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THE PLASMA RESONANCE IN INCOHERENT SCATTERING OF RADIO WAVES FROM A FULLY IONIZED PLASMA

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PREFACE

This report is part of RAND's theoretical study of high temperature plasmas. The results are of interest for application to the determination of densities and temperatures of high altitude plasmas. This report describes a continuation of the research reported in RM-3440-PR, Incoherent Scattering of Radio Waves By a Plasma.

Several radar installations are under construction to measure the backscatter from powerful radar beams directed vertically upward. At least one such installation, the large Cornell University radar at Arecibo, Puerto Rico, is planning experiments in the area described in this report.

Comparison of the experimental and theoretical results will lead to a better understanding of plasma physics.

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SUMMARY

The detailed shape of the plasma resonance in the spectrum of radio waves incoherently scattered from a hot plasma is presented. The calculation includes exactly the lowest order effects of close collisions in the limit of long wavelength of the radiation and high plasma temperature. However, the results are applicable to some experiments to be performed on incoherent scattering of radar beams from the ionosphere. The magnetic field is neglected but this is a good approximation at high frequencies. The effect of collisions with neutral atoms has also been neglected and this restricts the validity of the results to altitudes above, say, 200 km.

We employ a diagrammatic description of electrical interactions of charged particles and we assume a weakly coupled, high-temperature plasma in thermodynamic equilibrium.

CONTENTS

I. INTRODUCTION

It is very likely that within the next year or two an experiment will be performed to detect the plasma resonance in a warm plasma by measuring the incoherent backscatter of radio waves from the ionosphere.[1] This is a very exciting prospect because after all these years of plasma physics research there is still no direct measurements of the electron plasma resonance in a classical plasma (or even any clear experimental proof of its existence*). Everyone knows it's there but nobody has seen it.

The backscattered radiation from a particular altitude consists of a large sharp central resonance[2] (which really consists of two acoustic resonances very close together) and two smaller resonances separated from the central line by the electron plasma frequency. In this report we shall restrict ourselves to a discussion of these two outside resonances.

The shape of the plasma resonances in the scattered radiation has been worked out in the random phase approximation by Salpeter[3], Dougherty and Farley[4], and more recently by Rosenbluth and Rostoker[5]. We shall extend these calculations to one higher order to include the effects of close collisions. We employ the formalism described in Ref. 2 and the reader will find it helpful to refer to that report for certain basic results.

In Section II we perform the calculation for the plasma resonance in the random phase approximation in order to show how the standard results are obtained by our method of calculation. In Section III we continue the calculation to the next order to include the effects of close collisions. The applicability of the results to scattering from the ionosphere is discussed in Section IV.

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* Note, however, that the low frequency acoustic resonance has been confirmed by the experiments of Bowles: K.W. Bowles, Phys. Rev. Letter, 1, 454 (1958).

II. RANDOM PHASE APPROXIMATION.

We shall describe here the shape of the spectrum in the neighborhood of the plasma resonance to the accuracy of the random phase approximation.

The two lowest order physical processes which contribute in this region are shown in Fig. 1. Since both processes have the same final state, the total scattering rate is proportional to the square of the sum of the individual amplitudes. The meaning of these diagrams and the notation we shall use are described in Ref. 2. Roughly speaking, Fig. 1b can be described as follows: an incident photon of momentum k_a is scattered into a state of momentum k_b and excites a plasma wave (braided line) which subsequently gives up its energy to a single particle.

