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(6) THEORY OF INVERSE MOMENTS

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The characteristic function for inverse moments called the inverse characteristic function (inverse c.f.) is defined. Liafounov's inequality for inverse moments is obtained. Various properties, in particular, the uniqueness property (also called the inversion formula), limit and continuity properties, are studied. Illustrations are provided wherever necessary. Many results admit multivariate generalizations. Inverse order statistics are defined. Also, some applications of inverse moments are considered.

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THEORY OF INVERSE MOMENTS *

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0. Summary. The characteristic function for inverse moments called the inverse characteristic function (inverse c.f.) is defined. Liafounov's inequality for inverse moments is obtained. Various properties, in particular, the uniqueness property (also called the inversion formula), limit and continuity properties, are studied. Illustrations are provided wherever necessary. Many results admit multivariate generalizations. Inverse order statistics are defined. Also, some applications of inverse moments are considered.

1. Introduction. Inverse moments of the binomial, Poisson, negative binomial and the hypergeometric distributions truncated at zero have been studied in [3], [4], [5], [6], [8], [10], [11] and [12]. Also, inverse moments of the gamma and the beta distributions have been used as approximations to the inverse moments of the positive discrete random variables. (See [4], [6] and [9]). Hence, it is of interest to study the inverse moments and the characteristic function for inverse moments of an arbitrary distribution.

2. Notation. Let X denote an arbitrary random variable (r.v.) and $F(x)$ denotes its cumulative distribution function (c.d.f.). Let α_k and β_k respectively denote the k^{th} inverse moment and k^{th} absolute moment of X ($k = 0, 1, \dots$). $c(t) = E(e^{it/X})$ denotes the inverse characteristic

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function (inverse c.f.) of X . α_1 can be interpreted as the reciprocal of the mean and α_k can be interpreted as the expected attraction on a particle, the law according to which this particle is attracted by another particle distant d is proportional to d^{-k} ($k = 1, 2, \dots$). Also, let Y be defined as the reciprocal of X , having $G(y)$ for its c.d.f. The characteristic function for regular moments will be called the regular c.f. Throughout i denotes square root of -1 . Let $[.]$ denote the largest integer contained in $(.)$.

3. Assumptions. We follow the convention that all c.d.f.'s are continuous from the right. Throughout, it is assumed that

$$(3.1) \quad F(0^+) - F(0^-) = 0.$$

Notice that the above condition is satisfied if $F(x)$ is continuous at $x = 0$.

4. Results. In this section, we state and provide proofs for the properties of the inverse moments and the characteristic function for the inverse moments.

Result 4.1. Let X and Y be any random variables with $F(x)$ and $G(y)$ respectively as c.d.f.'s. If

$$Y = X^{-1} \quad \text{and} \quad F(0^+) - F(0^-) = 0, \quad \text{then}$$

$$(4.1) \quad G(y) = 1 - F(1/y) + P(X = 1/y)$$

and the inverse characteristic function $c(t)$ of X will be the regular characteristic function of Y .

Proof:
$$G(y) = P(Y \leq y) = P(X \geq 1/y) \\ = 1 - F(1/y) + P(X = 1/y).$$

If $1/y$ is a continuity point of X then $P(X = 1/y) = 0$.

Consider

$$\begin{aligned}
 c(t) &= \int_{-\infty}^{\infty} e^{it/x} dF(x) = \int_{-\infty}^{0^-} + \int_{0^-}^{0^+} + \int_{0^+}^{\infty} e^{it/x} dF(x) \\
 &= \int_{0^-}^{-\infty} + \int_{\infty}^{0^+} e^{ity} dF(1/y) \\
 &= \int_{-\infty}^0 + \int_0^{\infty} e^{ity} dG(y) \\
 &= \int_{-\infty}^{\infty} e^{ity} dG(y) .
 \end{aligned}$$

This completes the proof of the assertion.

Example 4.1.1. Let X be defined as follows:

$$P(X = k) = 2^{-k}, \quad k = 1, 2, \dots .$$

$$\text{If } Y = 1/X, \text{ then } G(y) = \sum_{k=[1/y]+1}^{\infty} 2^{-k} = 2^{-[1/y]} .$$

Remark 4.1.1. Result 4.1 plays an important role in providing simple proofs for most of the foregoing results, which are extensions of results that hold for a regular characteristic function.

Definition 4.2.1. A sequence of c.d.f.'s $\{F_n(x)\}$ is said to be convergent, if there is a non-decreasing function $F(x)$ such that $F_n(x) \rightarrow F(x)$ at every continuity point of $F(x)$. [See Cramer [1], p.60].

Result 4.2. A sequence of distributions with c.d.f.'s $F_1(x), F_2(x), \dots$ converges to a distribution if and only if there is a c.d.f. $F(x)$ such that $F_n \rightarrow F$ at every continuity point of $F(x)$ - when such a function $F(x)$ exists, $F(x)$ is called the c.d.f. corresponding to the limiting distribution of the sequence, and we can briefly say that the sequence $\{F_n(x)\}$ or $\{X_n\}$ converges in distribution to $F(x)$.

