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FINE STRUCTURE OF THE DOPPLER SHIFT
IN HETEROGENEOUS MEDIA

by

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PREFACE

This paper is the logical continuation of the research started with Report ORA-63-5. In the introduction of that report the author expressed his hope that he would be able to investigate the influence of such phenomena as boundary layers and shock fronts on the Doppler shift. The deeper the author penetrated into those problems the more it became obvious that first the problem of the influence of the "duration of observation" on the Doppler shift needs to be clarified. This is the time required to measure the frequency shift. The present report is an analysis of the influence of the duration of observation on the Doppler shift. The boundary layer problem and the shock problem are deferred.

Frequent references are made to Report ORA-63-5. The use of equations from that report is indicated by an asterisk. For example, (3.7)* means equation (3.7) of ORA-63-5.

ABSTRACT

Formulae are developed for the Doppler shift at signals that either go directly from the emitter E to the missile M or that reach the receiver R after they have been reflected from M. E, M, and R may have any velocity relative to the heterogeneous atmosphere.
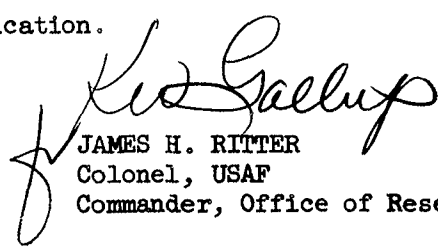
The derivation of the formulae accounts for the influence of the time needed for the frequency measurement on the Doppler shift. As a consequence, the refractive index, the velocities or velocity components that enter the formulae, are mean values over well defined intervals in space or time.

The relation between the error in the velocity measurement and the time measurement is established.

Keywords:

Doppler Effect
Doppler Radar
Refractive Index

This report is approved for publication.

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Colonel, USAF
Commander, Office of Research Analyses

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FINE STRUCTURE OF THE DOPPLER SHIFT

IN HETEROGENEOUS MEDIA

INTRODUCTION

The method employed in ORA-63-5 consisted in the repeated application
of the principle of the invariance of the phase function under the Lorentz
transformation and of the principle of the constancy of the frequency in
distinguished frames of reference. The power, usefulness, and mathemati-
cal elegance of that method need no further demonstration. Nevertheless,
the method proves rather inadequate for the treatment of the following
two aspects of the Doppler shift:

1. The influence of the time needed to measure the frequency shift.

2. The influence of such disturbances as boundary layers or shock
   fronts.

To account for those genuinely physical details the method of ORA-63-5
is too formal and too mathematical. What is needed here is a description
of the space-time geometry of the light and missile trajectories during
the time of observation. From there one can calculate the frequency ratio
in terms of proper mean values of the refractive index and the velocities.

TABLE I

DEFINITIONS OF QUANTITIES AND SUBSCRIPTS

$E \equiv$ emitter, $\quad M \equiv$ missile, $\quad R \equiv$ receiver, $\quad T \equiv$ atmosphere

Symbol	Physical Quantity	Measured by an observer who is at rest relative to	Location in T where this quantity holds	Object that has this velocity
$\vec{k}_{te}$	wave } original wave	T	E	
$\vec{k}_{tm}$	number }	T	M	
$\vec{\vec{k}}_{tr}$	"vector" } reflected wave	T	R	
$\vec{\vec{k}}_{tm}$	wave vector }	T	M	
ω_{ee}	} original wave	E	E	
ω_{mm}	}	M	M	
$\bar{\omega}_{rr}$	angular frequency } reflected wave	R	R	
$\bar{\omega}_{ee}$	}	E	E	
τ_{ee}	period of vibration of the original wave	E	E	
τ_{mm}		M	M	
n_e		T	E	
n_m	refractive index	T	M	
n_r		T	R	
$\vec{v}_e$		T		E
$\vec{v}_m$	velocity	T		M
$\vec{v}_r$		T		R

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CHAPTER I

DESCRIPTION OF THE PROBLEM

Throughout this report we shall assume that the atmosphere◊ T con-
stitutes an inertial body of reference. By this we mean:

1. There are no motions between parts of the atmosphere. With this
assumption we neglect the existence of boundary layers and suppose that
the light trajectories do not pass through the shock front ahead of the
missile or through the exhaust plume behind the missile.

2. The atmosphere is at rest relative to the earth.

3. The earth itself constitutes an inertial body of reference.

Of course, the last assumption is not strictly realistic. The in-
fluence of the non-linear motion of the earth on the frequency shift is
probably always neglectable, but the influence of the gravitational field
might become perceivable under certain conditions. This problem must be
deferred to later investigations. However, we can safely say that the
gravitational frequency shift destroys itself if emitter and receiver are
on the same gravitational potential. This is, for instance, the case
when the signal is emitted on ground level, reflected from the missile,
and received on ground level.

The used electromagnetic radiation is always assumed to be as mono-
chromatic as possible. Since we consider only very narrow sections of

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◊We use the symbol T for the atmosphere in order to be consistent with
ORA-63-5 where it means "transparent medium."

the waves (called beams), the waves may be considered as plane. A mono-chromatic, plane wave can be described by a wave number "vector" $\vec{k}$ and an angular frequency ω. Like the ordinary (classical) velocity $\vec{v}$, $\vec{k}$ is not a vector under the Lorentz transformation (LT). And like the refractive index n, the frequency ω is not a scalar under the LT. All these quantities depend on the frame of reference. Further, $\vec{k}$, n, and ω depend on the location in the atmosphere. Finally, the velocities must be specified as to the objects (emitters, missiles, mirrors, receivers) that have these velocities. Hence, all of the four quantities $\vec{k}$, ω, n, and $\vec{v}$ need two specifications; one for the frame of reference and one for the location or the object that has the respective velocity.

However, in this report (as in ORA-63-5) the velocities and the refractive index refer always to the atmosphere T so that the subscript for the frame of reference becomes dispensable for velocities and refractive index. We therefore adopt here the same subscript notation as in ORA-63-5. This is outlined in more detail in Table I, page 2. There, we introduce at the same time some other quantities.

Let now k_{te}, k_{tm}, $\bar{k}_{tm}$, $\bar{k}_{tr}$, v_e, v_m, v_r denote the absolute amounts of the respective "vectors" as defined in Table I. We then define the angles ψ_e, ψ_m, $\bar{\psi}_m$, $\bar{\psi}_r$ as observed from T at the respective locations, and the "normal components" v_{en}, v_{mn}, $\bar{v}_{mn}$, $\bar{v}_{rn}$ of the respective velocities as follows:

$$\vec{k}_{te} \cdot \vec{v}_e = k_{te} \, v_e \cos \psi_e = k_{te} \, v_{en} \qquad (1.1)$$

$$\vec{k}_{tm} \cdot \vec{v}_m = k_{tm} \, v_m \cos \psi_m = k_{tm} \, v_{mn} \qquad (1.2)$$

$$\vec{k}_{tm} \cdot \vec{v}_m = \bar{k}_{tm} \, v_m \cos \bar{\psi}_m = k_{tm} \, \bar{v}_{mn} \qquad (1.3)$$

$$\vec{k}_{tr} \cdot \vec{v}_r = \bar{k}_{tr} \, v_r \cos \bar{\psi}_r = k_{tr} \, \bar{v}_{rn} \qquad (1.4)$$

Figure 1 is a perspective illustration of these definitions. The word "perspective" may remind the reader that the "vectors" do not all lie in one plane.

