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THE MOTION OF ELECTRONS IN  
A RADIOFREQUENCY FIELD,

⑤ *by*

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PREFACE

This Memorandum discusses the motion of electrons in a particular radiofrequency field to determine whether confinement is likely in this environment. The problem is studied for two cases: one in which collisions of electrons with other particles are negligible (the collisionless case), and one in which electrons collide only with neutrals (the weakly-ionized case). In the first instance, the motion of the electrons is such that they will always remain in the interior of a certain region determined by the value of the magnetic field, initial electron velocity, and tube radius, as long as the magnetic field strength and field frequency satisfy certain inequalities. In the second case, the electrons will remain within the tube for all values of magnetic field strength; however, since diffusion is neglected in the basic equations, this result does not imply that the electrons are confined.

The results of this study are applicable to the fields of electric propulsion, controlled thermonuclear fusion, and gaseous electronics.

SUMMARY

The motion of electrons in a radiofrequency field is investigated for a cylindrical geometry and an electromagnetic-field configuration consisting of a spatially constant axial magnetic field and an azimuthal electric field which varies linearly with distance from the center of the cylinder. This field configuration, which approximates that found in the electrodeless ring discharge near breakdown, has also been suggested as a plasma-confinement scheme. When the motion is collisionless, the equations can be solved analytically to show that stable (time-bounded) motion exists for certain values of the ratio of electron cyclotron frequency to applied field frequency. Confinement of the orbits to a region entirely inside the cylinder is completely assured only for those particles which start from rest; in all other cases the maximum radius of extent depends on initial position and velocity.

Collisions are qualitatively accounted for in a crude manner by use of the Langevin equation, which includes a drag force due to collisions in the equations of motion. Even with this simple model the resulting equation is apparently intractable except by machine solution. It is therefore not possible, without analytic solutions, to discuss the effect of collisions of this type on the stability of the motion, although it appears from several of the examples in this Memorandum that an initially unstable or divergent motion quickly damps out when the friction force is added, and that for large friction forces, the mean motion is always stable. Confinement is not ensured, however, because diffusion is not included in the basic equations.

Some numerical examples of the motion are presented for both the collisionless case and the Langevin equation. An approximate solution is obtained for the case with collisions which adequately describes the mean motion whenever the ratio of collision frequency to field frequency is larger than three.

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CONTENTS

PREFACE .....	111
SUMMARY .....	v
ACKNOWLEDGMENT .....	vii
SYMBOLS .....	xi
Section	
I. INTRODUCTION .....	1
II. THE EQUATIONS OF MOTION .....	2
The Collisionless Motion .....	9
Motion With Collisions .....	20
An Approximate Solution .....	22
Numerical Examples .....	25
Appendix:	
A. ON SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS .....	29
B. APPROXIMATE SOLUTION OF EQ. (15) .....	36
REFERENCES .....	41

SYMBOLS

- $A_1$  = constants of integration  
 $A(t)$  = statistical collision function  
 $a$  = parameter in Mathieu equation (Eq. (A-3))  
 $B, B_0$  = magnetic-field strength  
 $C_1$  = initial angular velocity  
 $c$  = velocity of light  
 $ce$  = Mathieu function  
 $e$  = unit of charge  
 $F$  = force  
 $F_1, F_2$  = functions defined by Eqs. (33) and (34)  
 $J_0, J_1$  = Bessel functions  
 $j$  = current  
 $k$  = characteristic root (Eq. (A-10)), dummy variable  
 $m$  = mass, dummy variable  
 $P(\tau)$  = periodic real function of time  
 $P_\theta$  = conjugate angular momentum  
 $p(\tau)$  = periodic complex function of time  
 $q$  = parameter in Mathieu equation (Eq. (A-3))  
 $R$  = dimensionless radius  
 $R_0$  = dimensionless radius of container  
 $r$  = radial coordinate  
 $se$  = Mathieu function  
 $t$  = time  
 $V$  = initial radial velocity  
 $v$  = velocity

- w = auxiliary function
- $x = \frac{1}{\beta} \frac{w'}{w}$  (Eq. (B-1))
- $y = 2 \operatorname{Re}(\rho) = \gamma \sin \tau$  (Eq. (B-8))
- z = position coordinate
- $\alpha_{ij}$  = coefficients in general solution (Eq. (A-6))
- $\beta$  = ratio of collision frequency to field frequency
- $\gamma = \Omega/\beta$ , ratio of cyclotron frequency to collision frequency
- $\theta$  = angular coordinate
- $\lambda$  = real parameter (Eq. (A-2))
- $\mu = \kappa + i\alpha = k/2\pi$ , characteristic exponent
- $\nu$  = collision frequency
- $\rho = (\gamma \sin \tau + i)/2$  (Eq. (B-1))
- $\tau = \omega t$ , dimensionless time
- $\Phi$  = periodic function of  $\tau$  (Eq. (A-11))
- $\varphi = \Omega \sin^2 \tau / 2$
- $\Omega$  = ratio of cyclotron frequency to field frequency
- $\omega$  = field frequency

## I. INTRODUCTION

Several recent papers<sup>(1-7)</sup> have been concerned with the confinement of plasmas by radiofrequency fields. Some of these treat the problem from the continuum standpoint; others have used the single-particle-orbit theory. In the latter category, Weibel<sup>(6)</sup> has investigated the radial bounds for electrons in a  $TE_{01}$  mode waveguide at cutoff and has concluded that the orbits are stably bound for certain values of the ratio of cyclotron frequency to applied field frequency.

The equations of motion in Weibel's study were complicated by the electromagnetic-force terms, which are Bessel functions for this waveguide. He investigated the solution in two regions where the equations became tractable: one where the field frequency is much larger than the cyclotron frequency, and another for small radii, valid in regions close to the waveguide axis. In neither case was the actual motion found, nor were the effects of initial conditions discussed. Further, in the second case an apparently unnecessary assumption was made in the analysis, which yielded an erroneous angular-momentum equation. (This assumption did not affect the radial motion.)

The present analysis will consider the motion of electrons for a situation similar to Weibel's second approximation. A more complete investigation will be made of the trajectories, the effects of boundary conditions, and, qualitatively, the effects of collisions on the radial bounds of the motion.

## II. THE EQUATIONS OF MOTION

The electromagnetic field for the  $TE_{01}$  waveguide of Ref. 6 is given by

$$\begin{aligned} E_{\theta} &= -B_0 J_1\left(\frac{\omega r}{c}\right) \cos \omega t \\ B_z &= B_0 J_0\left(\frac{\omega r}{c}\right) \sin \omega t \end{aligned} \tag{1}$$

where  $r$ ,  $\theta$ ,  $z$  are cylindrical coordinates,  $\omega$  is the field frequency, and  $c$  is the velocity of light. The Weibel approximation  $(\omega r/c)^2 \ll 1$  is equivalent to replacing the Bessel functions by their values for small argument

$$\begin{aligned} E_{\theta} &= -\left(\frac{\omega B_0}{2c}\right) r \cos \omega t \\ B_z &= B_0 \sin \omega t \end{aligned} \tag{2}$$

One can visualize this approximation as a restriction to small radial distances, which is necessary if the frequencies are in the microwave regime. On the other hand, if the frequencies are much lower than microwave, the radius need not be restrictively small. This particular interpretation of Weibel's approximation permits the coupling of a plasma-confinement scheme to an operable discharge device of current interest--the electrodeless ring discharge.

A cross-sectional view of an electrodeless discharge is sketched in Fig. 1, in which an axial magnetic field is induced by an excitation coil of several turns, and a Faraday cage can be placed around the tube to isolate the interior from the axial electric field of the coil. When the discharge is operating, the currents in the gas give rise to an intense circumferential glow, which accounts for the "ring"

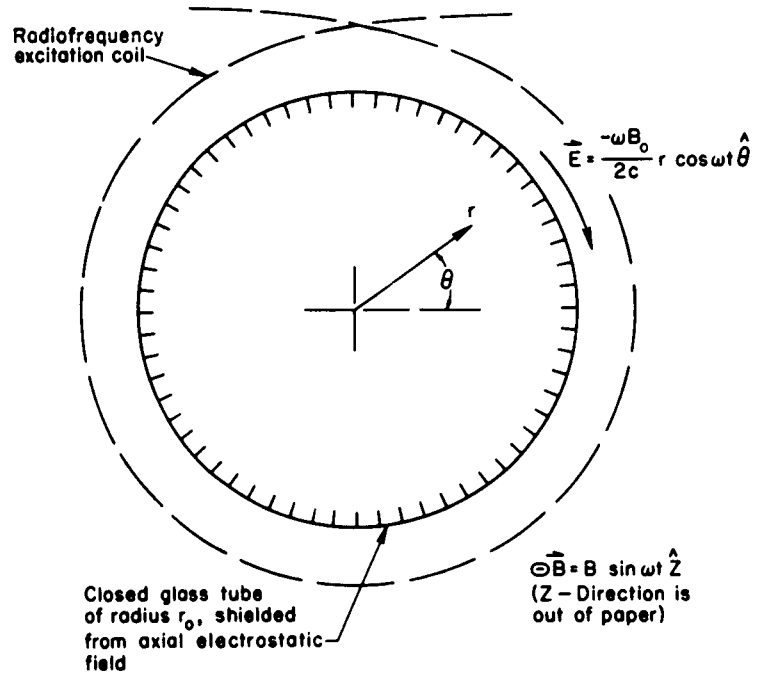


Fig. 1—Geometry of the electrodeless ring discharge

terminology; the use of the word "electrodeless" differentiates this case from that discharge where the electrodes come into contact with the gas.

At breakdown, just as the discharge becomes self-sustaining, the electromagnetic fields in the interior are given by Eq. (2);<sup>(8)</sup> as the electron density increases, the induced currents in the interior distort these fields, and their determination can be quite difficult.<sup>(9)</sup> Therefore, the use of Eq. (2) in this analysis precludes large electron concentrations, and the results will apply to a gas in which the electric conductivity is small. A treatment of the case in which the densities are higher is given by Zauderer.<sup>(7)</sup>

To obtain the motion in the case of a weakly ionized plasma, we will use the Langevin equation. This equation has been used extensively to describe the ac conductivity of a plasma (see, e.g., Ref. 10) when the only important interactions are those occurring between electrons and neutrals.

