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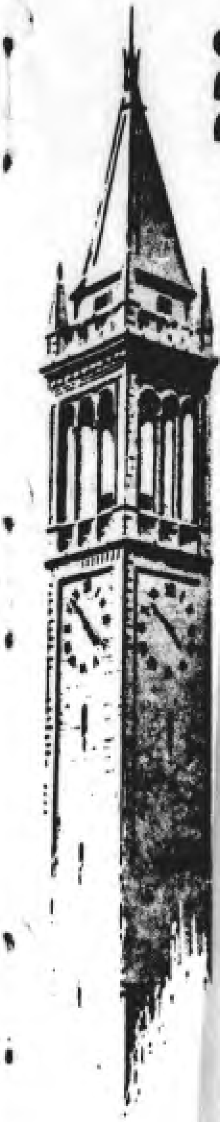

RECENT DEVELOPMENTS IN FREE-SURFACE FLOWS

by

John V. Wehausen

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INSTITUTE OF ENGINEERING RESEARCH

UNIVERSITY OF CALIFORNIA
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TABLE OF CONTENTS

ABSTRACT

The following report attempts to present a self-contained exposition of the developments in the theory of gravity waves and of jets and cavities which have taken place during the last five or six years, i.e., since the manuscripts were finished for the article Jets and Cavities, by D. Gilbarg and the article Surface Waves, by E. Laitone and the author for volume 9 of the Encyclopaedia of Physics [Springer, Berlin, 1960].

Introduction

Surface phenomena will be interpreted here to include those associated with fluid flows in which either a single fluid has a free boundary or else two or more different fluids have well defined interfaces. Since a stratified flow of fluids of, say, different densities is a kind of approximation to the flow of a single fluid with continuously varying density, it seems reasonable to include also certain cases of this sort. Emphasis is, of course, upon flow phenomena connected with the presence of the free boundary, the interface or the density variation.

The topics which may be included under the above definition are still too numerous to be usefully discussed in a relatively small number of pages. We omit entirely any discussion of engineering applications, of oceanographic problems connected with tides, ocean currents or the statistics of ocean waves, and of many interesting surface phenomena studied by chemists in their investigations of surface forces. The last neglect is perhaps the most serious, for many surface phenomena are easily observable in the laboratory and even upon the surface of natural streams which cannot be explained by the simple model of surface force which will be used below.

There exist several relatively recent expositions of the subjects of this chapter. In Volume 9 of the Encyclopaedia of Physics [Springer, Berlin, 1960] are the articles

Jets and cavities by D. Gilbarg and Surface waves by E. V. Laitone and the author. Each of these expositions contains a fairly comprehensive summary of results obtained up to about 1958, and each is accompanied by an extensive bibliography. We shall lean heavily upon these articles and only more recent results or ones neglected in those articles will be reported here. We also use this opportunity to complete and bring up to date the bibliographies in the articles mentioned above. Further, we call attention to three other extended accounts of the subject matter: Jets, wakes and cavities [Academic Press, New York, 1957] by G. Birkhoff and E. H. Zarantonello, Water waves [Interscience, New York, 1957] by J. J. Stoker, and a recent book in Russian, Theory of jets in an ideal fluid [Moscow, 1961] by M. I. Gurevich. All the cited expositions suffer from a tendency to report only theoretical developments. Although R. Miche's Propriétes des trains d'ondes océanique et de laboratoire [Imprimerie Nationale, Paris, 1954] records considerable information about physical waves, there is still need in this subject for a combined treatment of theoretical prediction and actual observation, preferably controlled laboratory experiments.

In the following, after developing enough of the fundamental theory in order to be able to formulate a variety of problems, we shall discuss in a discursive way some of the recent advances in the study of surface phenomena and point out some areas still needing investigation. We shall consider first solutions satisfying the exact equations and boundary conditions (a conventional use of the word "exact") and then approximate methods of solution. As one might anticipate, the effort devoted to approximate methods bulks much larger than that devoted to exact ones.