The points E, M, and R are the instantaneous (not simultaneous!) positions of emitter, missile, and receiver. The directed curves S and $\bar{S}$ are the light trajectories from E to M and from M to R relative to T.

We shall always assume that the refractive index - as far as its geometrical dependency is concerned - depends only on the distance from the center of the earth C and not on the azimuth angle. An immediate consequence of this assumption is that all light trajectories are plane curves. A light trajectory going from E to M, for instance, lies in the plane EMC.

For our later purposes we define now four cases of Doppler shift and list the formulae as they have been derived in ORA-63-5.

CASE I: Emitter E stationary on the ground $(\vec{v}_e = 0)$ and missile M moving with $\vec{v}_m$. Signal emitted by E and received by M. The corresponding formula is obtained from the more general formula (3.7)* if one specializes on $\vec{v}_e = 0$ and replaces the subscript
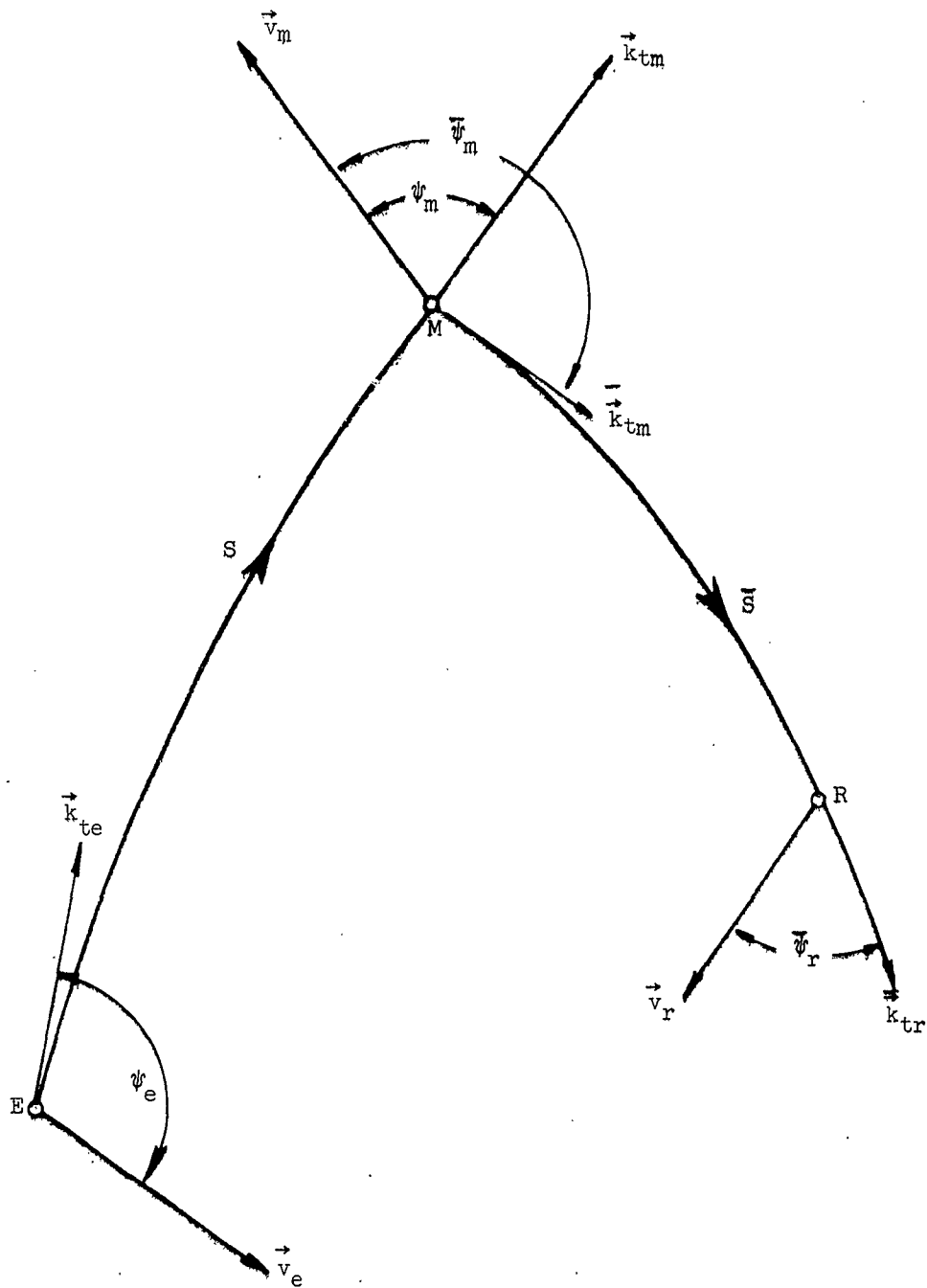
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Figure 1. Perspective illustration of "vectors" and angles relative to T.

r by m. This yields

$$\frac{\omega_{mm}}{\omega_{ee}} = \frac{1 - n_m \, \beta_{mn}}{\sqrt{1 - \beta_m^2}} \tag{1.5}$$

with

$$\beta_m = \frac{v_m}{c} \tag{1.6}$$

$$\beta_{mn} = \frac{v_{mn}}{c} = \beta_m \, \cos \psi_m \tag{1.7}$$

Case II: Again $\vec{v}_e = 0$ and $\vec{v}_m \neq 0$, but signal emitted by E, reflected from M and received by E. This is the case for which formula (4.8)* has been developed. It is

$$\frac{\omega_{ee}}{\overline{\omega}_{ee}} = \frac{1 + n_m \, \beta_{mn}}{1 - n_m \, \beta_{mn}} \tag{1.8}$$

Case III: $\vec{v}_e \neq 0$, $\vec{v}_m \neq 0$, signal emitted by E and received by M. Slightly altering formula (3.7)* yields

$$\frac{\omega_{mm}}{\omega_{ee}} = \sqrt{\frac{1 - \beta_e^2}{1 - \beta_m^2}} \; \frac{1 - n_m \, \beta_{mn}}{1 - n_e \, \beta_{en}} \tag{1.9}$$

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with

$$\beta_e = \frac{v_e}{c} \qquad (1.10)$$

$$\beta_{en} = \frac{v_{en}}{c} = \beta_e \cos \psi_e \qquad (1.11)$$

Case IV: This is the most general case, with $\vec{v}_e \neq 0$, $\vec{v}_m \neq 0$, $\vec{v}_r \neq 0$, where the signal is emitted by E, reflected from M and received by R. For this case refer to formula (4.6)*. It is:

$$\frac{\omega_{ee}}{\bar{\omega}_{rr}} = \sqrt{\frac{1 - \beta_r^2}{1 - \beta_e^2}} \; \frac{(1 - n_m \bar{\beta}_{mn})(1 - n_e \beta_{en})}{(1 - n_m \beta_{mn})(1 - n_r \bar{\beta}_{rn})} \qquad (1.12)$$

with

$$\beta_r = \frac{v_r}{c} \qquad (1.13)$$

$$\bar{\beta}_{rn} = \frac{\bar{v}_{rn}}{c} = \beta_r \cos \bar{\psi}_r \qquad (1.14)$$

It is now our aim to rederive these formulae in such a way that we can account for the influence of the "time of observation." This will at the same time provide for a more detailed description of such quantities as n_e, n_m, v_e, v_{en}, ψ_e, ψ_m, which, as of now, are "local" or "instantaneous values." All these quantities will be described as mean values over well-defined intervals in space or time.