The photon scattering rate $\Gamma(k,\omega)$ is given by Eq. (2.3) of Ref. 2:

$$\Gamma(k,\omega) = \tfrac{1}{2}\sum_{e_b, e_a} \sum_2 \sum_1 \rho_1 \frac{c}{8\pi^2} \frac{\omega_b}{\omega_a} \left| \langle 2; k_b, e_b \left| M \right| 1; k_a\, e_a \rangle \right|^2 \tag{2.1}$$

$$(2\pi)^3\, \delta^3(P_2 - P_1 + \hbar k)\; 2\pi\delta(E_2 - E_1 - \hbar\omega) \quad .$$

This is the familiar Golden Rule ($\omega = \omega_b - \omega_a$, $k = k_b - k_a$). The subscripts a and b refer to the initial and final states of the radiation (k_a is the initial wave number, e_a is the initial polarization). In Eq. (2.1) we sum over all final states and average over all initial states. The correct average over the initial particles states is obtained by using a suitable Gibbs factor,

$$\rho_1 = e^{\beta\Omega}\, e^{-\beta(E_1 - \mu N_1)} \tag{2.2}$$

where $\beta^{-1} = kT$, μ is the chemical potential, N_1 is the number of particles in the state 1, and $\exp \beta\Omega$ is the normalizing factor fixed by $Tr\rho_1 = 1$.

We shall deal only with unpolarized radiation, so we sum and average over the polarizations, which appear in the form $(\hat{e}_b \cdot \hat{e}_a)^2$:

$$\frac{1}{2} \sum_{e_b, e_a} (\hat{e}_b \cdot \hat{e}_a)^2 = \frac{1}{2} (1 + \cos^2\theta) \tag{2.3}$$

where $k_b \cdot k_a = k_b k_a \cos\theta$. For backscattered radiation $\cos^2\theta = 1$.

To obtain the amplitude M we simply draw all possible modes for scattering a photon in state (k_a, e_a) to a state (k_b, e_b). The amplitude for any process is readily computed from the corresponding diagram by use of the calculating rules presented in Ref. 6.

From Fig. 1 and Ref. 6 we immediately have

$$M_{rpa} = \sqrt{\lambda} - \sqrt{\lambda}\, \frac{1}{\lambda} Q_e^o(k,\omega) \frac{\lambda}{k^2 + Q_e^o(k,\omega)} \tag{2.4}$$

Using Eqs. (2.1)-(2.4) we obtain the following expression for the scattering rate

$$\Gamma_{rpa}(k,\omega) = n r_o^2 c \left| 1 - \frac{Q_e^o(k,\omega)}{k^2 + Q_e^o(k,\omega)} \right|^2 \frac{1}{\sqrt{2\pi}\, k} e^{-\frac{1}{2}\frac{\omega^2}{k^2}} \tag{2.5}$$

where $r_o = e^2/mc^2$, the classical electron radius. The first term within the absolute value signs in Eq.(2.5) represents the scattering from free electrons and leads to the familiar Doppler spectrum. Near the resonance, however, this term is quite unimportant.

From Eqs. (6.12) and (6.13) of Ref. 6 we have (with $z = \omega/k$)

$$Q_e^o(z) = 1 - z\, e^{-\frac{1}{2} z^2} \int_0^z dt\, e^{\frac{1}{2} t^2} + i(\pi/2)^{\frac{1}{2}}\, z\, e^{-\frac{1}{2} z^2} \tag{2.6}$$

The spectrum near the plasma resonance in the random phase approximation (Eq. 2.5) is discussed at some length in Ref. 3.

In the next section we shall turn to the contributions of one higher order in the coupling constant λ.

III. COLLISION TERMS

The important lowest order diagrams which involve close collisions are shown in Fig. 2. There are many more diagrams of the same order in the interaction λ but they are discarded for one of two reasons. (1) Diagrams with two like particles in the final state, for example, Fig. 3a, are smaller by a factor of order k^2 and are unimportant in the long wavelength limit, and (2) diagrams with virtual ion lines (instead of electron lines), for example, Fig. 3b are of order α^2 and vanish in the limit of infinite ion mass.

