

Technical Report

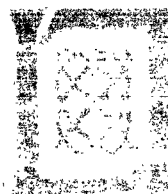
No.

G. B. Vignani

Computation of the Spreading
of an Electron Beam
Under Acceleration
and Space-Charge Repulsion

Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY



409 705

NO. 07

FOR ERRATA

AD 409705

THE FOLLOWING PAGES ARE CHANGES

TO BASIC DOCUMENT

AD 409705

409705

ERRATA SHEET for TECHNICAL REPORT 308

The following errors have been detected in Technical Report 308, "Computation of the Spreading of an Electron Beam Under Acceleration and Space-Charge Repulsion." Please make the appropriate changes in your copy of the report.

Page 12, Fig. 4. Substitute the corrected illustration and caption shown below.

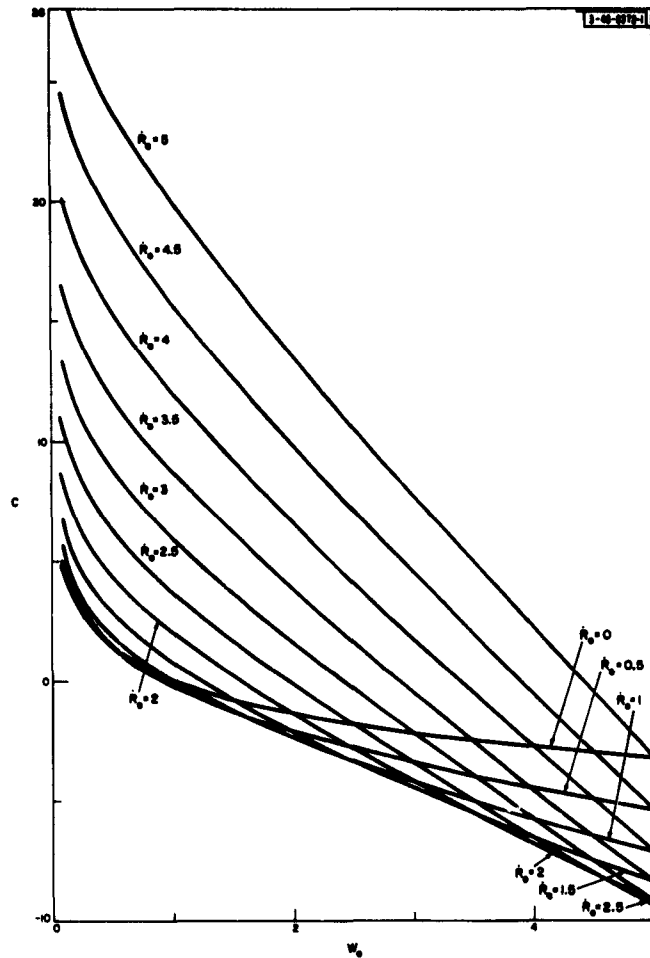


Fig. 4. C vs W_0 for several $\dot{R}_0$.

Page 13, third equation. Should read as follows:

$$\dot{R}_0 = \dot{r}_0 \sqrt{\frac{B}{K}} = r_0' \sqrt{\frac{\dot{z}_0^3}{2\pi\epsilon_0 m} \frac{2\pi\epsilon_0 m}{eI}}$$