Fundamental Equations

In a fixed coordinate system let the y-axis be directed upwards, i.e. opposite to the force of gravity in situations in which gravity is taken into account, the x-axis to the right and the z-axis coming out of the paper. We shall also write when convenient x_1, x_2, x_3 for x, y, z. Suppose a given region of this space to be filled with moving fluid. The motion may be described by giving the velocity field $\underline{v}(x, y, z, t) = (u, v, w)$, or equivalently $u_i(x_1, x_2, x_3, t)$, $i = 1, 2, 3$. The fluid itself may be described by giving the density ρ, viscosity coefficients λ, μ, etc. and various equations of state expressing these as functions of, say, pressure p and temperature T. However, we shall not generally need this full description of the fluid, for it usually suffices to assume it to be incompressible.

The following facts may be found in most text books on fluid mechanics. Conservation of mass in the flow is expressed by the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0\,, \qquad (1)$$

and the condition of incompressibility by the vanishing of the material derivative of ρ,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} = 0\,, \qquad (2)$$

so that for an incompressible fluid

$$\frac{\partial u_i}{\partial x_i} = 0\,. \qquad (3)$$

In this case the stress tensor for a fluid obeying the Navier-Stokes equation is

$$s_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4}$$

and the Navier-Stokes equations themselves are

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + F_i + \nu\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}, \tag{5}$$

where ν is the kinematic viscosity μ/ρ and generally $F_i = -\partial U/\partial x_i$, where $U = g x_2$. It will be convenient to have this also in vector notation in the form

$$\frac{\partial \underline{v}}{\partial t} + \operatorname{grad}\tfrac{1}{2}v^2 - \underline{v}\times\operatorname{curl}\underline{v} = -\frac{1}{\rho}\operatorname{grad}p + \underline{F} + \nu\Delta\underline{v}. \tag{5'}$$

Let the coordinate system $O'x'y'z'$ be moving with respect to $Oxyz$ in such a way that a point $\underline{r}'$ fixed in $O'x'y'z'$ moves with velocity $\underline{v}_c = \underline{v}_0 + \underline{\omega}\times\underline{r}'$. One may next introduce the velocity $\underline{v}'$ of a fluid particle relative to $O'x'y'z'$ where $v = \underline{v}' + \underline{v}_c$. Then the equations for the absolute motion in the moving coordinate system are

$$\frac{\partial \underline{v}(x',y',z',t)}{\partial t} + \operatorname{grad}\left[\tfrac{1}{2}v^2 - \underline{v}\cdot\underline{v}_c\right] - (\underline{v}-\underline{v}_c)\times\operatorname{curl}\underline{v} = -\frac{1}{\rho}\operatorname{grad}p + \underline{F} + \nu\Delta\underline{v} \tag{6a}$$

and for the relative motion $\underline{v}'$ become

$$\frac{\partial \underline{v}'}{\partial t} + \operatorname{grad}\tfrac{1}{2}v'^2 - \underline{v}'\times\operatorname{curl}\underline{v}' + 2\,\underline{\omega}\times\underline{v}' \tag{6b}$$

$$= -\frac{1}{\rho}\operatorname{grad}p + \underline{F} + \nu\Delta\underline{v}' - \frac{d\underline{v}_0}{dt} - \frac{d\underline{\omega}}{dt}\times\underline{r}' - \underline{\omega}\times(\underline{\omega}\times\underline{r}').$$

The last equations will be useful in discussing the motion of a fluid in a moving container.

Next consider the boundary conditions. If two different fluids have a common bounding surface, i.e. an interface, it is well known that there is a thin film between the two fluids in which surface forces act. These have the effect of causing a discontinuity in the stress as one passes from one side of the film to the other. Let us denote the characteristics of one of the fluids by the superscript 1 and of the other by 2, and let $\underline{n}$ be the normal to the interface, pointing into fluid 2. Then in the simplest theory of surface forces, the limiting values of $\mathcal{A}_{ij}$ as one approaches the surface from either side must satisfy

$$[\mathcal{A}_{ij}^{(2)} - \mathcal{A}_{ij}^{(1)}]n_j = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)n_i\ , \tag{7}$$

