Generation of Secondary Motions in the Field of a Vortex

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The investigation is concerned with motions in the region of the core of a vortex which exhibit peripheral vorticity. From theoretical reasoning it appears that development of such motions is favored as a zonal maximum of axially vorticity components is produced, e.g., by divergence of the vortex core. Several experiments devised to verify theoretical conclusions furnish evidence that concentrations of vorticity periodic about the perimeter actually do occur as expected.

IN his paper of 1880, "Vibrations of a Columnar Vortex",[1] Lord Kelvin analyzed axially symmetric as well as peripherally periodic oscillations of rectilinear vortices. By methods of small perturbations he treated the well-known wave patterns which occur on the surface of the hollow core of a vortex in a liquid and also the corresponding inertial oscillations in a Rankine vortex. The question which arises and which will be considered here pertains to the particular conditions under which peripherally periodic deformations occur in the region of the core of a vortex. It is proposed to show that zonal concentration of axial vorticity components created by deformation of, and derived from, the core of the parent vortex gives rise to motions exhibiting peripheral periodicity.

Use was made both in guiding the course of experiments and in interpreting experimental results, of theoretical investigations of axially symmetric deformations of the vortex core, namely for steady flow of those by Burgers[2] and Fraenkel[3] and, for the case of oscillations, of those by Görtler,[4] Oser,[5] Kiesling,[6] and by Morrison and Morgan,[7,8] and of the pioneer work on rotating fluids by G. I. Taylor.[9]

Certain aspects of this theory pertinent to the present inquiry will be presented first.

EFFECT OF AXIAL CONTRACTION OF A VORTEX CORE

Burgers[2] and Fraenkel[3] have considered the stretching of a vortex core of radius R in an incompressible nonviscous fluid with initially constant rotation $\omega = \frac{1}{2}\gamma_z$ and constant axial velocity V_{z1}. Starting with Lamb's equation grad $h = \mathbf{V} \times \mathbf{\gamma}$ they derived the following Bessels equation for the axial velocity distribution V_{z2} created by convergence or divergence:

$$\frac{d^2 V_{z2}}{dr^2} + \frac{1}{r}\frac{dV_{z2}}{dr} + \frac{4\omega^2}{V_{z1}^2}(V_{z2} - V_{z1}) = 0, \quad (1)$$

with the solution

$$V_{z2} = V_{z1}\{1 + aJ_0(2\omega R_1/V_{z1})\}, \quad (2)$$

where

$$a = \frac{1-n^2}{n^2}\frac{n\alpha}{2J_1(n\alpha)},$$

$$n = R_2/R_1,$$

$$\alpha = 2\omega R_1/V_{z1}.$$

Because of the proportionality between circulation Γ and stream function $\psi = (V_{z1}/2\omega)\Gamma$ derived from initial conditions, the axial component of vorticity likewise is proportional to the axial velocity component,

$$\gamma_z = (2\omega/V_{z1})V_z. \quad (3)$$

When computing the flow patterns for a given parameter α, it is found that the axial velocity on centerline is reversed when stretching the core beyond a certain value $n_p < 1$, and likewise if, in a diverging flow in the core, n exceeds a certain value $n_q > 1$, see Sec. 3 of Fig. 1, and it may be shown readily that thereupon the flow is radially unstable.