Page 19, Appendix. The corrected printout is shown below.

```

C                               ELECTRON BEAM
  DIMENSION DELSO(2000),SO(2000),V(2000),ROP(2000),RP(2000),CAPRP(20
100),CAPR(2000),RHO(2000),RHOP(2000),FIRHO(2000),FLOG(2000),FLSS(20
200)
16 READ INPUT TAPE 2,1,SO(1),DELSO(1),RHO(1),F1,U,RHOP(1),SEND
  WRITE OUTPUT TAPE 3,5,SO(1),DELSO(1),RHO(1),F1,U,RHOP(1),SEND
  WRITE OUTPUT TAPE 3,3
  FM=0.434294
  PI=3.14159265
  V(1)=SO(1)*U
  I=1
13 I=I+1
  DELSO(I)=2.0*DELSO(I-1)
12 SO(I)=SO(I-1)+DELSO(I)
  V(I)=(SO(I)+SO(I-1))*U/2.0
  ROP(I)=F1/SQRTF(V(I)**3)*(DELSO(I)/RHO(I-1))*2*0.220+1.0
  IF (ROP(I)-2.0) 10,11,11
11 DELSO(I)=DELSO(I)/2.0
  GO TO 17
10 RP(I)=0.44*DELSO(I)*F1/(RHO(I-1)*SQRTF(V(I)**3))
  CAPRP(I)=RHOP(I-1)*SQRTF(V(I-1)/V(I))
  CAPR(I)=CAPRP(I)*DELSO(I)
  RHO(I)=ROP(I)*RHO(I-1)+CAPR(I)
  RHOP(I)=RP(I)+CAPRP(I)
  FIRHO(I)=F1/(PI*RHO(I)**2)
  FLOG(I)=LOGF(FIRHO(I))*FM*0.2
  FLSS(I)=LOGF(SO(I)-SO(1))*FM*0.2
  WRITE OUTPUT TAPE 3,2,SO(I),RHO(I),FIRHO(I),FLOG(I),FLSS(I)
  WRITE OUTPUT TAPE 14,4,FLSS(I),FLOG(I)
  IF (I-2000) 14,15,15
14 IF (SO(I)-SEND) 13,15,15
15 GO TO 16

C                               FORMAT STATEMENTS
1  FORMAT (5E10.4,2F6.2)
2  FORMAT (3E20.6,2F20.6)
3  FORMAT (10H          S16X,4H RHO14X,6H I/RHOBX,12H LOGF(I/RO)10X,1
10H LOG(S=SO)/)
4  FORMAT (2F20.6)
5  FORMAT (4H SO=E10.4,7H DELTA=E10.4,5H RHO=F10.4,4H I=F10.4,4H U=F
110.4,6H RHOP=F6.2,6H SEND=F6.2)
  END

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3 January 1964

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

COMPUTATION OF THE SPREADING OF AN ELECTRON BEAM
UNDER ACCELERATION AND SPACE-CHARGE REPULSION

G. E. VIBRANS

Group 48

TECHNICAL REPORT NO. 308

18 APRIL 1963

ABSTRACT

The known differential equation, which describes the spreading of a circular electron beam in an accelerating field under space-charge repulsion, is derived in a simplified form so that the general solution can be precomputed. The solution is discussed and is given in a diagram. All auxiliary calculations are listed in order that the beam radius can be found for any set of initial conditions. Only the diagrams and a slide rule are needed. Another independent method of computation for the same purpose, which may be useful under special conditions, is also included.

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LIST OF SYMBOLS

r	Beam radius = $r(z)$
z	Coordinate along the beam axis
t	Flight time of an electron, starting at $t = 0$ at $z = 0$
r'_0	$(dr/dz)_0$
r_0	Beam radius at $z = 0$
$\dot{z}$	Velocity of the electron $\dot{z} = \dot{z}(t)$ or $\dot{z}(z)$
$\dot{z}_0$	Velocity of the electron at $z = 0$, $t = 0$
U	Potential which corresponds to the initial velocity $\dot{z}_0$
I	Beam current
e	Charge of an electron
m	Mass of an electron
ϵ_0	$8.8543 \cdot 10^{-12} \frac{\text{amp sec}}{\text{V} \cdot \text{m}}$
E_z	Accelerating electric field
K	$\frac{eI}{\epsilon_0 2\pi m}$
A	$\frac{e}{m} E_z$
B	$\dot{z}_0$
R	$r \cdot \frac{A}{\sqrt{BK}}$ dimensionless radius
T	$t \frac{A}{B}$ dimensionless time
W	$\frac{R}{\sqrt{T+1}}$
$\dot{W}$	$\frac{dW}{dT}$
Z	$z \frac{A}{B^2}$ dimensionless coordinate in axial direction

COMPUTATION OF THE SPREADING OF AN ELECTRON BEAM UNDER ACCELERATION AND SPACE-CHARGE REPULSION

I. INTRODUCTION

In electron optical devices in which there is a high electron current density, as in cathode-ray tubes, the field due to the electrons themselves, i.e., space charge, is an important factor in the determination of the shape of any electron beam. The diverging effect of space charge in a region which is field free is a standard presentation in many books on electron optics. The equation of motion is usually attributed to Watson (cf. Ref. 3)

The space charge effect in an accelerated electron beam is also important but does not seem to have been treated extensively in the literature. We will investigate a circular electron beam in an accelerating field. The electrons have a certain initial velocity, and enter the axial electrical field -- for example, through a hole in a plane electrode. The electrons are now subjected to constant acceleration and mutual repulsion. The equation of motion has been derived and solved by H. Moss¹ who gives several examples for the conditions found in cathode-ray tubes.

