VIBRATION FREQUENCIES OF A CIRCULAR CYLINDER
OF FINITE LENGTH IN AN INVISCID FLUID

Introduction

The customary procedure for taking into account the effect of the surrounding fluid on the vibration of a free-free beam is to increase the magnitude of the mass per unit length of the beam by the added mass at each section, considered as a two-dimensional form, with a correction factor for the three-dimensionality of the flow. In the present work there are compared, for a particular case, The natural frequencies of vibration obtained by ~~this~~ the "strip-theory" procedure are compared with the values given by a more exact formulation of the problem, in which the effect of the presence of the fluid is incorporated properly into the vibration equation, ~~[1]~~.* For this purpose the flexural vibration of a uniform circular cylinder of finite length was selected for study.

General Procedure

The displacement of the beam in a mode of frequency ω will be assumed to be of the form

$$\Delta(z, t) = v(z) \sin \omega t \qquad (1)$$

in which z denotes distance along the axis of the cylinder, which extends from $z = -1$ to $z = 1$. The amplitude of the vibration, $v(z)$, is called the displacement function of the beam. By differentiating (1) with respect to time one obtains $\omega v(z)$ as the corresponding velocity function.

The velocity potential for the motion of the fluid, Φ, may similarly be expressed in the form

$$\Phi(r, \theta, z, t) = \phi(r, \theta, z) \sin \omega t \qquad (2)$$

in terms of cylindrical coordinates with origin at the center of the beam. Both Φ and ϕ satisfy Laplace's equation.

* Numbers in [ ] indicate references at end of this report.

The maximum kinetic energy of the beam is given by

$$T_s = \frac{\pi}{2} a^2 \rho_s \omega^2 \int_{-1}^{1} v^2(z)\, dz \tag{3}$$

where a is the radius of the cylinder and ρ_s its density, assumed to be a constant. For flexural vibrations the maximum potential energy V is [2]

$$V = \frac{\pi}{8} a^4 \rho_s E \int_{-1}^{1} [v''(z)]^2\, dz \tag{4}$$

where E is the modulus of elasticity. Furthermore it will be seen that the amplitude of the kinetic energy of the fluid due to the vibration of the beam can also be expressed in terms of the displacement function in the form

$$T_f = \frac{1}{2} \rho_f \omega^2 \int_{-1}^{1}\int_{-1}^{1} A(\xi, z)\, \underline{v(\xi)}\, v(z)\, d\xi\, dz \tag{5}$$

where ρ_f is the density of the fluid. Thus the amplitude of the total kinetic energy is expressible in the form

$$T = T_s + T_f = \frac{\omega^2}{2} \int_{-1}^{1}\int_{-1}^{1} \left(\pi a^2 \rho_s \delta(\xi, z) + \rho_f A(\xi, z)\right) v(\xi)\, v(z)\, d\xi\, dz \tag{6}$$

where $\delta(\xi, z)$ is the Dirac delta function

$$\delta(\xi, z) = 0 , \quad \xi \neq z ; \qquad \int_{-\ell}^{\ell} \delta(\xi, z)\, dz = 1 \tag{7}$$

If the "added-mass" function $A(\xi, z)$ were known, one could obtain the vibration frequencies by the Rayleigh-Ritz method from the expressions for V and T in (4) and (6). Because the solution of the potential-flow problem is obtained from an integral equation which may be solved by the use of quadrature formulas, it was convenient to approximate the integrals for T_s and V also by means of quadrature formulas. The potential and kinetic energies are thus expressed as quadratic forms in a finite number of terms from which the natural frequencies of the body-fluid system can be obtained as the eigenvalues of the potential-energy matrix with respect to the total-energy matrix, these matrices being composed of the coefficients of the respective quadratic forms [1].

Quadratic Forms for V and T_s

An unsuccessful initial attempt to solve the problem will first be briefly described. A quadrature formula was used to replace the integral for T_s in (1). Next the interpolation polynomial corresponding to the quadrature formula was differentiated twice. The resulting polynomial was squared and a quadrature formula was then used to approximate the second integral. This attempt was not successful, because the resulting quadratic form for the potential energy derived from the second integral was not positive definite. It is believed that the difficulty was caused by the numerical approximation of $v''(\zeta)$, the second derivative of $v(\zeta)$.