From Allis,<sup>(11)</sup> the Langevin equation is written

$$\frac{d\vec{v}}{dt} = \vec{F} + \vec{A}(t) \quad (3)$$

where  $\vec{F}$  is the Lorentz force on the electron

$$\vec{F} = -\frac{e}{m} \left( \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right) \quad (4)$$

and  $\vec{A}(t)$  is a fluctuating force due to the collisions. In order to solve Eq. (3), the force  $\vec{A}(t)$  must be defined. If we assume that  $\vec{A}(t)$  is independent of  $\vec{v}$ , and that  $\vec{A}(t)$  varies extremely rapidly compared to the variations in  $\vec{v}$ , then it is possible to define  $\vec{A}(t)$  statistically and solve Eq. (3).

The second assumption made for  $\vec{A}(t)$  is tantamount to assuming that the motion of the electron is taking place in two ways; that is, there exists a small time interval  $\Delta t$  during which the variations in  $\vec{v}(t)$  are small while  $\vec{A}(t)$  may oscillate several times. Furthermore,

since  $\vec{A}(t)$  is a random force, there is no correlation between  $\vec{A}(t)$  and  $\vec{A}(t + \Delta t)$ . (Such a process is called a Markoff process,<sup>(12)</sup> in which the occurrence of an event at time  $t$  depends only on the state of the system at time  $t$ .) Even so, the velocity defined by Eq. (3) must vary in a continuous way so that a solution of Eq. (3) will have meaning in a physical sense. This smoothing process is a serious assumption, and the use of the Langevin equation in this particular case has been suggested only because of the a posteriori success of this equation in predicting with reasonable accuracy certain transport phenomena in electromagnetic fields.<sup>(11, 13)</sup>

Because the Langevin equation yields the motion of an average electron, i.e., a drift velocity, it is more suitable for the determination of mobility and conductivity than for the present study of confinement, where we are concerned with the loss of individual electrons from the plasma. The inability of the Langevin to produce this information occurs because diffusion, which is a primary mechanism for loss in any plasma, cannot be taken into account in this "smoothed" picture of the motion. What the Langevin equation will show, diffusion excluded, is how the average electron moves about in the container under the influences of the electromagnetic forces and collisions with neutral particles. Naturally, if this motion is toward the container walls, or if it grows large with time, one would suspect that the forces are not favorable for confinement. However, because diffusion, reattachment, and recombination are neglected, one cannot infer that a stable Langevin orbit implies confinement.

With the understanding that  $\vec{A}(t)$  must have a large frequency compared to the variations in  $\vec{v}(t)$ , then, we can define the fluctuation term as<sup>(11)</sup>

$$\vec{A}(t)_{ave} = -\nu \vec{v} \quad (5)$$

where  $\nu$  is a collision frequency which must by assumption be higher than any of the other frequencies in the equations of motion. It can be identified with the collision frequency for momentum transfer  $\nu_c$

if the particle-distribution function is Maxwellian. In this case  $\nu$  is independent of the energy of the particle, and the first assumption for  $\vec{A}(t)$  is met. Molmud<sup>(13)</sup> has investigated the solution of Eq. (3) when  $F = E_0 e^{i\omega t}$  and has shown that when the distribution function is non-Maxwellian,  $\nu$  may be complex and a function of temperature. When  $|\nu| \ll \omega$ , or  $|\nu| \gg \omega$ , the imaginary part of  $\nu$  is negligible. We will restrict ourselves to real values of  $\nu$  in this analysis.

Combining Eqs. (2) through (5) gives the basic equation of motion for the system under consideration:

$$\frac{d\vec{v}}{dt} = -\frac{e}{m} \left( \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right) - \nu \vec{v} \quad (6)$$

The forces in Eq. (2) are such that the motion of the electron is entirely in the cross-sectional or  $r\theta$  plane. It is convenient to write Eq. (6) in complex form, letting  $z = re^{i\theta}$ . One obtains

$$z'' - i \left[ \frac{\Omega \cos \tau}{2} z + (\Omega \sin \tau + i\beta) z' \right] = 0 \quad (7)$$

where  $z$  has been made dimensionless with respect to the initial position  $r(0)$ , the primes denote differentiation with respect to time  $\tau = \omega t$ , and the dimensionless parameters  $\Omega$  and  $\beta$  are defined as

$$\Omega = \frac{eB_0}{mc\omega} = \frac{\text{cyclotron frequency}}{\text{field frequency}} \quad (8)$$

and

$$\beta = \frac{\nu}{\omega} = \frac{\text{collision frequency}}{\text{field frequency}} \quad (9)$$

The initial conditions in Eq. (7) are  $z(0) = 1$  and  $z'(0) = V + iC_1$ , where  $V$  and  $C_1$  are dimensionless velocities in the radial and angular directions.

Under the conditions of averaging discussed previously,  $z$  is the position of an average electron at time  $\tau$  when  $\beta$  is larger than the associated frequency of  $z'$ . Before discussing the formal solution of Eq. (7), let us determine the restrictions placed on  $\beta$  and  $\Omega$  by the Langevin approximation. Allis<sup>(11)</sup> discusses the case in which Eq. (6) is solved for a constant magnetic field and a sinusoidal electric field. Although Allis did not specify the value of the time average in comparison to the various frequencies, it can be deduced from his treatment that

$$\frac{1}{\beta} \lesssim \frac{1}{\Omega} \ll \omega \Delta t \ll 1 \quad (10)$$

i.e., he averaged over a time  $\Delta t$  greater than the cyclotron period but less than the period of the field. Since the energy is constant over a cyclotron period in a static magnetic field, the energy assumption is satisfied, and the velocity varies as the field frequency  $\omega$ .

On the other hand, Molmud<sup>(13)</sup> discusses the same situation and treats both the case of  $\beta \gg 1$  and that  $\beta \ll 1$ . On the basis of the discussion by both Chandrasekhar<sup>(12)</sup> and Allis<sup>(11)</sup> that the time  $\Delta t$  must be long compared to  $1/\nu$  and, furthermore, that the energy must be constant during  $\Delta t$ , it is difficult to see the significance of the case of  $\beta \ll 1$ , which is equivalent to the inequality

$$\frac{1}{\omega} \ll \frac{1}{\nu} \quad (11)$$

or, since the collision time must be shorter than the averaging time,

$$\frac{1}{\omega} \ll \Delta t \quad (12)$$

This inequality indicates that the average has been taken over the period of the applied field, or, equivalently, that the energy is constant over the period of the applied field. This averaging process may be used to define an effective electric field<sup>(11)</sup> which is inde-

pendent of the periodic terms and may be treated as a dc field in the breakdown theory for an ac discharge. In the present case, however, an averaging process such as the above would not yield meaningful results.

Therefore, solutions of Eq. (7) will be discussed for the case given by Eq. (10) and the case in which the average time is shorter than the cyclotron period:

$$\frac{1}{\beta} \ll \omega \Delta t \ll \frac{1}{2} \lesssim 1 \quad (13)$$

We will also obtain solutions of Eq. (7) for  $\beta = 0$ .

The general solution of Eq. (7) is discussed completely in Appendix A. For convenience, the solution and subsidiary equations are repeated below. After several transformations, one obtains for the position of the average electron

$$z = w(\tau) e^{(i\Omega \sin^2 \tau/2 - \beta\tau/2)} \quad (14)$$

where  $w$  is the general solution of the differential equation

$$w'' + \left( \frac{\Omega \sin \tau + i\beta}{2} \right)^2 w = 0 \quad (15)$$

with initial conditions

$$\begin{aligned} w(0) &= 1 \\ w'(0) &= V + iC_1 + \beta/2 \end{aligned} \quad (16)$$

Equation (15) reduces to the Mathieu equation when  $\beta = 0$ , which is consistent with Weibel's solution.<sup>(6)</sup> When  $\beta \neq 0$ , a general solution is more difficult to obtain. The collisionless motion will be discussed first, so that the general behavior of the undamped motion may be ascertained.

### THE COLLISIONLESS MOTION

Although the collisionless case has been previously derived by Weibel,<sup>(6)</sup> neither explicit trajectory information nor initial conditions were discussed. Since we are interested here in both the motion and the possibility of containment (compared to boundedness), we will investigate Eqs. (14) through (16) more thoroughly in this section.

When  $\beta = 0$ , Eq. (15) becomes the Mathieu equation, which is discussed more thoroughly in Appendix A. Let  $X(\tau)$  and  $Y(\tau)$  be solutions to Eq. (15); then Eq. (14) becomes

$$R^2 = |z|^2 = (X + vY)^2 + C_1^2 Y^2 \quad (17)$$

and

$$\theta = \arg z = \tan^{-1} \left( \frac{C_1 Y}{X + vY} \right) + \Omega \sin^2 \tau / 2 \quad (18)$$

in polar coordinates, where  $R$  is dimensionless with respect to the initial radial position, and  $\theta(0) = 0$ . Because the collision term is zero, it is not necessary to average Eq. (3), so that  $R$  and  $\theta$  describe the actual position of the electron.

It is of interest to compare the above solution with the analysis made by Weibel. If one solves the differential equations given in Ref. 6, which are written in Cartesian coordinates, it is seen that the radial motion is identical to that of Eq. (17), although Weibel did not specify the initial conditions. On the other hand, the angular motion resulting from Weibel's analysis lacks the periodic term  $\Omega \sin^2 \tau / 2$ , due to an assumption made by Weibel while linearizing the equations of motion. Because the centrifugal-force term becomes singular at the origin, Weibel saw that the problem would become tractable if the electromagnetic forces were approximated by a scalar potential which removed the singularity and permitted linearization

of the equations. This scalar potential affects the angular motion, but it does yield values of the radial motion which are correct. Since Weibel was concerned primarily with the radial bounds of the orbits, the conclusions of Ref. 6 are not affected by the error in the angular motion.