[1] Lord Kelvin, "Vibrations of a Columnar Vortex," *Mathematical and Physical Papers*, (Cambridge University Press, Cambridge, England, 1910), pp. 152–165.
[2] J. M. Burgers, "The Effect of Stretching of a Vortex Core," University of Maryland Inst. for Fluid Dynamics and Applied Mechanics Technical Report BN-80 (1956).
[3] L. E. Fraenkel, Proc. Roy. Soc. (London) **A233**, 506 (1956).
[4] H. Görtler, Z. Angew. Math. U. Mech. **24**, 210 (1944).
[5] H. J. Oser, Z. Angew. Math. U. Mech. **38**, 386 (1958).
[6] I. Kiesling, "Erzwungene Schwingungen in einer aus Rotation and Potentialwirbel Zusammengeretzten Grundstroemung," Deutsche Versuchsanstalt für Luftfahrt Report 154, 1961.
[7] J. A. Morrison and G. W. Morgan, "Small Forced Oscillations in an Ideal Rotating Liquid," Division of Applied Mechanics, Brown University Rept. A11-121/76 (1955).
[8] J. A. Morrison, "Small Forced Oscillations at Low Frequencies in an Ideal Rotating Liquid," Division of Applied Mechanics Brown University Rept. 265/07/6 (1956).
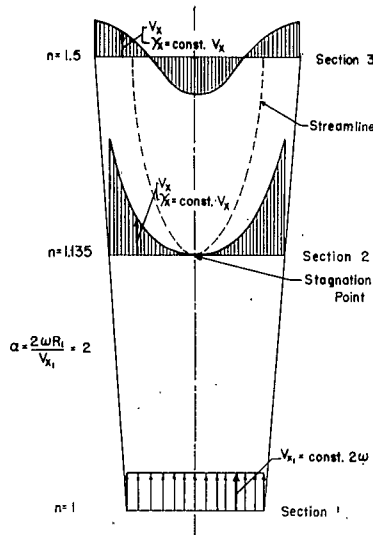[9] G. I. Taylor, Proc. Roy. Soc. (London) **A102**, 180 (1923).

FIG. 1. Profiles of axial velocity and vorticity components in a diverging stream tube of initially constant axial velocity and rotation.

These conditions, however, are not of interest here but instead we consider flows within the limiting case $1 < n \leq n_q$, when by divergence, i.e. axial contraction of the core, the velocity at the center is reduced to zero with increasing n.

For a value $\alpha = 2$ this limiting case occurs at $n_q = 1.135$, see Sec. 2 of Fig. 1. For $1 < n < 1.135$ and for $\gamma_z > 0$ we have $\partial \gamma_z / \partial r > 0$ for $0 < r < R$. The vorticity γ_z reaches a maximum at the edge of the core. We shall return to this result after treatment of the case of oscillations which will be considered next.

DEFORMATION OF THE CORE OF A VORTEX BY OSCILLATIONS

Let us consider the incompressible nonviscous axially symmetric flow produced by oscillations in a rectilinear Rankine vortex extending at right angles between two plane parallel bounding surfaces, see Fig. 2.

The conditions of zero amplitude are

$$V_\phi = r\omega = (\Gamma/2\pi R)r/R \quad \text{for} \quad 0 \leq r \leq R, \quad (4a)$$

$$V_\phi = \Gamma/2\pi r \quad \text{for} \quad r \geq R, \quad (4b)$$

$$V_x = V_r = 0; \quad (4c)$$

and the oscillations periodic in x and t are given by

$$V_r = \bar{V}_r(r) \exp i(\alpha x - \beta t), \quad (5a)$$

$$V_x = \bar{V}_x(r) \exp i(\alpha x - \beta t), \quad (5b)$$

$$p = \bar{p}(r) \exp i(\alpha x - \beta t). \quad (5c)$$

These motions are governed by the condition of continuity and by Euler's equations, which are, in linearized form,

$$\frac{DV_r}{Dt} - 2\omega V_\phi = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \quad (6a)$$

$$\frac{DV_\phi}{Dt} + 2\omega V_r = 0, \quad (6b)$$

$$\frac{DV_x}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \quad (6c)$$

By substitution of (5a) to (5c) into (6a) to (6c) and elimination of p are obtained, within the restrictions of small perturbation theory, the two Bessels equations, one for the amplitudes of the radial, the other for the axial velocity components:

$$\frac{d^2V_r}{dr^2} + \frac{1}{r}\frac{dV_r}{dr} + \left[\frac{4\omega^2\alpha^2}{\beta^2} - \alpha^2 - \frac{1}{r^2}\right]V_r = 0, \quad (7)$$

$$\frac{d^2V_x}{dr^2} + \frac{1}{r}\frac{dV_x}{dr} + \left[\frac{4\omega^2\alpha^2}{\beta^2} - \alpha^2\right]V_x = 0. \quad (8)$$