To avoid the numerical differentiation, it was decided to find a suitable interpolation polynomial for $v''(\zeta)$ and then by integration to obtain an expression approximating $v(\zeta)$. This is the procedure which will now be described in detail.

The conditions to be met by the interpolation polynomial, $p(\zeta)$, are that it be equal to $v''(\zeta)$ at certain prescribed points, ζ_j, and also that it satisfy the same end conditions as the function $v(\zeta)$ For a free-free beam - that is, one which is not restrained at either end - these conditions are

$$v''(\zeta)\Big|_{\zeta=\pm 1} = 0 \tag{8}$$

$$v'''(\zeta)\Big|_{\zeta=\pm 1} = 0 \tag{9}$$

Therefore, the conditions prescribed for $p(\zeta)$ are

(a) $$P(\zeta_j) = d_j = \frac{d^2 v(\zeta)}{d\zeta^2}\Big|_{\zeta=\zeta_j}$$

(b) $$P(\pm \ell) = 0$$

(c) $$P'(\pm \ell) = 0$$

If

$$\prod^{n}(\zeta-\zeta_k) = (\zeta-\zeta_1)(\zeta-\zeta_2)\cdots(\zeta-\zeta_n)$$

and

$$\prod_{k=1}^{n}{}'(z_j - z_k) = \prod_{k=1}^{n}{}'(z - z_k)\Big|_{z=z_j}$$

where the prime indicates differentiation with respect to z in the right member and omission of the factor with $k = j$ in the left one, then the expression

$$\sum_{j=1}^{n} \frac{\prod_{k=1}^{n}(z - z_k)}{(z - z_j)\prod_{k=1}^{n}{}'(z_j - z_k)}\, d_j$$

satisfies condition (a) because at $z = z_i$

$$\frac{\prod_{k=1}^{n}(z_i - z_k)}{(z_i - z_j)\prod_{k=1}^{n}{}'(z_j - z_k)} = \delta_{ij}$$

where $\delta_{ij} = 0$ or 1 according as $i \neq j$ or $i = j$.

For an interpolation polynomial which does not include the end points among the z_j , the end conditions may be satisfied by including a factor $(z^2 - 1)^2/(z_j{}^2 - 1)^2$ which reduces to unity when $z = z_j$, and is zero when $z = \pm 1$. Also, the first derivative is zero when $z = \pm 1$. The desired interpolation polynomial will then be

$$P(z) = \sum_{j=1}^{n} \frac{(z^2-1)^2}{(z_j^2-1)^2}\, \frac{\prod_{k=1}^{n}(z - z_k)}{(z - z_j)\prod_{k=1}^{n}{}'(z_j - z_k)}\, d_j \tag{10}$$

The next step is to determine an approximation, $q(z)_1$ to $v(z)$ in terms of the quantities d_j . This may be done by taking

$$v(z) \doteq q(z) = \int_0^z \int_0^z P(u)\, du\, dv + C_1 z + C_2 \tag{11}$$

The polynomial $p(u)$ may be taken in the form

$$P(u) = \sum_{j=1}^{n} \frac{(u^2-1)^2}{(z_j^2-1)^2}\, \frac{d_j}{\prod_{k=1}^{n}{}'(z_j - z_k)} \sum_{p=1}^{n} E_{pj}\, u^{p-1}$$

If the coefficients of the polynomial $\prod_{k=1}^{n}(z - z_k)$ are known, the coefficients E_{pj} may be most easily found by synthetic division of Π by z_j . Since z_j is a root of Π , the remainder should be zero, which serves as a check. For instance, if the z_k are the Gaussian quadrature points, Π will be the Legendre polynomial of degree n with a leading coefficient of unity.