In order to introduce the collision terms into the scattering formula let us rewrite Eq. (2.5) entirely in terms of Q_e.

$$\Gamma_{rpa}(k,\omega) = \frac{n r_o^2 c}{\pi\omega} \left| 1 - \frac{Q_e^o(k,\omega)}{k^2 + Q_e^o(k,\omega)} \right|^2 \operatorname{Im} Q_e^o(k,\omega) \qquad (3.1)$$

From the structure of the diagrams in Fig. 2 we can see, by comparison with the calculation of the high frequency plasma conductivity in Ref. 6, that the natural extension of Eq. (3.1) to one higher order is

$$\Gamma(k,\omega) = \frac{n r_o^2 c}{\pi\omega} \left| 1 - \frac{Q_e(k,\omega)}{k^2 + Q_e(k,\omega} \right|^2 \operatorname{Im} Q_e(k,\omega) \qquad (3.2)$$

where the imaginary part of $Q_e(k,\omega)$ contains the lowest order collision terms beyond the random phase approximation. That is,

$$Q_e(k,\omega) = Q_e^o(k,\omega) + i \frac{\lambda}{6\sqrt{2}\,\pi^{3/2}} \frac{k^2}{\omega^3} K_a(\omega) \qquad (3.3)$$

where

$$K_a(\omega) = \int dq \; q^3 \exp(-\tfrac{1}{8}\hbar^2 q^2) \exp(-\tfrac{1}{2}\frac{\omega^2}{q^2}) \frac{q^2+1}{q^2+2} \left| \frac{1}{q^2 + Q^o(\omega/q)} \right|^2 \qquad (3.4)$$

In Fig. 4 we plot $\Gamma(k,\omega)/nr_o^2$ for a sample value of k^2 and we compare with the corresponding random phase approximation result $\Gamma_{rpa}(k,\omega)/nr_o^2$. The main effect of collisions is to broaden the resonance. We have ignored the change in the real part of $Q_e(k,\omega)$ which results in a frequency shift.

Near the resonance we can put Eq. (3.2) into a more familiar form. First, we can neglect the non-resonant term,

$$\Gamma(k,\omega) = \frac{nr_o^2c}{\pi\omega}\,\frac{|Q_e(k,\omega)|^2}{k^4}\,\frac{1}{\left|1+\frac{1}{k^2}Q_e(k,\omega)\right|^2}\,\mathrm{Im}\;Q_e(k,\omega). \tag{3.5}$$

At the resonance $Q_e(k,\omega) \approx -k^2$ (so $|Q_e(k,\omega)|^2/k^4 \approx 1$) and $\omega = \omega_2 \approx 1$. We then have

$$\Gamma(k,\omega) = nr_o^2c\frac{1}{\pi}\,\frac{1}{\left|1+\frac{1}{k^2}Q_e(k,\omega)\right|^2}\,\mathrm{Im}\;Q_e(k,\omega) \tag{3.6}$$

The denominator is just the absolute square of the dielectric constant. From Eqs. (3.8)and(3.10), and (4.30) of Ref. 6 we have

$$E^{-1}(k,\omega) = \frac{\omega_L}{2}\,\frac{z_L}{\omega-\omega_L+\frac{1}{2}\,i\,\gamma_L} \tag{3.7}$$

$$\gamma_L(k) = z_L\,\frac{\omega_L}{k^2}\,\mathrm{Im}\;Q_e(k,\omega_L) \tag{3.8}$$

Furthermore, in the approximation where we neglect the effect of collisions on the frequency of oscillation we have (from Eq. (4.37) of Ref. 6)

$$z_L^{-1} = 1 + O(k^2) \tag{3.9}$$

so we can now write Eq. (3.6) in the form

$$\Gamma(k,\omega) = n\, r_o^2 c \frac{k^2}{2} \frac{1}{\pi} \frac{\frac{1}{2}\gamma_L}{(\omega-\omega_L)^2 + (\frac{1}{2}\gamma_L)^2} \qquad (3.10)$$

The total area under the resonance is $\frac{1}{2}k^2 n\, r_o^2 c$. The area under the central resonance is approximately $\frac{1}{2}\, n\, r_o^2 c$ so that the total contribution of the plasma resonances is rather small. The integral over the resonance is, of course, unaffected by collisions.

