theorem 15, is a modification of a lemma of M. A. Naimark [5]. By the use of a density argument we have dispensed with the assumption of Naimark of finite dimensionality of the extension.

Remark 2. By use of theorem 15, M. A. Naimark [5] has shown that every finite dimensional extension H of a closed symmetric operator H_1 defines an extremal spectral function of H_1 . This implies, in particular, that the orthogonal spectral functions of H_1 are extremal.

Theorem 16. If H is a finite dimensional extension of a closed symmetric operator H_1 , then H_1 must have equal deficiency indices.

Proof. Suppose that H is defined by H_2 and V . Then H_2 is a Hermitian operator in the finite dimensional space $\mathcal{H}_2$. Since $U_2 : \mathcal{L}_2(\bar{\lambda}) \xrightarrow[\text{onto}]{\text{isom.}} \mathcal{L}_2(\lambda)$, it follows that $\dim \mathcal{L}_2(\bar{\lambda}) = \dim \mathcal{L}_2(\lambda)$. Hence, $\dim \mathcal{M}_2(\bar{\lambda}) = \dim \mathcal{M}_2(\lambda)$, i. e., $m_2 = n_2$. By theorem 10, (7), $m_1 = n_1$. #

In the next section we shall consider symmetric operators with unequal deficiency indices. The results of the present section show that certain statements cannot hold when the deficiency indices are unequal. We present some of these in the following theorem.

Theorem 17. Suppose that H is a self adjoint extension of the closed symmetric operator H_1 . Let H be defined by H_2 and V . If $m_1 \neq n_1$, then

- (1) $m_2 \neq n_2$;
- (2) $\mathcal{M}_2(\lambda) \neq \mathcal{M}_2(\bar{\lambda})$;
- (3) $\mathcal{H}_2$ is not finite dimensional;
- (4) $\mathcal{L}(\mathcal{H}_2) \neq \{0\}$;
- (5) $\mathcal{M}_2(\lambda) \neq \mathcal{H}_2$ and $\mathcal{M}_2(\bar{\lambda}) \neq \mathcal{H}_2$.

Proof. (1) follows from theorem 10, (7). (2) follows from (1). (3) follows from theorem 16 and the hypothesis $m_1 \neq n_1$. (4) follows from (2) and theorem 14.

We prove (5) as follows: Suppose $\mathcal{M}_2(\bar{\lambda}) = \mathcal{H}_2$ and therefore $\mathcal{L}_2(\bar{\lambda}) = \{0\}$. Since $U_2 : \mathcal{L}_2(\bar{\lambda}) \xrightarrow[\text{onto}]{\text{isom.}} \mathcal{L}_2(\lambda)$, $\mathcal{L}_2(\lambda) = \{0\}$ and therefore $\mathcal{M}_2(\lambda) = \mathcal{H}_2$ also. Hence, $\mathcal{M}_2(\bar{\lambda}) = \mathcal{M}_2(\lambda)$, which contradicts (2). A similar argument holds if $\mathcal{M}_2(\lambda) = \mathcal{H}_2$. #

5. Extremal spectral functions of symmetric operators with unequal deficiency indices

We first introduce the notion of a partial isometry. (See Murray and von Neumann [2].)

Definition 7. A bounded linear operator W in a Hilbert space $\mathcal{H}$ is called a partial isometry if it maps a subspace $\mathcal{E}$ isometrically onto another subspace $\mathcal{F}$, while it maps $\mathcal{H} \ominus \mathcal{E}$ onto $\{0\}$. $\mathcal{E}$ is called the initial set of W , and $\mathcal{F}$ is called the final set of W .

If W is a partial isometry, then the following statements hold:

(1) If $P(\mathcal{E})$ is the operator of orthogonal projection on $\mathcal{E}$ and $P(\mathcal{F})$ is the operator of orthogonal projection on $\mathcal{F}$, then $P(\mathcal{E}) = W^*W$, $P(\mathcal{F}) = WW^*$.

(2) U^* is a partial isometry with initial set $\mathcal{F}$ and final set $\mathcal{E}$.