On combining all constant terms, one can write p(u) as

$$P(u) = \sum_{j=1}^{n} \sum_{p=1}^{n} F_{pj} \left(u^{p+3} - 2u^{p+1} + u^{p-1} \right) d_j \tag{12}$$

with

$$F_{pj} = \frac{E_{pj}}{(z_j^2-1)^2 \prod_{k=1}^{n}{}' (z_j - z_k)}$$

Performing the integration indicated in (11) gives

$$q(z) = \sum_{j=1}^{n} \sum_{p=1}^{n} F_{pj} \left(\frac{z^{p+5}}{(p+4)(p+5)} - \frac{2z^{p+3}}{(p+2)(p+3)} + \frac{z^{p+1}}{p(p+1)} \right) d_j + C_1 z + C_2 \tag{13}$$

and

$$q(z_i) = C_1 z + C_2 + \sum_{j=1}^{n} G_{ij} d_j \tag{14}$$

with

$$G_{ij} = \sum_{p=1}^{n} Z_{ip} F_{pj}$$

and with

$$Z_{ip} = \frac{z_i^{p+5}}{(p+4)(p+5)} - \frac{2z_i^{p+3}}{(p+2)(p+3)} + \frac{z_i^{p+1}}{p(p+1)}$$

The constants C_1 and C_2 are functions of the quantities d_j. In order to evaluate them it will be assumed that the bar has (a) zero linear momentum and (b) zero angular momentum. Condition (a) gives

$$\int_{-1}^{1} q(z)\, dz = 0 \tag{15}$$

Substituting (13) into the integral and evaluating it give

$$C_2 = \sum_{j=1}^{n} f_j d_j$$

where

$$f_j = -\sum_{m=1}^{n/2} F_{2m-1,j} \left(\frac{1}{(2m+5)(2m+4)(2m+3)} - \frac{2}{(2m+3)(2m+2)(2m+1)} + \frac{1}{(2m+1)2m(2m-1)} \right)$$

It is assumed that n is even.

Using condition (b) of zero angular momentum gives

$$\int_{-1}^{1} \zeta\, q(\zeta)\, d\zeta = 0 \qquad (16)$$

and in a similar way C_1 is found to be

$$C_1 = \sum_{j=1}^{n} e_j d_j$$

where

$$e_j = -3\sum_{m=1}^{n/2} F_{2m,j} \left(\frac{1}{(2m+7)(2m+5)(2m+4)} - \frac{2}{(2m+5)(2m+3)(2m+2)} + \frac{1}{(2m+3)(2m+1)2m} \right)$$

Thus $q(x_i)$ can finally be expressed as

$$q(x_i) = \sum_{j=1}^{n} C_{ij}\, d_j \qquad (17)$$

with

$$C_{ij} = G_{ij} + \zeta_i e_j + f_j$$

The potential energy of the beam (2) can now be expressed as a quadratic form in the quantities d_i. Use of the interpolation polynomial $P(\zeta)$ to approximate $v''(\zeta)$ gives for the potential energy

$$V \doteq \frac{\pi a^4 E}{8} \int_{-1}^{1} P^2(\zeta)\, d\zeta$$

Using a quadrature formula with quadrature points ζ_j ($j = 1, 2, \cdots, n$) gives

$$V = \frac{\pi a^4 E}{8} \sum_{i=1}^{n} R_i d_i^2$$

since $P(\zeta_i) = d_i$. The quantities R_i are the weighting factors of the quadrature formula. In matrix form V can be written as

$$V = \frac{\pi a^4 E}{8}\, \underline{d}^T \underline{A}\, \underline{d} \qquad (18)$$

with $\underline{d}$ a column matrix composed of the d_i and A a diagonal, n x n , matrix composed of the weights of the quadrature formula. The symbol T denotes the transpose of the matrix.

The kinetic energy of the beam is obtained from (3) by substituting $q(\zeta)$ for $v(\zeta)$ and applying a quadrature formula. The result is

$$T_s = \frac{\pi \rho_s \omega^2 a^2}{2} \sum_{i=1}^{n} R_i \sum_{j=1}^{n} C_{ij} d_j \sum_{k=1}^{n} C_{ik} d_k$$
$$= \frac{\pi \rho_s \omega^2 a^2}{2} \sum_{j,k=1}^{n} B_{jk} d_j d_k$$

with

$$B_{jk} = \sum_{i=1}^{n} R_i C_{ij} C_{ik}.$$

This expression can also be written in matrix form as

$$T_s = \frac{\pi \rho_s \omega^2 a^2}{2} \underline{d}^T \underline{B_s} \underline{d} \tag{19}$$

where

$$B_s = (B_{jk})$$

For the kinetic energy of the fluid we have [3]

$$T_f = -\frac{\rho_f}{2} \int_S \phi \frac{\partial \phi}{\partial n} dS$$

where the integration extends over the surface of the cylinder. In terms of cylindrical coordinates this assumes the form

$$T_f = -\frac{1}{2} \rho_f a \int_{-1}^{1} \int_{0}^{2\pi} \phi \frac{\partial \phi}{\partial r} d\theta \, dz \; ; \quad r = a \tag{20}$$