where R_1 and R_2 are the two principal radii of curvature of the interface, and where T, the surface-tension coefficient, is a characteristic of an interface between the two fluids in question, one of which must be a liquid. The signs of R_1 and R_2 will be positive, if the concave sides of the associated curves are toward fluid 1. Here T will be taken to be constant over the interface. However, a variation in T is often observed, even without change of temperature, and results in an additional term equal to the gradient of T along the surface. This does not exhaust the possibilities. For a more detailed treatment of the phenomenological theory of surface forces one may consult the article of F. P. Buff in vol. 10 of the Encyclopaedia of Physics [1960], papers by Scriven [1960], Goodrich [1961] and Dorrestein [1951], and chapter 10 of a recent book by Aris [1962].

For viscous fluids the boundary condition above is supplemented by an additional one, continuity of velocity at the interface:

$$u_i^{(1)} = u_i^{(2)}. \tag{8}$$

This boundary condition is also taken to hold at solid boundaries.

If fluid 2 is absent, i.e., if one may assume that its presence has only a negligeable effect upon the motion, the boundary is called a "free boundary". In this case condition (8) is deleted.

One may evidently rephrase (7) by saying that the dynamical boundary condition at an interface is continuity of the velocity and of the tangential stress components and a discontinuity in the normal component given by $T(R_1^{-1} + R_2^{-1})$; at a free boundary the tangential stresses vanish.

At any physical boundary the normal component of the velocity of the fluid must equal the velocity of the boundary in the direction of its normal:

$$\underline{v} \cdot \underline{n} = V_n \,. \tag{9}$$

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This is sometimes called the "kinematic boundary condition". If the boundary surface is given by $F(x, y, z, t) = 0$ with $\underline{n} = \operatorname{grad} F$, then

$$V_n = -F_t \Big/ \sqrt{F_x^2 + F_y^2 + F_z^2}$$

so that (9) becomes

$$\frac{DF}{Dt} = uF_x + vF_y + wF_z + F_t = 0. \tag{9'}$$

Since inviscid fluids will occupy us chiefly, we note that for such fluids (7) simplifies to

$$p^{(1)} - p^{(2)} = T(R_1^{-1} + R_2^{-1}) \tag{10}$$

and that (8) is replaced by the kinematic condition

$$\underline{v}^{(1)} \cdot \underline{n} = \underline{v}^{(2)} \cdot \underline{n} , \tag{11}$$

which must, of course, hold in any case.

We shall derive or state several consequences and special cases which will be more or less useful later. Take the scalar product of (5') with $\underline{v}$ and suppose $\underline{F} = -\operatorname{grad} U(x, y, z)$. Then after making use of (2) and carrying out some simple manipulations, one finds

$$\frac{D}{Dt}\left(\frac{1}{2}v^2 + \frac{p}{\rho} + U\right) = \frac{1}{\rho}\frac{\partial p}{\partial t} + \nu\, \underline{v} \cdot \Delta \underline{v} . \tag{12}$$

In particular, if the fluid is inviscid and the motion steady, $\frac{1}{2}v^2 + U + \frac{p}{\rho}$ must be constant on each streamline. Again, if the fluid is inviscid and if there is no surface tension, then on a free surface $p = const.$ and it follows from (12) that

$$\frac{D}{Dt}\left(\frac{1}{2}v^2 + U\right) = -\frac{1}{\rho}\frac{\partial p}{\partial t} \tag{13}$$

on the free surface. If the motion is steady, then $\frac{1}{2}V^2 + U$ is constant along each surface streamline. We note especially that these theorems do not require the fluid to be homogeneous or the flow irrotational. If both of these conditions are assumed, as well as inviscidity, then there exists a velocity potential $\Phi(x, y, z, t)$ such that

$$\Delta\Phi = 0, \qquad \underline{V} = \operatorname{grad}\Phi, \tag{14}$$

and further, the pressure is given by the Bernoulli integral,

$$\Phi_t(x, y, z, t) + \frac{1}{2}V^2 + U + \frac{p}{\rho} = F(t), \tag{15a}$$

or, corresponding to (6a),

$$\Phi_t(x', y', z', t) + \frac{1}{2}V^2 - \underline{V}_0 \cdot \operatorname{grad}\Phi + U + \frac{p}{\rho} = F(t). \tag{15b}$$