Such an example is defined by a set of six independent parameters. By Moss' method, every example is solved by computing a function numerically and then integrating it. For a given beam length, the limits of integration must be found by trial and error. The mathematical difficulties have apparently prevented any wider application of this solution.

In the course of voltage breakdown studies we became interested in field emission as an initiating mechanism. The high current density of field emission makes the space-charge effect especially important. In this connection it was desirable to compute many cases which would have been laborious by Moss' method. A simplified method has been found for computing the shape of an accelerated circular electron beam with space charge, and we believe that it may be useful for other applications. This method is presented here in detail.

In Sec. II the mathematical derivation and a discussion of the resolving function are given. Section III covers the numerical computation of the resolving function. The function is represented in Figs. 2 and 3.

The reader who wants to solve a specific problem may omit Secs. II and III and simply follow the instructions given in Sec. IV where all necessary steps for the application of the resolving function are listed. A numerical example is also given in Sec. IV.

In Sec. V a step-by-step procedure is described which is based upon the formula for a beam in a field-free space. This is a second independent solution of the same problem.

II. MATHEMATICAL ANALYSIS

A. Equation of Motion

For the sake of completeness, we will derive our simplified procedure from the fundamentals, following closely the original deductions of H. Moss. After our Eq. (12) we deviate from Moss' analysis.

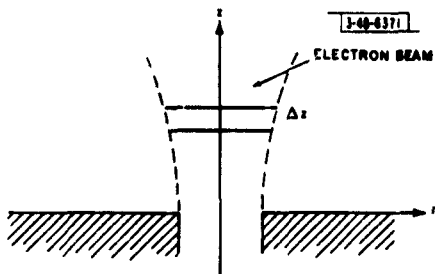


Fig. 1. Geometry for space-charge spreading of a beam in an electric field.

We look at a thin slice (thickness Δz) of a circular electron beam (Fig. 1) and assume that in any plane perpendicular to the beam axis the electrons are monoenergetic. The total electrostatic flux due to the space charge in the slice is

$$\int D \, dS = \rho = \frac{I \Delta z}{z} \quad (1)$$

where

- D = flux (displacement density),
- S = surface of the cylindrical slice,
- ρ = charge within the slice,
- I = beam current,
- z = velocity of the beam electrons.

Since we assume that the field strength and the flux are constant in axial direction, the end discs do not contribute and Eq. (1) becomes

$$D \cdot 2\pi r \Delta z = \frac{I \Delta z}{z} \quad (2)$$

The radial electrical field is then

$$E_{\text{rad}} = \frac{D}{\epsilon_0} = \frac{1}{\epsilon_0} \frac{I}{2\pi r z} \quad (3)$$

and the radial force on the electrons is

$$F = e E_{\text{rad}} = \frac{e I}{\epsilon_0 2\pi r z} \quad (4)$$

The equations of motion of the electrons are with an external axial field E

$$\frac{d^2 r}{dt^2} = \frac{1}{\epsilon_0 2\pi r z} \frac{e}{m} \quad , \quad (5)$$

$$\frac{d^2 z}{dt^2} = \frac{e}{m} E \quad . \quad (6)$$

We integrate Eq. (6)

$$\dot{z} = \frac{e}{m} E t + \dot{z}_0 \quad , \quad (7)$$

$$z = \frac{e}{m} \frac{E}{2} t^2 + \dot{z}_0 t = A \frac{t^2}{2} + B t \quad . \quad (8)$$

Equation (7) inserted in Eq. (5) yields

$$\frac{d^2 r}{dt^2} = \frac{eI}{\epsilon_0 2\pi r m \left(\frac{e}{m} E t + \dot{z}_0 \right)} \quad (9)$$

This is Eq. (6) of H. Moss¹ except for a factor $1/4 \pi \epsilon_0$ due to the use of different units. In order to simplify the differential equation, we introduce the abbreviations

$$K = \frac{eI}{\epsilon_0 2\pi m} \quad , \quad \left[\frac{m^3}{\text{sec}^3} \right] \quad ; \quad (10)$$

$$A = \frac{e}{m} E \quad , \quad \left[\frac{m}{\text{sec}^2} \right] \quad ; \quad (11)$$

$$B = \dot{z}_0 \quad , \quad \left[\frac{m}{\text{sec}} \right] \quad ; \quad (12)$$

and the dimensionless variables

$$R = r \frac{A}{\sqrt{BK}} \quad , \quad (13)$$

$$T = t \frac{A}{B} \quad , \quad (14)$$

$$Z = z \frac{A}{B^2} \quad . \quad (14a)$$

We insert Eqs. (13), (14) and (14a) into Eqs. (8) and (9) and obtain, respectively,

$$Z = \frac{1}{2} T^2 + T \quad , \quad (14b)$$

and

$$R \frac{d^2 R}{dT^2} = \frac{1}{T+1} \quad (15)$$

The dimensionless Eq. (15) could be solved as it stands by some numerical method, but it would still give a two parametric set of curves, which is difficult to compute and to represent.

B. Formal Solution

We write instead

$$(T+1)^{3/2} \frac{d^2 R}{dT^2} = \left(\frac{R}{\sqrt{T+1}} \right)^{-1} \quad (16)$$