The potential ϕ is assumed to be due to an axial distribution of doublets $\mu(z)$ oriented in the direction $\theta = \pi/2$. Thus the velocity potential is

$$\phi = -y \int_{-1}^{1} \frac{\mu(\zeta)}{R^3} d\zeta \tag{21}$$

where

$$R^2 = x^2 + y^2 + (z-\zeta)^2 = (z-\zeta)^2 + r^2$$

and

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

At $r = a$ we obtain then

$$\phi = -a \sin\theta \int_{-1}^{1} \frac{\mu(\zeta)}{R_0^3(z,\zeta)}\, d\zeta$$

in which $R_0^2 = (z-\zeta)^2 + a^2$. Also the boundary condition at the surface of the cylinder yields

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \omega\, v(z) \sin\theta \tag{22}$$

The kinetic energy of the fluid is then

$$T_f = -\frac{\rho_f}{2}\omega a^2 \int_0^{2\pi} \sin^2\theta \int_{-1}^{1} v(z) \int_{-1}^{1} \frac{\mu(\zeta)}{R_0^3(z,\zeta)}\, d\zeta\, dz\, d\theta$$

The kinetic energy can hence be written as

$$T_f = -\frac{\pi\rho_f\omega a^2}{2} \int_{-1}^{1}\int_{-1}^{1} v(z)\mu(\zeta)\, m(z,\zeta)\, d\zeta\, dz \tag{23}$$

where

$$m(z,\zeta) = \frac{1}{[(z-\zeta)^2 + a^2]^{3/2}}$$

Since $m(z,\zeta)$ is small except in the neighborhood of $\zeta = z$, where it rises to a sharp peak, it is desirable to reduce the rapid change in the integrand in that neighborhood so that representation by a quadrature formula is more accurate. This may be done by writing the expression as [4]

$$T_f = -\frac{\pi\rho_f\omega a^2}{2}\left\{\int_{-1}^{1}\int_{-1}^{1} v(z)\,[\mu(\zeta) - \mu(z)]\, m(z,\zeta)\, d\zeta\, dz + \int_{-1}^{1} v(z)\,\mu(z) \int_{-1}^{1} m(z,\zeta)\, d\zeta\, dz\right\}$$

Substituting quadrature formulas for the integrals with the exception of $\int_{-1}^{1} m(z_i,\zeta)\, d\zeta$ gives

$$T_f = -\frac{\pi\rho_f\omega a^2}{2}\left\{\sum_i\sum_j R_i v_i (\mu_j - \mu_i) M_{ij} R_j + \sum_i R_i v_i \mu_i m_i\right\}$$

with

$$v_i = v(z_i) \quad , \quad \mu_j = \mu(\zeta_j) \quad , \quad M_{ij} = m(z_i, \zeta_j)$$

and

$$m_i = \int_{-1}^{1} m(z_i, \zeta)\, d\zeta = \frac{z_i + 1}{a^2[(z_i+1)^2 + a^2]^{1/2}} - \frac{z_i - 1}{a^2[(z_i-1)^2 + a^2]^{1/2}}$$

The kinetic energy can be written as

$$T_f = -\frac{\pi \rho_f \omega a^2}{2}\left\{\sum_i v_i R_i \left(\sum_j \mu_j M_{ij} R_j - \sum_j \delta_{ij} \mu_j \sum_k M_{ik} R_k + m_i \sum \delta_{ij} \mu_j\right)\right\}$$

$$= -\frac{\pi \rho_f \omega a^2}{2}\left\{\sum_i v_i R_i \sum_j [M_{ij} R_j + \delta_{ij}(m_i - n_i)]\mu_j\right\} \tag{24}$$

where

$$n_i = \sum_k m_{ik} R_k$$

In matrix form

$$T_f = -\frac{\pi \rho_f \omega a^2}{2}\, \underline{v}^T \underline{W}\, \underline{\mu} \tag{25}$$

with

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad , \quad \underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$\underline{W} = (W_{ij}) = \{R_i M_{ij} R_j + R_i \delta_{ij}(m_i - n_i)\}; \; i,j = 1, \ldots, n$$