The singularity still exists in the present analysis, since both  $R'$  and  $\theta'$  become infinite as  $R \rightarrow 0$  except when the initial angular velocity  $C_1 = 0$ . However, this singularity will never be attained; inspection of the Hamiltonian of the system<sup>(6)</sup> shows that the conjugate angular momentum is given by

$$P_\theta \equiv \frac{mr^2}{\omega} \theta' - \frac{eB_0 c}{\omega} r J_1\left(\frac{\omega r}{c}\right) \sin \tau \quad (19)$$

where the first term is the angular momentum, and the last term is the azimuthal component of the vector potential for the field configuration of Eq. (1). Because we defined  $C_1$  to be the value of  $\theta'$  when  $\tau = 0$ , and since the second term vanishes when  $\tau = 0$ , it is immediately obvious that when  $C_1 = 0$ , the quantity  $P_\theta = 0$  for all times, and the singularity does not exist. Similarly, when  $C_1 \neq 0$ , then  $P_\theta \neq 0$  and the orbit will never pass through the singular point at  $R = 0$ . This can be seen in Figs. 2 through 5.

Therefore, by linearizing Eq. (1) before insertion into the equations of motion, and by utilizing the initial conditions of the problem, it is possible to obtain solutions of the motion even when the singularity at the origin is not removed. This refinement is not immediately obvious if one works in cylindrical coordinates (the natural coordinate system), because then the equations are nonlinear.<sup>(14)</sup>

The behavior of  $R$  and  $\theta$  is obtained by investigating the properties of the Mathieu functions. The Mathieu equation, as discussed in Appendix A, is a special case of the Hill differential equation. As such, there exist regions of stable and unstable solutions, depending on the parameter  $\Omega$ . The regions of stability are tabulated in Ref. 15

and plotted in Fig. 9 (Appendix A), where it is shown that time-bounded solutions exist for  $\Omega$  in the following regions:  $0 < \Omega < 2.29$ ;  $3.78 < \Omega < 5.45$ ;  $6.96 < \Omega < 8.60$ ; etc. Thus for these values of  $\Omega$ , the motion defined by Eqs. (17) and (18) will be bounded for all values of time, i.e.,

$$|z| = \sqrt{(X + VY)^2 + C_1^2 Y^2} < M \quad (20)$$

where the bound  $M$  is some function of  $\Omega$ ,  $V$ , and  $C_1$ . Now, in order for the condition given by Eq. (20) to ensure containment of the particles within a given region, it must be specified further that

$$1 < M < R_0 \quad (21)$$

where  $R_0$  is the dimensionless radius of the tube.

Unfortunately, it is not possible to determine the maximum values of  $|z|$  for a given set of initial conditions without resorting to numerical calculations, because the Mathieu functions  $X$  and  $Y$  do not, as a rule, have extensive recursion relations between themselves and their derivatives. The difficulty of obtaining a least upper bound for  $|z|$  is illustrated if we write  $R$  in terms of the actual Mathieu functions.

For  $\Omega$  in the range of stable solutions given above, Appendix A shows that it is possible to have, by appropriate choice of  $\Omega$ , periodic solutions of Eq. (15). For convenience, let us choose solutions so that the Mathieu functions are of fractional order; when the fraction is rational, say  $\alpha = p/s < 1$ , then the period is  $2\pi s$ . The problem of determining  $\alpha$  for a given  $\Omega$  is discussed at length in Ref. 15. The Mathieu functions corresponding to these parameters are

$$\begin{aligned}
 X \equiv ce_k(\tau, \Omega) &= \frac{\sum_{m=-\infty}^{\infty} A_m^{(k)} \cos(m + \alpha)\tau}{\sum_{m=-\infty}^{\infty} A_m^{(k)}} \\
 Y \equiv se_k(\tau, \Omega) &= \frac{\sum_{m=-\infty}^{\infty} A_m^{(k)} \sin(m + \alpha)\tau}{\sum_{m=-\infty}^{\infty} A_m^{(k)}}
 \end{aligned}
 \tag{22}$$

The subscript  $k = m + \alpha$ , where  $m$  is either even or odd, depending on the value of  $\Omega$ .

Inserting Eq. (22) into the position equations

$$R^2 = [ce_k + Vse_k]^2 + C_1^2 se_k^2
 \tag{23}$$

and

$$\theta = \tan^{-1} \left( \frac{C_1 se_k}{(ce_k + Vse_k)} \right) + \Omega \sin^2 \tau / 2
 \tag{24}$$

From Eq. (21) we can write the restriction on  $R$  as

$$R^2 = [ce_k + Vse_k]^2 + C_1^2 se_k^2 \leq 1 \leq M^2 < R_0^2
 \tag{25}$$

It is seen from Eq. (25) that the bound  $M$  on  $R$  will be a function of  $\Omega$  (through the Mathieu functions),  $V$ , and  $C_1$ . The straightforward procedure of differentiating Eq. (25) to find the maxima of  $R$  in the region  $0 \leq \tau \leq 2\pi$  does not permit a solution, because of the lack of analytic relationships between  $ce$ ,  $se$ , and their derivatives.

Since the mathematical determination of the radial bounds is difficult, let us examine the physical situation at  $\tau = 0$ . Starting from  $R = 1$  at  $\tau = 0$ , the only force acting on an electron is the

azimuthal electric field, which will cause an angular acceleration which decreases in magnitude as  $\tau$  increases. Suppose the electron had an outward velocity at  $\tau = 0$  which was larger than the azimuthal velocity obtained from the electric field. Then, if the particle is initially near the outer radius (i.e., if  $R_0 \sim O(1)$ ), the trajectory could intersect the wall of the tube. This would imply that the only particles always contained in the plasma for a stable solution of Eq. (15) are those which start from rest. This is unlikely in a gas with nonzero thermal energy, so it appears that some particles will always be lost to the wall.

Some examples have been prepared to illustrate this collisionless motion--particularly the effect of the initial conditions and cyclotron-frequency parameter  $\Omega$  on the behavior of the orbits. The solutions were obtained on an IBM 7090 computer, since the Mathieu functions desired for the solutions are not generally available in tabulated form. (Earlier solutions<sup>(14)</sup> using the series were later found to have convergence problems; in particular, the simple solution for the angular motion was not recognized, and the numerical integration for the angle  $\theta$  incurred cumulative errors. In some cases, however, the series solution agreed well with the machine solution, as will be seen in Fig. 4.)

Figures 2 and 3 show the motion for a natural period of  $4\pi$  in the case of low and high cyclotron frequency. Only the motion for  $2\pi$  is presented in Fig. 3; during the second half of the period the electron retraces its path back to the starting position. There is one value of  $\Omega$  in each stable region of the Mathieu equation which will yield a period of  $4\pi$ . The lowest value,  $\Omega = 1.387$ , is given in Fig. 2. For a discharge at 30 mc, this corresponds to a weak magnetic field on the order of 10 gauss.

At the other extreme, Fig. 3 gives the motion for  $\Omega = 17.242$ , where  $B_0 \sim 175$  gauss at 30 mc. The effect of increased magnetic field is directly evident here, where the electron makes 5 circuits in a quarter-period. The order of occurrence of the circuits is marked by number in Fig. 3. Notice also that for a large portion of the trajec-

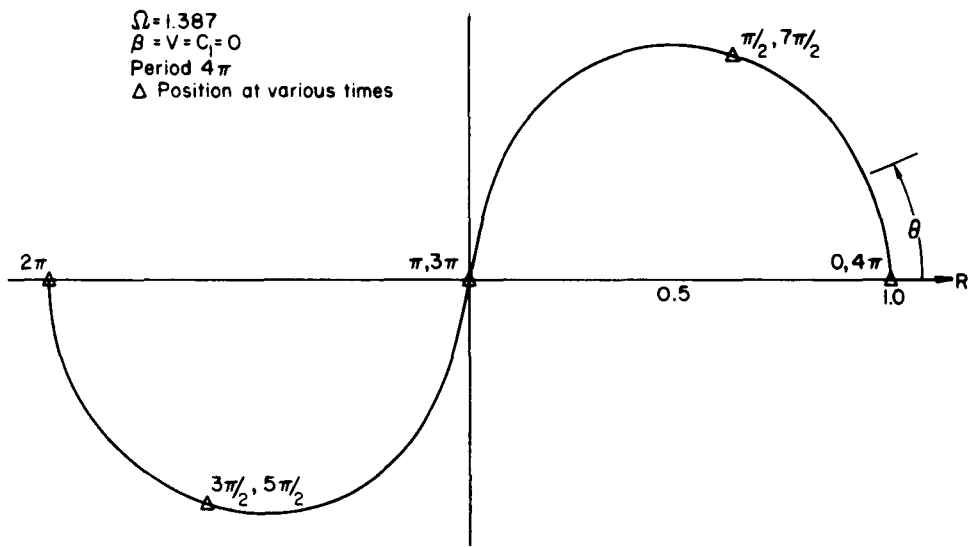


Fig. 2—Collisionless motion for small magnetic field

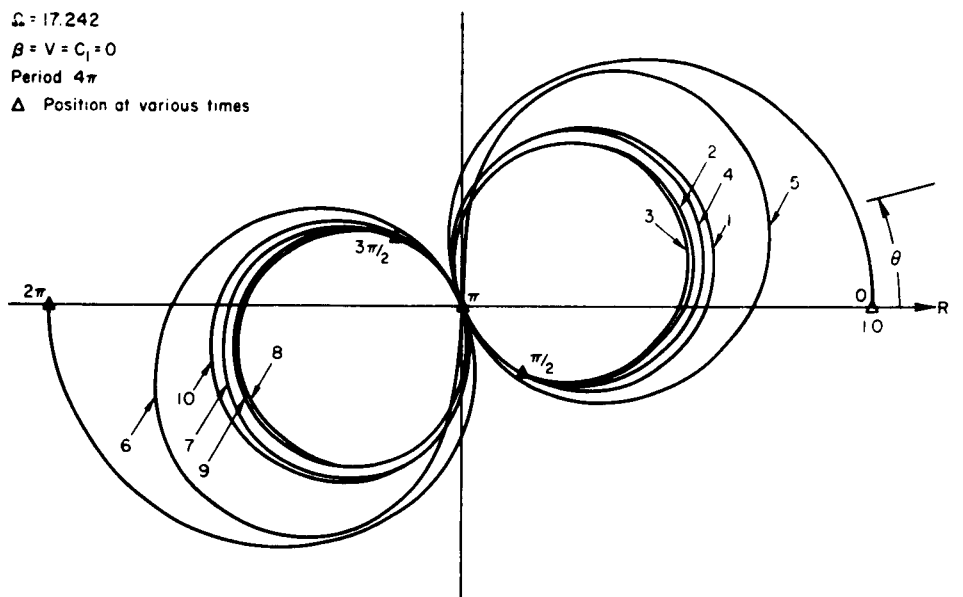


Fig. 3 — Collisionless motion for large magnetic field

tory, the orbit is nearly circular. Here, the guiding-center approximation is valid,<sup>(11, 16)</sup> for the electron can make several gyrations in an essentially constant magnetic field. For values of  $\Omega$  larger than 20, the guiding-center approximation could be used for that portion of the trajectory where  $B \neq 0$ .