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CHAPTER II

THE DURATION OF OBSERVATION

Every frequency measurement needs a minimum of time.$^{\diamond}$ If the frequency is supposed to be constant in the time, the minimum time required to measure it is the period of vibration. However, to enhance the accuracy with which the frequency is to be determined one better observes over a much longer time interval Δt and counts the number of vibrations. This time interval Δt which, within certain limits can be selected at liberty, is the "duration of observation."

It is now our aim to determine the duration of observation which is necessary to measure the Doppler shift. It is sufficient if we confine our consideration in this chapter to the technologically most important case, which is Case II as described in Chapter I.

The techniques of measurement are always more or less interference methods. A part of the original wave is branched off before emission and "stored" in a time delay device. The reflected wave is grossly amplified and then superimposed on the stored wave. If A, $\bar{A}$, ϕ, and $\bar{\phi}$ denote the amplitudes and the phase functions of the original and the reflected wave, the superposition can be mathematically described by

$$\bar{A} \sin \bar{\phi} + A \sin \phi = \left(\bar{A} + A \right) \cos \left(\frac{\bar{\phi} - \phi}{2} \right) \sin \left(\frac{\bar{\phi} + \phi}{2} \right)$$

$$+ \left(\bar{A} - A \right) \sin \left(\frac{\bar{\phi} - \phi}{2} \right) \cos \left(\frac{\bar{\phi} + \phi}{2} \right)$$

$^{\diamond}$A similar statement does not hold, for instance, for length or mass measurement.

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If one then filters out the term with the amplitude $(\bar{A} - A)$ or if one amplifies such that $\bar{A} - A = 0$, the superposition results in one single modulated wave with the angular modulation frequency

$$\omega^* = \frac{1}{2} \, |\bar{\omega}_{ee} - \omega_{ee}| \tag{2.1}$$

In order to measure ω^* one has to observe at least one period of modulation which is

$$\tau^* = \frac{2\pi}{\omega^*} = \frac{4\pi}{|\bar{\omega}_{ee} - \omega_{ee}|} \tag{2.2}$$

We call τ^* the "minimum duration of observation." In order to reduce the error of measurement one may observe during a much longer period of time Δt. However, if then ω^* and τ^* are not constant, one measures their mean value over the period Δt. If we denote these mean values by $\langle \omega^* \rangle$ and $\langle \tau^* \rangle$ and if

$$\sigma \geq 1 \tag{2.3}$$

is any real number not smaller than unity, we may write

$$\Delta t = \sigma \, \langle \tau^* \rangle = \frac{2\pi \, \sigma}{\langle \omega^* \rangle} = \frac{4\pi \, \sigma}{\langle |\bar{\omega}_{ee} - \omega_{ee}| \rangle} \tag{2.4}$$

This result shows that there are two antagonistic interests. The first one calls for a high accuracy in measuring $\langle \omega^* \rangle$. To achieve this one would have to make Δt and σ great. The second interest is to

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obtain information about ω^* which is as detailed as possible. This, in turn, calls for making Δt and σ small. Consequently, one has to seek a compromise between these two interests; that is, one has to select the number σ or the duration of observation Δt within certain ranges. In the last chapter of this report we shall return to this interesting subject.

It is now our concern to estimate the order of magnitude of the minimum time of observation τ^*. To this end we employ formula (1.8) which has been derived in ORA-63-5. The fact that we disregarded there the duration of observation as well as the boundary layer certainly does not affect the magnitude relations. Slightly rearranging equation (1.8) yields

$$\omega_{ee} - \overline{\omega}_{ee} = \beta_{mn} (\overline{\omega}_{ee} + \omega_{ee}) \qquad (2.5)$$

For all technically achievable missile velocities, $\overline{\omega}_{ee}$ and ω_{ee} are very close. Further, n_m is very close to unity. Hence we obtain from equations (2.1) and (2.5)

$$\omega^* = \omega_{ee} \, |\beta_{mn}| = \omega_{ee} \, \beta_m \, |\cos \psi_m| \qquad (2.6)$$

However, we need $\langle \omega^* \rangle$ rather than ω^*. Therefore, we formally introduce the mean value $\langle \beta_{mn} \rangle$ of β_{mn} but leave its exact definition to Chapter IV. Hence, we replace equation (2.6) by

$$\langle \omega^* \rangle = \omega_{ee} \, |\langle \beta_{mn} \rangle| = \omega_{ee} \, |\langle \beta_m \cos \psi_m \rangle| \qquad (2.7)$$

Substituting this into equation (2.4) yields then

$$\Delta t = \sigma \, \langle \tau^* \rangle = \frac{2\pi \, \sigma}{\omega_{ee} \, |\langle \beta_{mn} \rangle|} = \frac{\sigma \, \tau_{ee}}{|\langle \beta_{mn} \rangle|} \qquad (2.8)$$

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Here, τ_{ee} is the period of vibration of the original signal as observed by and at E (see also Table I).

Equation (2.8) shows that the duration of observation is far greater than the period of vibration. To have an intuitive example let us assume a typical radar frequency of some 10^9 cps which corresponds to a period of vibration

$$\tau_{ee} \approx 10^{-9} \text{ s}$$

If we then assume

$$v_m \approx 3000 \frac{m}{s}$$

and exclude extreme cases with $|v_{mn}| \ll v_m$, we have also

$$|v_{mn}| \approx 3000 \frac{m}{s}$$

With $c = 3.0 \times 10^8 \frac{m}{s}$ this yields

$$|\beta_{mn}| \approx 10^{-5}$$

It then follows from equation (2.8)

$$\langle \tau^* \rangle \approx 10^{-4} \text{ s}$$

which means

$$\Delta t \gtrsim \tau^* \gg \tau_{ee}$$

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CHAPTER III

THE TWO PEAKS OF VIBRATION
LIMITING THE DURATION OF OBSERVATION

As we mentioned in the introduction, we shall derive the Doppler shift
formulae for various cases in a certain geometrical manner. This method
is rather laborious, so that we limit its detailed outline to Case I of
Chapter I. There, E is at rest relative to T. Hence, clocks that rest
in E (E-clocks) and clocks that rest in T (T-clocks) measure the same
times. If we do not specify otherwise the times in this and the next chap-
ter are measured by E-clocks or T-clocks.