Their solutions give the perturbation amplitudes

$$V_r = a_r J_1(Ar) \quad (9)$$
$$\text{for } 0 \leq r \leq R$$
$$V_x = a_x J_0(Ar) \quad (10)$$

for (7) and (8), respectively, with

$$A = (\alpha/\beta)(4\omega^2 - \beta^2)^{\frac{1}{2}}. \quad (11)$$

The corresponding solutions for $r \geq R$ given in reference 1 are not of interest here. Particular solutions, obtained by substitution from (2) and (10) into (5a) and (5b), respectively, subject to the given boundary conditions and the integral continuity condition

$$\int_{x=0}^{\lambda/4} V_r 2\pi R \, dx = \int_{r=0}^{R} V_x 2\pi r \, dr \quad (12)$$
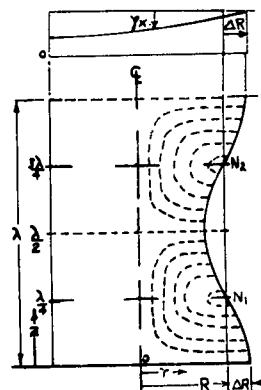

FIG. 2. Meridional streamlines of oscillations in the core of a vortex.

with choice of N_1 and N_2 as nodal points (see Fig. 2), are, admitting the first harmonic only,

$$V_r = -(\alpha/A)J_1(Ar) \cos{(\alpha x)} \cos{(\beta t)}, \qquad (13)$$

$$V_x = aJ_0(Ar) \sin{(\alpha x)} \cos{(\beta t)}, \qquad (14)$$

where $a = a_x = a_r$ by Eq. (12).

The circulatory component of perturbation velocities obtained by integration of (6b) upon substitution of (13) is

$$V_\phi = -2\omega \int_0^t V_r \, dt$$

$$= \frac{2\omega}{\beta} \frac{\alpha}{A} aJ_1(Ar) \cos{(\alpha x)} \sin{(\beta t)}. \qquad (15)$$

The axial vorticity component which for the case of rotationally symmetric flow is given by $\gamma_x = \partial V_\phi/\partial r + V_\phi/r$, may now be derived from (15)

$$\gamma_x = (2\omega/\beta)\alpha a J_0(Ar) \cos{(\alpha x)} \sin{(\beta t)}. \qquad (16)$$

Finally the oscillatory variations of the radius of stream surfaces which in the undisturbed state are cylinders can be determined by integration of Eq. (13) with respect to time for any radius r

$$\Delta r(r_1 x_1 t) = \int_0^t V_r \, dt$$

$$= -(\alpha/A\beta)aJ_1(Ar) \cos{(\alpha x)} \sin{(\beta t)}, \qquad (17)$$

and, in particular, for the core radius R

$$\Delta R(x_1 t) = -(\alpha/A\beta)aJ_1(AR) \cos{(\alpha x)} \sin{(\beta t)}, \qquad (18)$$

where $J_1(AR)$ is taken to be the first maximum of the $J_1(Ar)$ function.

By inspection of Eqs. (17) and (18) it is seen that since, within the range of (Ar) here applicable, both J_0 and J_1 are positive, ΔR and γ_x must be of opposite sign. From this it follows that there occurs a decrease of axial vorticity component within the core except at $r = R$ as the core radius is increased as shown in Fig. 2. According to this result obtained from a linearized treatment of the problem of oscillations in the core of a vortex, a maximum of axial vorticity occurs at the edge of the core as is also the case according to the corresponding result obtained previously for steady divergence of the core.

We shall inquire whether this condition favors the development of peripherally periodic motions. The nature of such motions will be indicated next.