The solution of this Eq. (16) is given in the literature, e.g., by Kamke.² We may call this solution a formal one since it is given in terms that have no direct physical meaning. We introduce the new variable into Eq. (15)

$$W = \frac{R}{\sqrt{T+1}} = r \frac{A/\sqrt{K}}{\sqrt{At+B}} \quad (17)$$

and obtain from Eq. (15) after one integration

$$(T+1)^2 \dot{W}^2 = \frac{1}{4} W^2 + 2 \int \frac{1}{W} dW = \frac{1}{4} W^2 + 2 \ln W + C \quad (18)$$

Separation of variables yields

$$\int_{W_0}^W \frac{dW}{\sqrt{\frac{1}{4} W^2 + 2 \ln W + C}} = \ln(T+1) + C_1 \quad (19)$$

The integral represents our resolving function. Before we compute it numerically, we will make it independent of the parameter W_0 [Eq. (28)]. The boundary conditions are

$$t = 0 \quad , \quad r = r_0 \quad , \quad \dot{r} = \dot{r}_0 = r_0' \dot{z}_0 \quad ,$$

$$T = 0 \quad , \quad R = R_0 \quad , \quad \dot{R} = \dot{R}_0 \quad .$$

A dot on a capital letter means differentiation by T ; a dot on a lower case letter means differentiation by t .

The boundary conditions inserted in Eqs. (18) and (19) give

$$C = \dot{W}_0^2 - \frac{1}{4} W_0^2 - 2 \ln W_0 \quad , \quad (20)$$

$$C_1 = 0 \quad , \quad (20a)$$

where

$$\dot{W}_0 = \dot{R}_0 - \frac{1}{2} R_0 - \dot{r}_0 \sqrt{B/K} - \frac{1}{2} r_0 \frac{A}{\sqrt{BK}} \quad (21)$$

$$W_0 = R_0 = r_0 \frac{A}{\sqrt{BK}} \quad (21a)$$

C. Discussion of the Resolving Function

The different possible signs in Eq. (19) require special care. We rewrite Eq. (18)

$$\dot{W} = \frac{\pm \sqrt{\frac{1}{4} W^2 + 2 \ln W + C}}{T + 1} \quad (22)$$

The denominator $(T + 1)$ is always positive. $\dot{W}_0$ is known from the boundary conditions and determines which sign must be applied at the start of the computation. $\dot{W}_0$ also indicates the direction in which we have to integrate since T in Eq. (19) must be positive and can only increase. Therefore,

if $\dot{W}_0 < 0$, the negative sign in Eq. (19) applies and $W < W_0$,

if $\dot{W}_0 > 0$, the positive sign in Eq. (19) applies and $W > W_0$.

It remains to be investigated whether at any later time T there may be a change of the sign of $\dot{W}$, i.e., the point $\dot{W} = 0$ needs special attention. We introduce the index n and write

$$\dot{W}_n = 0 \quad (23)$$

and also hold the equations ready for use

$$\dot{W} = \frac{\dot{R}}{\sqrt{T+1}} - \frac{1}{2} \frac{R}{(T+1)^{3/2}} \quad (24)$$

$$\dot{W}_n = \frac{\dot{R}_n}{\sqrt{T_n+1}} - \frac{1}{2} \frac{R_n}{(T_n+1)^{3/2}} = 0 \quad (25)$$

We insert Eq. (23) into Eq. (22) and conclude that for $\dot{W}_n = 0$ either

$$T = \infty$$

or

$$\frac{1}{4} W_n^2 + 2 \ln W_n + C = 0 \quad (26)$$

The last case is the interesting one; it means that at W_n the integrand of Eq. (19) goes to infinity. An investigation of the limit of the integral in Eq. (19) is given in Eq. (31), and it is shown that the integral plotted vs W remains finite and has only an infinite slope at $W = W_n$.

At this point, $W = W_n$, we have to reconsider the question of signs in Eq. (19). It is physically evident that R is always positive and increases monotonically or first decreases to a

minimum and then increases.* With this in mind, it can be seen from Eq. (24) that if $\dot{W}$ started negative and became zero at W_n , it can only be positive afterward. That means that from W_n on, the positive sign applies in Eq. (19), and the direction of integration is reversed so that W increases from this point on.

With W_n determined by Eq. (26) we can define the integral independent of the parameter W_0 , with only the parameter C left. We write

$$f(W) = \frac{1}{\sqrt{\frac{1}{4} W^2 + 2 \ln W + C}} \quad (27)$$

$$F(W) = \int_{W_n}^W \frac{1}{\sqrt{\frac{1}{4} W^2 + 2 \ln W + C}} dW \quad (28)$$

In order to compute $F(W)$ from $f(W)$ it is necessary to discuss the limit

$$\lim_{W \rightarrow W_n} f(W)$$

because at this point the integrand becomes infinite

$$f(W) = \frac{1}{\sqrt{\frac{1}{4} (W_n + \epsilon)^2 + 2 \ln (W_n + \epsilon) + C}} \\ = \frac{1}{\sqrt{\frac{1}{2} \epsilon W_n + \frac{1}{4} \epsilon^2 + 2 \ln \left(1 + \frac{\epsilon}{W_n}\right)}}$$

since $1/4 W_n^2 + 2 \ln W_n + C = 0$ according to Eq. (26),

