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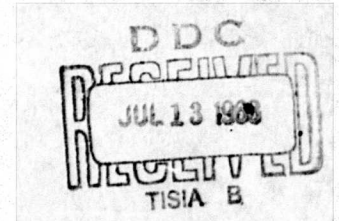
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**NEW ANALYTIC AND
COMPUTATIONAL METHODS FOR
TREATING NONLINEAR CIRCUITS**

Richard Bellman



**PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND**

The **RAND** *Corporation*
SANTA MONICA • CALIFORNIA

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PREFACE

Part of the Project RAND research program consists of basic supporting studies. The mathematical research presented here has applications in control theory, circuit synthesis, and numerical analysis.

SUMMARY

Nonlinearities in circuit theory arise in several important ways. First of all, they arise from genuine nonproportional behavior on the part of circuit components such as vacuum tubes, transistors or oscillators, and secondly they enter in conjunction with various types of control, for example, adaptive control.

Over the last ten years, a number of new mathematical techniques have been developed to treat various classes of nonlinear problems. In this paper we wish to briefly discuss three theories with which we have been personally associated, invariant imbedding, dynamic programming, and quasilinearization, and to provide a selected list of references for further reading in these areas.

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NEW ANALYTIC AND COMPUTATIONAL METHODS FOR
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1. INTRODUCTION

Nonlinearities in circuit theory arise in several important ways. First of all, they arise from genuine nonproportional behavior on the part of circuit components such as vacuum tubes, transistors or oscillators, and secondly they enter in conjunction with various types of control, for example, adaptive control.

Over the last ten years, a number of new mathematical techniques have been developed to treat various classes of nonlinear problems. In this paper we wish to briefly discuss three theories with which we have been personally associated, invariant imbedding, dynamic programming, and quasilinearization, and to provide a selected list of references for further reading in these areas.

We shall focus our attention upon a fundamental question, that of providing analytic and computational treatments of a nonlinear vector equation of the form

$$(1.1) \quad \frac{dx}{dt} = g(x),$$

where x is determined by two-point boundary conditions of the form

$$(1.2) \quad \begin{aligned} (x(0), b_1) &= c_1, \quad i = 1, 2, \dots, k, \\ (x(T), b_1) &= c_1, \quad i = k + 1, \dots, N. \end{aligned}$$

Equations of this type arise throughout circuit analysis and mathematical physics, and are not easily resolved analytically or numerically. A basic feature of our different approaches is that the numerical aspects are intimately linked to the conceptual and analytic ideas.

In using digital computers to resolve (1.1) subject to (1.2), we immediately encounter the fact that these devices are designed to perform iterative arithmetic calculations. They do not readily adapt to solving equations, even linear equations. Can we then bypass the two-point aspect of the equation in (1.2)? As we shall see, we can do this in two ways, one involving a transformation of the problem in an initial value problem, using invariant imbedding and dynamic programming, and the other involving a transformation of the problem into a sequence of problems in each of which $g(x)$ is linear.

2. INVARIANT IMBEDDING

To simplify the subsequent analysis so that it will not obscure the basic ideas, let us restrict our discussion to a second order nonlinear system

$$(2.2) \quad \begin{aligned} u' &= g(u,v), & u(0) &= 0, \\ v' &= h(u,v), & v(x) &= c, \quad 0 < t < x. \end{aligned}$$

This equation may be considered to arise from a nonlinear transmission line where the current is known at one end and the voltage at the other.

We wish to determine the missing initial values, $v(0)$ or $u(x)$, which will enable us to obtain a routine computational solution of (2.1) as an initial value problem. It is clear that $u(x)$, for example, depends upon c and T , and thus may be written $f(c,x)$. What is rather surprising is that $f(c,x)$ satisfies a quasi-linear partial differential equation of simple form, as we shall show in what follows.

In our initial work in this area, this equation, (2.2), was obtained from physical considerations using a transport process as a generator of the equations in (2.1); see [1], [2]. Subsequently [3], it was shown that equations of this type could be derived in a straightforward fashion by means of perturbation techniques.

The equation we wish to obtain for $f(c,x)$ is

$$(2.2) \quad \frac{\partial f}{\partial x} = -h(f,c) \frac{\partial f}{\partial c} + g(f,c), \quad f(c,0) = 0.$$

To derive this equation we argue in the following fashion. Let U, V be the solutions of

$$(2.3) \quad \begin{aligned} U' &= g(U,V), \quad U(0) = 0, \\ V' &= h(U,V), \quad V(x + \Delta) = c, \quad 0 < t < x + \Delta, \end{aligned}$$

where Δ is an infinitesimal. This equation can be

converted to an equation over $0 < t < x$ with the new boundary conditions

$$\begin{aligned}(2.4) \quad V(x) &= V(x + \Delta) - \Delta V'(x + \Delta) + o(\Delta) \\ &= c - \Delta V'(x) + o(\Delta) \\ &= c - h(U(x), V(x))\Delta + o(\Delta) \\ &= c - h(u(x), v(x))\Delta + o(\Delta).\end{aligned}$$

We have used the fact that in the common interval of definition, $0 < t < x$,

$$\begin{aligned}(2.5) \quad U &= u + w(t)\Delta + o(\Delta), \\ V &= v + q(t)\Delta + o(\Delta),\end{aligned}$$

where $w(t)$ and $q(t)$ satisfy the linear perturbation equations

$$\begin{aligned}(2.6) \quad \frac{dw}{dt} &= wg_u + qg_v, \quad w(0) = 0, \\ \frac{dq}{dt} &= wh_u + qh_v, \quad q(x) = -h(u(x), v(x)).\end{aligned}$$