Proof: See Cramer [1], pp.59-60.

Result 4.3. If a sequence of random variables $\{Y_n\}$ is such that its sequence of c.d.f.'s $\{G_n(y)\}$ converges to a c.d.f. $G(y)$ in the sense of Definition 4.2.1, then the sequence of c.d.f.'s $\{F_n(x)\}$ of the random variables $X_n = 1/Y_n$ $n = 1, 2, \dots$ also converges to the c.d.f. $F(x)$ where $F(x) = 1 - G(1/x)$ at every continuity point of $G(y)$, and conversely.

Proof: Assume that $G_n(y) \rightarrow G(y)$ at every continuity point of G . We will show that $F_n(x) \rightarrow F(y)$ at every continuity point of F . Since $G_n(y) \rightarrow G(y)$, we can find n large such that

$$|G_n(y) - G(y)| \leq \epsilon, \quad \text{for every given } \epsilon > 0.$$

Now, consider

$$F_n(1/y) - F(1/y) = G(y) - G_n(y),$$

after using Result 4.1. Consequently

$$|F_n(1/y) - F(1/y)| = |G_n(y) - G(y)| \leq \epsilon.$$

Hence the assertion. The above inequality can be used to prove the converse also.

Result 4.4. If $|X| \geq \epsilon > 0$, then $\beta_k \leq 1/\epsilon^k$ ($k > 0$).

where β_k is the k^{th} absolute inverse moment of X . The proof is trivial.

Result 4.5. For any real a, b, c such that $a \geq b \geq c \geq 0$, we have

$$(4.2) \quad \beta_b^{a-c} \leq \beta_c^{a-b} \beta_a^{b-c}.$$

Proof: β_k will be the regular absolute moment of Y which is the reciprocal of X . Appealing to the Liafounov's inequality for the regular absolute moments (See [12], pp.264-67) the above result readily follows.

Corollary 4.5.1. Putting $b = (c + a)/2$ in (4.2) we get

$$\beta_{(c+a)/2}^2 \leq \beta_a \beta_c .$$

Corollary 4.5.2. Put $c = 2k$ and $a = 2m$ (k and m are two positive integers such that $k \leq m$) in Corollary 4.1 and obtain

$$\beta_{k+m}^2 \leq \beta_{2k} \beta_{2m}, \text{ or } \alpha_{k+m}^2 \leq \alpha_{2k} \alpha_{2m} ,$$

where α_k denotes the k^{th} inverse moments of X . The inequality for the α 's follows because

$$|\alpha_{k+m}| \leq \beta_{k+m}, \quad \alpha_{2j} = \beta_{2j}, \quad \text{all integral } j .$$

Corollary 4.5.3. With $c = 0$ in (4.2) we obtain

$$\beta_b^a \leq \beta_a^b \text{ or } (\beta_b)^{1/b} \leq (\beta_a)^{1/a} ,$$

which implies that $(\ln \beta_k)/k$ is an increasing function of k .

Corollary 4.5.4. We have

$$\beta_1 \leq \beta_2^{1/2} \leq \beta_3^{1/3} \leq \dots \leq \beta_k^{1/k} \dots .$$

Proof: This result follows from a repeated application of Corollary 4.5.3. However, one can establish this result directly by considering the quadratic form

$$\begin{aligned} & \int_{-\infty}^{\infty} [u |x|^{(1-k)/2} + v |x|^{-(1+k)/2}] dF(x) \\ &= \beta_{k-1} u^2 + 2\beta_k uv + \beta_{k+1} v^2 \end{aligned}$$

which is obviously non-negative. Hence, the determinant of the quadratic form is non-negative. That is

$$\beta_k^{2k} \leq \beta_{k-1}^k \beta_{k+1}^k .$$

Writing the preceding inequality successively for $k = 1, 2, \dots, m$ and multiplying them, we obtain

$$\beta_m^{m+1} \leq \beta_{m+1}^m \quad \text{or} \quad \beta_m^{1/m} \leq \beta_{m+1}^{1/(m+1)},$$

from which Corollary 4.4 follows. The above method of proof is due to Cramer (See [1], p. 176).

Result 4.6. An inverse c.f. is uniformly continuous on the entire real line and satisfies the conditions

$$c(0) = 1, \quad |c(t)| \leq 1 \quad (-\infty < t < \infty).$$

Proof: Follows from the fact that $c(t)$ is the regular c.f. of Y and hence is uniformly continuous on the entire real line and satisfies the above conditions.

Result 4.7. If β_k , the k^{th} absolute inverse moment of X exists, then all inverse absolute moments β_m for $m < k$ exist.

Proof: This follows from Corollary 4.5.4.