Thus the condition in (10) may in this case be referred to the velocity potentials in the two fluids:

$$\rho^{(2)}\left\{\Phi_t^{(2)} + \frac{1}{2}|\operatorname{grad}\Phi^{(2)}|^2\right\} - \rho^{(1)}\left\{\Phi_t^{(1)} + \frac{1}{2}|\operatorname{grad}\Phi^{(1)}|^2\right\} + (\rho^{(2)} - \rho^{(1)})U = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{16}$$

If one restricts one's attention to <u>two-dimensional motion</u>, certain reformulations are possible. There exists a stream function Ψ such that $u = \Psi_y$, $v = -\Psi_x$, $\zeta = v_x - u_y = -\Delta\Psi$. If the motion is steady, the incompressibility condition implies that $\rho = \rho(\Psi)$ It follows from (12) that for steady motion

$$\frac{1}{2}V^2 + U + \frac{p}{\rho} = f_1(\Psi). \tag{17}$$

Further, for this situation one may also derive easily

$$\frac{\rho'}{\rho}\left(\tfrac{1}{2}v^2+U\right)-\zeta = f_2(\Psi), \tag{18}$$

where $\rho' = d\rho/d\Psi$. The functions $f_1(\Psi)$ and $f_2(\Psi)$ are arbitrary. The density distribution may be assumed given in the formulation of the problem. If (18) is rewritten as

$$\Delta\Psi + \frac{\rho'}{\rho}\left[\tfrac{1}{2}(\Psi_x^2+\Psi_y^2)+U\right] = f_2(\Psi), \tag{18'}$$

it is evident that this provides an equation for the determination of Ψ with boundary conditions $\Psi = const.$ on rigid surfaces, and $\frac{1}{2}(\Psi_x^2+\Psi_y^2)+U = const.$ on free surfaces without surface tension. We shall return to this later.

If the two-dimensional motion is irrotational, one may introduce a complex velocity $w(z,t) = u-iv$ and a complex velocity potential $F(z,t)=\Phi+i\Psi$, each an analytic function within regions of smooth fluid flow and related by $w = F'$. (The ambiguous use of the letter w should cause no confusion since in the present sense it occurs only in two-dimensional flows.) If two inviscid fluids have an interface and each is moving irrotationally, then the respective complex velocities must satisfy [cf. (16)]

$$\rho^{(2)}\left[\tfrac{1}{2}|F^{(2)\prime}|^2+\mathrm{Re}\,F_t^{(2)}\right]-\rho^{(1)}\left[\tfrac{1}{2}|F^{(1)\prime}|^2+\mathrm{Re}\,F_t^{(1)}\right]+(\rho^{(2)}-\rho^{(1)})U = T/R. \tag{19}$$

The above collection of equations and formulas from the fundamentals of fluid mechanics has not displayed a characteristic property of liquids, namely, that a liquid will not under ordinary circumstances allow a pressure less than the vapor pressure p_c of the liquid without changing to its gaseous phase. Hence a flow which might be possible according to the equations above will ordinarily not occur if the absolute pressure becomes less than p_c in some region. When such a situation occurs one must look for a different solution, one which will be partly bounded by a region of vapor. At the vapor-liquid interface one will expect to try to satisfy the boundary condition (16), with or without $T = 0$. However, since the density of the vapor is so small compared to that of the liquid, it is customary to neglect the inertial part of the boundary condition and to replace it by

$$p_c - p = T(R_1^{-1} + R_2^{-1}) . \qquad (20)$$

(We recall here a theorem from the theory of irrotational flow which states that a minimum of the pressure must always occur on a boundary.) Flows satisfying this boundary condition are usually called <u>cavitational flows</u>. The situation is formally similar to that of waves at an air-water interface where the atmospheric pressure p_A replaces p_c in (20). In fact, in water tunnels it is possible to realize cavitational flows in which the vapor pressure may be replaced by any other pressure between p_c and the pressure in the tunnel upstream of an obstacle.