Figure 4 demonstrates that particles with outward initial velocity can move toward the container wall for a portion of their trajectory. The dimensionless velocities of Fig. 4 compare to a thermal velocity of about 8V (or  $8C_1$ ) ev in a 30-mc discharge.

Rather complicated trajectories can be observed when a particle with initial velocity is subjected to a high magnetic field. Figure 5 shows the complex motion undergone by an electron during one-half period when  $\Omega = 17.242$  and  $V = C_1 = 1$ . Again, we have indicated the order in which the electron traverses the tube. For a stable solution the energy gained in the first quadrant must be lost by deceleration during the second quarter of its trajectory. As a result the electron spends a considerable portion of its trajectory in a small region of the tube. This behavior is favorable for either confinement or breakdown. During the remainder of the period the motion takes place primarily in the third and fourth quadrants.

Although this discussion of collisionless motion has been brief, the pattern of electron motion in this type of a radiofrequency field has been established: Stable solutions exist when the ratio of cyclotron frequency to field frequency is within certain limits. When  $\Omega \gg 1$ , the motion over a portion of the trajectory is adequately described by the guiding-center method.<sup>(11,16)</sup> If  $\Omega$  is outside the region for stable solutions, the motion is divergent.

An example of divergent motion is given in Fig. 6, where a solution originally obtained by Brault<sup>(17)</sup> has been extended for larger times to indicate the increase in maximum radius as  $\tau$  increases. The equation derived by Brault was not recognized as the Mathieu equation (in cylindrical coordinates this is not easy to see<sup>(14)</sup>), and as a result, some of his solutions were inadvertently in the unstable region. One of these unstable solutions occurs, as evident in Fig. 9,



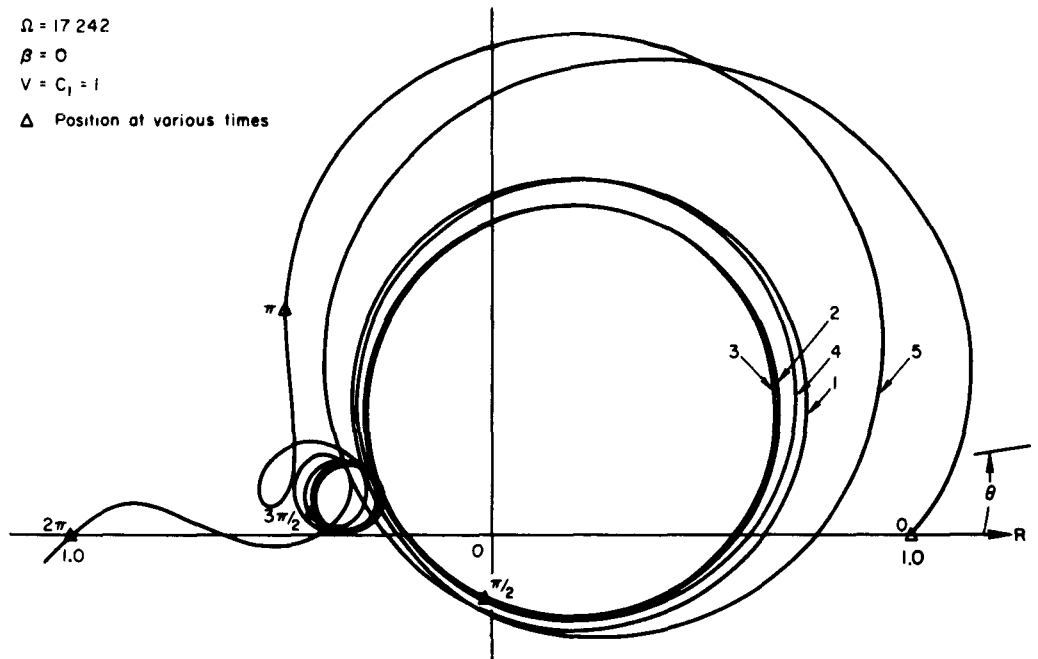


Fig. 5 — Collisionless motion in large magnetic field with initial velocity

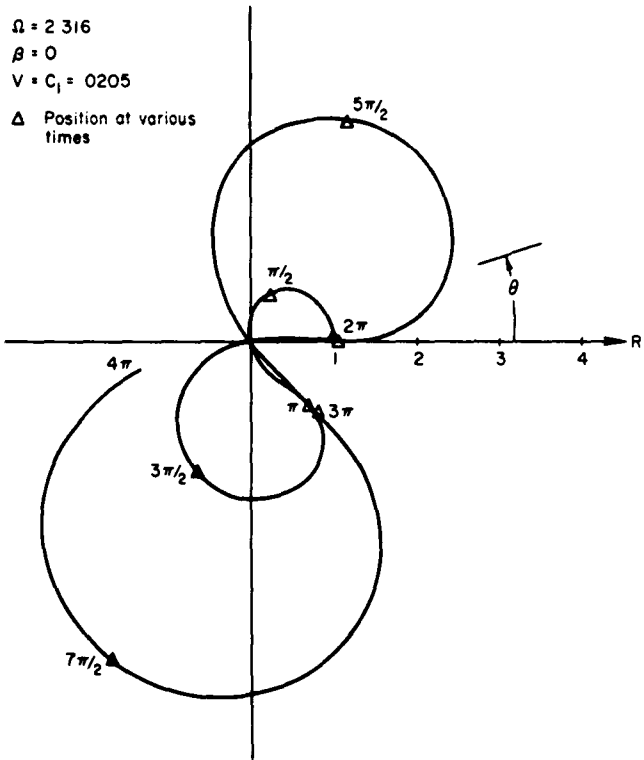


Fig. 6 — Unstable collisionless motion

when  $\Omega = 2.316$ ; this is the case plotted in Fig. 6 for which the initial conditions were taken as  $V = C_1 = 0.0205$ . It can be seen that the radius of gyration increases every time the electron passes near the origin (because  $V = C_1 \neq 0$ , the trajectory never passes through the origin, but since  $V = C_1 \ll 1$ , the distance of the path from the origin is small). The kinetic energy also exhibits an undamped increase, because each time the particle passes near the origin it gains energy, as compared to the stable motion shown in Figs. 2 through 5, where a periodic gain and loss of energy is achieved as the electron traverses the tube.

#### MOTION WITH COLLISIONS

When  $\beta \neq 0$ , Eq. (14) gives the position of an average electron in the tube, rather than the motion of individual electrons as it did in the preceding section. This occurs because the collision term in Eq. (6) has been averaged, and while momentum is conserved during collisions, no changes in direction are made as a result of collisions; hence if one were to follow an individual electron as we did in the last section, the position given by Eq. (14) would not describe the correct motion. If, however, one were to measure a current  $j = nev_{ave}$ , Eq. (14) would indicate the location and behavior of this current resulting from the group behavior of electrons in this radiofrequency field.

Keeping this in mind, let us discuss briefly the solution of the system of Eqs. (14) through (16). A more complete discussion is given in Appendix A. When  $\beta \neq 0$ , Eq. (15) is no longer the Mathieu equation, and while Eq. (15) admits a series solution of a type similar to the Mathieu functions, a search of the literature has failed to locate any treatment in detail of the solution or characteristic numbers associated with Eq. (15). Therefore, as shown in Appendix A, all we can say about Eq. (15) without going into the details of a series solution is that the solution  $w = A_1 w_1 + B_1 w_2$  is of the form where

$$w_i = e^{\mu_i(\tau)} \phi_i(\tau) \quad (i = 1, 2) \quad (26)$$

where the characteristic numbers  $\mu_i$  are related to the roots of the quadratic characteristic equation (Eq. (A-10)) and, as functions of the free parameters  $\Omega$  and  $\beta$ , admit a complete spectrum of real, imaginary, and complex values. The functions  $\phi_i(\tau)$  are periodic, of period  $2\pi$ , and are hence bounded. Therefore, the solutions  $w_i$  are bounded for all  $\tau$  only when both characteristic numbers  $\mu_i$  are imaginary. Let us write these solutions in the expression for  $z$  given by Eq. (14):

$$z = \left\{ A_1 e^{\mu_1(\tau)} \phi_1(\tau) + A_2 e^{\mu_2(\tau)} \phi_2(\tau) \right\} e^{i\varphi} e^{-\beta\tau/2} \quad (27)$$

It is immediately obvious from Eq. (27) that  $|z|$  will be bounded for all time  $\tau$  if and only if

$$|\text{Re}(\mu_i)| < \beta/2 \quad (28)$$

That is, if  $\mu_i$  is complex, the aperiodic portion of  $e^{\mu_i\tau}$  will cause  $|z|$  to become infinite unless the damping factor  $e^{-\beta\tau/2}$  predominates.

Let us examine this criterion in comparison with the solution of the previous section where  $\beta = 0$ . There, the instability of  $w_1$  or  $w_2$  implied immediately that  $|z|$  would be unbounded as  $\tau \rightarrow \infty$ , because the damping term  $e^{-\beta\tau/2}$  was lacking in the solution. Here, we have introduced a term which would apparently increase the stability of the system. It is intuitively obvious, but difficult to prove without a series solution, that this is indeed the case. For, since  $\mu_i$  will depend on both  $\beta$  and  $\Omega$ , the regions of stability of Eq. (15) may be less extensive than those of the Mathieu equation, and Eq. (27) might have regions of stability which are less extensive than those of Eq. (17) and (18).

Therefore, the only definite conclusion that can be made about Eq. (27) is that stable solutions exist. If the characteristic numbers  $\mu_i$  are such that  $\mu_1$  and  $\mu_2$  are complex conjugate, and if  $\text{Re}(\mu_1) = \beta/2$  then  $|z|$  is a periodic function of  $\tau$ . Otherwise, when  $|\text{Re}(\mu_1)| < \beta/2$ ,  $|z|$  will approach zero as  $\tau \rightarrow \infty$ . (This is true when the characteristic numbers  $\mu_i$  are imaginary as well.)

