UNCLASSIFIED

MEMORANDUM

RM-3634-PR

JUNE 1963

RELATIVISTIC SURFACE DYNAMICS OF AN ISOLATED WORLD TUBE OF PERFECT FLUID

Dominic G. B. Edelen

This research is sponsored by the United States Air Force under Project RAND—contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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1700 MAIN ST • SANTA MONICA • CALIFORNIA

PREFACE

This Memorandum constitutes another step in a continuing investigation of the physics and mathematics of general relativity. Here, we are concerned with the boundary values which are appropriate to those cosmological models which assume that the material medium is a perfect fluid. In the past, it has usually been assumed that the entire space is filled by the fluid. The results reported here make such an assumption unnecessary and provide the basis for a cosmological model in which there are a finite number of isolated world tubes of fluid which would hopefully correspond to galaxies.

SUMMARY

The problem of constructing consistent boundary data for an isolated world tube of perfect fluid is examined. The results show that one obtains a unique representation of all limiting quantities through simple constructions in a subsidiary three-dimensional hyperbolic-normal metric space. It also follows that the geodesic hypothesis can be invalidated only by internal dynamical processes. If the bounding surface is not known, a consistent procedure is given for the construction of the first and second fundamental forms and the boundary data such that a bounding surface is realizable in the underlying Einstein-Riemann space.

RELATIVISTIC SURFACE DYNAMICS OF AN ISOLATED WORLD TUBE OF PERFECT FLUID

1. The dynamics of discontinuity surfaces in general relativity [1] has been used to examine the properties of the bounding surfaces of galaxies in the Hubble E-Series [2, 3]. In view of the fact that most previous relativistic galactic models have been based on the assumption that the momentum-energy tensor interior to a galaxy has a form characteristic of a perfect fluid, it seemed appropriate to examine the surface dynamics of an isolated world tube of perfect fluid. The results show that one obtains a unique representation of all limiting quantities through simple constructions in a subsidiary three-dimensional hyperbolic-normal metric space. It is also shown that the geodesic hypothesis can be invalidated only by internal dynamic processes. If the bounding surface is not known, a consistent procedure is given for the construction of the first and second fundamental forms and the boundary data such that a bounding surface is realizable in the underlying Einstein-Riemann space.

2. Let E be an Einstein-Riemann space whose metric structure is defined by the quadratic differential form

$$(2.1)\qquad ds^2 = h_{AB}\, dx^A\, dx^B \quad (A,\ B,\quad = 0,\ 1,\ 2,\ 3)$$

having signature -2. We denote by Σ a regular, time-like hypersurface in E which is defined by the parametric equations

$$x^A = f^A(u^0, u^1, u^2).$$

The normal vector to Σ, defined by

$$x^A_\alpha N_A = 0, \quad (x^A_\alpha \overset{\text{def}}{=} \partial f^A(u)/\partial u^\alpha),$$

can be normalized by the requirement $N_A N^A = -1$. We shall also assume that any "time-slice" of Σ yields a closed two-surface. Let $a_{\alpha\beta}\, du^\alpha\, du^\beta$ denote the first fundamental form on Σ, then

$$a_{\alpha\beta} = \bar{h}_{AB}\, x^A_\alpha\, x^B_\beta$$

where the bar is used to denote evaluation on Σ. It is also evident that the four vectors N^A, x^A_α for $\alpha = 0, 1, 2$ are linearly independent on Σ in E and

(2.2) $$a^{\alpha\beta}\, x^A_\alpha\, x^B_\beta = \bar{h}^{AB} + N^A N^B .$$

(It goes without saying that the array of functions $a_{\alpha\beta}$ form a nonsingular, symmetric form with signature -1.) If we denote the coefficients of the second fundamental form by $b_{\alpha\beta}$, we have $b_{\alpha\beta} = b_{\beta\alpha}$ and

(2.3) $$x^A_{\alpha;\beta} = b_{\alpha\beta}\, N^A ,$$

(2.4) $$N^A{}_{;\alpha} = b_{\alpha\beta}\, a^{\beta\gamma}\, x^A_\gamma .$$

Throughout this paper, Latin indices have the range 0, 1, 2, 3, Greek indices have the range 0, 1, 2, and the semicolon is used to denote covariant differentiation--the comma being reserved for the case of coordinate differentiation.