(3) As a mapping of $\mathcal{F}$ onto $\mathcal{E}$, U^* is the inverse of U as a mapping of $\mathcal{E}$ onto $\mathcal{F}$.

Theorem 18. Suppose that W is a partial isometry with initial set $\mathcal{M}$ and final set $\mathcal{N}$. Let $\mathcal{H} = \mathcal{N} \ominus \mathcal{M}$. Then, $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$, where

(i) $W : \mathcal{M}'' \xrightarrow[\text{onto}]{\text{isom.}} \mathcal{M}''$;

(ii) if $f \in \mathcal{H} \oplus \mathcal{M}'$, $\lim_{p \rightarrow \infty} W^p f = 0$.

Proof. Let $\mathcal{M}_i = (W^*)^i \mathcal{H}$, $i = 0, 1, 2, \dots$. Then the following statements are true:

(a) $\mathcal{M}_i \subset \mathcal{M}$ for $i = 1, 2, \dots$. This is clear because W^* is a partial isometry with initial set $\mathcal{H}$ and final set $\mathcal{M}$.

(b) If $f \in \mathcal{M}_n$, where n is a positive integer then $W^p f \in \mathcal{M}_{n-p}$ for $p = 1, 2, \dots, n$, and $W^p f = 0$ for $p > n$. For, if $f \in \mathcal{M}_n$, $f = (W^*)^n g$ for some $g \in \mathcal{H}$. Since $WW^* = E$, $W^p f = (W^*)^{n-p} g \in \mathcal{M}_{n-p}$, $1 \leq p \leq n$. If $p > n$, $W^p f = W^{p-n} g = 0$.

(c) If $f \in \mathcal{M}_i$, $i = 0, 1, 2, \dots$, and if n is a positive integer, then

$(W^*)^n f \in \mathcal{M}_{i+n}$. For, if $f \in \mathcal{M}_i$, $f = (W^*)^i g$, where $g \in \mathcal{N}$. Therefore,
 $(W^*)^n f = (W^*)^{i+n} g \in \mathcal{M}_{i+n}$.

(d) $\mathcal{M}_i \perp \mathcal{M}_j$ if $i \neq j$. For, suppose $i < j$, and let $f \in \mathcal{M}_i, g \in \mathcal{M}_j$. Then, there exists $f_1 \in \mathcal{N}$ and $g_1 \in \mathcal{N}$ such that $f = (W^*)^i f_1, g = (W^*)^j g_1$. Hence, $(f, g) = ((W^*)^i f_1, (W^*)^j g_1) = (W^i (W^*)^i f_1, (W^*)^{j-i} g_1) = (f_1, (W^*)^{j-i} g_1) = 0$, since $f_1 \in \mathcal{N}, (W^*)^{j-i} g_1 \in \mathcal{M}_{j-i} \subset \mathcal{M}$.

Now let $\mathcal{M}' = \sum_{i=1}^{\infty} \mathcal{M}_i$. $\mathcal{M}'$ is a subspace of $\mathcal{M}$. Let

$\mathcal{M}'' = \mathcal{M} \ominus \mathcal{M}'$. We shall show that $\mathcal{M}'$ and $\mathcal{M}''$ satisfy (i) and (ii).

Since $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ and $\mathcal{H} = \mathcal{N} \oplus \mathcal{M}' + \mathcal{M}''$, and since

$W : \mathcal{M} \xrightarrow{\text{isom. onto}} \mathcal{H}$, in order to prove (i) it is sufficient to show that

$W : \mathcal{M}' \xrightarrow{\text{isom. onto}} \mathcal{N} \oplus \mathcal{M}'$. Suppose $f \in \mathcal{M}'$. Then, $f = \sum_{i=1}^{\infty} f_i$, where

$f_i \in \mathcal{M}_i$, and $Wf = \sum_{i=1}^{\infty} Wf_i$. Since $Wf_i \in \mathcal{M}_{i-1}$, by (b), we see that

