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AN ALTERNATE DERIVATION OF
THE POLLACZEK-KHINTCHINE FORMULA

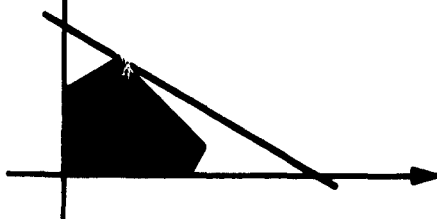
by

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THE POLLACZEK-KHINTCHINE FORMULA**

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AN ALTERNATE DERIVATION OF THE POLLACZEK-KHINTCHINE FORMULA

A derivation of the Pollaczek-Khintchine formula follows from the results obtained by John D. C. Little ⁽¹⁾ in his proof that

$$(1) \qquad L = \lambda W$$

where λ is the average rate of arrivals into a single-channel queue W is the average delay, and L is the average number in the queueing system at a random instant of time. It is assumed that the queueing process is strictly stationary.

In this derivation we also assume that inter-arrival times and service times are independently sampled positive random variables. We use the notation

π_0 = Pr (empty queueing system at a random instant of time)

λ = Average arrival rate (a number)

$\frac{1}{\mu}$ = Average service time (a number)

σ^2 = Variance of the service time (a number)

X = Delay (≥ 0) in queue of a customer arriving at a random instant of time (a random variable)

W_x = Expected value of X (a number)

Y = Time (≥ 0) required to complete the service of a customer in service at a random instant of time (a random variable)

W_y = Expected value of Y (a number)

Z = Time (≥ 0) to service the customers in queue at a random instant of time (a random variable)

W_z = Expected value of Z

From the definitions it follows that

$$(2a) \quad X = Y + Z$$

and that

$$(2b) \quad W_x = W_y + W_z$$

Introducing the subscript x to denote averages in queue (i.e., exclusive of services) in Equation (1) and the assumption that service times are independent of the number in queue, the average time to service the number in queue at a random instant of time is

$$(3) \quad W_z = \frac{1}{\mu} (\lambda W_x)$$

If the service facility is empty at the instant a customer arrives, $Y = 0$. If the service facility is busy, the expected value of the remaining service time of the customer in service is the expected value of the length-biased sampling distribution of service times.* Unconditionally,

$$(4) \quad W_y = (1 - \pi_0) \frac{\mu(\sigma^2 + \mu^{-2})}{2} = \frac{\lambda(\sigma^2 + \mu^{-2})}{2}$$

Substituting (3) and (4) into (2b) and solving for W_x gives the Pollaczek-Khintchine formula,

$$(5) \quad W_x = \frac{\lambda(\sigma^2 + \mu^{-2})}{2(1 - \lambda/\mu)}$$

This derivation can be extended to those cases where mixed streams feed the service channel with different priorities of service.

*Reference 2, page 65.

REFERENCES

1. Little, John D.C., "A Proof for the Queueing Formula: $L = \lambda W$,"
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2. Cox, D.R., "Renewal Theory," Methuen Monograph, John Wiley and
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