On the boundary of the solid, $r = a$, we have from (22)

$$\omega\, v(z) \sin\theta = \left.\frac{\partial \phi}{\partial r}\right|_{r=a}$$

Hence, substituting for ϕ from (21), we obtain

$$\omega\, v(z) \sin\theta = \left[\frac{\partial}{\partial r}\left(-y \int_{-1}^{1} \frac{\mu(\zeta)}{R^3}\, d\zeta\right)\right]_{r=a}$$

and thus

$$\omega\, v(z) = -\int_{-1}^{1} \mu(\zeta)\left(\frac{1}{R_0^3} - \frac{3a^2}{R_0^5}\right) d\zeta \tag{26}$$

with

$$v_i = v(z_i) \quad , \quad \mu_j = \mu(\zeta_j) \quad , \quad M_{ij} = m(z_i, \zeta_j)$$

and

$$m_i = \int_{-1}^{1} m(z_i,\zeta)\, d\zeta = \frac{z_i+1}{a^2[(z_i+1)^2+a^2]^{1/2}} - \frac{z_i-1}{a^2[(z_i-1)^2+a^2]^{1/2}}$$

The kinetic energy can be written as

$$T_f = -\frac{\pi \rho_f \omega a^2}{2}\left\{\sum_i v_i R_i \left(\sum_j \mu_j M_{ij} R_j - \sum_j \delta_{ij}\mu_j \sum_k M_{ik} R_k + m_i \sum \delta_{ij}\mu_j\right)\right\}$$

$$= -\frac{\pi \rho_f \omega a^2}{2}\left\{\sum_i v_i R_i \sum_j [M_{ij} R_j + \delta_{ij}(m_i - n_i)]\mu_j\right\} \tag{24}$$

where

$$n_i = \sum_k m_{ik} R_k$$

In matrix form

$$T_f = -\frac{\pi \rho_f \omega a^2}{2}\, \underline{v}^T \underline{W}\, \underline{\mu} \tag{25}$$

with

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad , \quad \underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$\underline{W} = (W_{ij}) = \{R_i M_{ij} R_j + R_i \delta_{ij}(m_i - n_i)\}; \; i,j = 1,\dots,n$$

On the boundary of the solid, $r = a$, we have from (22)

$$\omega\, v(z) \sin\theta = \left.\frac{\partial \phi}{\partial r}\right|_{r=a}$$

Hence, substituting for ϕ from (21), we obtain

$$\omega\, v(z) \sin\theta = \left[\frac{\partial}{\partial r}\left(-y \int_{-1}^{1} \frac{\mu(\zeta)}{R^3}\, d\zeta\right)\right]_{r=a}$$

and thus

$$\omega\, v(z) = -\int_{-1}^{1} \mu(\zeta)\left(\frac{1}{R_0^3} - \frac{3a^2}{R_0^5}\right) d\zeta \tag{26}$$

This is the integral equation for the unknown distribution of doublets, $\mu(\zeta, t)$. A solution to it will now be approximated by use of a quadrature formula.

If a modification in the integrand is introduced as before, the result is

$$\omega v(z) = -\left\{\int_{-1}^{1}(\mu(\zeta)-\mu(z))\, k(z,\zeta)\,d\zeta + \mu(z)\int_{-1}^{1} k(z,\zeta)\,d\zeta\right\}$$

with

$$k(z,\zeta) = \frac{1}{[(z-\zeta)^2+a^2]^{3/2}} - \frac{3a^2}{[(z-\zeta)^2+a^2]^{5/2}}$$

Substitution of a quadrature formula for the first integral gives

$$\omega v_i = -\left[\sum_j R_j \mu_j K_{ij} - \mu_i \sum_j R_j K_{ij} + k_i \mu_i\right] \tag{27}$$

in which

$$K_{ij} = k(z_i, \zeta_j)$$

and

$$k_i = k(z_i) = \int_{-1}^{1} k(z_i,\zeta)\,d\zeta$$

$$= -\left\{ \frac{1-z_i}{a^2[(z_i-1)^2+a^2]^{1/2}} + \frac{1-z_i}{[(z_i-1)^2+a^2]^{3/2}} + \frac{1+z_i}{a^2[(z_i+1)^2+a^2]^{1/2}} + \frac{1+z_i}{[(z_i+1)^2+a^2]^{3/2}}\right\}$$