We then consider a certain peak of vibration of the monochromatic radi-
ation emitted by E and received by M. We call this peak the "zero-peak."
It may leave E at the time t_{eo} and arrive at M at the time t_{mo}. Its
travel time is then

$$\Delta t_o = t_{mo} - t_{eo} \qquad (3.1)$$

We then assume that the number σ in equation (2.8) is selected such
that the number

$$\rho = \frac{\Delta t}{\tau_{ee}} = \frac{\sigma}{|\langle \beta_{mn} \rangle|} \qquad (3.2)$$

is an integer. This assumption serves only to make the derivations of
this and the next chapter as transparent as possible and is in no way
necessary nor even essential. In particular, this assumption has no

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impact on the technique of measurement even though σ is generally no longer an integer if ρ is an integer since $\dfrac{1}{|\langle\beta_{mn}\rangle|}$ is generally not an integer. When performing actual measurements one is not bound by the assumptions someone made in his theoretical derivations.

In all practical cases, ρ is a very great number. Under our special assumption it is now a very great integer. Hence, Δt is a very great multiple of τ_{ee}. A consequence of this is that at the time

$$t_{e\rho} = t_{eo} + \Delta t \qquad (3.3)$$

another peak of vibration leaves E. We call it the "ρ-peak." The ρ-peak is not consecutive to the zero-peak but separated from it by a very great number $(\rho-1)$ of peaks. Let the ρ-peak arrive at M at the time $t_{m\rho}$. Its travel time is then

$$\Delta t_\rho = t_{m\rho} - t_{e\rho} \qquad (3.4)$$

The period of vibration as observed by and at E is simply

$$\tau_{ee} = \frac{\Delta t}{\rho} = \frac{t_{e\rho} - t_{eo}}{\rho} \qquad (3.5)$$

However, the period of vibration as observed by and at M is not presented by the analogous expression, because it is measured by M-clocks while t_{mo} and $t_{m\rho}$ are measured by E-clocks. Let $d\tau_m$ denote the element of the eigen time$^\Diamond$ for bodies that are at rest relative to M.

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$^\Diamond$Here only one subscript makes sense. It is the one that indicates the frame of reference.

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If then dt denotes the time elements of the E-clocks it holds according to the special theory of relativity

$$d\tau_m = dt \sqrt{1 - \beta_m^2(t)}$$

We wrote $\beta_m(t)$ in order to indicate that it may vary with the time.

The period of vibration τ_{mm} as observed by and at M becomes then

$$\tau_{mm} = \frac{1}{\rho} \int_{t_{mo}}^{t_{m\rho}} \sqrt{1 - \beta_m^2(t)} \; dt \qquad (3.6)$$

By using the mean value theorem we may write

$$\tau_{mm} = \langle \sqrt{1 - \beta_m^2} \rangle \; \frac{t_{m\rho} - t_{mo}}{\rho} \qquad (3.7)$$

with

$$\langle \sqrt{1 - \beta_m^2} \rangle = \frac{1}{(t_{m\rho} - t_{mo})} \int_{t_{mo}}^{t_{m\rho}} \sqrt{1 - \beta_m^2} \; dt \qquad (3.8)$$

τ_{ee} and τ_{mm} are both eigen times in Minkowski's sense. Therefore, their ratio is the inverse of the frequency ratio. Hence, we have

$$\frac{\tau_{mm}}{\tau_{ee}} = \frac{\omega_{ee}}{\omega_{mm}} = \langle \sqrt{1 - \beta_m^2} \rangle \frac{t_{m\rho} - t_{mo}}{t_{e\rho} - t_{eo}} \qquad (3.9)$$

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If we now combine equations (3.1), (3.4), (3.5), (3.7), and (3.9) we obtain

$$\frac{\omega_{mm}}{\omega_{ee}} = \frac{1}{\left\langle \sqrt{1 - \beta_m^2} \right\rangle} \left\{ 1 - \frac{\Delta t_\rho - \Delta t_o}{t_{m\rho} - t_{mo}} \right\} \qquad (3.10)$$

This is the basic equation we shall use in the next chapter to calculate the ratio ω_{mm}/ω_{ee} as a function of the missile velocity.

CHAPTER IV

DOPPLER SHIFT FOR STATIONARY EMITTER

Equation (3.10) at the end of the last chapter sets the task for this chapter: to express the difference $\Delta t_\rho - \Delta t_0$ of the two travel times in terms of the missile velocity. This is a geometrical task which is greatly facilitated by using T as frame of reference. We hope that this will become clear in the course of this chapter; however, we would like to point out now that employing T as body of reference is not a necessity, but rather a convenience. On the other hand, a convenience can amount to a necessity when the difficulties that arise from disregarding it become unsurmountable. This best describes the present situation.

In the preceding chapter we introduced the zero-peak and the ρ-peak. In this chapter we have to consider their trajectories. Now, the trajectory of one single peak of vibration might be a questionable concept since electrodynamics as well as classical mechanics or thermodynamics are macroscopical theories and as such statistical in nature. Hence, the concept "trajectory" might be legitimate only if it refers to a sufficiently great number of waves. However, this does not constitute a serious problem in the present case since the number ρ of vibrations during the time of observation is very great as compared with unity. (In Chapter VI we shall show that it is of the orders 10^7, 10^8, or 10^9.) Consequently, it is always possible to select two further numbers ρ_1 and ρ_2 such that

$$1 \ll \rho_1 \ll \rho$$
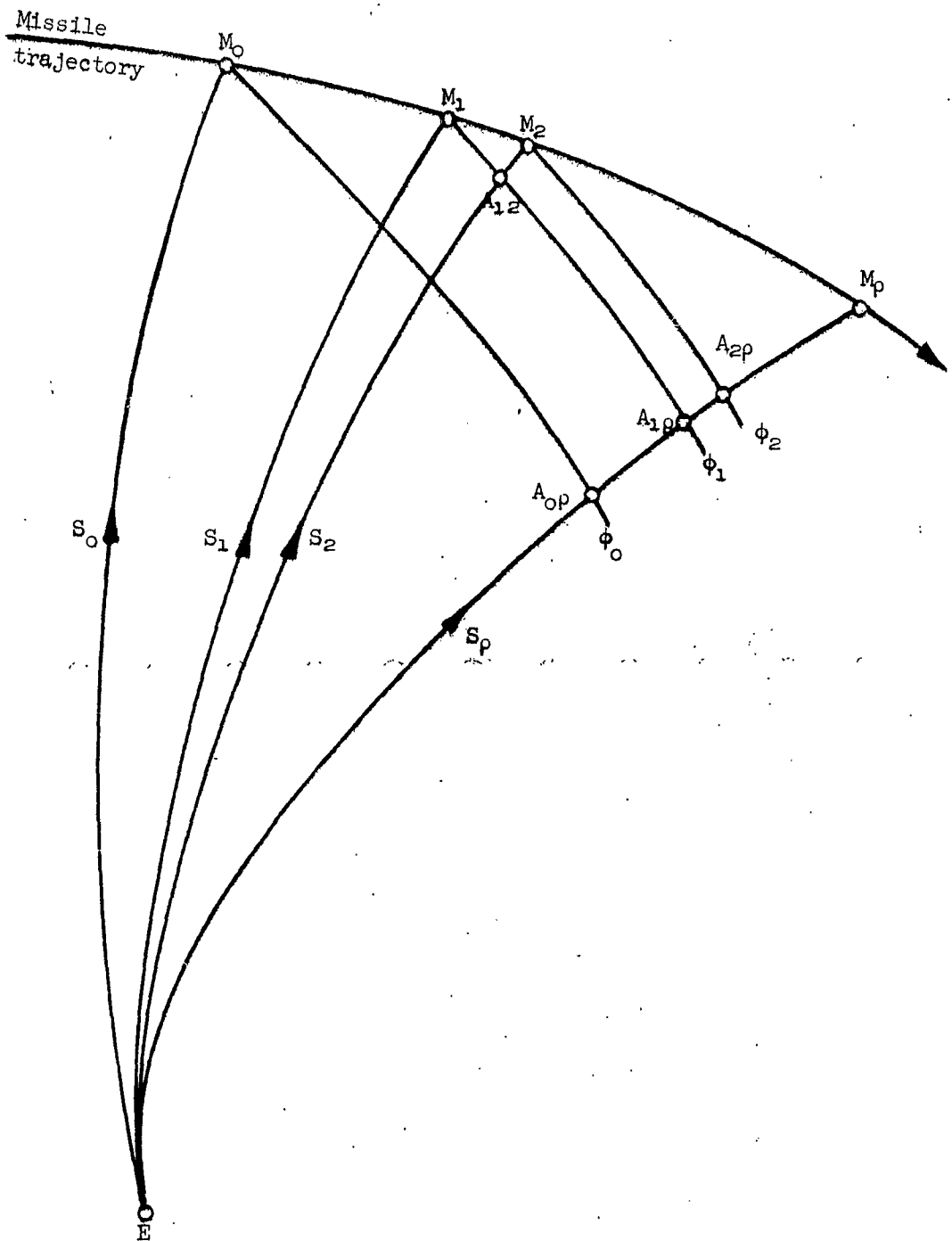$$1 \ll \rho_2 \ll \rho$$