Result 4.8. If X has an inverse moment of order k , then its characteristic function $c(t)$ has continuous derivatives of order up to and including k . Also,

$$\alpha_m = i^{-m} \left[\frac{d^m}{dt^m} c(t) \right]_{t=0} \quad (m = 1, 2, \dots, k).$$

Consequently, we have

$$c(t) = \sum_{m=0}^k \alpha_m (it)^m / m! + o(t^k),$$

where $o(t^k)$ denotes the error term which when divided by t^k tends to zero as $t \rightarrow 0$.

Proof: By definition we have

$$c(t) = \int e^{it/x} dF(x)$$

$$\frac{d^k}{dt^k} c(t) = i^k \int x^{-k} e^{it/x} dF(x)$$

since by hypothesis

$$\int |x|^{-k} dF(x) < \infty ,$$

the differentiation underneath the integral sign is permissible. Putting $t = 0$ we get

$$i^{-k} c^{(k)}(0) = \alpha_k .$$

Rest follows by expanding $c(t)$ as Taylor series.

Example 4.8.1. Let $f(x) = e^{-x} x^{p-1}/\Gamma(p)$, $x, p > 0$.

Then

$$\alpha_k = (p-1) (p-2) \dots (p-k), \quad k < p.$$

Hence

$$c(t) = 1 + \sum_{k=1}^{[p]} \frac{(it)^k}{k!} (p-1) (p-2) \dots (p-k) .$$

Example 4.8.2. Consider

$$\begin{aligned} f(x) &= x^{-2} e^{-1/x} , & x > 0, \\ &= 0 & \text{otherwise.} \end{aligned}$$

Then,

$$c(t) = 1 + \sum_{k=1}^{\infty} (it)^k = (1-it)^{-1}$$

Example 4.8.3. Consider

$$\begin{aligned} f(x) &= x^{-2}, & 1 \leq x < \infty , \\ &= 0 & \text{otherwise.} \end{aligned}$$

Then

$$c(t) = 1 + \sum_{k=1}^{\infty} (it)^k / (k+1)! = (it)^{-1} (e^{it} - 1) .$$

Example 4.8.4. Let

$$\begin{aligned} F(x) &= 0 , & x < a \\ &= 1 & x \geq a , & a \neq 0 . \end{aligned}$$

Then

$$c(t) = e^{it/a} = 1 + \sum_{k=1}^{\infty} (it)^k / a^k k! .$$

Result 4.9. If the inverse c.f. $c(t)$ has a finite derivative of even order $2k$ at $t=0$, then the $2k^{\text{th}}$ inverse moment α_{2k} exists and consequently all moments of order $m < 2k$.

Proof: This is a converse of Result 4.8. The method of proof will be similar to the one used by Cramer (See [1], p.89).

Consider

$$\begin{aligned} c^{(2k)}(0) &= \lim_{t \rightarrow 0} \int_{-\infty}^{\infty} \left(\frac{e^{it/x} - e^{-it/x}}{2t} \right)^{2k} dF(x) \\ &= (-1)^k \lim_{t \rightarrow 0} \int_{-\infty}^{\infty} \left(\frac{\sin t/x}{t} \right)^{2k} dF(x) . \end{aligned}$$

Also, for any finite interval (a, b) we have

$$\int_a^b x^{-2k} dF(x) = \lim_{t \rightarrow 0} \int_a^b \left(\frac{\sin t/x}{t} \right)^{2k} dF(x) \leq |c^{(2k)}(0)| .$$

Now, using Corollary 4.5.4, we can complete the proof. This result exhibits the relationship between the differentiability properties of $c(t)$ and the behavior of $F(x)$ for small values of x , because the behavior of $F(x)$ for small values of x determines the existence of the inverse moments α_k .

Result 4.10. A bounded and continuous function $c(t)$ is an inverse characteristic function of a distribution if and only if $c(0) = 1$ and that the function

$$\psi(x, A) = \int_0^A \int_0^A c(t-u) e^{i(t-u)/x} dt du$$

is real and non-negative for all real x and all $A > 0$.

Proof: Cramer has established this result for the regular characteristic function of an arbitrary random variable. (See [1], p.91 and [2]). Hence, the result can be applied to the regular c.f. of Y namely $c(t)$. This completes the proof.

Result 4.11. (Levy's inversion formula). If x_1 and x_2 are continuity points of $F(x)$ then,

$$F(x_2) - F(x_1) = (2\pi)^{-1} \lim_{T \rightarrow \infty} \int_{-T}^T \left[e^{-it/x_1} - e^{-it/x_2} \right] (it)^{-1} c(t) dt$$

Proof: The above result can be established directly as follows:

Assume $x_1 < x_2$ (without loss of generality). Write

$$\begin{aligned} I_c &= (2\pi)^{-1} \int_{-T}^T \left[e^{-it/x_1} - e^{-it/x_2} \right] (it)^{-1} c(t) dt \\ &= \pi^{-1} \int_{-\infty}^{\infty} \int_0^T [\sin t(z^{-1} - x_1^{-1}) - \sin t(z^{-1} - x_2^{-1})] t^{-1} dt dF(z), \end{aligned}$$

after changing the order of integration.