Let S_{AB} denote the jump strengths in the components of the momentum-energy tensor, that is,

$$(2.5) \qquad S_{AB} = [T_{AB}] = T_{AB}(+) - T_{AB}(-) \ ,$$

where $T_{AB}(+)$ $(T_{AB}(-))$ denotes the limiting value of T_{AB} as Σ is approached from the exterior (interior). The results of reference [1] show that the functions

$$(2.6) \qquad S_{\alpha\beta} \stackrel{\text{def}}{=} S_{AB}\, x^A_\alpha\, x^B_\beta$$

serve to determine the S_{AB}'s uniquely, this being a consequence of the existence requirements

$$(2.7) \qquad S_{AB}\, N^B = 0.$$

In addition, the $S_{\alpha\beta}$'s must satisfy the fundamental system of equations

$$(2.8) \qquad S^\alpha_{\beta;\alpha} = F_\beta \ ,$$

$$(2.9) \qquad S^{\alpha\beta}\, b_{\alpha\beta} + \chi = 0 \ ,$$

where

$$(2.10) \qquad F_\alpha = F_A\, x^A_\alpha \ , \ \chi = F_A\, N^A \ ,$$

and

$$(2.11) \qquad F_A = [T^B_{A;C}]\, N^C\, N_B \ .$$

3. We assume that Σ is the boundary of a world tube of perfect fluid whose immediate neighborhood is empty. The components of the momentum-energy tensor interior to Σ are thus given by

(3.1) $$T_{AB} = \mu W_A W_B - p(h_{AB} - W_A W_B) = \rho W_A W_B - p h_{AB} ,$$

where

(3.2) $$\rho = \mu + p, \; W_A W^A = 1,$$

while in the immediate neighborhood exterior to Σ all components of the momentum-energy tensor vanish. With a bar used to signify evaluation as Σ is approached from the interior, (3.1) yields

(3.3)* $$S_{AB} = -\bar{\mu} \bar{W}_A \bar{W}_B + \bar{p}(\bar{h}_{AB} - \bar{W}_A \bar{W}_B) ,$$

in view of the fact that $T_{AB} = 0$ immediately exterior to Σ. Since the functions S_{AB} must satisfy the four conditions (2.7), we have

$$0 = \bar{\mu} \bar{W}_A \bar{W}_B N^B - \bar{p}(N_A - \bar{W}_A \bar{W}_B N^B) .$$

In view of the fact that S_{AB} admits $\bar{W}_A$ as an eigen vector, the above equations can be satisfied for $S_{AB} \neq 0$ if and only if

(3.4) $$\bar{W}_A N^A = 0, \quad \bar{p} = 0.$$

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*The change in sign occurs owing to the definition of S_{AB} as the exterior limit minus the interior limit.

We thus have

$$(3.5) \qquad S_{AB} = -\ \overline{\rho}\ \overline{W}_A\ \overline{W}_B$$

as the most general form of the jump strengths of a locally isolated world tube of perfect fluid, the vanishing of $\overline{p}$ being used to obtain $\overline{\rho} = \overline{\mu}$.

4. In view of the first of (3.4) and the properties of the vectors $(N^A,\ x^A_\alpha)$, we have

$$(4.1) \qquad \overline{W}^A = W^\alpha\ x^A_\alpha\ ,$$

the quantities W_α being defined by

$$W_\alpha = W_A\ x^A_\alpha\ .$$

Surface covariant differentiation of (4.1) and use of (2.3) gives

$$(4.2) \qquad \overline{W}^A_{;\alpha} = W^\beta{}_{;\alpha}\ x^A_\beta + W^\beta\ b_{\beta\alpha}\ N^A\ .$$

Now,

$$\overline{W}^A_{;\alpha} = \overline{(W^A{}_{;B})}\ x^B_\alpha\ ,$$

so that a contraction with $x^C_\gamma\ a^{\alpha\gamma}$ together with (2.2) and (4.2) yields

$$(4.3) \qquad \overline{(W^A_{;D})} = (W^\beta{}_{;\alpha}\ x^A_\beta\ x^C_\gamma\ a^{\alpha\gamma} + W^\beta b_{\beta\alpha} a^{\alpha\gamma} x^C_\gamma N^A)\overline{h}_{CD} - \overline{(W^A_{;B})} N^B N_D\ .$$

When this result is combined with (3.4), we obtain the acceleration vector on Σ, namely

$$(4.4) \qquad \overline{(W^A_{;D} W^D)} = W^\beta{}_{;\alpha} W^\alpha x^A_\beta + W^\alpha W^\beta b_{\alpha\beta} N^A .$$