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} f(W) &= \frac{1}{\sqrt{\epsilon \left(\frac{W_n}{2} + \frac{2}{W_n}\right) + \epsilon^2 \left(\frac{1}{4} - \frac{2}{2W_n^2}\right) + \frac{2}{3} \frac{\epsilon^3}{W_n^3} - + \dots}} \\ &= \frac{1}{\sqrt{\epsilon \left(\frac{W_n}{2} + \frac{2}{W_n}\right) \left[1 + \epsilon \frac{(1/4) - (1/W_n)^2}{(W_n/2) + (2/W_n)}\right]}} \\ &= \left[\frac{1}{\sqrt{\epsilon \left(\frac{W_n}{2} + \frac{2}{W_n}\right)}} \right] \left[1 - \frac{\epsilon}{2} \left(\frac{\frac{1}{4} - \frac{1}{W_n^2}}{\frac{W_n}{2} + \frac{2}{W_n}} \right) \right] \end{aligned} \quad (29)$$

* No crossover is possible under the assumed conditions.

The second term in the expression is kept only to estimate the error. We insert Eq. (29) into Eq. (28)

$$\begin{aligned}
 F(W_n + \epsilon) &= \int_0^\epsilon f(W_n + \epsilon) d\epsilon \\
 &= 2\sqrt{\epsilon} \frac{1}{\sqrt{\frac{W_n}{2} + \frac{2}{W_n}}} - \frac{\epsilon^{3/2}}{3} \left[\frac{\frac{1}{4} - \frac{1}{W_n^2}}{\left(\frac{W_n}{2} + \frac{2}{W_n}\right)^{3/2}} \right]
 \end{aligned} \quad (30)$$

We take as an approximation

$$F(W_n + \epsilon) = 2\sqrt{\epsilon} \frac{1}{\sqrt{\frac{W_n}{2} + \frac{2}{W_n}}} \quad (31)$$

for which the relative error will be smaller than

$$\Delta = \frac{\epsilon}{6} \left(\frac{\frac{1}{4} - \frac{1}{W_n^2}}{\frac{W_n}{2} + \frac{2}{W_n}} \right) \quad (32)$$

If a maximum error Δ is allowed (and known), Eq. (32) gives the ϵ up to which we apply Eq. (31), i.e., Eq. (28) becomes

$$F(W) = 2\sqrt{\epsilon} \frac{1}{\sqrt{\frac{W_n}{2} + \frac{2}{W_n}}} + \int_{W_n + \epsilon}^W \frac{dW}{\sqrt{\frac{1}{4}W^2 + 2 \ln W + C}} \quad (33)$$

This equation is the basis for the numerical computation of $F(W)$ as it is explained below.

It is worthwhile to note that the minimum value of W , which we have called W_n , does not correspond to the minimum of the beam radius r or R . The minimum beam radius must satisfy the equation

$$\frac{dR}{dT} = \dot{R} = 0$$

With Eq. (24) we can write

$$\dot{W} = \frac{\dot{R}}{\sqrt{T+1}} - \frac{1}{2} \frac{R}{(T+1)^{3/2}}, \quad \dot{W}(R_{\min}) = -\frac{1}{2} \frac{R_{\min}}{(T_{\min}+1)^{3/2}}$$

and with Eqs. (17) and (22),

$$\dot{W}(R_{\min}) = -\frac{1}{2} \frac{W(R_{\min})}{T+1} = \frac{\pm \sqrt{\frac{1}{4} W^2(R_{\min}) + 2 \ln W(R_{\min}) + C}}{T+1}$$

This reduces to the equation

$$2 \ln W(R_{\min}) + C = 0, \quad W < W_0,$$

which determines $W(R_{\min})$.

III. NUMERICAL COMPUTATION OF THE RESOLVING FUNCTION

We have chosen a set of values for the parameter C between the limits -20 and $+20$. This covers all practical applications we can anticipate. For every C we first compute W_n by solving Eq. (26) with the Newton-Raphson method. The residual error is made smaller than 10^{-3} percent. Next we determine the initial step width ϵ from Eq. (32) assuming $\Delta = 10^{-4}$ and check whether $\epsilon < W_n$ which is the condition for Eq. (32) being valid. If $\epsilon > W_n$ we take $\epsilon^* = 0.9 W_n$ instead as initial step width. Now we compute $F(W)$ with a recursion formula:

from Eq. (31)

$$F_1 = 2\sqrt{\epsilon} \frac{1}{\sqrt{\frac{W_n}{2} + \frac{2}{W_n}}}; \quad (34a)$$

from Eq. (29)

$$f_1 = \frac{1}{\sqrt{\epsilon}} \frac{1}{\sqrt{\frac{W_n}{2} + \frac{2}{W_n}}}. \quad (34b)$$

$$f_i = \frac{1}{\sqrt{\frac{1}{4} W_i^2 + 2 \ln W_i + C}}, \quad (35a)$$

$$W_i = W_{i-1} + \epsilon, \quad (35b)$$

$$F_i = \begin{cases} F_{i-1} + \frac{1}{2} \epsilon f_{i-1} + \frac{1}{2} \epsilon f_i & \text{if } i \text{ is even} \\ F_{i-1} - \frac{\epsilon}{6} f_{i-2} + \frac{5}{6} \epsilon f_{i-1} + \frac{1}{3} \epsilon f_i & \text{if } i \text{ is odd} \end{cases} \quad (36)$$

$$(37)$$

Equation (36) is a simple extension of the integral over the next step with the trapezoidal rule; Eq. (37) has the following appearance if we insert successively the preceding equations for $i-1$, $i-2$, etc.

$$F_i = F_1 + \frac{1}{3} \epsilon f_1 + \frac{4}{3} \epsilon f_2 + \frac{2}{3} \epsilon f_3 + \dots + \frac{4}{3} \epsilon f_{i-1} + \frac{1}{3} \epsilon f_i. \quad (38)$$

This is Simpson's Rule. After every second step (i odd) we may adjust the step width ϵ . Then we have instead of Eq. (38)

$$F_1 = F_1 + \frac{1}{3} \epsilon_1 f_1 + \frac{4}{3} \epsilon_1 f_2 + \frac{1}{3} \epsilon_1 f_3 + \frac{1}{3} \epsilon_3 f_3 + \frac{4}{3} \epsilon_3 f_4 + \frac{1}{3} \epsilon_3 f_5 \\ + \dots + \frac{1}{3} \epsilon_{i-4} f_{i-2} + \frac{1}{3} \epsilon_{i-2} f_{i-2} + \frac{4}{3} \epsilon_{i-2} f_{i-1} + \frac{1}{3} \epsilon_{i-2} f_i \quad (39)$$

This is again Simpson's Rule with varying step width.

Equations (34) through (37) have been coded for the IBM 7090. The computed $F(W)$ is plotted in Figs. 2 and 3.