The relation between the velocity of the boundary and the doublet distribution can be written more conveniently as

$$\omega v(z) = -\sum_j H_{ij}\mu_j$$

where

$$(H_{ij}) = \left(R_j K_{ij} + \delta_{ij}(k_i - s_i)\right)$$

and

$$s_i = \sum_k R_k K_{ik}$$

and

$$\omega \underline{v} = -\underline{H}\underline{\mu} \tag{28}$$

Thus,

$$\underline{\mu} = -\omega \underline{H}^{-1} \underline{v} \tag{29}$$

When (29) for μ is substituted into (25) for the kinetic energy, the result is

$$T_f = \frac{\pi \rho_f \omega^2 a^2}{2} \underline{v}^T \underline{W} \underline{H}^{-1} \underline{v}$$

But from (17) we have

$$v(z) \doteq q(z) = \sum_{j=1}^{n} C_{ij}\, d_j$$

so that we obtain

$$T_f = \frac{\pi \rho_f \omega^2 a^2}{2} \underline{d}^T \underline{C}^T \underline{W} \underline{H}^{-1} \underline{C} \underline{d}$$

$$= \frac{\pi \rho_f \omega^2 a^2}{2} \underline{d}^T \underline{B}_f \underline{d} \tag{30}$$

with

$$\underline{B}_f = \underline{C}^T \underline{W} \underline{H}^{-1} \underline{C}$$

Thus, in summary, for the potential energy we have

$$V = \frac{\pi a^4 E}{8} \underline{d}^T \underline{A} \underline{d} \tag{18}$$

and for the kinetic energy, combining T_f and T_s, we have

$$T = \frac{\pi \rho_s \omega^2 a^2}{2} \underline{d}^T \underline{B} \underline{d} \tag{31}$$

where

$$\underline{B} = \underline{B}_s + \frac{\rho_f}{\rho_s} \underline{B}_f$$

To complete the analysis it is necessary to compute the frequencies so that they can be compared with those obtained by the two-dimensional assumption. From a physical view, what has been done is to replace the continuous cylindrical beam, a system with an infinite number of degrees of freedom, by a system with n degrees of freedom, n being the number of ordinates used for the quadrature formula. The partial differential equation for the beam vibration is then replaced by n ordinary differential equations. By treating the quantities d_i as the amplitudes of a system of generalized coordinates, one obtains the frequency determinant equation for the system [1]

$$\left| \underline{A} - \lambda \underline{B} \right| = 0 \tag{32}$$

where

$$\lambda = \frac{4 \rho_s \omega^2}{E a^2} \tag{33}$$

An alternative method which could be used to obtain (32) is the Rayleigh-Ritz procedure. The expressions for the potential and kinetic energies have been expressed as quadratic forms in terms of n unknown quantities, the d_i. Since the sum V + T is constant for a free vibration of the system, we have

$$\underline{d}^T A \underline{d} - \lambda \underline{d}^T B \underline{d} = \text{const.} \tag{34}$$

In this quadratic form, the coefficients d_i are to be chosen so that the frequency is a minimum. Partial differentiation of (32) with respect to each of the d_i yields dynamical equations from which (32) is again obtained.

A simple example will be worked to illustrate the method. If n = 2, then the Gaussian quadrature points are (-0.57735027, 0.57735027). The calculations for this case follow

$$\underline{E} = \begin{pmatrix} -.57735027 & .57735027 \\ 1 & 1 \end{pmatrix}$$

$$F_{pj} = \frac{E_{pj}}{(z_j^2 - 1)^2 \prod_{k=1}^{\prime} (z_j - z_k)}$$

$$\underline{F} = \begin{pmatrix} 1.1250000 & 1.1250000 \\ -1.9485716 & 1.9485716 \end{pmatrix}$$

$$Z_{ip} = \gamma_i^{p+1} \left(\frac{\gamma_i^4}{(p+5)(p+4)} - \frac{2\gamma_i^2}{(p+3)(p+2)} - \frac{1}{p(p+1)} \right)$$