The preceding discussion has been for arbitrary  $\beta$  and  $\Omega$ . However, the averaging process described earlier demands that certain restrictions on  $\beta$  and  $\Omega$ , given by Eqs. (10) and (13), be met. Within this structure, it is possible to average the motion over times long compared to the cyclotron frequency if  $\Omega \gg 1$  and to obtain the motion of the guiding center; hence one case of interest will be that in which  $\Omega/\beta \lesssim 1$ ,  $\Omega \gg 1$ . On the other hand, if one averages over times short compared to the cyclotron period, then  $\Omega$  is not restricted, and Eqs. (10) and (13) combine to give  $\Omega/\beta \ll 1$ ,  $\Omega \gtrsim 1$ .

#### AN APPROXIMATE SOLUTION

The fact that  $\beta \gg 1$  leads one to consider an approximate solution of Eq. (15). Rewrite Eq. (15) as

$$w'' + \beta^2 \left( \frac{\gamma \sin \tau + 1}{2} \right)^2 w = 0 \quad (29)$$

where  $\gamma = \Omega/\beta$  is the ratio of the cyclotron frequency to the collision frequency. From Eq. (10),  $\gamma \sim 1$  when the Langevin time average is taken over a cyclotron period; the second case, in which the time average is much less than a cyclotron period, is given by Eq. (13), and  $\gamma \ll 1$ .

The method of solution when  $\beta \gg 1$  (sometimes called the WKB method) is discussed fully in Appendix B. The results are reproduced here for convenience. The general solution of Eq. (29) when terms of the order  $1/\beta^2$  are neglected is

$$w = \frac{1}{\sqrt{1 - i\gamma \sin \tau}} \left[ A_1 e^{i\varphi - \beta\tau/2} + A_2 e^{-i\varphi + \beta\tau/2} \right] \quad (30)$$

where  $\varphi = \Omega \sin^2 \tau / 2$ .

It is obvious from the undamped exponential in the second term that  $w$  is not bounded as  $\tau \rightarrow \infty$ . We should note also that  $w(\tau)$  has the general appearance predicted by Eq. (26) of an exponential times a periodic function of  $\tau$ .

The fact that  $w(\tau)$  is unbounded does not preclude a bounded solution of Eq. (27): a comparison of the exponential terms in Eqs. (27) and (30) shows that the undamped exponent will vanish when the two equations are combined, viz.

$$z = \frac{1}{\beta} \sqrt{\frac{1 + i\gamma \sin \tau}{1 + \gamma^2 \sin^2 \tau}} \left\{ (V + i(C_1 - \frac{\gamma}{2})) (1 - e^{2i\varphi - \beta\tau}) + \beta \right\} \quad (31)$$

where the initial conditions on  $z$  have been used to determine the constants of integration  $A_1$  and  $A_2$ . After considerable algebra,  $z$  can be written in comparatively simple form:

$$z = \frac{\sqrt{F_1^2 + F_2^2}}{\beta(1 + \gamma^2 \sin^2 \tau)^{1/4}} \exp \left[ \frac{i}{2} \tan^{-1}(\gamma \sin \tau) + i \tan^{-1} \frac{F_2}{F_1} \right] \quad (32)$$

where

$$F_1 = V + \beta - e^{-\beta\tau} \left[ V \cos 2\varphi - (C_1 - \frac{\gamma}{2}) \sin 2\varphi \right] \quad (33)$$

$$F_2 = C_1 - \frac{\gamma}{2} - e^{-\beta\tau} \left[ (C_1 - \frac{\gamma}{2}) \cos 2\varphi + V \sin 2\varphi \right] \quad (34)$$

remembering that  $\varphi = \Omega \sin^2 \tau / 2$ , and  $\gamma = \Omega / \beta$ .

The components of motion are thus

$$R^2 = |z|^2 = \frac{F_1^2 + F_2^2}{\beta^2 (1 + \gamma^2 \sin^2 \tau)^{1/2}} \quad (35)$$

and

$$\theta = \arg z = \frac{1}{2} \tan^{-1} \left\{ \frac{(F_1^2 - F_2^2) \gamma \sin \tau + 2F_1 F_2}{(F_1^2 - F_2^2) - 2F_1 F_2 \gamma \sin \tau} \right\} \quad (36)$$

It is apparent from Eqs. (35) and (36) that the orbit will be bounded for all time  $\tau > 0$ , and if

$$\sqrt{\frac{F_1^2 + F_2^2}{\beta}} < R_0 \quad (37)$$

where  $R_0$  is the dimensionless radius of the container, that the motion will be confined to regions inside the tube for  $\tau > 0$ . This is true regardless of the stability conditions on Eq. (29), since the damping term  $e^{-\beta\tau}$  removes the instability present in Eq. (30).

Although Eqs. (35) and (36) are complicated, the motion in reality is quite simple. Since  $\beta \gg 1$ , the damping terms in  $F_1$  and  $F_2$  will vanish for relatively small times, and if we further assume that  $V = C_1 = 0$  and  $\gamma \leq 1$ , then

$$R^2 = \frac{1 + \gamma^2 / 4\beta^2}{\sqrt{1 + \gamma^2 \sin^2 \tau}} \approx \frac{1}{\sqrt{1 + \gamma^2 \sin^2 \tau}} \quad (38)$$

and

$$\tan 2\theta = \frac{\gamma \sin \tau \left(1 - \frac{\gamma^2}{4\beta^2}\right) - \frac{\gamma}{\beta}}{\left(1 - \frac{\gamma^2}{4\beta^2}\right) + \frac{\gamma^2}{\beta} \sin \tau} \approx \gamma \sin \tau \quad (39)$$

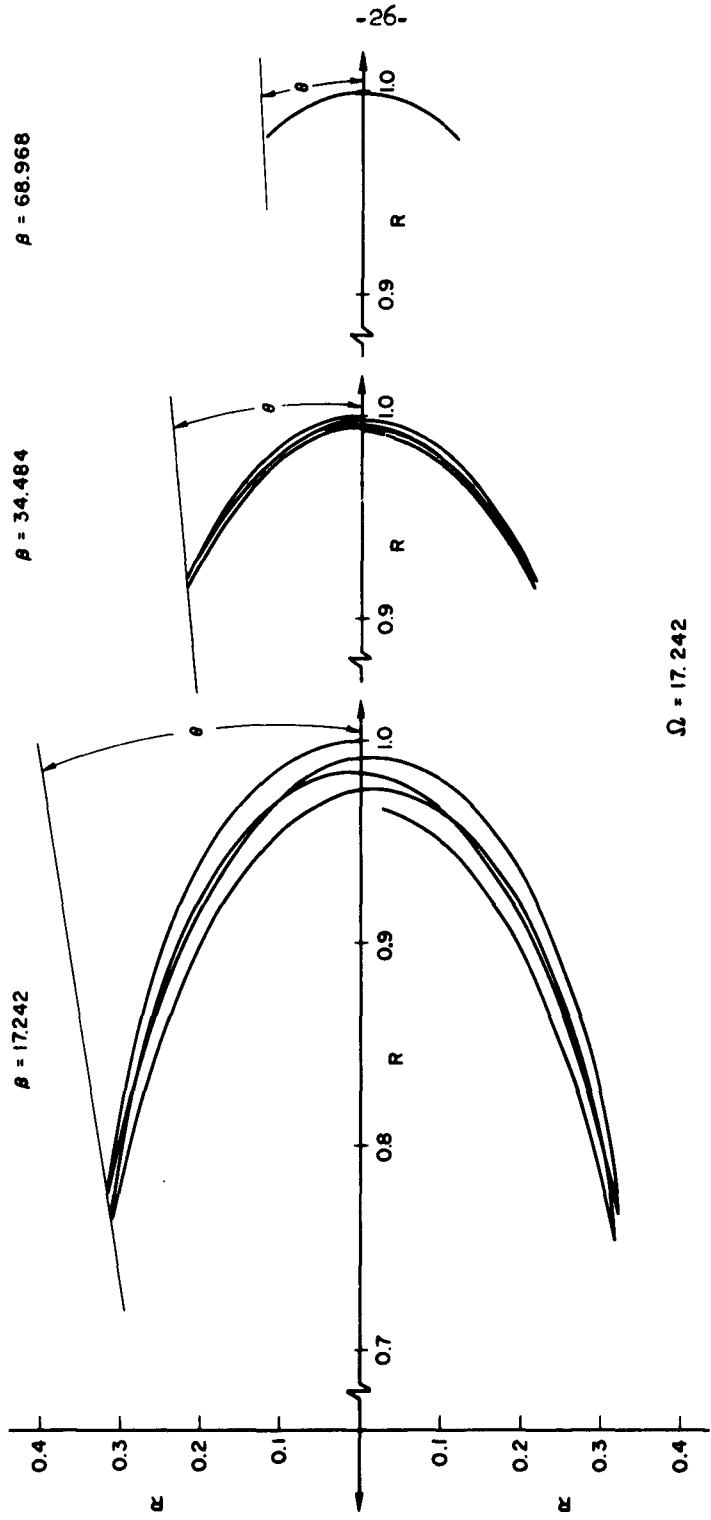
when  $\gamma/\beta = \Omega/\beta^2 \leq 1$ . This shows that the motion is a simple oscillation in which the electron position varies in a periodic manner, reaching its maximum extent when the electric field is zero. Highly damped motion corresponds to ohmic resistivity, as would be expected. The maximum extent of the motion is at a radial position of  $R = 0.84$ , and the maximum angular displacement is  $\gamma$  radians.

One would suspect from Eqs. (38) and (39) that the electron is experiencing very little acceleration in its trajectory. In fact, if one were to solve the original differential equation, Eq. (7), with the acceleration term  $z'' = 0$ , the result would be identical with the approximate versions of Eqs. (37) and (38), in which terms of the order  $\gamma/\beta$  are neglected. This result is not unexpected from the Langevin equation, because the drag term there is proportional to the velocity. When the collisions are predominant, the particle has no chance to accelerate before the drag decelerates it again.

#### NUMERICAL EXAMPLES

A few trajectories have been numerically computed to illustrate the motion of an average electron when collisions predominate. The values of  $\Omega$  chosen for these calculations are the same as those in the collisionless motion. The parameter  $\gamma$  (the ratio of cyclotron frequency to collision frequency) was varied from  $\gamma = 1$  to  $\gamma = 1/4$ . These results are presented in Figs. 7 and 8.