IV. APPLICATION TO SCATTERING FROM THE IONOSPHERE

The main result we have derived, Eq. (3.2), is strictly correct only in the high temperature limit, that is, when kT >> Rydberg. The reason for this is that we used the Born approximation in deriving the collisional contribution to Im $Q_e(k,\omega)$ in Eq. (3.3) so that the high wave number (or short distance) cut-off in Eq. (3.4) is the de Broglie wave number of the particle. The integral K_a can be written in the form

$$K_a(\omega) = \ln \left[C(\omega)\ k_T/k_D\right] = \ln \left[C(\omega)\ r_D/r_T\right]$$

where $k_T^2 = m/\hbar^2\beta = r_T^{-2}$, $k_D^2 = 4\pi e^2 n\beta = r_D^{\ 2}$, $(\beta = 1/kT)$, and $C(\omega)$ is some function of ω which we shall evaluate at the plasma frequency.

In considering the scattering of radio waves from high altitude plasmas we notice that the high temperature approximation fails to apply because the temperatures are typically about 1000°K - 2000°K. At these lower temperatures we expect that our results can be used if one replaces $r_T = \hbar(\beta/m)^{\frac{1}{2}}$ by the classical distance of closest approach $r_C = e^2\beta$. The parameters in the argument of the logarithm then combine to form $\lambda = k_D^{\ 3}/n$ times some numerical factor of the order of one. At any rate, there is no other way of computing this quantity at the present time. Authors who use completely classical methods are forced to introduce an arbitrary cut-off to keep certain integrals finite. This again leads to the logarithm indicated above with an unknown numerical factor in the argument.

REFERENCES

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3. E. E. Salpeter, Phys. Rev. 120, 1528 (1960).

4. J. P. Dougherty and D. T. Farley, Proc. Roy. Soc. (London) A259, 79, (1960).

5. M. N. Rosenbluth and N. Rostoker, Phys. Fluids 5, 776 (1962).

6. D. F. DuBois, V. Gilinsky, and M. G. Kivelson, Phys. Rev. 129, 2376 (1963); see also RM-3224. Note, however, that the arguments of the logarithms in the expressions for the conductivity in Section VI of RM-3224 are incorrect because two contributing diagrams were overlooked. This affects the result near the plasma frequency, though the results at very low and very high frequencies are unchanged.

Fig. 1a

Fig. 1b

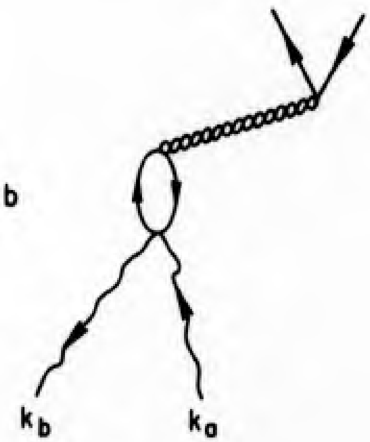

Fig. 1 — Lowest order scattering processes in the neighborhood of the plasma resonance