Exact Solutions

The word "exact" in this context customarily indicates that, after one has decided upon a fluid model, external force field, and type of flow, one treats the posed problem without further approximation. We shall consider first gravity waves with vorticity, then irrotational gravity waves, and finally free-surface flows without gravity. It seems appropriate to point out here that little attention seems to have been given to the exact theory of waves in heterogeneous fluids since the work of Dubreil-Jacotin [1937]. However, there are notable exceptions: Long [53, 58] and Yih [1960b] have given several explicit solutions to the equations (18'), some of which fortunately turn out to be physically significant. Further, there appear to be no investigations of the exact theory of waves in viscous fluids, a not surprising fact in view of the intractability of the equations.

Gravity waves with vorticity. Only two-dimensional steady motion of a homogeneous fluid will be considered. Let the x-axis be at the mean water level, and a flat horizontal bottom at $y=-h$. Let the free surface $y=\eta(x)$ correspond to $\psi=0$ and the bottom to the streamline $\psi=-Q$. This corresponds then to a steady flow to the right with discharge rate Q and a mean velocity $c_1=Q/h$. We shall use the letter c to represent the mean velocity along the streamline $y=-h$. From (18'), with $\rho'=0$ since the fluid is homogeneous, we have then the following equation and boundary conditions for ψ :

$$
\begin{aligned}
&\Delta\psi = f(\psi), \\
&\psi_x^2(x,\eta(x)) + \psi_y^2 + 2g\eta(x) = \text{const.}\ , \qquad (21) \\
&\psi(x,\eta(x)) = 0,\ \psi(x,-h) = -Q,
\end{aligned}
$$

where $f(\psi)$ is some given function. The problem is awkward as posed because $\eta(x)$ is to be determined as part of the solution. Let us further suppose that ψ is a monotonic increasing function of y (so that $u>0$). Then we may take x and ψ as independent variables, following Gouyon [1958]. One finds easily the following equations which must be satisfied by $u(x,\psi)$, $v(x,\psi)$:

$$
\begin{aligned}
&\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial \psi} - v\frac{\partial u}{\partial \psi} = 0, \\
&-\frac{\partial v}{\partial x} + u\frac{\partial u}{\partial \psi} + v\frac{\partial v}{\partial \psi} = f(\psi), \qquad (22)
\end{aligned}
$$

$$
u u_x(x,0) + v v_x + g\, v/u = 0, \qquad v(x,-Q) = 0.
$$

Finally, if the motion is to be periodic, one must also impose

$$
u(x+\lambda,\psi) = u(x,\psi), \qquad v(x+\lambda,\psi) = v(x,\psi).
$$

With regard to this condition one should keep in mind that a flow with period λ/n, n an integer, is also one of period λ. Hence, a statement asserting uniqueness of solution carries the proviso that the motion is not also of period λ/n, $n > 1$.

Gouyon [1958] has considered this problem for infinite as well as finite depth in a paper whose results include the classical ones of Levi-Civita, Struik, and Dubreil-Jacotin. It is only possible to indicate his procedure, which in principle, however, is straightforward. He introduces new variables $\rho = \exp(2\pi\psi/c\lambda)$, $\theta = 2\pi x/\lambda$ which map the rectangle $0 \le x \le \lambda$, $-Q \le \psi \le 0$ onto a ring-shaped region in the (ρ, θ)-plane, and also writes

$$u = c(1+U), \quad v = cV, \quad \varphi(\rho) = -\frac{\lambda}{2\pi c\rho} f(\psi). \tag{23}$$

Next he introduces a parameter α and assumes that φ depends analytically upon α, in particular,

$$\varphi(\rho, \alpha) = \sum_{n=0}^{\infty} \varphi_n(\rho)\alpha^n, \tag{24}$$

and that one may find solutions depending analytically upon α:

$$U = \sum_1^{\infty} u_n(\rho,\theta)\alpha^n, \quad V = \sum v_n(\rho,\theta)\alpha^n, \quad \frac{\lambda g}{2\pi c^2} = p_0 + \sum_{n=1}^{\infty} p_n \alpha^n. \tag{25}$$