For purposes of comparison, one value of  $\Omega$  was selected so that when  $\beta = 0$  the trajectory became identical with that computed for the collisionless case. Figure 7, for example, shows the motion for  $\Omega = 17.242$  and various values of  $\beta$ ; the collisionless motion for this case



$\Omega = 17.242$

Fig. 7—Average motion of electrons for  $\Omega = 17.242$

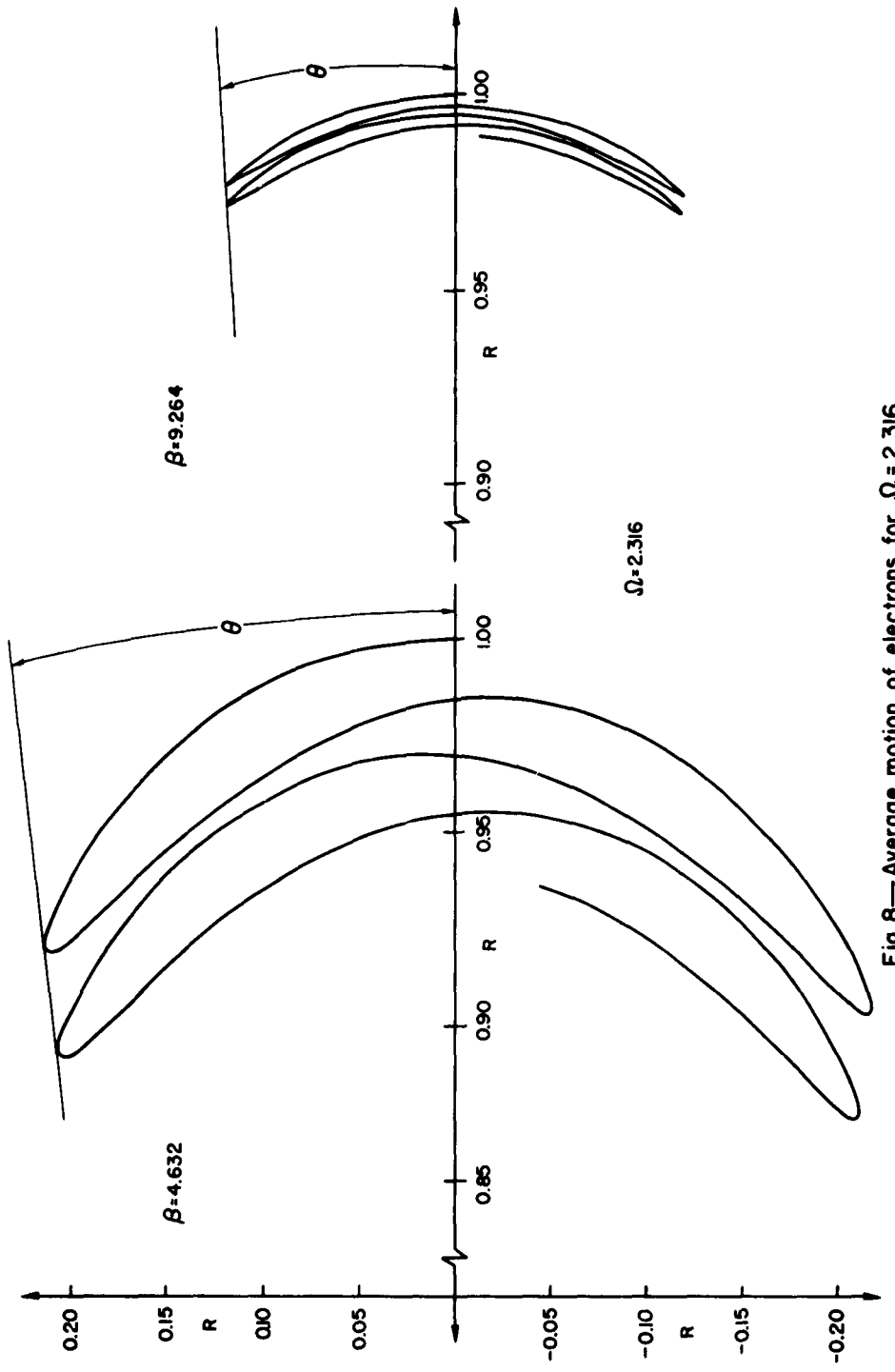


Fig. 8—Average motion of electrons for  $\Omega = 2.316$

is given in Fig. 3. The unstable collisionless motion for  $\Omega = 2.316$  becomes stable when  $\beta \gg 1$ . An example of this is given in Fig. 8.

The motion given in Figs. 7 and 8 agrees well with the approximate solution given by Eqs. (38) and (39) when  $\tau$  is large enough so that  $e^{-\beta\tau} \rightarrow 0$ . The complete approximate solution given by Eqs. (35) and (36) should, in fact, more than adequately describe the motion as long as  $\beta$  is larger than the limits prescribed in Appendix B.

In conclusion, one can say that the average motion of an electron in a weakly ionized gas is bounded in the sense that the position of the electron oscillates 90 deg out of phase with the applied field and will not intersect the container wall if Eq. (37) is satisfied. However, since diffusion is neglected, some electrons will always leave the high-density region for the wall, and thus the mathematical existence of bounded solutions does not imply confinement.

Appendix A

ON SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS  
WITH PERIODIC COEFFICIENTS

The basic equation of this study, Eq. (15), is of the form

$$\frac{d^2 w}{d\tau^2} + p(\tau)w = 0 \quad (A-1)$$

where  $p(\tau)$  is a continuous, periodic, complex function of the real variable  $\tau$  over the range  $-\infty < \tau < \infty$ . The function  $p(\tau)$  contains two general parameters,  $\Omega$  and  $\beta$ .

The most familiar form taken by Eq. (A-1) is that for the case in which  $p(\tau)$  is a real, even, periodic function  $P(\tau)$ ; Eq. (A-1) is then generally called the Hill equation:

$$\frac{d^2 w}{d\tau^2} + (P(\tau) + \lambda)w = 0 \quad (A-2)$$

where  $\lambda$  is some real parameter. Here  $\tau$  can be either real or complex. A special case of the Hill equation is the Mathieu equation

$$\frac{d^2 w}{d\tau^2} + (a - 2q \cos 2\tau) w = 0 \quad (A-3)$$

where  $a$  and  $q$  are again parameters in the equation.

The theory of the general solution and its behavior for the case in which  $p(\tau)$  is real, such as in Eq. (A-2) or (A-3), is discussed at some length in Ince<sup>(18)</sup> and McLachlan.<sup>(15)</sup> The fact that the present equation has the same type of general solution was not known to the author until it was found in Coddington and Levinson.<sup>(19)</sup> Because

the nature or stability of the general solution of Eq. (A-1) is essential to the discussion in the text, a short discussion of its existence and properties is included here.

It can be shown (theorems 2.3 - 5.1 of Ref. 18) that Eq. (A-1) admits a solution of the form

$$w(\tau) = A_1 e^{\mu_1 \tau} \phi_1(\tau) + A_2 e^{\mu_2 \tau} \phi_2(\tau) \quad (\text{A-4})$$

where  $\phi(\tau)$  is periodic, of the same period of  $p(\tau)$ , and where the exponent  $\mu$  is determined by the roots of the characteristic equation and is a function of the parameters  $\Omega$  and  $\beta$ . An outline of the proof is as follows: The set of all solutions of Eq. (A-1) forms a two-dimensional vector space over the complex field. Two of these solutions are linearly independent and hence form a basis for the space. These are called the fundamental set. Any linear combination of these is also a solution. If  $w_1(\tau)$  and  $w_2(\tau)$  are the two solutions, then

$$w(\tau) = A_3 w_1(\tau) + A_4 w_2(\tau) \quad (\text{A-5})$$

is a solution. Now  $p(\tau + 2\pi) = p(\tau)$ , since  $2\pi$  is the period of  $p(\tau)$  in this case. Therefore  $w_1(\tau + 2\pi)$  and  $w_2(\tau + 2\pi)$  satisfy Eq. (A-1) and can also be written in terms of the fundamental set as

$$w_1(\tau + 2\pi) = \alpha_{11} w_1(\tau) + \alpha_{12} w_2(\tau) \quad (\text{A-6})$$

$$w_2(\tau + 2\pi) = \alpha_{21} w_1(\tau) + \alpha_{22} w_2(\tau)$$

Writing Eq. (A-5) at  $\tau + 2\pi$ , and combining this with Eq. (A-6), one has, for the general solution at  $\tau + 2\pi$

$$w(\tau + 2\pi) = (A_3 \alpha_{11} + A_4 \alpha_{21}) w_1(\tau) + (A_3 \alpha_{12} + A_4 \alpha_{22}) w_2(\tau) \quad (\text{A-7})$$

i.e.,  $w(\tau + 2\pi)$  is a linear combination of the fundamental solutions  $w_1(\tau)$  and  $w_2(\tau)$ . We want to determine a general solution  $w(\tau)$  such that

$$w(\tau + 2\pi) = kw(\tau) \quad (\text{A-8})$$

This is possible if  $k$  is related to the coefficients in Eq. (A-7) in the following way:

$$\begin{aligned} kA_3 &= A_3\alpha_{11} + A_4\alpha_{21} \\ kA_4 &= A_3\alpha_{12} + A_4\alpha_{22} \end{aligned} \quad (\text{A-9})$$

which have a solution if and only if

$$\begin{vmatrix} \alpha_{11} - k & \alpha_{21} \\ \alpha_{12} & \alpha_{22} - k \end{vmatrix} = 0 \quad (\text{A-10})$$

The determinantal Eq. (A-10) is called the characteristic equation, and  $k$  the characteristic roots of this equation. If we let  $k = e^{2\pi\mu}$ , then  $\mu$  is the characteristic exponent, which is determined to within modulo  $2\pi$  for the imaginary part. The real part of  $\mu$  is determined uniquely.