$Wf \in \mathcal{N} \oplus \mathcal{M}'$. Thus, $W : \mathcal{M}' \xrightarrow{\text{isom.}} \mathcal{N} \oplus \mathcal{M}'$. To show that the map is

onto, let $g \in \mathcal{N} \oplus \mathcal{M}'$. Then, $g = \sum_{i=0}^{\infty} f_i$, where $f_i \in \mathcal{M}_i$. If $f = W^* g$,

$f = \sum_{i=0}^{\infty} W^* f_i \in \mathcal{M}'$, by (c). Further, $Wf = WW^* g = g$. Hence,

$W : \mathcal{M}' \xrightarrow{\text{isom. onto}} \mathcal{N} \oplus \mathcal{M}'$. This proves (i).

We now prove (ii). Let $f \in \mathcal{N} \oplus \mathcal{M}'$. Then, $f = \sum_{i=0}^{\infty} f_i, f_i \in \mathcal{M}_i$.

By (b), $W^p f = \sum_{i=0}^{\infty} W^p f_i = \sum_{i=p}^{\infty} W^p f_i$. Hence, $\|W^p f\|^2 = \sum_{i=p}^{\infty} \|W^p f_i\|^2 = \sum_{i=p}^{\infty} \|f_i\|^2$. Thus, $\lim_{p \rightarrow \infty} \|W^p f\|^2 = 0$, and (ii) is proved. #

Theorem 19. Let λ be a fixed non-real number. Suppose that H_1 is a closed symmetric operator in $\mathcal{H}_1$ with deficiency indices (m, n) (with respect to λ), $m \neq n$. Let H be a self adjoint extension of H_1 defined by H_2 and V , where H_2 is a closed Hermitian operator with deficiency indices $(0, m - n)$ if $m > n$ and $(n - m, 0)$ if $m < n$. Then the spectral function defined by H is extremal.

Proof. Assume that $m > n$. The case $m < n$ then follows by interchanging the roles of $\bar{\lambda}$ and λ in theorem 9 and defining H by H_2 and V^* .

By theorem 11 there exists a minimal self adjoint extension H_0 of H_1 such that $\mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}$, $H_1 \subset H_0 \subset H$, and H_0 and H define the same spectral function of H_1 . By theorem 12, H_0 is defined by V and a Hermitian operator H_3 with the same deficiency subspaces as H_2 . Since we can always consider H_0 instead of H , if necessary, it follows that without loss of generality we can consider H to be a minimal self adjoint extension.

Since $\mathcal{M}_2(\bar{\lambda}) = \{0\}$ and $\mathcal{L}_2(\bar{\lambda}) = \mathcal{H}_2$, we have that if $f \in \mathcal{H}_2$,

$Uf \in \mathcal{L}_2(\lambda) \subset \mathfrak{H}_2$. Hence,

(a) $U_{12}f = 0$ for all $f \in \mathfrak{H}_2$.

Further, $Uf = U_{22}f$ for all $f \in \mathfrak{H}_2$; whence

(b) $U_{22} : \mathfrak{H}_2 \xrightarrow[\text{onto}]{\text{isom.}} \mathcal{L}_2(\lambda)$. U_{22} is thus a partial isometry in $\mathfrak{H}_2$

with initial set $\mathfrak{H}_2$ and final set $\mathcal{L}_2(\lambda)$, while U_{22}^* is a partial isometry with initial set $\mathcal{L}_2(\lambda)$ and final set $\mathfrak{H}_2$. We have that

$$E = P(\mathfrak{H}_2) = U_{22}^* U_{22}, \text{ while } P(\mathcal{L}_2(\lambda)) = U_{22} U_{22}^*.$$

Now let A be a bounded self adjoint operator in $\mathfrak{H}$ with matrix representation

$$A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix},$$

where $B : \mathfrak{H}_2 \rightarrow \mathfrak{H}_1$, $C : \mathfrak{H}_2 \rightarrow \mathfrak{H}_2$, C is self adjoint, and suppose that

A commutes with H . By theorem 13, if we can show $B \equiv 0$, we are through.