(4.1)

Figure 2.   Missile trajectory, light trajectories, and phase surfaces
with respect to T   as body of reference. (Perspective
picture)

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Instead of considering the zero-peak and ρ-peak one may then consider the first ρ_1 and the last ρ_2 waves. For these two wave-groups the concept of a trajectory certainly makes sense. On the other hand, the number $\{\rho - (\rho_1 + \rho_2)\}$ of waves or peaks that separate the two wave-groups is still approximately equal to ρ. Yet this is the only essential we shall use as far as the zero-peak and the ρ-peak are concerned. Figure 2 is a grossly distorted, perspective picture of the missile trajectory, the trajectories S_O and S_ρ of the zero-peak and the ρ-peak, and of two auxiliary light trajectories S_1 and S_2. The function of these two trajectories will be seen later in this chapter.

Of course, the light trajectories and the missile trajectory do not generally lie in one plane. Generally, the missile trajectory does not lie in a plane at all.

The points M_O, M_1, M_2, M_ρ denote the positions of the missile at the times t_{mo}, t_{m1}, t_{m2}, $t_{m\rho}$ (all measured by E-clocks or T-clocks). The curves ϕ_O, ϕ_1, ϕ_2 symbolize surfaces of constant phase with respect to E as origin of radiation. The intersections of the surfaces ϕ_O, ϕ_1, ϕ_2 with the curve S_ρ are the points $A_{O\rho}$, $A_{1\rho}$, $A_{2\rho}$. The intersection of ϕ_1 with S_2 is the point A_{12}.

In Figure 2 all geometrical quantities like trajectories, surfaces, points, and angles refer to T. This results in two advantages which we shall now discuss by assuming temporarily that T as frame of reference is replaced by Σ^* relative to which T is in motion. We indicate this by adding an asterisk to those geometrical quantities we deal with.

The heterogeneity of T has then the consequence that the refractive index n^* is an explicit function of the time. If we then consider the two light trajectories from E^* to M_0^* along S_0^* and from E^* to $A_{0\rho}^*$ along S_ρ^* we can make the following statement:

Identical waves or peaks need equal times for both trajectories because M_0^* and $A_{0\rho}^*$ lie on one surface of constant phase, but different waves or peaks need different times even for one and the same trajectory because the refractive index has changed in the meantime.

Only if T itself serves as frame of reference need different peaks or waves take the same time for the same trajectory. If we now return to T as frame of reference, Δt_0 is the time the zero-peak need from E to M_0 as well as from E to $A_{0\rho}$. On the other hand, the ρ-peak needs, from E to $A_{0\rho}$, the same time as the zero-peak. If we denote the time of the ρ-peak from E to $A_{0\rho}$ by $\Delta t_{0\rho}$, we have

$$\Delta t_0 = \Delta t_{0\rho} \qquad\qquad (4.2)$$

This is the first advantage of employing T as frame of reference.

The second advantage is that n does not depend on the direction, while n^* does because of the drag effect. Hence, T is isotropical only with respect to itself but not with respect to Σ^*. However, in such frames of reference the light trajectories and the surfaces of constant phase are not orthogonal. Hence, the orthogonality of the light

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trajectories and the phase surfaces in Figure 2 is the second advantage of T as frame of reference.

These two advantages will greatly facilitate the following derivations. Moreover, it is necessary for the understanding of the entire complex of problems that one is conscious of these facts.

We have now to calculate the difference $\Delta t_\rho - \Delta t_o$ of the travel times occurring in the basic equation (3.10). The travel times can be expressed by line integrals along the respective light trajectories. If dS_o, dS_1, dS_2, and dS_ρ denote the line element on S_o, S_1, S_2, and S_ρ we may write

$$\Delta t_o = \frac{1}{c} \int_E^{M_o} n_o \, dS_o \tag{4.3}$$

$$\Delta t_\rho = \frac{1}{c} \int_E^{M_\rho} n_\rho \, dS_\rho \tag{4.4}$$

$$\Delta t_1 = \frac{1}{c} \int_E^{M_1} n_1 \, dS_1 \tag{4.5}$$

$$\Delta t_2 = \frac{1}{c} \int_E^{M_2} n_2 \, dS_2 \tag{4.6}$$

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n_0, n_ρ, n_1, n_2 denote the refractive index as function of position along S_0, S_ρ, S_1, S_2.

By construction we have further

$$\frac{1}{c} \int_E^{M_0} n_0 \, dS_0 = \frac{1}{c} \int_E^{A_{0\rho}} n_\rho \, dS_\rho \qquad (4.7)$$

$$\frac{1}{c} \int_{A_{12}}^{M_2} n_2 \, dS_2 = \frac{1}{c} \int_{A_{1\rho}}^{A_{2\rho}} n_\rho \, dS_\rho \qquad (4.8)$$

If we now build the difference

$$\Delta t_\rho - \Delta t_0$$

and observe equations (4.3), (4.4), (4.7) and Figure 2 we find

$$\Delta t_\rho - \Delta t_0 = \frac{1}{c} \int_{A_{0\rho}}^{M_\rho} n_\rho \, dS_\rho \qquad (4.9)$$

Thus we have to express this line integral in terms of the missile velocity. To facilitate this we introduced the auxiliary light trajectories S_1 and S_2. We assume that the time difference

$$dt = t_2 - t_1 \qquad (4.10)$$

is sufficiently small so that the triangle $M_1 A_{12} M_2$ can be considered as

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composed of straight lines. Since the angle at A_{12} is 90 degrees by construction we have then

$$\overline{A_{12}M_2} = \overline{M_1M_2} \cos \psi_m \tag{4.11}$$

where ψ_m is the angle between the two "vectors" $\vec{k}_{tm}$ and $\vec{v}_m$ at the position under consideration and as introduced in Chapter I.