Let us write the general solution as

$$w(\tau) = e^{\mu\tau} \phi(\tau) \quad (\text{A-11})$$

where  $\phi(\tau)$  is unspecified for the moment. Writing Eq. (A-11) at  $\tau + 2\pi$ , we have

$$\phi(\tau + 2\pi) = e^{-\mu\tau - 2\pi\mu} w(\tau + 2\pi) \quad (\text{A-12})$$

and, using Eq. (A-8) and the definition of  $k$

$$\phi(\tau + 2\pi) = e^{-\mu\tau} w(\tau) \equiv \phi(\tau) \quad (\text{A-13})$$

Therefore  $\phi(\tau)$  is a periodic function of  $\tau$  with the same period ( $2\pi$ ) as the coefficient  $p(\tau)$ .

To conclude, we have shown that Eq. (A-1) has the solutions

$$w_1 = e^{\mu_1 \tau} \phi_1(\tau) \quad (i = 1, 2) \quad (\text{A-14})$$

where the two characteristic exponents  $\mu_1$  are given by the solution of Eq. (A-10). The coefficients  $\alpha_{ij}$  in Eq. (A-10) are found from the initial data and the solution evaluated at  $\tau = 2\pi$  and are hence functions of the parameters  $\Omega$  and  $\beta$ . The general solution of Eq. (A-1) is therefore given by Eq. (A-4).

#### THE NATURE OF THE SOLUTION

First, we can write the coefficients of the characteristic equation in terms of the initial conditions and the solutions. Let us pick

$$\begin{aligned} w_1(0) &= 1 & w_2(0) &= 0 \\ w_1'(0) &= 0 & w_2'(0) &= 1 \end{aligned} \quad (\text{A-15})$$

and let the constants  $A_1, A_2$  in Eq. (A-4) take care of the initial conditions on Eq. (15) of the text. Then Eq. (A-6), differentiated and evaluated at  $\tau = 0$ , yields

$$\begin{aligned} w_1(2\pi) &= \alpha_{11} \\ w_2(2\pi) &= \alpha_{22} \end{aligned} \quad (\text{A-16})$$

Expanding the determinant given by Eq. (A-10), we have

$$k^2 - \left( w_1(2\pi) + w_2'(2\pi) \right) k + 1 = 0 \quad (\text{A-17})$$

where use has been made of the fact that the Wronskian of the fundamental set  $w_1 w_2' - w_2 w_1' = 1$ . Equation (A-17) shows that the values of  $k$  depend on the values of the solution at the point  $\tau = 2\pi$ . When  $k$  is such that  $\mu$  is purely imaginary, the solution remains finite as  $\tau \rightarrow \infty$ . On the other hand, if  $\mu$  is real or complex, then  $e^{\mu\tau}$  becomes infinite for either  $\tau \rightarrow \infty$  or  $\tau \rightarrow -\infty$ . In this case the solution is unstable.

While the theory just presented is very simple, it is not easy to determine the characteristic exponents. Ince<sup>(19)</sup> discusses a method for doing so which involves repeated integration of  $p(\tau)$  with the terms of a power series expansion of the fundamental set of solutions. These solutions could be obtained by a series expansion. Because of the nature of  $p(\tau)$ , however, the undertaking of such a task is not warranted. Even with the simple Mathieu equation, a whole book<sup>(15)</sup> is devoted to the various series solutions, stability diagrams, and numerical techniques.

#### THE MATHIEU EQUATION

The Mathieu equation emerges from Eq. (A-1) when  $\beta = 0$ :

$$\frac{d^2 w}{d\tau^2} + \left( \frac{\Omega^2}{4} \sin^2 \tau \right) w = \frac{d^2 w}{d\tau^2} + \frac{\Omega^2}{8} (1 - \cos 2\tau) w = 0 \quad (\text{A-18})$$

Notice that now  $p(\tau)$  is an even function of  $\tau$  and permits us to write for a general solution

$$w(\tau) = Ae^{\mu\tau} \phi(\tau) + Be^{-\mu\tau} \phi(-\tau) \quad (\text{A-19})$$

Let  $\phi(\pm\tau) = \sum_{-\infty}^{\infty} c_m e^{\pm im\tau}$  and let  $\mu = \kappa + i\alpha$ . Then

$$w(\tau) = A_1 e^{\kappa\tau} \sum_{-\infty}^{\infty} c_m e^{i(\alpha+im)\tau} + A_2 e^{-\kappa\tau} \sum_{-\infty}^{\infty} c_m e^{-i(\alpha+im)\tau} \quad (\text{A-20})$$

This rather simple form for the series solution is due to the evenness and periodicity of  $p(\tau)$ . In the general case (Eq. (A-1)), a series solution would be much more complicated.

If  $\mu = i\alpha$ , when  $\alpha$  is nonintegral, Eq. (A-20) is stable. (If  $\alpha$  is integral, the solution is called neutral.) If  $\alpha$  is rational, say  $\alpha = p/s$ , then the period of  $w$  is  $2\pi s$ . If  $\alpha$  is irrational,  $w$  is aperiodic; i.e., oscillatory and bounded, but nonrepeating.

The characteristic roots of Eq. (A-10) have been found for this set of fundamental solutions, which are Mathieu functions. Figure 9 reproduces in part the regions of stability as a function of the parameter  $\Omega$ , along with some points indicating those values of  $\Omega$  which yield values of  $\alpha$  giving period of  $4\pi$ . The unstable solution illustrated in Fig. 6 is also indicated in Fig. 9. The legend "even" and "odd" in Fig. 10 (Appendix B) indicates whether the index in Eq. (A-20) is even or odd.

The numerical determination of  $\Omega$  for  $\alpha = 1/2$  so that  $\Omega$  lies in one of the stable regions is treated extensively in Ref. 15. The author found that the method of continued fractions (§ 5.11 of Ref. 15) was the simplest to perform.

The lines dividing regions of stable and unstable solutions correspond to Mathieu functions of integral order. Here  $\alpha = 0$  or  $\alpha = 1$ , and it is not possible to have two periodic solutions.<sup>(18)</sup> For that reason, the examples given in the text are for fractional values of  $\alpha$ .

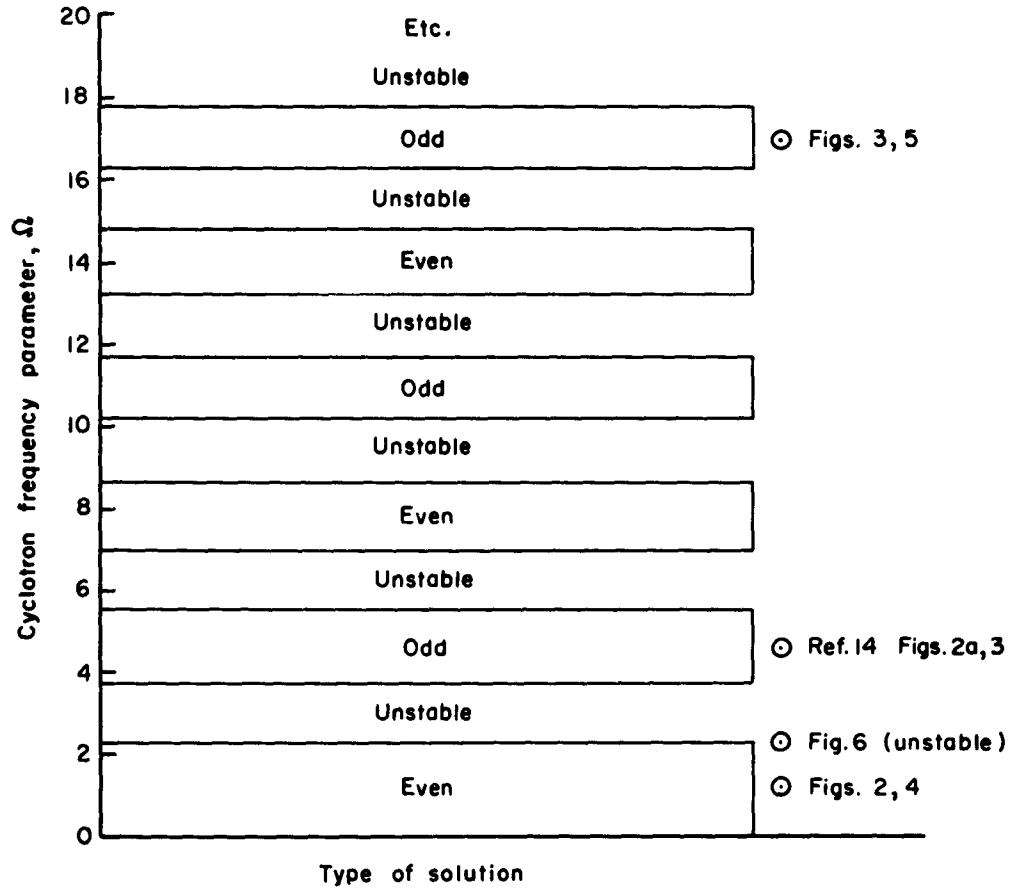


Fig. 9 — Stability chart for solutions of Mathieu equation, showing values of  $\Omega$  for which numerical solutions were obtained

Appendix B

APPROXIMATE SOLUTION OF EQ. (15)

Write Eq. (15) as

$$\frac{d^2 w}{d\tau^2} + \beta^2 \left( \frac{\gamma \sin \tau + 1}{2} \right)^2 w = 0 \quad (B-1)$$

where  $\gamma = \frac{\Omega}{\beta} \ll 1$  and  $\beta \gg 1$ . Let  $x = w'/\beta w$  and  $\rho = (\gamma \sin \tau + 1)/2$ ; then

$$\frac{1}{\beta} \frac{dx}{d\tau} + x^2 + \rho^2 = 0 \quad (B-2)$$

(This is the Riccati equation.) Assume that

$$x = x_0 + \frac{1}{\beta} x_1 + \frac{1}{\beta^2} x_2 + \dots \quad (B-3)$$

Inserting this expression for  $x$  into Eq. (B-2) and equating the coefficients of powers of  $\beta$  to zero gives

$$x = \pm i\rho - \frac{\rho'}{2\beta\rho} \mp \frac{1}{\beta^2} \left( \frac{2\rho\rho'' - 3\rho'^2}{8\rho^3} \right)^2 + \dots \quad (B-4)$$

so that

$$w = \sqrt{\frac{\rho(0)}{\rho}} e^{\pm(1\mp\beta\tau/2)} e^{\mp \frac{1}{\beta} \int_0^\tau \frac{2\rho\rho'' - 3\rho'^2}{8\rho^3} ds} + \dots \quad (B-5)$$

Neglecting the integral term for the time being, we note that  $w(\tau)$  as given by Eq. (B-5) is unstable, since the term  $e^{\beta\tau/2} \rightarrow \infty$  as  $\tau \rightarrow \infty$ .