By the remark to theorem 13, the following equations hold:

$$BU_{21} = U_{12}B^* = 0 \text{ and } BU_{22} = U_{11}B.$$

$$\text{On } \mathcal{M}_1(\bar{\lambda}), U_{21} = V_{21} \text{ and therefore } BV_{21}\mathcal{M}_1(\bar{\lambda}) = BU_{21}\mathcal{M}_1(\bar{\lambda}) = \{0\}.$$

Since by theorem 10, $V_{21}\mathcal{M}_1(\bar{\lambda})$ is dense in $\mathcal{M}_2(\lambda)$, $B\mathcal{M}_2(\lambda) = \{0\}$, i. e.,

$$B P(\mathcal{M}_2(\lambda)) = 0.$$

From $BU_{22} = U_{11}B$, we have that $BU_{22}U_{22}^* = U_{11}BU_{22}^*$, or by (b)

$BP(\mathcal{L}_2(\lambda)) = U_{11}BU_{22}^*$. Adding this with $BP(\mathcal{M}_2(\lambda)) = 0$, we obtain that $B = U_{11}BU_{22}^*$. Iterating this equation we obtain that $B = U_{11}^p B (U_{22}^*)^p$ for every positive integer p . Since $\|U_{11}\| \leq 1$, $\|Bf\| \leq \|B\| \| (U_{22}^*)^p f \|$ for each $f \in \mathcal{H}_2$.

By theorem 18, $\mathcal{L}_2(\lambda) = \mathcal{M}^1 \oplus \mathcal{M}^2$, where $U_{22}^* : \mathcal{M}^2 \xrightarrow{\text{isom. onto}} \mathcal{M}^2$, and if $f \in \mathcal{M}_2(\lambda) \oplus \mathcal{M}^1$, then $\lim_{p \rightarrow \infty} \| (U_{22}^*)^p f \| = 0$. But if $U_{22}^* : \mathcal{M}^2 \xrightarrow{\text{isom. onto}} \mathcal{M}^2$, then $U_{22} : \mathcal{M}^2 \xrightarrow{\text{isom. onto}} \mathcal{M}^2$ and $U : \mathcal{M}^2 \xrightarrow{\text{isom. onto}} \mathcal{M}^2$.

This means that U and therefore H is reduced by $\mathcal{M}^2$, a subspace of

$\mathcal{H}_2$. Since H is a minimal self adjoint extension of H_1 , $\mathcal{M}^2 = \{0\}$. Hence, $\mathcal{H}_2 = \mathcal{M}_2(\lambda) \oplus \mathcal{M}^1$, and therefore if $f \in \mathcal{H}_2$, $\lim_{p \rightarrow \infty} \| (U_{22}^*)^p f \| = 0$. Since $\|Bf\| \leq \|B\| \| (U_{22}^*)^p f \|$ for each $f \in \mathcal{H}_2$ and for every positive integer p , it follows that $B \equiv 0$ on $\mathcal{H}_2$. #

Remark. Since the operator H_2 in theorem 19 is a Hermitian operator with deficiency indices $(0, m-n)$ or $(n-m, 0)$, it may seem that we are dealing with a wider class of operators than the maximal symmetric operators. That this is not so is shown by theorem 20.

Lemma. If U is a unitary operator of $\mathcal{H}$ onto $\mathcal{H}$, then $(U - E)^{-1}$ exists if and only if $\mathcal{R}(U - E)$ is dense in $\mathcal{H}$.

Proof. Suppose $\mathcal{R}(U - E)$ is dense in $\mathcal{H}$. We wish to show that $(U - E)^{-1}$ exists. Suppose $(U - E)f = 0$. From this equation it follows

that $(U^* - E)f = 0$. Hence, for all $g \in \mathcal{H}$, $0 = ((U^* - E)f, g) = (f, (U - E)g)$, i.e., $f \perp \mathcal{R}(U - E)$. Thus, $f = 0$.