$$\underline{Z} = \begin{pmatrix} .14938272 & -.026169139 \\ .14938272 & .026169139 \end{pmatrix}$$

$$\underline{G} = (\underline{Z})(\underline{F})$$

$$\underline{G} = \begin{pmatrix} .21904800 & .11707312 \\ .11707312 & .21904800 \end{pmatrix}$$

$$f_j = -\sum_{m=1}^{n/2} F_{2m-1,j} \left(\frac{1}{(2m+5)(2m+4)(2m+3)} - \frac{2}{(2m+3)(2m+2)(2m+1)} - \frac{1}{(2m+1)2m(2m-1)} \right)$$

$$f_j = -.15535714 \qquad j = 1,2$$

$$e_j = -3\sum_{m=1}^{n/2} F_{2m,j} \left(\frac{1}{(2m+7)(2m+5)(2m+4)} - \frac{2}{(2m+5)(2m+3)(2m+2)} - \frac{1}{(2m+3)(2m+1)2m} \right)$$

$$e_j = (.12681086 \quad -.12681086)$$

$$Z_i e_j = \begin{pmatrix} -.073214827 & .073214827 \\ .073214827 & -.073214827 \end{pmatrix}$$

$$C_{ij} = Z_i e_j + f_j + G_{ij}$$

$$\underline{C} = \begin{pmatrix} -.00952397 & .03492081 \\ .03492081 & -.00952397 \end{pmatrix}$$

As a check on the calculations, the frequencies of vibration will be computed for a beam vibrating in vacuo. In this case

$$B_{ij} = \sum_{k=1}^{n} R_k C_{ki} C_{kj}$$

$$R_k = 1 \quad , \qquad k = 1, 2$$

$$\underline{B} = \underline{C}^T \underline{C}$$

$$\underline{B} = 10^{-4} \begin{pmatrix} 13.1069 & -6.65695 \\ -6.65695 & 13.1069 \end{pmatrix}$$

Since $\underline{A} = \underline{I}$ the problem is most easily solved for the inverse of the eigenvalues which are related to the frequencies. That is,

$$|\underline{B} - K\underline{I}| = 0$$

where

$$K = \frac{1}{\lambda}$$

Thus,

$$\begin{vmatrix} 13.1069\times10^{-4} - K & -6.65695\times10^{-4} \\ -6.65695\times10^{-4} & 13.1069\times10^{-4} - K \end{vmatrix} = 0$$

A convenient method of solving this determinant is first to add the second column to the first, and then to subtract the second row from the first. This gives

$$\begin{vmatrix} 0 & -19.7638\times10^{-4} + K \\ 6.4500\times10^{-4} - K & 13.1069\times10^{-4} - K \end{vmatrix} = 0$$

and then

$$K_1 = 6.4500\times10^{-4} \qquad \lambda_2 = 1550.39$$

$$K_2 = 19.7638\times10^{-4} \qquad \lambda_1 = 505.98$$

Comparable values are available from Timoshenko [2], where a solution for the one-dimensional flexural vibration of a free-free beam is given. The fourth root of λ here is comparable to the value given by Timoshenko. Table I presents a comparison of $\sqrt{\lambda}$ with the square of Timoshenko's value. This value is proportional to ω, the frequency as given by (33)

Table I - Comparison of Eigenvalues for Free-Free Beam Vibrating in Vacuo

	Timoshenko	Quadrature Method (n = 2)
1st mode	22.37	22.49
2nd mode	61.67	39.36

To include the effect of the fluid, it is first necessary to calculate the matrix $\underline{B}_f$. For this purpose the radius was taken to be $a = 0.10$. The ratio of fluid density to solid density used was 0.365, which is representative of aluminum in water.