Now, choose δ so large that $x_1^{-1} - \delta^{-1} > x_2^{-1} + \delta^{-1}$ and write

$$I_c = \int_{-\infty}^{(x_1^{-1} + \delta^{-1})^{-1}} (x_1^{-1} - \delta^{-1})^{-1} + \int_{(x_1^{-1} + \delta^{-1})^{-1}}^{(x_1^{-1} - \delta^{-1})^{-1}} (x_2^{-1} + \delta^{-1})^{-1} (x_2^{-1} - \delta^{-1})^{-1} + \int_{(x_2^{-1} + \delta^{-1})^{-1}}^{(x_2^{-1} - \delta^{-1})^{-1}} (x_1^{-1} + \delta^{-1})^{-1} (x_1^{-1} - \delta^{-1})^{-1} + \int_{(x_2^{-1} - \delta^{-1})^{-1}}^{\infty} G(T, z; x_1, x_2) dF(z)$$

where

$$G(T, z; x_1, x_2) = \pi^{-1} \int_0^T [\sin(z^{-1} - x_1^{-1})t - \sin(z^{-1} - x_2^{-1})t] t^{-1} dt.$$

From here on the proof will be identical to the one given by Gnedenko and Kolmogorov (See [7], pp.49-50).

Alternate proof: Since $c(t)$ is the regular c.f. of Y , we can apply Levy's inversion formula and obtain

$$G(y_2) - G(y_1) = (2\pi)^{-1} \lim_{T \rightarrow \infty} \int_{-T}^T \left[e^{-ity_2} - e^{-ity_1} \right] (it)^{-1} c(t) dt$$

where G is the c.d.f. of Y and y_1 and y_2 are points of continuity of $G(y)$. Now replace y_1 by $1/x_1$, y_2 by $1/x_2$ and use (4.1) and obtain

$$F(x_1) - F(x_2) = (2\pi)^{-1} \lim_{T \rightarrow \infty} \int_{-T}^T \left[e^{-it/x_2} - e^{-it/x_1} \right] (it)^{-1} c(t) dt.$$

This completes the proof.

Result 4.12. The distribution function is uniquely determined by its inverse characteristic function.

Proof: Using Result 4.11 it readily follows that at every continuity point x of $F(x)$ we have

$$F(x) = (2\pi)^{-1} \lim_{u \rightarrow -\infty} \lim_{t \rightarrow \infty} \int_{-T}^T [e^{-it/x} - e^{-it/u}] (it)^{-1} c(t) dt,$$

where the limit in u is taken over the set of continuity points of $F(u)$. Interchanging the limits we obtain

$$F(x) = (2\pi)^{-1} \lim_{T \rightarrow \infty} \int_{-T}^T (e^{-it/x} - 1) (it)^{-1} c(t) dt$$

Corollary 4.12.1. If $F'(x) = f(x)$ exists at x , then

$$f(x) = (2\pi x^2)^{-1} \int_{-\infty}^{\infty} e^{-it/x} c(t) dt ,$$

where the principal value of the integral is considered.

Remark 4.12.1. Results 4.11 and 4.12 will hold if $F(x)$ is an arbitrary right continuous function of bounded variation, subject to the restriction $F(-\infty) = 0$ (See Gnedenko and Kolmogorov [7], p.50).

Example 4.12.1. Find the density function of the random variable which has $1/(1-it)$ for its inverse c.f.

$$f(x) = (2\pi x^2)^{-1} \int_{-\infty}^{\infty} e^{-it/x} (1-it)^{-1} dt .$$

The residue at $t = -i$ is $e^{-1/x}$, when $x > 0$. Consequently,

$$f(x) = x^{-2} e^{-1/x} , \quad x > 0 .$$

Result 4.13. The inverse characteristic function of a random variable X is real if and only if X is symmetrical about zero; i.e. if for every x

$$F(x) = 1 - F(-x + 0) .$$

Proof: Assume that $F(x)$ is symmetric about zero. Then

$$\begin{aligned} c(t) &= \int e^{it/x} dF(x) = \int_{-\infty}^{0^-} + \int_{0^-}^{0^+} + \int_{0^+}^{\infty} e^{it/x} dF(x) \\ &= \int_{-\infty}^{0^-} e^{it/x} d[1-F(-x+0)] + \int_{0^+}^{\infty} e^{it/x} dF(x) . \end{aligned}$$

since $F(0^+) - F(0^-) = 0$ by our assumption. Therefore

$$\begin{aligned} c(t) &= \int_{0^+}^{\infty} (e^{it/x} + e^{-it/x}) dF(x) \\ &= 2 \int_{0^+}^{\infty} \cos t/x dF(x) . \end{aligned}$$

Result 4.14. Assume that $\alpha_0 = 1, \alpha_1, \dots$, the inverse moments are finite and that the series $\sum_0^{\infty} \alpha_k r^k/k!$ is absolutely convergent for some $r > 0$.