Suppose, on the other hand, that $(U - E)^{-1}$ exists. We wish to prove that $\mathcal{R}(U - E)$ is dense in $\mathcal{H}$. Suppose that $0 = (f, (U - E)g)$ for all $g \in \mathcal{H}$. Then, $0 = ((U^* - E)f, g)$ for all $g \in \mathcal{H}$ and therefore $(U^* - E)f = 0$. Hence, $(U - E)f = 0$ and $f = 0$. #

Theorem 20. If H is a Hermitian operator with deficiency indices $(0, n)$ or $(n, 0)$, then H is a maximal symmetric operator. If H is a Hermitian operator with deficiency indices $(0, 0)$, then H is a self adjoint operator.

Proof. The second statement follows immediately from the first, since a symmetric operator with deficiency indices $(0, 0)$ is self adjoint.

Suppose that H has deficiency indices $(n, 0)$. We shall show that H is a symmetric operator (and therefore maximal symmetric.) From this it follows that if H has deficiency indices $(0, n)$, then since $-H$ has deficiency indices $(n, 0)$, $-H$ and therefore H is symmetric in this case also.

If U is the Cayley transform of H , $U: \mathcal{L}(\bar{\lambda}) \xrightarrow[\text{onto}]{\text{isom.}} \mathcal{L}(\lambda) = \mathcal{H}$.

Suppose we define W by the equations:

$$Wf = Uf \text{ for all } f \in \mathcal{L}(\bar{\lambda}),$$

$$Wf = 0 \text{ for all } f \in \mathcal{M}(\bar{\lambda}).$$

Then W is a partial isometry, and by theorem 18 $\mathcal{L}(\bar{\lambda}) = \mathcal{M}^1 \oplus \mathcal{M}^2$, where $W: \mathcal{M}^2 \xrightarrow[\text{onto}]{\text{isom.}} \mathcal{M}^2$ and if $f \in \mathcal{M}(\bar{\lambda}) \oplus \mathcal{M}^1$, $\lim_{p \rightarrow \infty} W^p f = 0$. We have that $U: \mathcal{M}^2 \xrightarrow[\text{onto}]{\text{isom.}} \mathcal{M}^2$ and $U: \mathcal{M}^1 \xrightarrow[\text{onto}]{\text{isom.}} \mathcal{M}(\bar{\lambda}) \oplus \mathcal{M}^1$. Since U is the Cayley transform of H , $(U - E)^{-1}$ exists by theorem 2, and therefore $(U - E)\mathcal{M}^2$ is dense in $\mathcal{M}^2$ by the preceding lemma.

We claim further that $(U - E)\mathcal{M}^1$ is dense in $\mathcal{M}(\bar{\lambda}) \oplus \mathcal{M}^1$.

Suppose $g \in \mathcal{M}(\bar{\lambda}) \oplus \mathcal{M}^1$ and $0 = (g, (U - E)f)$ for all $f \in \mathcal{M}^1$. Letting $g = g^1 + g^2$, where $g^1 \in \mathcal{M}(\bar{\lambda})$, $g^2 \in \mathcal{M}^1$, we have that for all $f \in \mathcal{M}^1$, $0 = (g, (U - E)f) = ((U^* - E)g, f) = (U^*g - g^2, f)$. Since $U^*g - g^2 \in \mathcal{M}^1$, $U^*g = g^2$. Therefore, $g = Ug^2 = Wg$. Iterating this equation, $g = W^p g$, and therefore $g = \lim_{p \rightarrow \infty} W^p g = 0$. Thus, $(U - E)\mathcal{M}^1$ is dense in $\mathcal{M}(\bar{\lambda}) \oplus \mathcal{M}^1$.

Since $(U - E)\mathcal{M}^2$ is dense in $\mathcal{M}^2$ and $(U - E)\mathcal{M}^1$ is dense in $\mathcal{M}(\bar{\lambda}) \oplus \mathcal{M}^1$, $(U - E)\mathcal{L}(\bar{\lambda})$ is dense in $\mathcal{H}$. Because by theorem 2, $\mathcal{U}(H) = \mathcal{K}(U - E)$, H is a symmetric operator. #

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