Fig. 2a

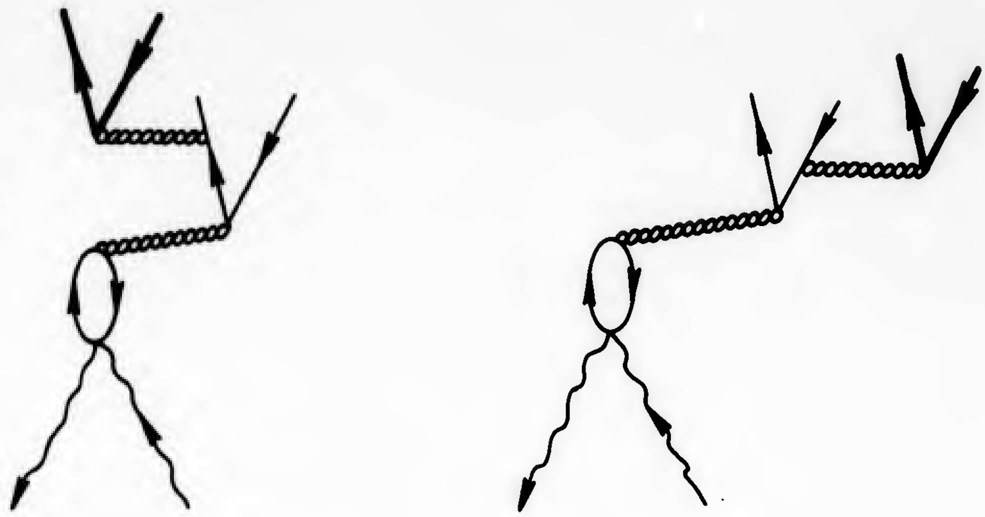

Fig. 2b

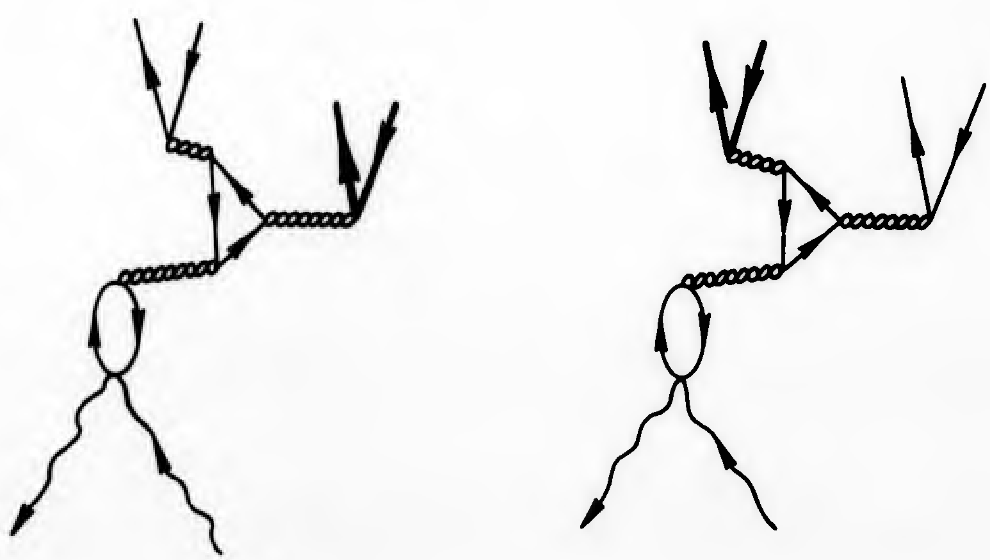

Fig. 2—Lowest order diagrams which contain close collisions

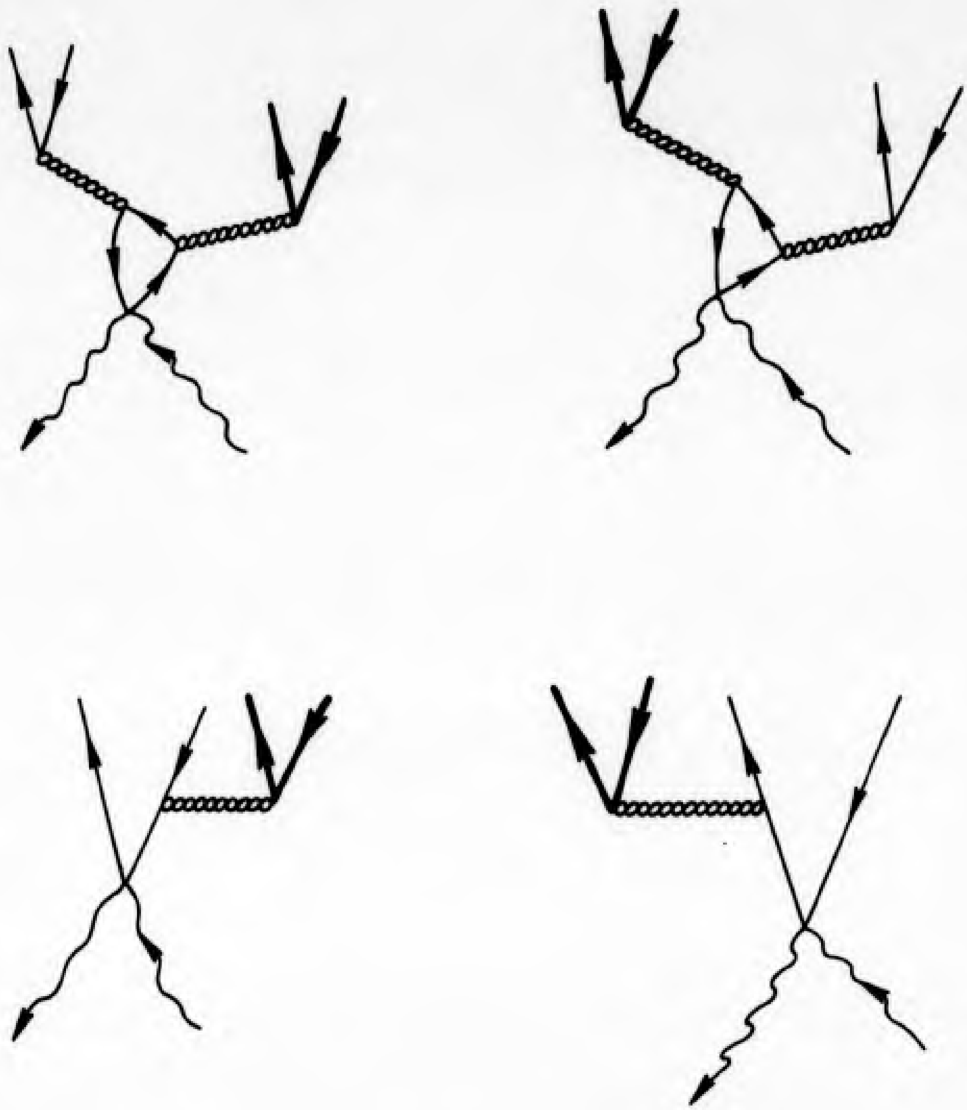