Let us now obtain the linear equations satisfied by the functions u_c and v_c , which is to say, the perturbations of the solution with respect to the boundary value. We have, in standard fashion,

$$\begin{aligned}(2.7) \quad \frac{d}{dt} (u_c) &= u_c g_u + v_c g_v, \quad u_c(0) = 0, \\ \frac{d}{dt} (v_c) &= u_c h_u + v_c h_v, \quad v_c(x) = 1.\end{aligned}$$

Assuming that this linear system, which is identical with (2.6), apart from the boundary conditions, possesses a unique solution, we see that

$$(2.8) \quad w = -h(u,v)u_c,$$
$$q = -h(u,v)v_c.$$

Since

$$(2.9) \quad U(x + \Delta) = U(x) + U'(x)\Delta + o(\Delta)$$
$$= U(x) + q(u,v)\Delta + o(\Delta)$$
$$= u(x) + w\Delta + g(u,v)\Delta,$$

we see that $f(c,x) \equiv u(x)$ satisfies the equation in (2.2).

Setting $v(0) = t(c,x)$, we obtain similarly the equation

$$(2.10) \quad \frac{\partial t}{\partial x} = -f(r,c) \frac{\partial t}{\partial c}.$$

3. DISCUSSION

The equation of (2.2) is an initial value problem, which can be solved computationally by means of a number of different techniques ideally suited to a digital computer; see [4,5,6,7]. Furthermore, we are led to new analytic and conceptual approaches; see [8], [9].

4. DYNAMIC PROGRAMMING

Two-point boundary-value problems arise in a natural fashion from variational questions. For example, if we wish to minimize the functional

$$(4.1) \quad J(u) = \int_0^T (u'^2 + 2g(u))dt$$

over all functions u satisfying the initial condition $u(0) = c$, we are led in the usual fashion to the Euler equation

$$(4.2) \quad u'' - g'(u) = 0,$$

with the new boundary constraint $u'(T) = 0$.

If we are able to formulate the original variational problem in such a way that it leads to an initial value problem, we possess a technique for treating the equation above, a two-point boundary-value problem, as an initial value problem.

To transform the original variational problem into an initial value problem, we use the theory of dynamic programming [10,11,12]. The minimum value of $J(u)$ is regarded as a function of c and T ,

$$(4.3) \quad \min_u J(u) = f(c,T).$$

Employing the principle of optimality [10,11,12], we obtain the nonlinear partial differential equation

$$(4.4) \quad f_T = \min_v [v^2 + 2g(c) + vf(c)],$$

or

$$(4.5) \quad f_T = 2g(c) - f_c^2/4,$$

with the initial condition $f(c,0) = 0$.

For numerical purposes, we can use (4.5), or (4.4), if the minimum is not readily obtainable, or the finite difference version

$$(4.6) \quad f(c, T + \Delta) = \min_v [(v^2 + 2g(c))\Delta + f(c + v\Delta, T)],$$

$T = 0, \Delta, \dots$, with $f(c,0) = 0$; see [11].

The foregoing algorithm is particularly useful in cases where constraints of the form $|u'(t)| \leq k$ are imposed. In place of (4.6), we have

$$(4.7) \quad f(c, T + \Delta) = \min_{|v| \leq k} [(v^2 + 2g(c))\Delta + f(c + v\Delta, T)].$$

5. DISCUSSION

We have shown that the two-point boundary-value problem arising from the variational problem can be replaced by an initial-value problem. There are, however, formidable difficulties that prevent the routine application of this technique to systems of differential equations of high order, see [11]. These can be partially overcome by means of polynomial approximation and successive

approximations [11], but much more work is needed before we can feel that we have mastery of these basic problems of mathematical physics.

6. QUASILINEARIZATION

A most important class of two-point boundary-value problems that can be treated with a very satisfactory degree of completeness is determined by the linear vector system

$$(6.1) \quad \frac{dx}{dt} = A(t)x + b(t),$$

with $x(t)$ subject to the conditions

$$(6.2) \quad \begin{aligned} (x(0), a_i) &= c_i, & i = 1, 2, \dots, k, \\ (x(T), a_i) &= c_i, & i = k + 1, \dots, N. \end{aligned}$$

Using the fact that the general solution of (6.1) is a linear combination of N particular solutions of $x' = A(t)x$ plus a particular solution, the solution of (6.1) is reduced to the solution of a system of linear algebraic equations. If N is not too large, say $N \leq 50$, this is a reasonably direct method of solution. If N is larger than this, we may have computational difficulties and be forced to use some other techniques [13].

The theory of quasilinearization replaced

$$(6.3) \quad \frac{dx}{dt} = \xi(x),$$

with $x(t)$ subject to (6.2), by a sequence of equations

$$(6.4) \quad \frac{dx_{n+1}}{dt} = g(x_n) + J(x_n)(x_{n+1} - x_n),$$

where $J(x)$ is the Jacobian matrix of g ($\partial g_i / \partial x_j$), and $x_0(t)$ is chosen judiciously.

It can be shown that one has quadratic convergence, under reasonable conditions on $g(x)$ and x_0 , and, in many cases, monotonicity; see [14,15,16,17,18,19].

Detailed discussions will be found in the cited references, together with a number of computational results.

The theory of quasilinearization ties together the fundamental technique of Newton-Raphson and an ingenious idea of Caplygin, see [15], concerning differential inequalities.

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