Thus a small value of α implies small values of the vorticity, small deviations of (u, v) from $(c, 0)$ and small surface slopes. Gouyon is able to give explicit formulas for u_n and v_n and to find an explicit radius of convergence for the series by finding a convergent series of majorants. Further, he shows that the solution is unique in the sense specified above, that the motion is symmetric about the crests and troughs, and that nowhere else are the slopes of the streamlines zero. $f(\psi)$ is required to be continuous. The velocity must be subcritical $(Q^2 \leq gh^3)$.

Later Moiseev [1960 b] treated the same equations (23) in a different way, deriving from them a pair of nonlinear integral equations. His existence theorem is somewhat different from Gouyon's. First, he allows more generality in $f(\psi)$, merely integrability with a bound on the integral to keep $u > 0$. Second, the exact solution is not a perturbation of a uniform flow as in Gouyon's theorem [cf.(24)], but rather of the shear flow

$$u = \left[2\int_{-Q}^{\psi} f(\psi)\,d\psi + c^2\right]^{\frac{1}{2}}, \qquad v = 0, \tag{26}$$

which is easily confirmed to be a solution of (22). The linearized theory associated with this perturbation requires that $\nu \equiv gh^3/Q^2$ have a definite value, say ν_1, for a given value of λ, just as in the irrotational case. Moiseev's theorem asserts the existence of unique solutions for all values of ν such that $\nu_1 - \nu < \varepsilon$, where ε is some sufficiently small but positive number.

Ter-Krikorov [1961] has proved the existence of a solitary wave on a flow with arbitrary continuously differentiable vorticity distribution. The method is generally like that of Moiseev above; however, he must also make a preliminary stretching transformation as in the Friedrichs and Hyers [1954] treatment of the irrotational case. The velocity of propagation must always exceed $\sqrt{gh}$, conforming with the situation for irrotational solitary waves. An explicit formula is given for the first approximation; this will be given later in the discussion of approximate solutions.

It is evident that there has been much progress in the theory of rotational waves along the direction already traversed in the theory of irrotational waves. Further progress in this direction seems possible: generalization of cnoidal waves [cf. Moiseev, 1960a], interfacial waves, etc.

Irrotational gravity waves. Recent advances are all for steady two-dimensional flow. The most exciting recent result is that of Krasovskii [1960, 1961]. We recall that the now classical papers of Nekrasov, Levi-Civita and Struik prove the existence of permanent periodic waves of finite amplitude satisfying the exact boundary conditions, but only when the ratio of amplitude to wave length is quite small. On the other hand, computational studies of Michell's (deep-water) periodic wave with a corner at each crest enclosing 120° indicate that this should serve as a wave of highest amplitude-to wave-length ratio, and it has been conjectured that there should exist periodic waves of smooth profile with maximum slope spanning all angles between 0 and 30°. Krasovskii has, in fact, proved just this and even somewhat more. Let the bottom be periodic, symmetric about its maxima and minima, and steadily decreasing from maximum to minimum but with finite slope. A further condition on the bottom slope cannot be briefly stated; however, in a somewhat cruder form of the theorem one may require that the maximum angle of inclination, say α_0, be less than $\pi/6$.

Further, let L be the arclength of a period of the bottom, Q the discharge rate and c the mean velocity along the bottom. Then Krasovskii's theorem states that for any angle β such that $\alpha_0 < \beta < \pi/6$ and for any value of Q/cL there exists a steady periodic flow with this value of Q/cL, with the maxima and minima of the free surface occurring directly over those of the bottom, and with maximum angle of inclination of the free surface equal to β. A further inequality must be satisfied by this flow: the value of gL/c^2 may not exceed a certain constant depending upon Q/cL but independent of amplitude; otherwise no flow of the type described, i.e. with matching maxima and minima, exists. (There may, however, exist then another type with its crest over the minima of the bottom [see Gerber, 1957].) If the bottom is horizontal, $\alpha_0 = 0$, $L = \lambda$ and $Q = c_1 h$. In this case $g\lambda/c^2$ may not exceed $2\pi \coth(2\pi h/\lambda)$. It is especially notable here that in Krasovskii's proof there is no limitation $c^2 < gh$ as in the existence proof of Struik. Thus his existence proof includes all classes of periodic waves shown on Fig. 1 with the exception of the limiting highest waves and the solitary waves. However, words associated with particular approximation methods such as "infinitesimal wave", "Stokes wave", "cnoidal wave", etc. do not enter into his proof. (Fig. 1 is a modified version of one appearing on p. 754 of the article by the author and Laitone [1960].)