Then $F(x)$ is the only c.d.f. having $\alpha_0, \alpha_1, \dots$, for its inverse moments.

Proof: Let $Y = X^{-1}$ and let $G(y)$ be the c.d.f. of Y . Then $\alpha_0 = 1, \alpha_1, \dots$ will be the regular moments of Y and by applying Cramer's theorem (See [1], pp. 176-177). We can assert that $G(y)$ is the only c.d.f. having $\alpha_0, \alpha_1, \dots$ for its regular moments. It now follows from Result 4.1 that $F(x) = 1 - G(1/x) + p(X=x)$ is the only c.d.f. having $\alpha_0, \alpha_1, \dots$ for its inverse moments. Also, one can proceed along the lines of Cramer and establish that a set of inverse moments uniquely determine the inverse c.f. and by the inversion formula (or uniqueness theorem) uniquely define the c.d.f.

Result 4.15. (Kendall & Rao's Theorem). Let $\alpha_{2,n}$ be the second inverse moments of X_n in the sequence $\{X_n\}$. If

$$\alpha_{2,n} < B < \infty, \quad n = 1, 2, \dots,$$

then, there is a subsequence of $\{X_n\}$ which converges in distribution.

Proof: Define $Y_n = X_n^{-1}$ and the corresponding sequence $\{Y_n\}$. $\alpha_{2,n}$ will be the regular second moment of Y_n . Since the regular second moment of the random variable Y_n in the sequence $\{Y_n\}$ is finite, we can apply the theorem due to Kendall and Rao (See Wilks [14], p. 127) and assert that there exists a subsequence $\{Y_{n_k}\}$ which converges in distribution. Now, from Result 4.2, we have the corresponding sequence $\{X_{n_k}\}$ where $X_{n_k} = Y_{n_k}^{-1}$ also converging in distribution. This completes the proof.

Result 4.16. Let $\{X_n\}$ be a sequence of random variables. Let $\alpha_{k,n}$, the k^{th} inverse moments of X_n be finite for all n and k . Let $\lim_{n \rightarrow \infty} \alpha_{k,n} = \alpha_k < \infty$, for all k . Then if $\{X_n\}$ converges in distribution to $F(x)$, $\alpha_0, \alpha_1, \dots$ is the inverse moment-sequence of $F(x)$. Conversely, if $\{\alpha_k\}$ uniquely determines a c.d.f. $F(x)$, it is the limiting c.d.f. of $\{X_n\}$.

Proof: Assume that $\{X_n\}$ converges in distribution to $F(x)$. We will show that $\{\alpha_k\}$ is the moment-sequence of $F(x)$. Define the sequence $\{Y_n\}$ such that $Y_n = X_n^{-1}$. Then $\{Y_n\}$ converges in distribution to $G(y) = 1 - F(1/y)$ at every continuity point y of G . Now, applying the result due to Kendall and Rao (See Wilks [14], pp. 128-129) we can assert that $\alpha_k (k = 1, 2, \dots)$ are the regular moments of $G(y)$. Consequently $\alpha_k (k = 1, 2, \dots)$ are the inverse moments of $F(x)$.

Proof for the converse: Proof will be similar to the one given in [14]. Assume that $\alpha_1, \alpha_2, \dots$ uniquely determine a c.d.f. $F(x)$. We will show that the sequence $\{F_n(x)\}$ converges to $F(x)$. From Result 4.15 we know that every convergent subsequence of $\{F_n(x)\}$ converges to some c.d.f.

and from the first part of Result 4.16, we know that the limiting c.d.f.'s for these subsequences must all have the same inverse moment sequence, namely $\{\alpha_k\}$. However, since the sequence $\{\alpha_k\}$ is assumed to determine a c.d.f. uniquely, the limiting c.d.f.'s for the subsequences of $\{F_n(x)\}$ are all identical to $F(x)$, having the inverse moments α_k ($k = 0, 1, 2, \dots$).

Limit and Continuity Property of the Inverse c.f. We have seen that a c.d.f. can be uniquely obtained by a certain transformation applied to the inverse c.f. and vice versa. In the following, we will show that this transformation is also continuous. Quite often, we are interested to know whether a sequence of c.d.f.'s converge to a c.d.f. Towards this it might be difficult to investigate the convergence of a sequence of c.d.f.'s, while the convergence of inverse c.f.'s may be easy to investigate. Hence, consider the following theorem which is a direct analogue of a result due to Cramer and Levy.

Result 4.17. Let $\{X_n\}$ be a sequence of random variables and $\{F_n(x)\}$ and $\{C_n(t)\}$ denote the corresponding sequences of c.d.f.'s and inverse c.f.'s. A necessary and sufficient condition for the sequence $\{X_n\}$ to converge in distribution to a random variable X is that for every t , the sequence $\{c_n(t)\}$ converges to a limit $c(t)$ which is continuous at $t = 0$. If the above conditions are satisfied, $c(t)$ is identical to the inverse c.f. of X .