In a similar fashion we obtain

$$(4.5) \qquad \overline{(\rho_{,C})} = \rho_{,\alpha} x^B_\beta a^{\alpha\beta} \bar{h}_{BC} - \overline{(\rho_{,A})} N^A N_C ,$$

$$(4.6) \qquad \overline{(p_{,C})} = -\overline{(p_{,A})} N^A N_C ;$$

the latter result stemming from the fact that $\bar{p} = 0$. We also have from (4.1) that

$$1 = \bar{W}^A \bar{W}^B \bar{h}_{AB} = W^\alpha W^\beta \bar{h}_{AB} x^A_\alpha x^B_\beta = W^\alpha W^\beta a_{\alpha\beta} = W_\alpha W^\alpha .$$

Covariant differentiation of (3.1) yields

$$T^B_{A;C} = \rho_{,C} W_A W^B + \rho W_{A;C} W^B + \rho W_A W^B{}_{;C} - p_{,C} \delta^B_A .$$

Hence, on combining (2.11) and (3.4) and remembering the sign convention for jump strengths, we obtain

$$(4.7) \qquad F_A = -\bar{\rho} \bar{W}_A \overline{(W^B_{;C})} N^C N_B + \overline{(p_{,C})} N^C N_A ,$$

so that

$$(4.8) \qquad F_\beta = -\bar{\rho} W_\beta \overline{(W^B_{;C})} N^C N_B ,$$

$$(4.9) \qquad \chi = -\overline{(p_{,C})} N^C .$$

We have thus obtained all significant differential expressions in terms of the geometry of Σ together with surface and normal differential expressions.

5. We are now in a position to apply the fundamental system (2.8), (2.9). From (2.6) and (3.5) we have

$$(5.1) \qquad S^{\alpha}_{\beta} = -\,\overline{\rho}\, W_{\beta}\, W^{\alpha}\ .$$

Hence, if we substitute (5.1) and (4.9) into (2.9) we arrive at

$$(5.2) \qquad \overline{\rho}\, W^{\alpha}\, W^{\beta}\, b_{\alpha\beta} = -\,(\overline{p_{,C}})\, N^{C}\ .$$

Similarly, if we substitute (5.1) into (2.8) and use (4.9), we obtain

$$(5.3) \qquad (\overline{\rho}\, W^{\alpha})_{;\alpha}\, W_{\beta} + \overline{\rho}\, W_{\beta;\alpha}\, W^{\alpha} = \overline{\rho}\, W_{\beta}\, (\overline{W^{B}_{;C}})\, N^{C}\, N_{B}\ .$$

Contracting (5.3) with W^{α} and using the fact that $W_{\alpha}\, W^{\alpha} = 1$, gives us

$$(5.4) \qquad (\overline{\rho}\, W^{\alpha})_{;\alpha} = \overline{\rho}\, (\overline{W^{B}_{;C}})\, N_{B}\, N^{C}\ .$$

Thus, eliminating the common term between (5.3) and (5.4) we are led to

$$(5.5) \qquad \overline{\rho}\, W_{\beta;\alpha}\, W^{\alpha} = 0\ .$$

In addition, with the aid of (4.3) we see that equation (5.4) is equivalent to the condition

$$(5.6) \qquad \left\{\overline{(\rho\, W^{A})_{;A}}\right\} = 0\ .$$

<u>The vector field</u> W_{α} <u>thus defines geodesics in the three-dimensional hyperbolic-normal metric space</u> Σ^{*} <u>with</u>

<u>metric differential form</u> $a_{\alpha\beta}\, du^\alpha\, du^\beta = d\lambda^2$ <u>and the</u> <u>quantity</u> $\overline{\rho}$ <u>is given by</u>

$$(5.7) \qquad \overline{\rho} = \rho_o \exp\left\{ \int (N_B\, N^C (\overline{W^B_{;C}}) - W^\alpha_{;\alpha})\, d\lambda \right\} .$$

If the motion is incompressible, that is $W^A_{;A} = 0$, we have $\overline{\rho} = \rho_o$ by (4.3).

6. Thomas [4] has pointed out that an isolated world tube of perfect fluid contains a fluid particle which describes a geodesic in E if $(\rho\, W^A)_{;A} = 0$. The above result that this condition is rigorously satisfied on the boundary of such a world tube indicates that there must be creation or annihilation of the flux $\rho\, W^A$ due to internal processes if a geodesic is not to result.