However, the motion itself is stable, which is seen if we substitute  $w(\tau)$  into Eq. (14), yielding

$$z = \frac{1}{\beta} \sqrt{\frac{1 + i\gamma \sin \tau}{1 + \gamma^2 \sin^2 \tau}} \left\{ (v + iC_1 - \frac{1\gamma}{2})(1 - e^{2i\varphi - \beta\tau}) + \beta \right\} \quad (B-6)$$

This is Eq. (31) of the text.

The neglect of the integral term introduces an error which is best seen by examining Eq. (B-4). Since  $\rho = (\gamma \sin \tau + i)/2$ , we can write the term containing  $\rho$ ,  $\rho'$ , and  $\rho''$  in terms of  $\rho$ . Letting the last term in Eq. (B-4) be given by  $f(\rho)$ , then

$$w = \sqrt{\frac{\rho(0)}{\rho}} \exp \left[ \pm \beta \int_0^\tau [i\text{Re}(\rho - f(\rho))] - [\text{Im}(\rho - f(\rho))] ds \right] \quad (B-7)$$

where

$$\text{Re}(\rho - f(\rho)) = \frac{y}{2} - \frac{y[y^4 - (9 + 3\gamma^2)y^2 + (9\gamma^2 + 2)]}{4\beta^2(y^2 + 1)^3} \quad (B-8)$$

$$\text{Im}(\rho - f(\rho)) = \frac{1}{2} + \frac{5y^4 - (7 + 9\gamma^2)y^2 + 3\gamma^2}{4\beta^2(y^2 + 1)^3} \quad (B-9)$$

and where  $y = 2\rho - i = \gamma \sin \tau$  is real. The contribution of the second terms to the integrand is largest (for all time) when  $\gamma \sim 1$ . When  $\gamma \ll 1$ , then  $y \ll 1$ , and the integrand becomes

$$I = i \left( \frac{y}{2} - \frac{y}{2\beta^2} \right) - \frac{1}{2} \quad (B-10)$$

in which the dominant term is the real part.

By computing values of Eqs. (B-8) and (B-9) for  $\gamma = 1$ , it is possible to estimate the lower limit on  $\beta$  below which the approximate solution is invalid. Figure 10 shows the two equations for  $\beta^2 = 1$ , 10, and  $\infty$  over a quarter-period of the function  $f(\rho)$ . It is obvious that the error induced in the periodic portion of the solution is small when  $\beta^2 > 10$ . The error in the real part of the integrand will average out over a period of  $f(\rho)$  for  $\beta^2 > 10$ . Hence  $\beta > 3$  would appear to satisfy the approximation in Eq. (B-6).

It is also interesting to note that the approximate solution given by Eq. (B-5), when substituted back into the differential equation Eq. (B-1), satisfies Eq. (B-1) whenever  $\beta^2 > 10$  (or  $f(\rho) \approx 0$ ). This is true whether the integral term in Eq. (B-5) is retained or not.

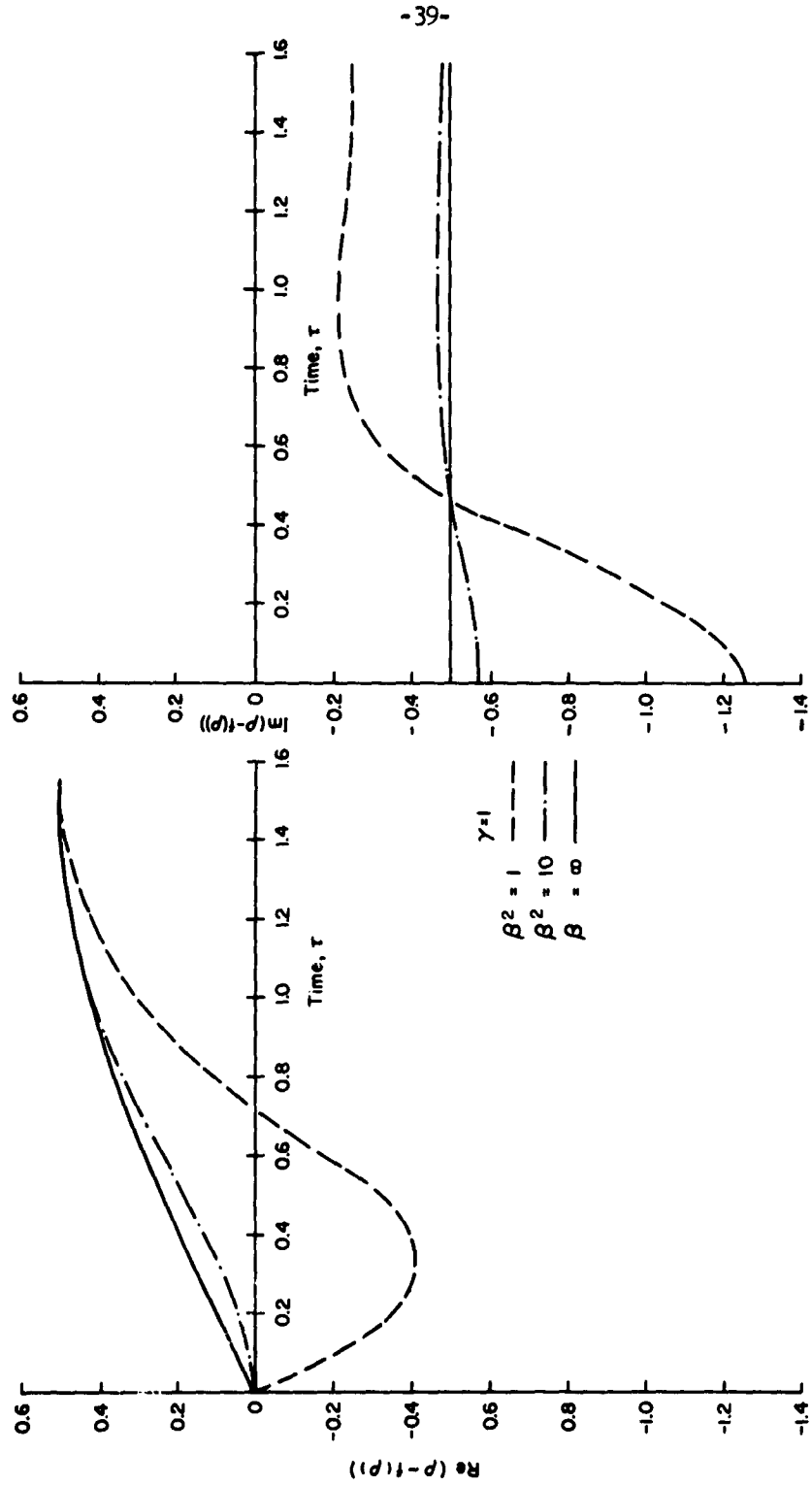


Fig. 10—Estimate of the error in the WKB approximation

REFERENCES

1. Boot, H.A.H., S. A. Self, and R.B.R.-S. Harvie, "Containment of a Fully Ionized Plasma by Radio Frequency Fields," J. Elect. Control, Vol. 4, 1958, p. 434.
2. Weibel, E. S., "A Note on the Confinement of a Plasma by R. F. Fields," J. Elect. Control, Vol. 5, 1958, p. 435.
3. Weibel, E. S., "Confinement of a Plasma Column by Radiation Pressure," in R. K. Landshoff (ed.), The Plasma in a Magnetic Field, Stanford University Press, 1958, pp. 60-76.
4. Dow, D. G., and R. C. Knechtli, "Plasma Containment by rf and dc Field Combinations," J. Elect. Control, Vol. 7, 1959, p. 316.
5. Gallop, J. W., T. L. Dutt, and H. Gibson, "The Behavior of a Plasma Under the Influence of a Standing Electromagnetic Wave in the Absence of a Constant Magnetic Field," J. Elect. Control, Vol. 10, 1961, p. 207.
6. Weibel, E. S., "Stable Orbits of Charged Particles in an Oscillating Electromagnetic Field," Phys. Rev., Vol. 114, April 1, 1959, p. 18.
7. Zauderer, B., "Confinement of a Partially Ionized Gas by a Periodic Magnetic Field," Phys. Fluids, Vol. 5, 1962, p. 720.
8. Eckert, H. U., Equations of the Electrodeless Ring Discharge and their Solution for the Breakdown Criterion, Convair Scientific Research Laboratory, San Diego, California, Research Report 5, August 1959; a condensed version is in Proceedings of the Fourth International Conference on Ionization Phenomena in Gases, Vol. I, North-Holland Publishing Co., Amsterdam, 1960, p. 320.
9. Eckert, H. U., "Diffusion Theory of the Electrodeless Ring Discharge," J. App. Phys., Vol. 33, 1962, p. 2780.
10. Mitra, S. K., The Upper Atmosphere, The Asiatic Society, Calcutta, India, 1952, pp. 623-629.
11. Allis, W. P., "Motion of Ions and Electrons," in S. Flugge (ed.), Handbuch Der Physik, Vol. 21, Springer-Verlag, Berlin, 1956, pp. 383-444.
12. Chandrasekhar, S., "Stochastic Problems in Physics and Astronomy," Rev. Mod. Phys., Vol. 15, 1943, p. 2.

13. Molmud, P., "Langevin Equation and the AC Conductivity of Non-Maxwellian Plasmas," Phys. Rev., Vol. 114, 1959, p. 29.
14. Romig, M. F., Motion of Charged Particles in a Radiofrequency Field, Convair Scientific Research Laboratory, San Diego, California, Research Note 44, January 1961.
15. MacLachlan, N. W., Theory and Application of Mathieu Functions, Clarendon Press, Oxford, 1947.
16. Kulsrud, R., "Adiabatic Invariant of the Harmonic Oscillator," Phys. Rev., Vol. 106, 1957, p. 205.
17. Brault, J. W., Orbits of Charged Particles in a Solenoid with Sinusoidal Current, Los Alamos Scientific Laboratory, LA-2080, June 1957.
18. Ince, E. L., Ordinary Differential Equations, Dover Publications, New York, 1926.
19. Coddington, E. A., and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Company, New York, 1955.