TWO-DIMENSIONAL PERIODIC CONCENTRATIONS OF VORTICITY AS A RESULT OF SMALL RADIAL PERTURBATIONS OF AN INITIALLY CONCENTRIC CIRCULAR VORTICITY DISTRIBUTION

Before dealing with a vorticity distribution, let us consider the kinematics of a fluid line in the field
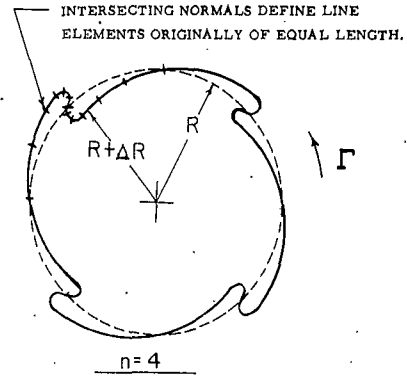

FIG. 3. Deformation of an initially circular streamline subjected to a small perturbation $R = R \cos{(\quad)}$.

of a plane potential vortex of circulation Γ which at the instant t_0 has the shape

$$R = R_0 + \Delta R, \qquad (19)$$

$$\Delta R = \langle \Delta R \rangle \cos{n\phi_0}; \, n = 1, 2, 3 \cdots ; \, \Delta R/R_0 \ll 1 \qquad (19a)$$

All particles on this line move in concentric circles but at different velocities governed by the relation of constancy of moment of momentum

$$V_\phi r = \text{const}. \qquad (20)$$

Their velocities relative to points on the circle $R = R_0$ are

$$\Delta V_\phi = V_\phi(R_0 + \Delta R) - V_\phi(R_0)$$

$$\approx -\frac{\Gamma}{2\pi} \frac{\langle \Delta R \rangle}{R_0^2} \cos{n\phi_0}. \qquad (21)$$

This produces corresponding peripheral displacements with respect to their original position ϕ_0, which are at $t \geq t_0$,

$$R_0(\phi - \phi_0) = \Delta V_\phi(t - t_0)$$

$$= -\frac{\Gamma}{2\pi} \frac{\langle \Delta R \rangle}{R_0^2} \cos{(n\phi_0)} (t - t_0), \qquad (22)$$

where ϕ marks the peripheral position at time t.

Of particular interest is the pattern of the fluid line at the instant $(t_0 + \Delta t)$ at which the particles of maximum radial displacement $\pm\langle \Delta R \rangle$ originally located at $n\phi_0 = 0, \pi, 2\pi$ have traversed the arc

$$\frac{\pi}{2n} = \frac{\Gamma}{2\pi} \frac{\langle \Delta R \rangle}{R_0^3} \Delta t \qquad (23)$$

relative to their original position. After the interval

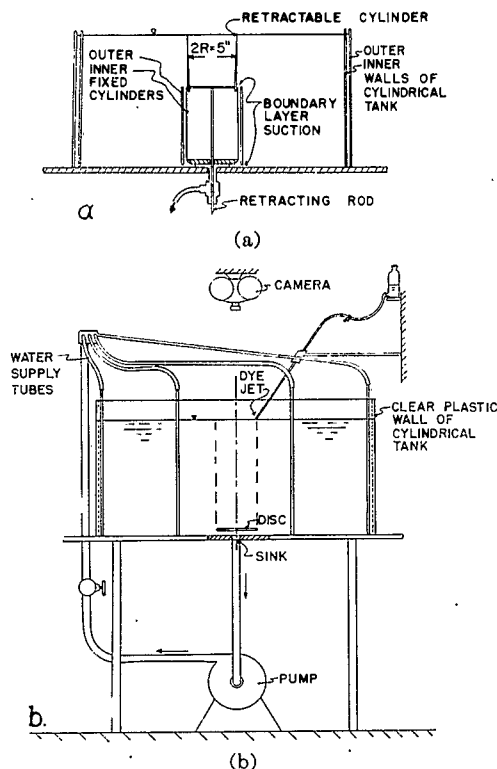$$\Delta t = (\pi^2/R_0^3/n\Gamma)\langle \Delta R \rangle, \qquad (24)$$