7. Under the customary assumption that the discontinuity problem is of second order in the h's, that is

$$[h_{AB}] = [h_{AB,C}] = 0 ,$$

jumps in the momentum-energy tensor imply jumps in the second coordinate derivatives of the h's. It has been shown [5] that there exists a symmetric tensor λ on Σ such that

$$(7.1) \qquad [h_{AB,CD}] = \lambda_{AB}\, N_C\, N_D .$$

If the Einstein field equations are to be soluable, it may be shown [1] that we must have

$$(7.2) \qquad \lambda_{\alpha\beta} = \kappa (2S_{\alpha\beta} - S\, a_{\alpha\beta}) ,$$

where

(7.3) $$\lambda_{\alpha\beta} = \lambda_{AB}\, x^A_\alpha\, x^B_\beta \, .$$

The inversion of (7.3), as given by equation (3.17) of [1], then determines the jump strengths λ_{AB} of the metric field to the extent that such determinations may be effected. For the case at hand, (5.1) and (7.2) give

(7.4) $$\lambda_{\alpha\beta} = \kappa\overline{\rho}(2W_\alpha\, W_\beta - a_{\alpha\beta}) \, .$$

8. If we assume that the hypersurface Σ is known, the above results give a complete statement of the boundary data for a world tube of perfect fluid. One takes a time-like geodesic in the metric space Σ^* with unit tangent vector (W^α). The limiting values of the velocity field are then uniquely determined by

(8.1) $$\overline{W}^A = W^\alpha\, x^A_\alpha \, ,$$

and the quantities $\overline{\rho}$ and $\overline{p}$ are given by (5.7) and $\overline{p} = 0$. The jump strength of the metric field are then given by (7.4).

9. Suppose now that the equation for Σ is not known. In this case, the results of the previous paragraph still give us the appropriate boundary data, but the quantities $a_{\alpha\beta}$, $b_{\alpha\beta}$ and x^A_α are unknown. Since the Gauss-Cadazzi equations can be shown to be valid for second order problems [5], we can still obtain a complete determination of the boundary data if we can find consistent means of

determining the quantities $a_{\alpha\beta}$. It has been shown [1] that the differential system (2.8) can be integrated in the space Σ^* under rather general circumstances. The results are

$$(9.1)\qquad S_{\alpha\beta} = Ⓗ\,(K_{\alpha\beta} - \tfrac{1}{2}(K + \Pi)a_{\alpha\beta}) + Q_{\alpha\beta}\ ,$$

where $K_{\alpha\beta}$ are the components of the Ricci tensor of Σ^*, K is the scalar curvature of Σ^*, Ⓗ and Π are constants, and $Q_{\alpha\beta}$ are the components of a symmetric tensor such that

$$(9.2)\qquad Q^{\beta}_{\alpha;\beta} = F_{\alpha}\ .$$

For the case of a perfect fluid, we therefore have

$$(9.3)\qquad -\overline{\rho}\,W_{\alpha}\,W_{\beta} = Ⓗ\,(K_{\alpha\beta} - \tfrac{1}{2}(K + \Pi)a_{\alpha\beta}) + Q_{\alpha\beta}$$

and

$$(9.4)\qquad Q^{\beta}_{\alpha;\beta} = -\ \overline{\rho}\,W_{\alpha}\emptyset\ ,$$

where

$$(9.5)\qquad \emptyset = \overline{(W^{A}{}_{;B})}\ N^{B}\ N_{A}\ .$$

Hence, for any particular solution of (9.4), the system (9.3) becomes a differential system for the determination of the quantities $a_{\alpha\beta}$. Any system of a's determined in this fashion can be combined with the Gauss-Cadazzi equations and the previous results to yield the remaining unknowns

$b_{\alpha\beta}$ and x_{α}^{A}. If one proceeds in this manner, cognizance of certain intrinsic properties of the bounding surface Σ can be included in a straightforward manner. For instance, if it is known that the hypersurface Σ admits a group of motions, this fact can be combined with (9.3) so that the resulting a's will exhibit such properties (see Sec. 6 of [1] for a particular example). In the particularly simple case in which $\phi = 0$, we can take $Q_{\alpha\beta} = 0$. The system (9.3) then shows that the space Σ^* is a three-dimensional analog of an Einstein-Riemann space with incoherent matter.

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