It is still an open question whether the Michell wave with corners at the crests really exists, whether Krasovskii's waves converge to it as $\beta \to \pi/6$ if it does exist, and whether there exist steady irrotational waves with maximum angle of inclination greater than $\pi/6$. Computational studies of Michell's wave in water of finite depth have been carried out by Chappelear [1959]. Figure 6 on page shows some points plotted on the basis of Chappelear's work. It is evident that they conform well with computations based upon the Stokes approximation of fifth order as long as $h/\lambda > 0.15$. Since the status of the other approximations near the same region is questionable for smaller values of h/λ, there is no computational evidence casting doubt upon the existence of Michell's waves.

It is somewhat remarkable that Krasovskii achieves his results without recourse to radically new analytic tools, although he needs to refine existing ones. After introducing the usual variables $\omega = \theta + i\tau$, $\zeta = \xi + i\eta = \rho e^{i\gamma}$ by

$$w = u - iv = c e^{-i\omega} = c e^{\tau} e^{-i\theta}, \quad \zeta = e^{-2\pi i f/Lc}, \tag{27}$$

where $f(z) = \varphi + i\psi$, the problem is reduced to finding a function $\omega(\zeta)$ analytic in a ring $\rho_0 = e^{-2\pi Q/Lc} < \rho < 1$ cut along the negative ξ-axis and satisfying

$$\begin{aligned} \frac{\partial\tau}{\partial\gamma} &= \frac{gL}{2\pi c^2} e^{-3\tau} \sin\theta \quad \text{for } \rho = 1, \quad \theta(\rho_0, \gamma) = \alpha(\lambda(\gamma)/L), \\ \frac{d\lambda}{d\gamma} &= \frac{L}{2\pi} e^{\tau(\rho_0, \gamma)}, \quad \frac{1}{2\pi}\int_0^{2\pi} e^{\tau(\rho_0, \gamma)}\, d\gamma = 1, \end{aligned} \tag{28}$$

where λ is arclength measured along the bottom from a maximum and α is the angle which the tangent at a given point makes with the horizontal. This is next reduced by a procedure