On the other hand, the distance $\overline{M_1M_2}$ is the distance the missile has traveled in the time $dt = t_2 - t_1$. Consequently, we have

$$\overline{M_1M_2} = v_m(t) \, dt$$

If we substitute this into equation (4.11) we obtain

$$\overline{A_{12}M_2} = v_m(t) \cos \psi_m(t) \, dt \tag{4.12}$$

That $(t_2 - t_1)$ is infinitesimal makes the integral signs in equation (4.8) dispensable. Moreover, the line element dS_2 can be replaced by the distance $\overline{A_{12}M_2}$. If we then write n_m instead of n_2 expressing that this is the value of n in the immediate neighborhood of M, equation (4.8) becomes

$$n_m(t) \, v_m(t) \cos \psi_m(t) = n_\rho \, dS_\rho$$

Integrating the left-hand side of this equation from t_{mo} to $t_{m\rho}$ corresponds to integrating the right-hand side from $A_{o\rho}$ to M_ρ. This yields

$$\frac{1}{c} \int_{t_{mo}}^{t_{m\rho}} n_m(t) \, v_m(t) \cos \psi_m(t) \, dt = \frac{1}{c} \int_{A_{o\rho}}^{M_\rho} n_\rho \, dS_\rho$$

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If we then substitute this into equation (4.9) we obtain

$$\Delta t_\rho - \Delta t_o = \frac{1}{c} \int_{t_{mo}}^{t_{m\rho}} n_m(t)\, v_m(t)\, \cos \Psi_m(t)\, dt$$

This is to be substituted into the basic equation (3.10) of the preceding chapter. The result is

$$\frac{\omega_{mm}}{\omega_{ee}} = \frac{1}{\langle \sqrt{1 - \beta_m^2} \rangle} \left\{ 1 - \frac{1}{(t_{m\rho} - t_{mo})} \int_{t_{mo}}^{t_{m\rho}} n_m(t)\, \beta_m(t)\, \cos \Psi_m(t)\, dt \right\}$$

$$(4.13)$$

It now suggests itself to apply again the mean value theorem and to write

$$\langle n_m\, \beta_m\, \cos \Psi_m \rangle = \frac{1}{(t_{m\rho} - t_{mo})} \int_{t_{mo}}^{t_{m\rho}} n_m(t)\, v_m(t)\, \cos \Psi_m(t)\, dt \quad (4.14)$$

which leads to

$$\frac{\omega_{mm}}{\omega_{ee}} = \frac{1 - \langle n_m\, \beta_m\, \cos \Psi_m \rangle}{\langle \sqrt{1 - \beta_m^2} \rangle} \qquad (4.15)$$

This result replaces the former result (1.5). It has the same algebraic structure but the advantage that the occurring combinations of the refractive index and the missile velocity are now mean values over well-defined time or space intervals.

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We defer all further discussions of this result to Chapter VI and proceed immediately to Case II of Chapter I where the signal is reflected from M and then received by E. This complication does not change the geometry, so that we need not go into any details but can confine ourselves to listing the final result:

$$\frac{\omega_{ee}}{\overline{\omega}_{ee}} = \frac{1 + \langle n_m \; \beta_m \; \cos \; \psi_m \rangle}{1 - \langle n_m \; \beta_m \; \cos \; \psi_m \rangle} \qquad (4.16)$$

This equation replaces the former result (1.8).
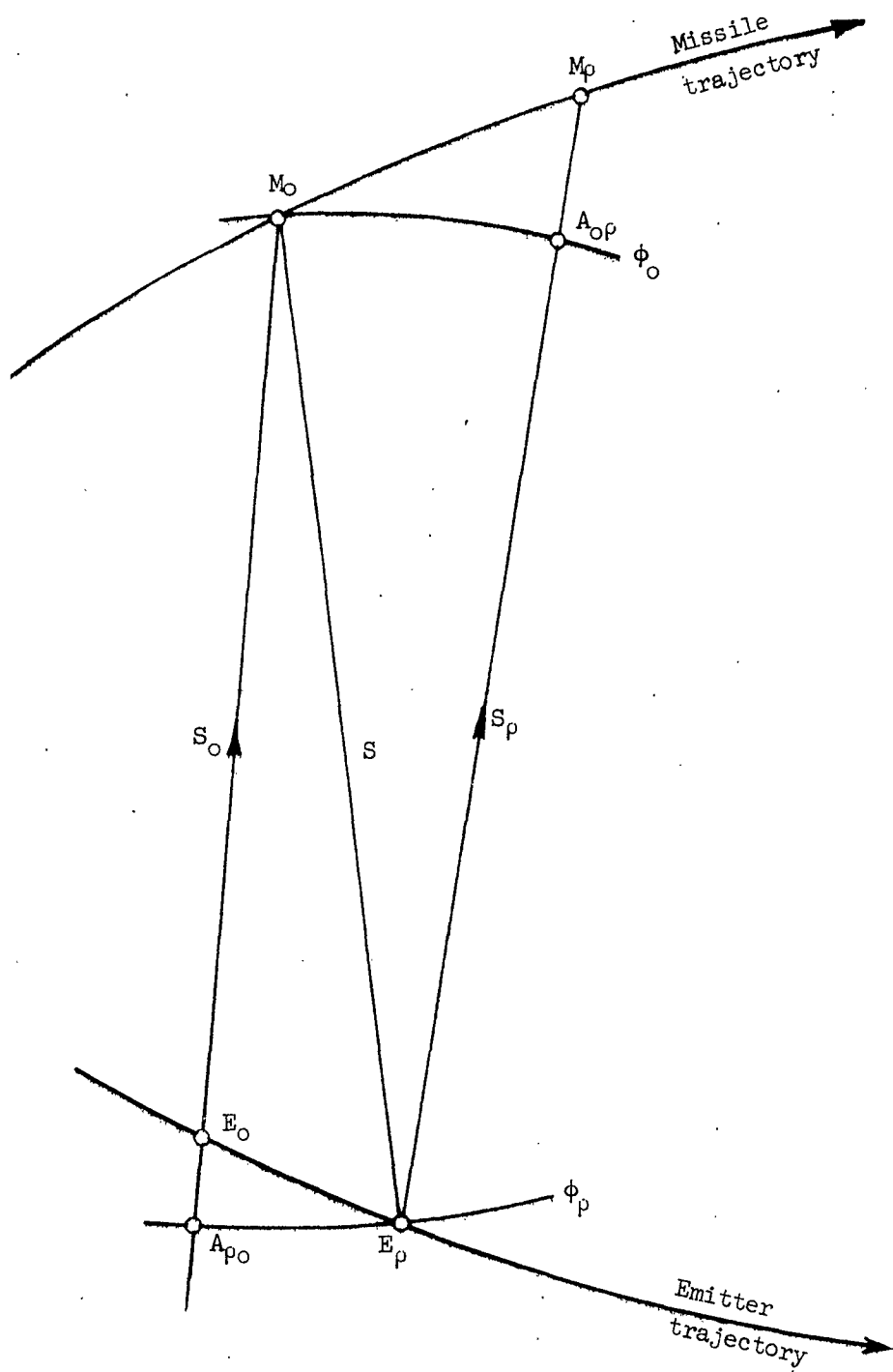
Figure 3. Moving emitter and moving missile.
(Perspective picture)

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CHAPTER V

MOVING EMITTER AND RECEIVER

We turn now to Cases III and IV of Chapter I. We need not go through the whole analysis but may confine ourselves to those special features that require some additional thoughts. This concerns primarily the geometry which is pictured in Figure 3 for Case III.