$$M_{ij} \quad \frac{1}{[(z_i - z_j)^2 + a^2]^{3/2}}$$

$$\underline{M} \quad \begin{pmatrix} 8000.0000 & .64769665 \\ .64769665 & 8000.0000 \end{pmatrix}$$

$$m_i = \frac{z_i + 1}{a^2[(z_i+1)^2 + a^2]^{1/2}} - \frac{z_i - 1}{a^2[(z_i-1)^2 + a^2]^{1/2}}$$

$$m_i = 797.02912 \qquad i = 1, 2$$

$$n_i = \sum_{k=1}^{n} M_{ik} R_k$$

$$R = 1 \quad , \quad k = 1, 2$$

$$n_i = 8000.64769655 \qquad i = 1, 2$$

$$W_{ij} = R_i M_{ij} R_j + R_i \delta_{ij} (m_i - n_i)$$

$$\underline{W} = \begin{pmatrix} 796.38150 & .64769655 \\ .64769655 & 796.38150 \end{pmatrix}$$

$$K_{ij} = \frac{1}{[(z_i - z_j)^2 + a^2]^{3/2}} - \frac{3a^2}{[(z_i - z_j)^2 + a^2]^{5/2}}$$

$$\underline{K} = \begin{pmatrix} -16000.0000 & .64406008 \\ .64406008 & -16000.0000 \end{pmatrix}$$

$$k_i = -\left(\frac{1 - z_i}{a^2[(z_i+1)^2 + a^2]^{1/2}} + \frac{1 - z_i}{[(z_i+1)^2 + a^2]^{3/2}} + \frac{1 + z_i}{a^2[(z_i+1)^2 + a^2]^{1/2}} + \frac{1 + z_i}{[(z_i+1)^2 + a^2]^{3/2}} \right)$$

$$k_i = -802.91310 \qquad i = 1, 2$$

$$S_i = \sum_{k=1}^{n} R_k K_{ik}$$

$$s_i = -15{,}999.356 \qquad i = 1, 2$$

$$H_{ij} = R_j K_{ij} + \delta_{ij} (k_i - s_i)$$

$$\underline{H} = \begin{pmatrix} -803.55714 & .64406008 \\ .64406008 & -803.55714 \end{pmatrix}$$

$$-\underline{H}^{-1} = 10^{-3} \begin{pmatrix} 1.2444673 & 9.9745459\times10^{-4} \\ 9.9745459\times10^{-4} & 1.2444673 \end{pmatrix}$$

$$\underline{B}_f = \underline{C}^T \underline{W}(-\underline{H}^{-1})\underline{C}$$

$$\underline{B}_f = 10^{-4}\begin{pmatrix} 12.9739 & -6.57119 \\ -6.57119 & 12.9739 \end{pmatrix}$$

$$\underline{B} = \frac{\rho_f}{\rho_s}\underline{B}_f + \underline{B}_s$$

$$\underline{B} = 10^{-4}\begin{pmatrix} 17.8370 & -9.05003 \\ -9.05003 & 17.8370 \end{pmatrix}$$

$$|\underline{B} - K\underline{A}| = 0$$

$$\underline{A} = \underline{I}$$

$$\begin{vmatrix} 17.8370\times10^{-4} - K & -9.05003\times10^{-4} \\ -9.05003\times10^{-4} & 17.8370\times10^{-4} - K \end{vmatrix} = 0$$

$$K_1 = 8.78698\times10^{-4} \qquad \lambda_2 = 1138.05$$

$$K_2 = 26.8870\times10^{-4} \qquad \lambda_1 = 371.926$$

A comparison with values obtained using two-dimensional strip theory can be made by dividing the values in Table I by $\sqrt{1 + \rho_f/\rho_s}$. This comparison is given in Table II.

Table II - Comparison of Eigenvalues for Free-Free Circular Aluminum Beam of Length-Diameter Ratio 10 Vibrating in Water

	Strip Theory		More Exact Theory
	Timoshenko	Quadrature Method (n = 2)	Quadrature Method (n = 2)
1st mode	19.15	19.25	19.38
2nd mode	52.78	33.70	33.78

Considering that the system has been allowed only two degrees of freedom (n = 2) , one sees that the agreement between the eigenvalues for the first mode is remarkably good. In this mode the frequency given by the more exact theory exceeds that by strip theory by 0.67 percent. The corresponding three-dimensional correction factor obtained from a two-node vibration of a spheroid of length-diameter ratio 10 [5] is 1.09, an increase in frequency of 9 percent. Calculations of the eigenvalues with larger values of n are presently under way, and it will not be clear whether the very small correction to the frequencies given by strip theory is a valid one until this work is completed.

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