Proof: Let us first show that the condition is necessary. Assume that the sequence $\{F_n(x)\}$ converges to $F(x)$, the c.d.f. of X , at every continuity point x of $F(x)$. We will show that $\lim_{n \rightarrow \infty} c_n(t) = c(t)$ for every t . Now,

$$c_n(t) = \int_{-\infty}^{\infty} \cos(t/x) dF_n(x) + i \int_{-\infty}^{\infty} \sin(t/x) dF_n(x)$$

The functions $\cos(t/x)$ and $\sin(t/x)$ are bounded on $(-\infty, 0^-) + (0^+, \infty)$ for every t , and $F(0^+) - F(0^-) = 0$. Now, since $\{F_n(x)\}$ converges

to the c.d.f. $F(x)$ at every continuity point x , we have by Helley-Bray theorem (See Loève [9], p. 182)

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \cos(t/x) dF_n(x) = \int_{-\infty}^{\infty} \cos(t/x) dF(x)$$

and

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \sin(t/x) dF_n(x) = \int_{-\infty}^{\infty} \sin(t/x) dF(x).$$

Consequently $\lim_{n \rightarrow \infty} c_n(t) = c(t)$.

Now, consider the sufficiency of the condition. Assume that $c_n(t) \rightarrow c(t)$ for every t and that $c(t)$ is continuous at $t = 0$. Consider the sequence of random variables $\{Y_n\}$, $Y_n = X_n^{-1}$ and the corresponding c.d.f. sequence $\{G_n(y)\}$, where $G_n(y) = 1 - F_n(y^{-1}) + P(X = y^{-1})$, $n = 1, 2, \dots$. Then $\{c_n(t)\}$ be the sequence of regular c.f.'s corresponding to the sequence $\{Y_n\}$. Now, it is known (See Cramer [1], pp. 60-62) that the sequence of c.d.f.'s $\{G_n(y)\}$ contains a subsequence $\{G_{n_k}(y)\}$ which converges to a non-decreasing and right continuous function $G(y)$. It can be shown (for instance, See Cramer [1], p. 97) that $G(y)$ satisfies the remaining criteria for a c.d.f. Also, $c(t)$, the limit of the sequence $\{c_n(t)\}$ is the regular c.f. of $G(y)$. If there is another subsequence of $\{G_n(y)\}$ which converges to a c.d.f. $G^*(y)$. Now, $G^*(y)$ also has $c(t)$ for its regular c.f. But by the uniqueness property (See Result 4.12) $G^*(y) = G(y)$. This implies that every convergent subsequence of $\{G_n(y)\}$ converges to the c.d.f. $G(y)$. That is $\{G_n(y)\}$ converges to the c.d.f. $G(y)$. Now applying Result 4.3, it follows that the corresponding sequence $\{F_n(x)\}$ converges to the c.d.f. $F(x) = 1 - G(1/x)$.

Corollary 4.17.1. A necessary and sufficient condition for a sequence $\{X_n\}$ of random variables having the sequence of inverse c.f.'s $\{c_n(t)\}$,

to converge to the constant $a \neq 0$ is that

$$\lim_{n \rightarrow \infty} c_n(t) = e^{it/a}$$

Result 4.18. A necessary and sufficient condition for the independence of two r.v.'s X and Z is that their joint inverse c.f. $c(t,u)$ can be written as

$$c(t,u) = c_1(t) c_2(u) ,$$

when c_1 and c_2 are the inverse c.f.'s of X and Z respectively.

Proof: The proof will be identical to the one given by Cramer [1] (See p.266).

5. Special Properties of the Inverse Characteristic Functions.

In this section we will present some special properties of the inverse c.f.'s.

Result 5.1. (i) If X is a r.v. having $c(t)$ for its inverse c.f., then the inverse c.f. of $Z = aX^{-1}$ where a is a real constant, is $c(t/a)$.

(ii) $X_1, X_2, \dots, X_n$ are independent r.v.'s having respectively $c_1(t), c_2(t), \dots, c_n(t)$ for their inverse c.f.'s, then the inverse c.f. of

$$Z = \sum_{k=1}^n a_k X_k^{-1} \text{ is given by}$$

$$c(t) = \prod_{k=1}^n c_k(t/a_k) .$$

Further, if the X 's are identical and $a_k = 1/n$ ($k = 1, 2, \dots, n$) then the inverse c.f. of Z is

$$c(t) = [c(t/n)]^n .$$

(iii) The squared modulus of an inverse c.f. is an inverse c.f.