The body of reference is T. The times t_{eo}, $t_{e\rho}$, t_{mo}, $t_{m\rho}$, $\bar{t}_{mo}$, $\bar{t}_{m\rho}$, $\bar{t}_{ro}$, $\bar{t}_{r\rho}$ are measured by T-clocks. These are the times when the zero-peak, the ρ-peak, the reflected zero-peak, or the reflected ρ-peak leave E, arrive at M, leave M, and arrive at R, respectively. To these events correspond the positions E_o, E_ρ, M_o, M_ρ, R_o, R_ρ of emitter, missile, and receiver. (R_o and R_ρ are not shown in Figure 3.)

In Figure 3, S_o and S_ρ are the light trajectories of the two peaks from E to M. S is an auxiliary light trajectory. ϕ_o is a surface of constant phase with E_ρ as origin. ϕ_ρ is a surface of constant phase with M_o as imagined origin of radiation. It then holds by definition

$$\int_{E_\rho}^{A_{o\rho}} n_\rho \, dS_\rho = \int_{E_\rho}^{M_o} n \, dS$$

$$\int_{M_o}^{A_{\rho o}} n_o \, dS_o = \int_{M_o}^{E_\rho} n \, dS$$

From these two equations follows

$$\int_{E_\rho}^{A_{o\rho}} n_\rho \, dS_\rho = \int_{A_{\rho o}}^{M_o} n_o \, dS_o \qquad (5.1)$$

The travel times for the two peaks are

$$\Delta t_o = \frac{1}{c} \int_{E_o}^{M_o} n_o \, dS_o = \frac{1}{c} \int_{A_{\rho o}}^{M_o} n_o \, dS_o - \frac{1}{c} \int_{A_{\rho o}}^{E_o} n_o \, dS_o \qquad (5.2)$$

$$\Delta t_\rho = \frac{1}{c} \int_{E_\rho}^{M_o} n_\rho \, dS_\rho = \frac{1}{c} \int_{E_\rho}^{A_{o\rho}} n_\rho \, dS_\rho + \frac{1}{c} \int_{A_{o\rho}}^{M_\rho} n_\rho \, dS_\rho \qquad (5.3)$$

It then follows from equations (5.1), (5.2), (5.3)

$$\Delta t_\rho - \Delta t_o = \frac{1}{c} \int_{A_{o\rho}}^{M_\rho} n_\rho \, dS_\rho + \frac{1}{c} \int_{A_{\rho o}}^{E_o} n_o \, dS_o \qquad (5.4)$$

This equation is the analog to equation (4.9). From now on the derivation is very similar to that in Chapters III and IV. The final result for Case III is

$$\frac{\omega_{mm}}{\omega_{ee}} = \frac{\langle \sqrt{1 - \beta_e^2} \rangle}{\langle \sqrt{1 - \beta_m^2} \rangle} \cdot \frac{1 - \langle n_m \, \beta_m \, \cos \Psi_m \rangle}{1 - \langle n_e \, \beta_e \, \cos \Psi_e \rangle} \qquad (5.5)$$

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The result for Case IV is

$$\frac{\omega_{ee}}{\overline{\omega}_{rr}} = \frac{\langle \sqrt{1 - \beta_r^2} \rangle}{\langle \sqrt{1 - \beta_e^2} \rangle} \cdot \frac{\{1 - \langle n_m \beta_m \cos \overline{\psi}_m \rangle\} \{1 - \langle n_e \beta_e \cos \psi_e \rangle\}}{\{1 - \langle n_m \beta_m \cos \psi_m \rangle\} \{1 - \langle n_r \beta_r \cos \overline{\psi}_r \rangle\}} \quad (5.6)$$

These equations replace the former results (1.9) and (1.12). The mean values containing subscript m have already been defined in Chapters III and IV. Those containing subscripts e and r are defined as follows:

$$\langle \sqrt{1 - \beta_e^2} \rangle = \frac{1}{(t_{e\rho} - t_{eo})} \int_{t_{eo}}^{t_{e\rho}} \sqrt{1 - \beta_e^2(t)} \; dt \qquad (5.7)$$

$$\langle \sqrt{1 - \beta_r^2} \rangle = \frac{1}{(\overline{t}_{r\rho} - \overline{t}_{ro})} \int_{\overline{t}_{ro}}^{\overline{t}_{r\rho}} \sqrt{1 - \beta_r^2(t)} \; dt \qquad (5.8)$$

$$\langle n_m \beta_m \cos \psi_m \rangle = \frac{1}{(t_{m\rho} - t_{mo})} \int_{t_{mo}}^{t_{m\rho}} n_m(t) \; \beta_m(t) \cos \psi_m(t) dt \quad (5.9)$$

$$\langle n_m \beta_m \cos \overline{\psi}_m \rangle = \frac{1}{(\overline{t}_{m\rho} - \overline{t}_{mo})} \int_{\overline{t}_{mo}}^{\overline{t}_{m\rho}} n_m(t) \; \beta_m(t) \cos \overline{\psi}_m(t) dt \quad (5.10)$$

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$$\langle n_r \, \beta_r \, \cos \overline{\psi}_r \rangle = \frac{1}{(\overline{t}_{r\rho} - \overline{t}_{ro})} \int_{\overline{t}_{ro}}^{\overline{t}_{r\rho}} n_r(t) \, \beta_r(t) \, \cos \overline{\psi}_r(t) dt \qquad (5.11)$$

Of course, it holds:

$$\overline{t}_{mo} = t_{mo}$$

$$\overline{t}_{m\rho} = t_{m\rho}$$

The bars are retained for formalistic fanaticism only. However, they prove quite useful in case of $\overline{t}_{ro}$ and $\overline{t}_{r\rho}$, for if we now identify E with R those times become $\overline{t}_{eo}$ and $\overline{t}_{e\rho}$ and these are the times when the reflected zero-peak and ρ-peak arrive at E. They are to be distinguished from t_{eo} and $t_{e\rho}$.

CHAPTER VI

ACCURACY CONSIDERATIONS

So far all formulae are exact (with the possible exception that the neglect of the boundary layer influence might result in certain inaccuracies). Paradoxical as it may sound, one starts to make approximations as soon as one indulges in accuracy considerations. We shall now estimate the error in the velocity measurement caused by the error in the measurement of the time of observation Δt.

We first mention that the time intervals in the definitions (3.8), (4.14), (5.7), (5.8), (5.9), (5.10), (5.11) for the respective mean values are all approximately equal:

$$\Delta t = t_{e\rho} - t_{eo} \approx t_{m\rho} - t_{mo} \approx \bar{t}_{r\rho} - \bar{t}_{ro}$$

Hence, all of those mean values which are basically mean values of velocity components are extended over approximately the same time interval Δt. In other words, Δt is the uncertainty of the time to which those velocity components refer.