Fig. 2c

Fig. 3a

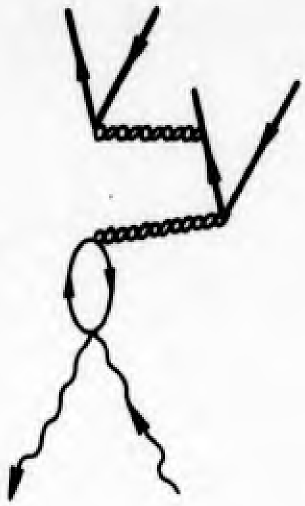

Fig. 3b

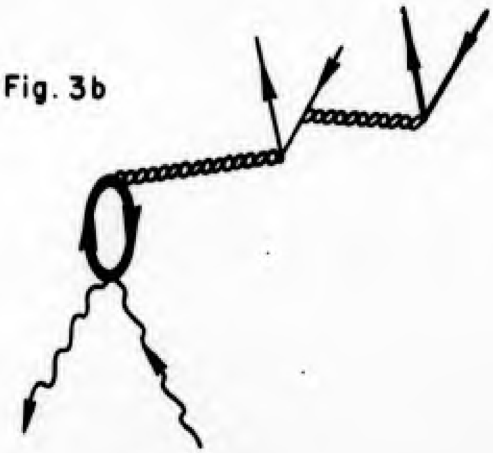

Fig. 3—Types of diagrams which are not included