Once $F(W, C)$ is known and a set of initial conditions is given, we apply the formal solution as follows. We insert Eqs. (20a) and (28) into Eq. (19)

$$F(W, C) - F(W_0, C) = \ln(T + 1) \quad (40)$$

and invert Eq. (14b) to

$$T + 1 = + \sqrt{1 + 2Z} \quad (41)$$

We insert Eq. (41) into Eq. (40) and write

$$F(W, C) = \ln \sqrt{1 + 2Z} + F(W_0, C) \quad (42)$$

Since W_0 , C and Z are defined by the initial conditions and F is known, Eq. (42) yields W . We write Eq. (17) with Eq. (41) inserted as

$$R = W \sqrt{T + 1} = W \sqrt{1 + 2Z} \quad (43)$$

In this equation we insert W and find R . By inserting Eqs. (13) and (21a) into Eq. (43) we obtain finally

$$\frac{r}{r_0} = \frac{R}{W_0} = \frac{W}{W_0} \sqrt{1 + 2Z} \quad (44)$$

We can now demonstrate the step-by-step application of the formal solution starting with specific initial conditions.

IV. APPLICATION OF THE FORMAL SOLUTION TO A PHYSICAL PROBLEM

To start the computation, we must know all the parameters which define the beam at the starting point. These are:

Beam current	I
Initial beam radius	r_0
Initial beam divergence	$r'_0 = \left(\frac{dr}{dz}\right)_0$
Initial potential of electrons	U
Accelerating electric field	E_z

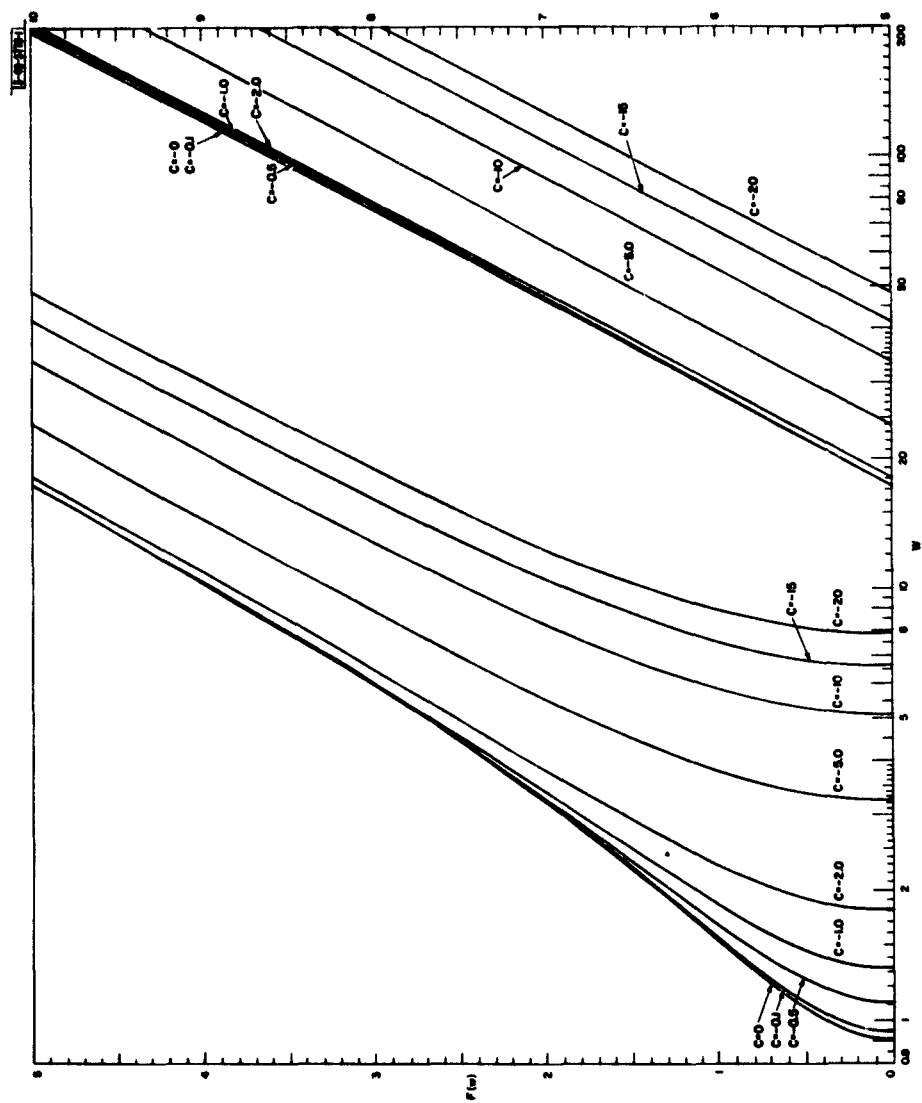


Fig. 2. $F(W)$ vs W for C negative.

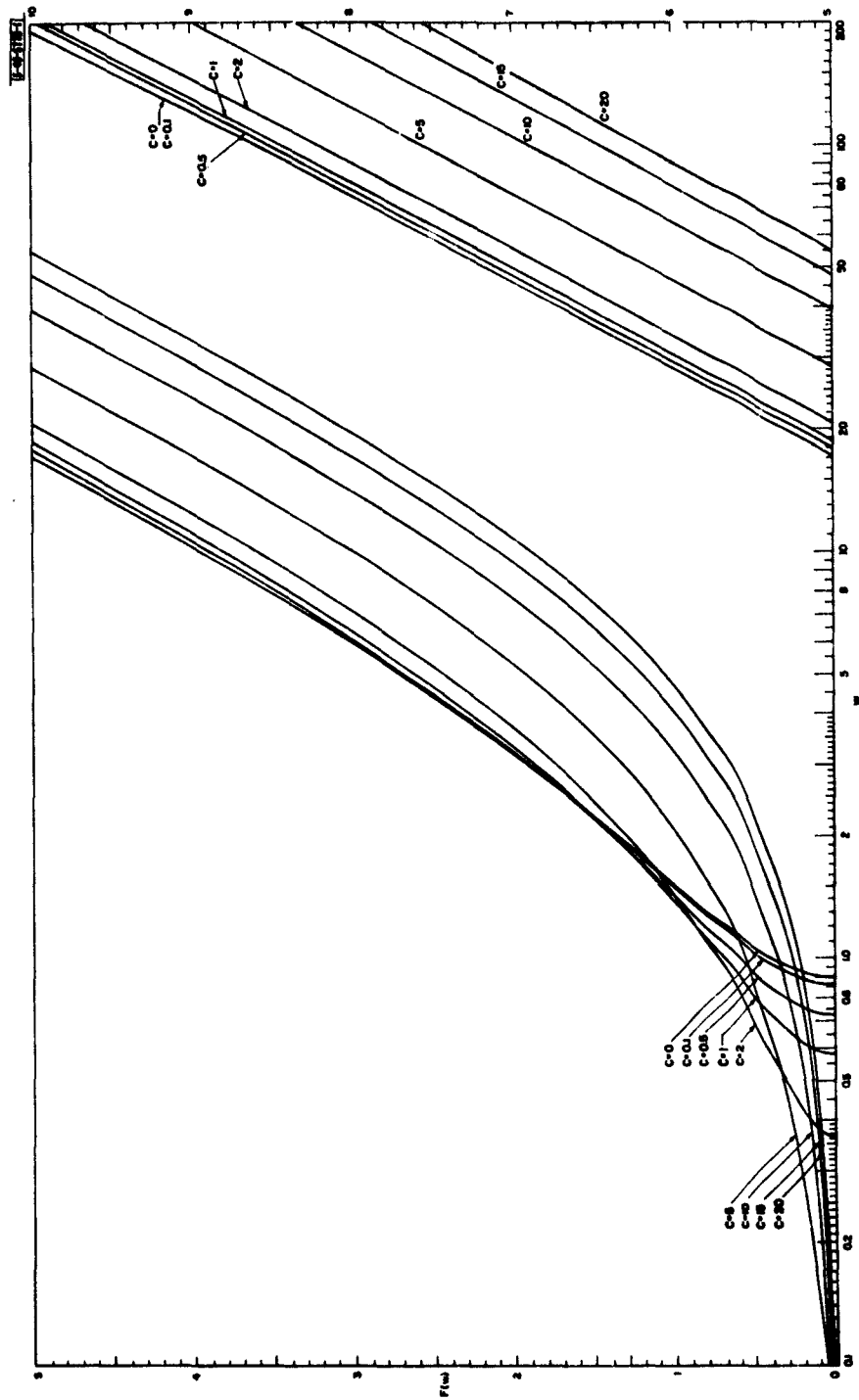


Fig. 3. $F(W)$ vs W for C positive.

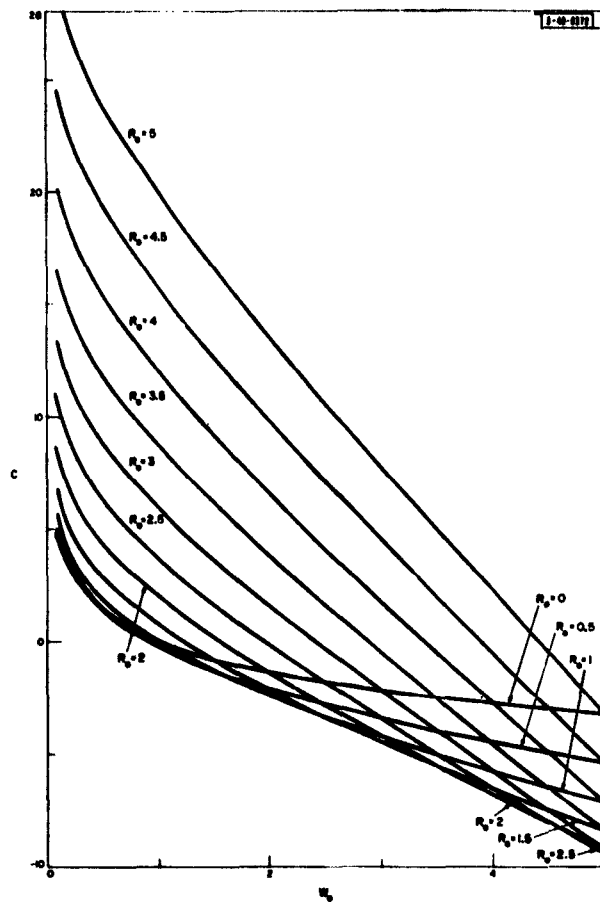


Fig. 4. C vs W_0 for several R_0 .

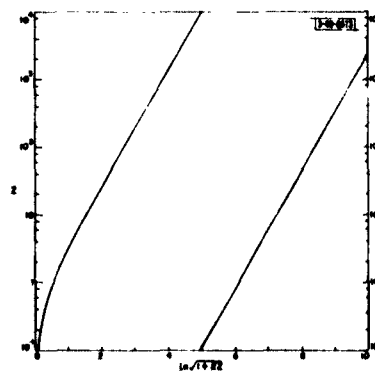


Fig. 5. Z vs $\ln \sqrt{1+2Z}$.

In order to find the beam radius r at a given beam length z we compute successively the following quantities:

$$\dot{z}_0 = \sqrt{2U \frac{e}{m}}$$

$$W_0 = r_0 \frac{A}{\sqrt{BK}} = r_0 \frac{e}{m} E_z \sqrt{\frac{\epsilon_0 2\pi m}{\dot{z}_0 e I}} \quad , \quad [\text{Eq. (21a)}]$$

$$\dot{R}_0 = \dot{r}_0 \sqrt{\frac{B}{K}} = r_0' \sqrt{\frac{\dot{z}_0^3 e I}{2\pi \epsilon_0 m}} \quad ,$$

$$C = \dot{R}_0^2 - W_0 \dot{R}_0 - 2 \ln W_0 \quad (\text{Fig. 4}) \quad ,$$

$$Z = z \frac{A}{B^2} = z \frac{E_z \frac{e}{m}}{\dot{z}_0^2} \quad [\text{Eq. (14a)}]$$

$|F(W_0, C)|$ is taken from Fig. 2 or Fig. 3. The sign is still indeterminate; we must take the sign of $\dot{W}_0$.

$$\dot{W}_0 = \dot{R}_0 - \frac{1}{2} W_0 \quad , \quad [\text{Eq. (21)}]$$

$$F(W_0, C) = \frac{\dot{W}_0}{|W_0|} \cdot |F(W_0, C)| \quad , \quad (45)$$

$$F(W, C) = F(W_0, C) + \ln \sqrt{1 + 2Z} \quad [\text{Fig. 5 and Eq. (42)}]$$

$W(F)$ is taken from Fig. 2 or Fig. 3

$$\frac{r}{r_0} = \frac{W}{W_0} \sqrt{1 + 2Z} \quad , \quad [\text{Eq. (44)}]$$

Let us look at two examples. In the first, the initial condition correspond to the focusing of a post-acceleration CRT; in the second, to field emission.

Initial Conditions

<u>Example 1</u>	<u>Example 2</u>
$I = 25 \cdot 10^{-6} \text{ amp}$	$1 \cdot 10^{-6} \text{ amp}$
$r_0 = 1 \cdot 10^{-3} \text{ m}$	$1.785 \cdot 10^{-9} \text{ m}$
$r'_0 = -\frac{1}{200}$	0
$U = 1 \cdot 10^3 \text{ v}$	5 v
$E_z = 5 \cdot 10^3 \text{ v/m}$	$5 \cdot 10^7 \text{ v/m}$
$z = 0.25 \text{ m}$	$1 \cdot 10^{-3} \text{ m}$

Parameters

<u>Example 1†</u>	<u>Example 2</u>
$\dot{z}_0 = 1.873 \cdot 10^7 \text{ m/sec}$	$1.236 \cdot 10^6 \text{ m/sec}$
$W_0 = 0.7223$	$2.43 \cdot 10^{-1}$
$\dot{R}_0 = -1.435$	0
$C = 3.781$	2.83
$Z = 0.628$	$5 \cdot 10^3$
$ F(W_0, C) = 0.5100$	0
$\dot{W}_0 = -1.796$	—
$F(W_0, C) = -0.5100$	0
$\ln \sqrt{1 + 2Z} = 0.4080$	4.62
$F(W, C) = -0.1030$	4.62
$W(F) = 0.1791$	20.5

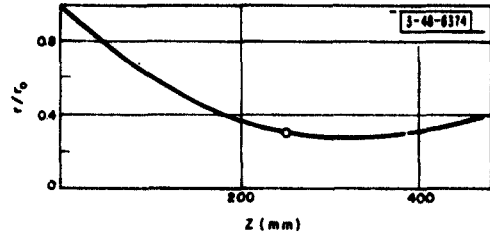
Results

<u>Example 1</u>	<u>Example 2</u>
$\frac{r}{r_0} = 0.3040$	845

The result in Example 1 is compared in Fig. 6 with that of H. Moss for the same conditions.

† These numerical values have more significant digits than one could read from the diagrams, and the result is more precise than usually needed.

Fig. 6. Relative beam radius vs length for conditions of Example 1. The curve is that of H. Moss. The point indicates our result for $Z = 250$ mm.



V. ANOTHER INDEPENDENT WAY TO COMPUTE BEAM SPREADING

Instead of supposing a homogeneous field, we now assume many slices perpendicular to the axis of the beam, where the potential is constant within every slice. At the cut the potential jumps to the potential of the next slice. Within each slice the beam spreads according to the formula for a field-free space. At every cut we have to apply the law of electron optical refraction because the angle of divergence is reduced by the jump of the potential.