Proof: (i) and (ii) are elementary. Towards the proof for (iii) consider X_1 and X_2 which are independent and identically distributed r.v.'s

having $c(t)$ as the common inverse c.f. Then, from (11) $c_{X_1-X_2}(t)$, the inverse c.f. of X_1-X_2 is

$$\begin{aligned} c_{X_1-X_2}(t) &= c_{X_1}(t) \cdot c_{-X_2}(t) \\ &= c(t) \overline{c(t)} = |c(t)|^2 . \end{aligned}$$

Result 5.2. Let X be a r.v. having $F(x)$ and $c(t)$ for its c.d.f. and inverse c.f. respectively. If

$$F(A) - F(-A) \leq \epsilon ,$$

then

$$|c(t_2) - c(t_1)| \leq A^{-1} |t_2 - t_1| + 2\epsilon .$$

Proof: Consider

$$|c(t_2) - c(t_1)| \leq \int_{|x| \leq A} + \int_{|x| > A} |e^{it_2/x} - e^{it_1/x}| dF(x) .$$

Since

$$|e^{iz_2} - e^{iz_1}| \leq |z_2 - z_1| , \quad \left| \frac{d}{dz} e^{iz} \right| = 1$$

and

$$|e^{iz_2} - e^{iz_1}| \leq 2 .$$

We have

$$\begin{aligned} |c(t_2) - c(t_1)| &\leq 2 \int_{|x| \leq A} dF(x) + |t_2 - t_1| A^{-1} \\ &\leq 2\epsilon + |t_2 - t_1| A^{-1} . \end{aligned}$$

Remark 5.2.1. This inequality can be used to establish directly the uniform continuity of an inverse c.f. (Result 4.7).

Result 5.3. With the notation as in Result 5.2, for $A, \tau > 0$, we have

$$F(A) - F(-A) \geq [1 - |(2\tau)^{-1} \int_{-\tau}^{\tau} c(t) dt|] [1 - A\tau^{-1}]$$

Proof: Consider

$$\begin{aligned} |(2\tau)^{-1} \int_{-\tau}^{\tau} c(t) dt| &= |(2\tau)^{-1} \int_{-\tau}^{\tau} \left\{ \int e^{it/x} dF(x) \right\} dt| \\ &= |(2\tau)^{-1} \int \left\{ \int_{-\tau}^{\tau} e^{it/x} dt \right\} dF(x)| \\ &= \left| \int (x/\tau) \sin(\tau/x) dF(x) \right| \\ &\leq \left| \int_{|x| \leq A} \right| + \left| \int_{|x| > A} (x/\tau) \sin(\tau/x) dF(x) \right|. \end{aligned}$$

Since $|\sin(\tau/x) / (\tau/x)| \leq 1$ and $|\sin(\tau/x)| \leq 1$ we obtain

$$\begin{aligned} |(2\tau)^{-1} \int_{-\tau}^{\tau} c(t) dt| &\leq A\tau^{-1} [F(A) - F(-A)] + 1 - F(A) + F(-A) \\ &= (A\tau^{-1} - 1) [F(A) - F(-A)] + 1, \end{aligned}$$

from which the desired inequality follows. Further, if $A = 2\tau$, the inequality becomes

$$F\left(\frac{\tau}{2}\right) - F\left(-\frac{\tau}{2}\right) \geq 2-2 \left| (2\tau)^{-1} \int_{-\tau}^{\tau} c(t) dt \right|.$$

Result 5.4. If $c(t)$ is an inverse c.f. then for every t , we have

$$1 - R_e c(2t) \leq 4[1 - R_e c(t)],$$

where $R_e(.)$ denotes the real part of $(.)$.

Proof: $R_e c(t) = \int \cos(t/x) dF(x).$

Hence:

$$\begin{aligned}
 1 - R_{\theta} c(2t) &= \int [1 - \cos(2t/x)] dF(x) \\
 &= 2 \int \sin^2(t/x) dF(x) \\
 &= 2 \int [1 - \cos(t/x)] [1 + \cos(t/x)] dF(x) \\
 &\leq 4 \int [1 - \cos(t/x)] dF(x) \\
 &= 4 [1 - R_{\theta} c(t)] .
 \end{aligned}$$

Note that if $c(t)$ is real, the preceding inequality becomes

$$1 - c(2t) \leq 4[1 - c(t)] .$$

Consequently, since, for an arbitrary $c(t)$, $|c(t)|^2$ is also an inverse c.f. (See Result 4.1, (iii)), we have

$$1 - |c(2t)|^2 \leq 4[1 - |c(t)|^2] .$$

Result 5.5. If $c(t)$ is an inverse c.f. and if for some sequence $t_1, t_2, \dots$ converging to 0,

$$|c(t)| = 1 ,$$

then there exists a real number $a \neq 0$ such that

$$c(t) = e^{ita} .$$

Proof: If $c(t)$ is the inverse c.f. corresponding to a r.v. X , then $c(t)$ will be the regular c.f. of a r.v. $Y = 1/X$. Since, for a sequence $\{t_k\}$, $t_k \rightarrow 0$, $|c(t_k)| = 1$, we can show (See Gnedenko and Kolmogorov [7], Theorem 2, p. 56) that there exists a real number $a \neq 0$ such that

$$c(t) = e^{iat} .$$

Notice that we can also give a direct proof along the lines of Gnedenko and Kolmogorov [7].