FIG. 4. Arrangement for Experiment No. 1.

the relative peripheral displacement of any other particle on the fluid line by Eq. (22) is

$$R_0(\phi - \phi_0) = -(\pi/2n)R_0 \cos (n\phi_0). \qquad (25)$$

From this equation and from the relation for the radial displacement may be constructed the shape of the fluid line at $t = t_0 + \Delta t$ as shown in Fig. 3.

Let us now impose a uniform vorticity distribution $\gamma_0 = \Delta\Gamma/2\pi R_0$ upon a fluid line with the same initial shape according to Eqs. (19) and (19a), and trace the subsequent deformation of this line. Since the remainder of the vortex field apart from the origin remains irrotational the circulation inside that line is Γ, outside it is $\Gamma + \Delta\Gamma$. It is seen that within the approximations of small perturbation theory Eqs. (20) to (25) remain applicable except in a small region about the S portion of the deformed fluid line, Fig. 3, provided $\Delta\Gamma/\Gamma \ll 1$, since outside this region the velocities induced by the vorticity distribution are negligible. It is seen that half of the initial fluid line forms the S portion. Consequently, one half of the original uniformly distributed vorticity is concentrated in equal parts in equidistant regions about the perimeter.

Let us now investigate experimentally the flow patterns which arise under conditions analogous to those for which a peripheral periodicity of concentration of vorticity has been predicted by linearized theory.

EXPERIMENTAL INVESTIGATION

The preceding analysis which has lead to the prediction of peripheral periodic concentrations of axial vorticity components naturally leaves open the question relative to the pattern of flow which will arise under the specified condition. These patterns were investigated by techniques of visualization of flow. A summary of the findings of the visual studies and of closely related measurements will be presented here while the comprehensive account of details of this work and further results not directly pertinent to the present topic have been given elsewhere.[10,11] Five series of experiments will be discussed which have in common conditions favoring a zonal maximum of vorticity distribution in the field of a vortex.

EXPERIMENT 1. NONSTEADY DECAY OF A DISCONTINUITY SURFACE

A rotationally symmetrical vorticity distribution with maximum occurring at some $R > 0$ was produced by creating a coaxial discontinuity surface in a vortex. To this end a retractable cylinder of 5-in. diameter was placed coaxially into a large upright circular cylindrical water tank, so that its upper edge in fully extended position barely protruded above the free surface, see Fig. 4(a). A vortex was maintained in the space around the cylinder by a loop circuit admitting water near-tangentially through the pervious inner cylindrical wall of the tank and removing it through a sink located at the center of its bottom, see Fig. 4(b). The circulation of this vortex, of the order of 0.5 ft²/sec was constant beyond the boundary layer on the outside wall of the retractable cylinder. This layer was kept thin by draining fluid from it to the sink, see Fig. 4(a). The fluid within the cylinder was stagnant.

The motions of decay of the discontinuity surface which took place within a few seconds after retraction of the cylinder were observable by dye techniques and readily recorded by a movie camera. The motion was essentially two dimensional. Figures 5(a to d) show a succession of flow patterns which

[10] J. R. Weske and T. M. Rankin, "Production of Secondary Vortices in the Field of a Primary Vortex," University of Maryland Institute for Fluid Dynamics and Applied Mathematics Tech. Note BN-244 (1961).

[11] T. M. Rankin, M.S. thesis, University of Maryland, Department of Aeronautical Engineering (1962).

are evidence of the generation of peripheral concentrations of vorticity at wavelengths of successively larger simple fractions of the perimeter. During transitional periods between rotationally symmetrical patterns, combinations of vorticity concentrations took place. The occurrence of the vortex patterns in this case may be regarded as a special case of Helmholtz instability,[12] special because it is subject to the restraints of peripheral periodicity.