The appropriate formula for beam spreading is taken from a recent résumé of Ivey³ and after correction of an obvious misprint it reads

$$\frac{S}{r_m} = 1.021 \frac{V^{3/4}}{I^{1/2}} \varphi(V) F\left(\frac{r}{r_m}\right), \quad (46)$$

where

r_m = minimum beam radius at the "waist",

S = axial distance from the "waist" of the beam,

r = beam radius,

V = potential in kv,

I = beam current in amperes,

$$F\left(\frac{r}{r_m}\right) = 2 \int_1^{\sqrt{\ln(r/r_m)}} e^{x^2} dx,$$

$\varphi(V) \approx 1$ (for our purpose).

The function $F(r/r_m)$ is taken from Jahnke-Emde⁴ and plotted in Fig. 7. We represent it within the range $r/r_m = 1 \dots 2$ by the simpler expression

$$F^*\left(\frac{r}{r_m}\right) = 2.09 \left(\frac{r}{r_m} - 1\right)^{1/2} \quad (47)$$

which is also plotted in Fig. 7.

With F^* inserted, Eq. (46) becomes

$$\frac{S}{r_m} = 2.14 \frac{V^{3/4}}{I^{1/2}} \left(\frac{r}{r_m} - 1\right)^{1/2} \quad (48)$$

or

$$\frac{r}{r_m} = 0.220 \frac{S^2}{r_m^2} \frac{1}{V^{3/2}} + 1 \quad (49)$$

The divergence of the beam is

$$\frac{dr}{ds} = 0.44 \frac{S}{r_m} \cdot \frac{1}{V^{3/2}} \quad (50)$$

Since for our conditions of field emission the spreading of the beam depends only upon the current density, we may choose a very small initial diameter and a correspondingly small current; then we have a very small angle of divergence. For a sufficiently small beam divergence, we can compute the radius at the end of the i^{th} slice by adding separate components:

- (a) The radius at the entrance of the slice,
- (b) The increment due to space-charge repulsion within the slice, computed for an initially parallel beam,
- (c) The increment due to the initial beam divergence which equals the beam divergence at the end of the preceding slice times the electron optical refraction index $\sqrt{V_{i-1}/V_i}$.

Thus we obtain

$$r_i = 0.22 \frac{(\Delta S_i)^2}{r_{i-1}} \frac{1}{V_i^{3/2}} + r_{i-1} + \left(\frac{dr}{ds}\right)_{i-1} \Delta S_i \sqrt{\frac{V_{i-1}}{V_i}} \quad (51)$$

$$\left(\frac{dr}{ds}\right)_i = 0.44 \frac{\Delta S_i}{r_{i-1}} \frac{1}{V_i^{3/2}} + \left(\frac{dr}{ds}\right)_{i-1} \sqrt{\frac{V_{i-1}}{V_i}} \quad (52)$$

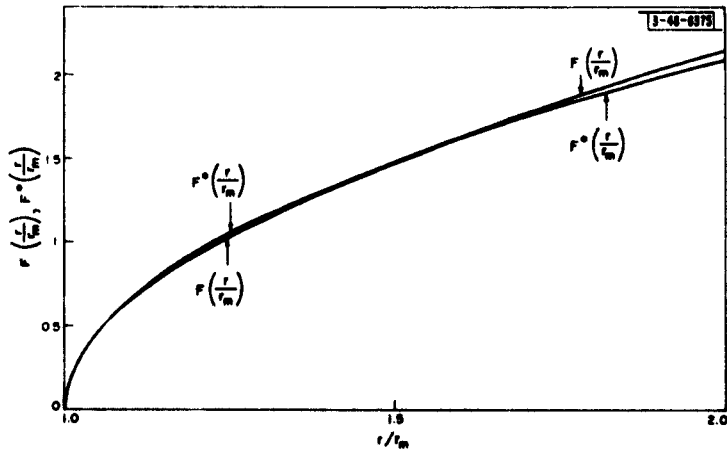


Fig. 7. Comparison of $F\left(\frac{r}{r_m}\right) = 2 \int_1^{\sqrt{\ln(r/r_m)}} e^{x^2} dx$ and $F^*\left(\frac{r}{r_m}\right) = 2.09 \left(\frac{r}{r_m} - 1\right)^{1/2}$.

Assuming initially $(dr/ds)_0 = 0$, we can compute r step by step. We have only to watch that $r_i < 2r_{i-1}$ because otherwise our representation of $F(r/r_m)$ fails. It is best to start with a very small ΔS , e.g., 10^{-5} mm and to double the stepwidth every cycle. In such a way for $E = 50$ kv/mm, $z = 1$ mm, $V_0 = 5$ ev, $r_0 = 1.785 \cdot 10^{-6}$ mm, $I = 10^{-6}$ amp we obtain

$$\frac{r}{r_0} = 845$$

This happens to be just the value that we have found with the graphical method in Example 2 (Sec. IV). (In general we have to allow for a reading error).

The coding of Eqs. (51) and (52) for the IBM 7090 is reproduced in the appendix.

VI. CONCLUSION

The two examples shown, and many more which have been computed, indicate that both methods give the same results within the limits of the reading error.

The first method, which uses the resolving function, is appropriate if one wants to quickly compute a few examples. For the exploration of a variety of conditions, the first method also offers a fairly simple way to obtain one result at a time, which is useful for the choice of the next input. Of course, the first method is always recommended if no electronic computer is at hand.

The second method automatically gives the complete beam shape up to the length under consideration, but it always involves the use of a high speed computer. It is therefore most suitable for longer series of examples as they occur by systematic variation of parameters in small steps.

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2. E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen*, Vol. 1 (Akademische Verlagsgesellschaft Geest and Portig, Leipzig, 1959), p. 568.
3. H.F. Ivey, "Space Charge Limited Currents," *Advances in Electronics and Electron Physics* **6**, 137 (1954).
4. E. Jahnke and F. Emde, *Funktionentafeln Mit Formeln und Kurven*, (Leipzig 1933), Ch. III.

APPENDIX
FORTRAN PROGRAM FOR SOLUTION OF EQS. (51) AND (52).

ELECTRON BEAM