Remark 5.5.1. The condition of Result 5.5 will certainly be satisfied if $|c(t)| = 1$ for $0 \leq t \leq b$ ($b > 0$).

Definition 5.6.1. A discrete variable is said to be a rational variable if there exists a real number $h > 0$ such that every possible value of the r.v. is of the form h/k where k takes integral values (not necessarily all of them) zero excepted.

Result 5.6. In order that a random variable be a rational r.v. it is necessary and sufficient that for some non-zero value of the argument the modulus of the inverse c.f. of the r.v. be equal to unity.

Proof: The necessary part. Let

$$h_k = P \{X = h/k\},$$

then the inverse c.f. of X is given by

$$c(t) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} e^{ikt/h} p_k$$

$$|c(t)| = 1.$$

The proof for the sufficiency part will be identical to the one in Gnedenko and Kolmogorov [7] (See p. 59).

6. Inverse Cumulants or Inverse Semi-invariants. Let x_k denote the k^{th} inverse cumulant of the distribution where x_k are defined by

$$x_k = \text{the coefficient of } (it)^k/k! \text{ in } \ln c(t).$$

That is

$$1 + \sum_{k=1}^{\infty} \alpha_k (it)^k/k! = \text{Exp} \left[\sum_{k=1}^{\infty} x_k (it)^k/k! \right]$$

By comparing the coefficients of powers on (it) we obtain

$$x_1 = \alpha_1, \quad x_2 = \alpha_2 - \alpha_1^2, \quad x_3 = \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3 \text{ etc.}$$

and x_1 and x_2 denote the expected value and the variance of $1/X$ where X is the r.v. under consideration. It can easily be shown that if k^{th} inverse moment of X exists, then x_k exists and conversely.

7. Multivariate Generalizations. Many results of Section 4 can be generalized to multivariate situations, especially Result 4.1, Taylor expansion of the inverse c.f. (Result 4.8), Kendall & Rao's results (Result 4.15 and 4.16), inversion formula (Result 4.11), and continuity theorem (Result 4.17) and Result 4.18. The formulation and the proof of any multivariate result will be a straight forward generalization of the corresponding result for the univariate case. These are left to the reader.

8. Inverse Order Statistics. Let $X_{1,n} < X_{2,n} < \dots < X_{n,n}$ be the order statistics in a sample of size n drawn from the population with c.d.f. $F(x)$ and let $Y_{1,n} < Y_{2,n} < \dots < Y_{n,n}$ be the order statistics in a sample of size n drawn from the population with c.d.f. $G(y)$, where the r.v. corresponding to $G(y)$ is the reciprocal of the r.v. corresponding to $F(x)$. We, further assume that either $F(x)$ or $G(y)$ is continuous. We have the following results.

Result 8.1. The distributions of $Y_{m,n}$ and $1/X_{n-m+1,n}$ are identical, $m = 1, 2, \dots, n$.

Proof. Result 8.1 follows from the probability density of $Y_{m,n}$ and Result 4.1.

Result 8.2. The distributions of $Y_{\ell,n} Y_{m,n}$ and $1/(X_{n-m+1,n} X_{n-\ell+1,n})$ are identical, $1 \leq \ell < m \leq n$.

Proof. Result follows from the joint probability density of $Y_{\ell,m}$ and $Y_{m,n}$ and Result 4.1.

Remark. Results 8.1 and 8.2 enable us to compute the moments of the inverse order statistics, $Y_{m,n}$ in terms of the inverse moments of $X_{m,n}$.

9. Application of the Inverse Moments.

9.1 Inverse chi-distribution. Consider a random sample $(x_1, x_2, \dots, x_n)$ drawn from a population having the following density function:

$$f(x; \theta) = (2/\pi\theta)^{\frac{1}{2}} x^{-2} e^{-1/(2\theta x^2)}, \quad x > 0 .$$
$$= 0 \quad \text{otherwise.}$$

It can be easily verified that the maximum likelihood estimate of θ is given by

$$\hat{\theta} = n^{-1} \sum_{k=1}^n x_k^{-2} .$$

The regular moments of $\hat{\theta}$ will involve the inverse moments of the population.

9.2 Inverse Weibull distribution. Consider the density

$$f(x; \theta) = m(\theta x^{m+1})^{-1} e^{-1/\theta x^m}, \quad x > 0, m > 0 ,$$
$$= 0 \quad \text{otherwise.}$$

Assume that m is known and we wish to estimate θ on the basis of a random sample $(x_1, x_2, \dots, x_n)$ drawn from the above population. The method of maximum likelihood provides an estimate given by

$$\hat{\theta} = n^{-1} \sum_{k=1}^n x_k^{-m} .$$

Again, the regular moments of $\hat{\theta}$ involve the inverse moments of the population. If $m = 1$, the density function will be called the inverse exponential density.

Conclusion. There might be some real situations in which the distributions mentioned above give a better fit than the classical distributions. However, the author has not explored such a situation.

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