Periodic concentrations of vorticity do not occur if the discontinuity surface enclosing a cylinder of stagnant fluid is close to a wall, as in the case of a cylindrical tank set to rotate while the water is at rest. This is demonstrated in Shapiro's film "Vorticity, Part I".[13] In this case the vorticity increases from zero to a maximum at the wall.

Evaluation of the film records indicate that within the range of observations the increase of the wavelength of vorticity concentrations was porportional to the increase of radial width of the mixing zone. The latter was calculated from basic relationships, see reference 9. For the succession of flow patterns, Figs. 5(a to d), therefore the ratio of width of mixing zone to wavelength was constant.

EXPERIMENT 2. PERIPHERAL CONCENTRATION OF VORTICITY IN A QUASISTEADY ZONAL VORTICITY DISTRIBUTION

In this experiment which was devised by the junior author (Rankin) a circular disk was mounted
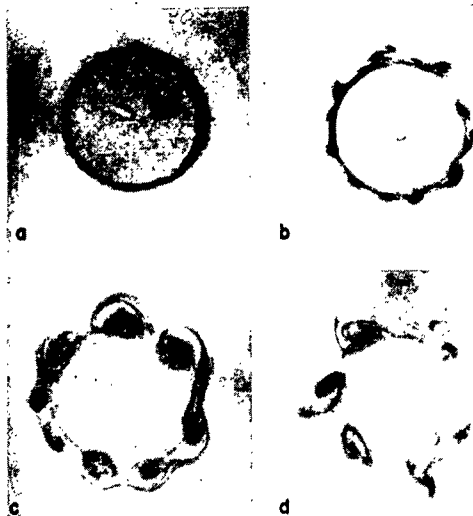

FIG. 5. (a) to (d) Successive flow patterns showing concentrations of vorticity. Experiment No. 1.

[12] V. Bierknes, *Physikalische Hydrodynamik* (Springer-Verlag, Berlin, 1933), p. 380 *et seq.*
[13] A. H Shapiro, "Vorticity, Part I," (Educational Services, Inc., Watertown, Massachusetts, 1962).
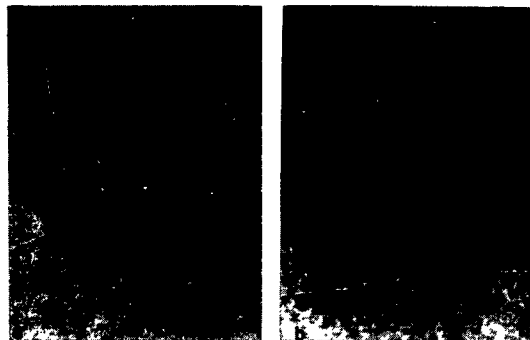
FIG. 6. (a) and (b) Flow patterns of Experiment No. 2. D is the diameter of disk, h the depth of water above disk.

concentrically $\frac{1}{2}$ in. above the sink of the tank, see Fig. 4(b). Fluid which having lost a part of its moment of momentum by friction along the upper surface of the disk was transported into the cylindrical space between the disk and the free surface to be returned to the main body of fluid through radial outward motion. Thereby the axial vorticity distributed over the surface of this cylinder was replenished to defray the loss due to diffusion. In this case the radial extent of the mixing zone was inherently wider than in Experiment 1 and the wavelength of peripheral concentrations of vorticity was a large fraction, $\frac{1}{2}$ to $\frac{1}{5}$ of the perimeter, Figs. 6(a) and (b). The formation of discrete concentrations of vorticity was aided by a factor extraneous to this inquiry, namely by drainage of fluid across the rim of the disk. These concentrations of vorticity were found to occur, however, in a short-lived form also when such drainage was eliminated.