```

C      DIMENSION DELSO(2000), SO(2000), V(2000), ROP(2000), RP(200),
CAPRP(2000), CAPR(2000), RHO(2000), RHOP(2000), FIRHO(2000),
FLOG(20), FLSS(2000)

16     READ INPUT TAPE 2, 4, SO(1), DELSO(1), RHO(1), FI, U, RHOP(1),
SEND WRITE OUTPUT TAPE 3, 3
FM=0.434294
PI=3.14159265
I=1

13     I=I+1
DELSO(I)=2.0* DELSO(I-1)

12     SO(I)=SO(I-1)+DELSO(I)
V(I)=(SO(I)+SO(I-1))* U/2.0
ROP(I)=FI/SQRTF(V(I)** 3)* (DELSO(I)/RHO(I-1))* 2* 0.220+1.0
IF(ROP(I)-2.0) 10, 11, 11

11     DELSO(I)=DELSO(I)/2.0
GO TO 12

10     RP(I)=0.44* DELSO(I)* FI/(RHO(I-1)* SQRTF(V(I)** 3))
CAPRP(I)=RHOP(I-1)* SQRTF(V(I-1)/V(I))
CAPR(I)=CAPRP(I)* DELSO(I)
RHO(I)=ROP(I)* RHO(I-1)+CAPR(I)
RHOP(I)=RP(I)+CAPRP(I)
FIRHO(I)=FI/(PI* RHO(I)** 2)
FLOG(I)=LOGF(FIRHO(I))* FM* 0.2
FLSS(I)=LOGF(SO(I)-SO(1))* FM* 0.2
WRITE OUTPUT TAPE 3, 2, SO(I), RHO(I), FIRHO(I), FLOG(I), FLSS(I)
WRITE OUTPUT TAPE 14, 4, FLSS(I), FLOG(I)
IF(I-2000) 14, 15, 15

14     IF(SO(I)-SEND) 13, 15, 15

15     GO TO 16

```

FORMAT STATEMENTS

```

1      FORMAT (5E10.4, 2F6.2)

2      FORMAT (3E20.6, 2F20.6)

3      FORMAT (10H S16X, 4H RHO14X, 6H I/RHO8X, 12H LOGF(I/RHO) 10X, 1
10H LOG(S-SO)/)

4      FORMAT (2F20.6)

END

```

EXPLANATION OF THE SYMBOLS USED IN THE FORTRAN CODE

Fortran	Equations (51) and (52)
SO(1)	S_0
SEND	S_∞
SO(I)	S_{i-1}
DELISO(I)	ΔS_{i-1}
V(I)	V_{i-1}
ROP(I)	$0.22 \frac{1}{V_{i-1}^{3/2}} \cdot \frac{(\Delta S_{i-1})^2}{r_{i-2}} + 1$
RP(I)	$0.44 \frac{\Delta S_{i-1}}{r_{i-2}} \frac{1}{V_{i-1}^{3/2}}$
CAPRP(I)	$(\frac{dr}{ds})_{i-2} \sqrt{\frac{V_{i-2}}{V_{i-1}}}$
RHO(I)	r_{i-1}
RHOP(I)	$(\frac{dr}{ds})_{i-1}$
FIRHO(I)	$\frac{1}{\pi r_{i-1}^2}$
FLAG(I)	$0.2 \log_{10} \frac{1}{\pi r_{i-1}^2}$
FLSS(I)	$0.2 \log_{10} (S_{i-1} - S_0)$

} for plotting purposes

The last three symbols are not explained in Eqs. (51) and (52). They are only used to plot the results in a convenient form.