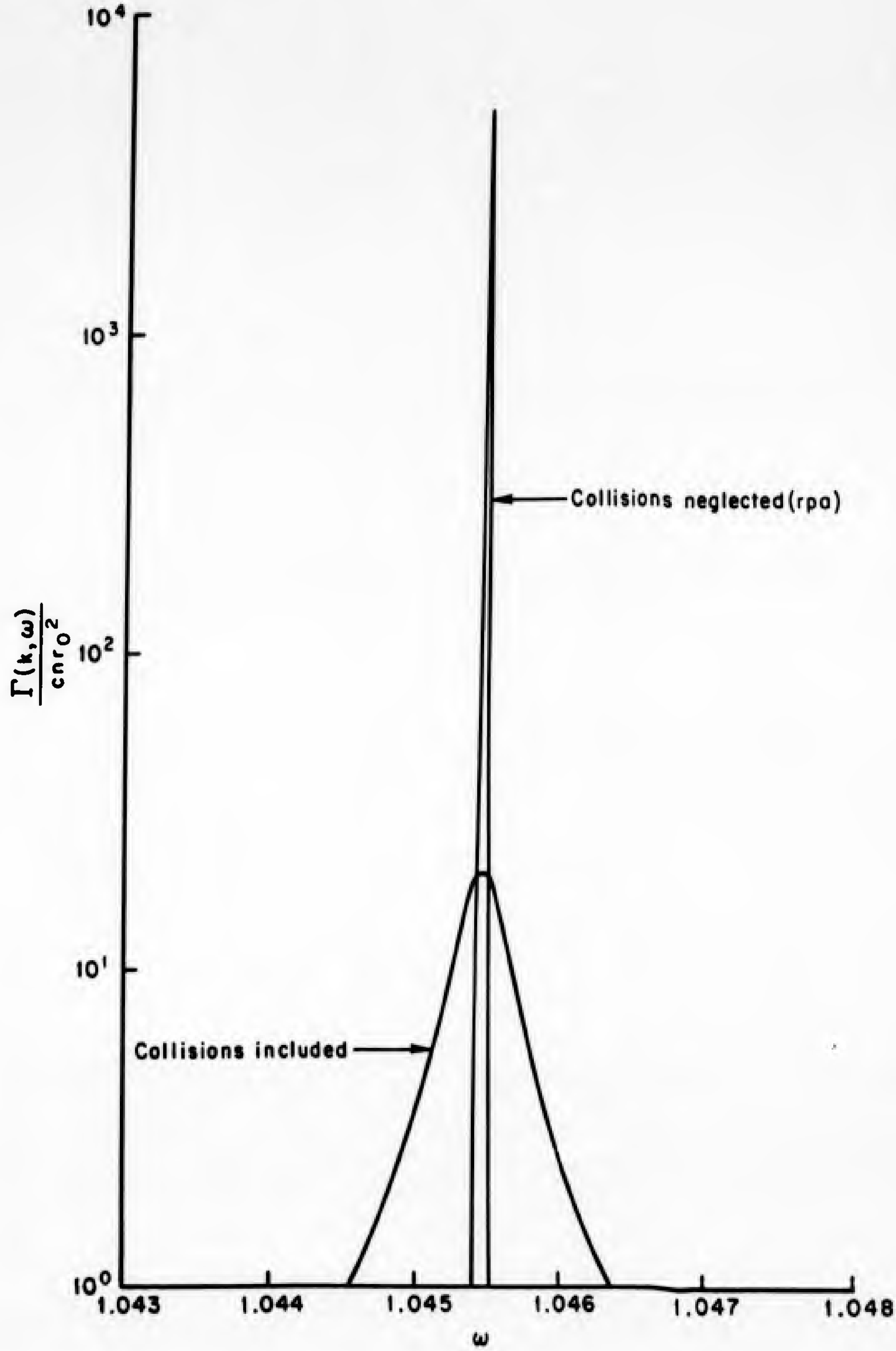

Fig. 4 — Scattering rate, $\Gamma(k,\omega)$ for $k = 0.17$, $\lambda = 0.01$, and $\hbar = 0.10$

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