EXPERIMENT 3. OSCILLATIONS IN THE VORTEX CORE

A vortex field with core was produced between two circular parallel disks of 5-in. diameter mounted concentrically an axial distance of 5 in. apart in the same tank, Fig. 4(b). The upper disk could be moved axially. Boundary layers along the surfaces of the disks facing each other were kept thin and the core regions were kept cylindrical by continuous removal of fluid.

Oscillations of the core were induced by instantaneous axial displacement of the disk, which initiated periodic deformations changing the originally cylindrical stream surfaces into what looked like truncated conic surfaces diverging alternately at the top and bottom. Flow patterns within the core taken through the transparent upper disk at the place and instant of maximum diameter of the core, Fig. 7,
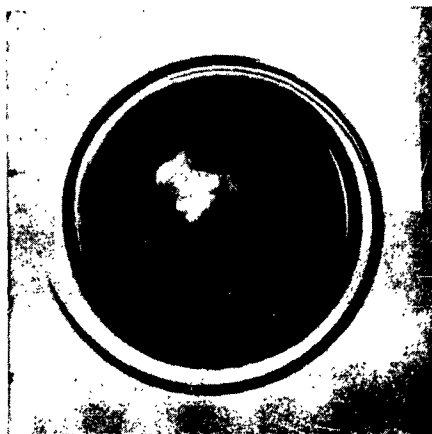
FIG. 7. Peripherally periodic motions generated by oscillations, Experiment No. 3.

show the development of peripherally periodic patterns also in this case. The occurrence during these oscillations of loop vortices ascribed to tangential vorticity components which were wound helically about the core, which in cross section had the appearance of a pair of Taylor vortices between rotating cylinders, was observed and investigated by the junior author (Rankin). These investigations will be presented in a separate report.

FIG 8. Evidence of peripheral periodicity in the wave of an axially propagated wave. Experiment No. 5.

EXPERIMENT 4. TRANSIENT DEFORMATION OF THE CORE OF A VORTEX SINK

Impulsive deformations of the core region of a vortex sink were produced by closing instantaneously the sink at the bottom of the vortex tank. In this case a wave created in this manner and readily rendered visible by dye technique traveled axially upward from the bottom of the tank. The head of this wave which had a characteristic pointed bullet shape was followed by a trail in which peripheral concentration of vorticity of the parent vortex core was indicated by the formation of two or more helical dye streamers, see Fig. 8.

EXPERIMENT 5. SECONDARY MOTION IN LOOP VORTICES CREATED WITHIN A LAMINAR SHEAR LAYER

This experiment was carried out in the laminar boundary layer of a stream of water of considerable depth flowing over a flat plate. The location was 14 ft from the leading edge, the free stream velocity $\frac{1}{8}$ to $\frac{1}{4}$ ft/sec, the Reynolds number $R_x = 140\ 000$ to $280\ 000$. A captive vortex was created in the lee of a ramp of $\frac{1}{2}$-in. height and of 5-in. width normal to the direction of undistrubed flow and stabilized by the suction of two sinks located in the plate close to the ends of the ramp. The ramp was then lowered so as to be level with the plate and shortly thereafter the two sinks were closed instantaneously. Thereby two pressure-wave pulses were set in motion converging through the legs of the vortex loop toward its apex where they collided. As in the preceding experiment helical patterns emerged in the core, see Fig. 9. This was taken as evidence of periodic concentration of vorticity occurring as a result of the pressure impulse. The nature of secondary motion created by encounter of the waves in the neighborhood of the apex was not investigated at this time because it appeared to play a minor role in the rapid disintegration of the vortex loop initiated by secondary motion in the legs. The retractable ramp and the sink holes served to control a process which was observed to take place in vortex loops

FIG. 9. Evidence of peripheral periodicity in the wake of waves propagated through the legs of a loop vortex.

arising in shear layers also without the aid of these devices.

These experimental results are presented as evidence of the occurrence of peripherally periodic concentration of vorticity derived from the core of the parent vortex. Their occurrence could be related to radial divergence of the core region as postulated by linearized theory.