We shall now investigate how the error in the measurement of the velocity components is related to the error in the measurement of Δt. We confine this consideration to Case II of Chapter I.

The basic assumption is that the error $d\Delta t$ (the symbol d in this chapter is used to denote errors) in the measurement of the duration of observation Δt does not depend on Δt itself. Within certain limits this is in good agreement with reality.

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Let Δt denote the "true value" and Δt^* the "measured value" of the duration of observation. We have then

$$\Delta t = \Delta t^* + d\Delta t \qquad (6.1)$$

Of course, we assume

$$\frac{|d\Delta t|}{\Delta t} \approx \frac{|d\Delta t|}{\Delta t^*} \ll 1 \qquad (6.2)$$

We now apply this to equation (2.8) which holds for Case II of Chapter I. This yields

$$|\langle \beta_{mn} \rangle| = \frac{2\pi \, \sigma}{\omega_{ee} \, (\Delta t^* + d\Delta t)}$$

or, because of relation (6.2)

$$|\langle \beta_{mn} \rangle| = \frac{2\pi \, \sigma}{\omega_{ee} \, \Delta t} \left\{ 1 - \frac{d\Delta t}{\Delta t} \right\}$$

for which we may write

$$|\langle \beta_{mn} \rangle| = \frac{2\pi \, \sigma}{\omega_{ee} \, \Delta t} + |d\langle \beta_{mn} \rangle|$$

with

$$|d\langle \beta_{mn} \rangle| = \frac{2\pi \, \sigma}{\omega_{ee} \, \Delta t} \, \frac{|d\Delta t|}{\Delta t}$$

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If we substitute here equation (2.8) we obtain

$$\left| \frac{d\langle \beta_{mn} \rangle}{\langle \beta_{mn} \rangle} \right| = \left| \frac{d\langle v_{mn} \rangle}{\langle v_{mn} \rangle} \right| = \frac{|d\Delta t|}{\Delta t} = \frac{\omega_{ee} |\langle \beta_{mn} \rangle|}{2\pi \, \sigma} |d\Delta t| \qquad (6.3)$$

This equation relates the relative velocity error to the relative error of the time measurement. If we rearrange it a little we also obtain

$$|d\langle v_{mn} \rangle| \; \Delta t = |d\Delta t| \cdot |\langle v_{mn} \rangle| \qquad (6.4)$$

In this form the error relation reminds one of Heisenberg's uncertainty relation which is, for instance,

$$\Delta E \; \Delta t = h \qquad . \qquad (6.5)$$

where ΔE and Δt are the uncertainties of the energy E of an elementary particle and the time t to which E refers.

The similarities between the two equations (6.4) and (6.5) or between the corresponding physical situations are intriguing even though not unlimited. On the left-hand sides of both equations stand the products of two uncertainties. In equation (6.4) these are the uncertainties of the missile velocity and of the time to which the velocity refers. On the right-hand sides of both equations stand quantities which are not unlimitedly at our disposal. However, it is here where the analogy eventually stops. While the natural constant h is not at all at our disposal, the quantity $|d\Delta t|$ is dictated by the technique of measurement.

The physical explanation for the existence of the two uncertainty relations is also rather analogous. To measure the energy of a particle means to measure its frequency according to Einstein's law

$$E = \frac{h}{2\pi} \omega$$

For measuring a frequency one needs a certain time of observation. The greater one selects this time of observation the smaller becomes the frequency or energy error, but the greater becomes the uncertainty of the time to which the energy refers. The situation is quite similar for the measurement of the missile velocity which now plays the role of the particle energy. Besides, particle energy and missile velocity can both be translated into frequencies.

This sheds a new light on our results (4.15), (4.16), (5.5), and (5.6). The two facts that $|d\Delta t|$ is dictated by the state of the art and that we cannot tolerate unlimited errors $|d\langle v_{mn}\rangle|$ mean that we cannot measure the missile velocity as a continuous function of time but only as a step function consisting of a sequence of mean values. This is an inherent limitation of the whole method of measuring velocities by means of the Doppler shift.

It is worthwhile to consider a numerical example. We shall vary a few parameters but keep the following parameters constant throughout the whole example:

The missile velocity $\vec{v}_m$ shall be such that

$$|\langle \beta_{mn}\rangle| = 10^{-5} \tag{6.6}$$

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The angular radar frequency shall be

$$\omega_{ee} = 8\pi \times 10^9 \text{ s}^{-1} \qquad (6.7)$$

This determines the minimum time of observation

$$\tau^* = \frac{1}{4} 10^{-4} \text{ s} \qquad (6.8)$$

In order to have a completely determined situation we have to make two further assumptions. We do this in two different ways. Once we assume a fixed value for the relative velocity error

$$\left| \frac{d\langle v_{mn} \rangle}{\langle v_{mn} \rangle} \right|$$

and vary the time error $|d\Delta t|$, and once we assume a fixed value for the duration of observation Δt and vary again $|d\Delta t|$. In the first case, Δt, σ, and ρ are functions of the assumptions, in the latter case $\left| \frac{d\langle v_{mn} \rangle}{\langle v_{mn} \rangle} \right|$, σ, and ρ are functions of the assumptions. The first case is presented in Table II, the second in Table III.

TABLE II

FIXED ASSUMPTION

$$\left| \frac{d\langle v_{mn} \rangle}{\langle v_{mn} \rangle} \right| = 10^{-6}$$

$\frac{\|d\Delta t\|}{s}$	$\frac{\Delta t}{s}$	σ	ρ
10^{-6}	10^{0}	4×10^{4}	4×10^{9}
10^{-7}	10^{-1}	4×10^{3}	4×10^{8}
10^{-8}	10^{-2}	4×10^{2}	4×10^{7}

TABLE III

FIXED ASSUMPTION

$$\Delta t = 10^{-1} \text{ s}$$

$\frac{\|d\Delta t\|}{s}$	$\left\| \frac{d\langle v_{mn} \rangle}{\langle v_{mn} \rangle} \right\|$	σ	ρ
10^{-6}	10^{-5}	4×10^{4}	4×10^{9}
10^{-7}	10^{-6}	4×10^{3}	4×10^{8}
10^{-8}	10^{-7}	4×10^{2}	4×10^{7}

We selected the three values 10^{-6}, 10^{-7}, and 10^{-8} s for $\left| d\Delta t \right|$ in order to satisfy the three psychological types of engineers - the conservatives, the even-tempered, and the optimists.

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ORA-63-8   Unclassified Report

Formulae are developed for the
Doppler shift at signals that either
go directly from the emitter E to
the missile M or that reach the
receiver R after they have been
reflected from M. E, M,
and R may have any veloc-
ity relative to the (over)

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Refractive Index

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heterogeneous atmosphere. The derivation of the formulae accounts for the influence of the time needed for the frequency measurement on the Doppler shift. As a consequence, the refractive index, the velocities or velocity components that enter the formulae are mean values over well-defined intervals in space or time. The relation between the error in the velocity measurement and the time measurement is established.

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