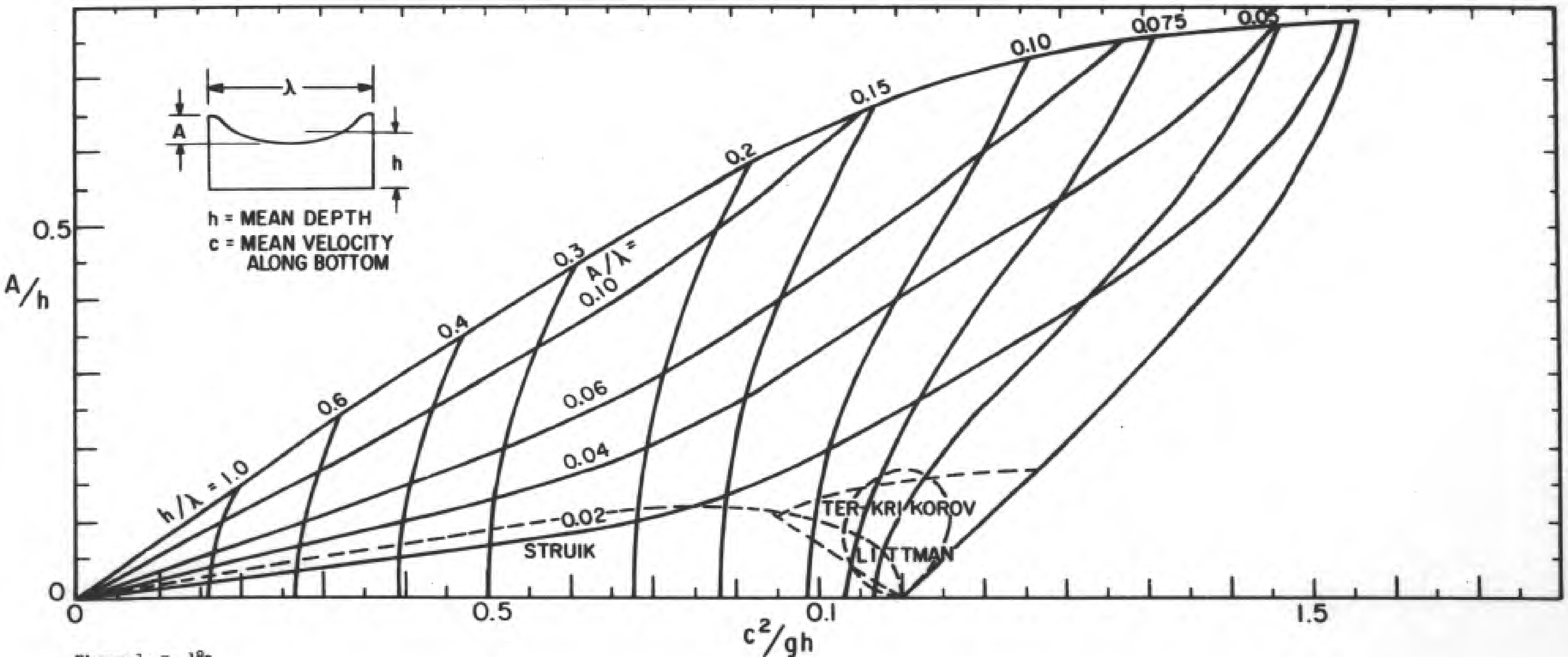


Figure 1, p. 18a

similar to that used earlier by Nekrasov and Moiseev to a nonlinear integral equation (see eq.(29) below). Krasovskii interprets this as a problem in finding for a certain nonlinear positive operator in a certain Banach space a fixed point lying in the positive cone. The hard part of the analysis consists, of course, in showing its existence under the desired circumstances.

In Fig. 1 are delineated regions showing schematically the extent of applicability of the existence proofs of Struik and of Littman. The size of the regions is not significant. However, certain geometrical characteristics are; the tangency of the Struik region to the vertical at $c^2/gh=1$ (oral communication from Littman) and of the Littman region to the two rays $k^2=0$ and $k^2=1$ emanating from $c^2/gh=1$. Although Littman's existence theorem for cnoidal waves is based upon the same approximation scheme as for the solitary wave, the theorem does not allow one to conclude the existence of solitary waves by taking a limit as $k \to 1$. A recent paper of Ter-Krikorov [1960] proves the existence of cnoidal waves in the region indicated on Fig. 1, where again only the shape of the region is intended to be correct. The method is functional-analytic. The problem is reduced to finding a fixed point of a nonlinear transformation in a certain Banach space and the existence of this is proved by showing that the transformation is contractive in a sufficiently small sphere.

The first proof of the existence of steady progressive waves, due to Nekrasov, was accomplished by first reducing the problem to the following integral equation for $\theta(\gamma)$:

$$\theta(\gamma) = \frac{1}{6\pi}\int_{-\pi}^{\pi} \frac{\mu \sin\theta(\beta)}{1+\mu\int_0^\beta \sin\theta(\alpha)\,d\alpha} K(\gamma,\beta)\,d\beta , \tag{29}$$

$$K(\gamma,\beta) = \sum_{n=1}^{\infty} \frac{2}{n} \tanh\frac{2\pi Q}{\lambda c} \sin\gamma \sin n\beta ,$$

$$\mu = \frac{3}{2\pi}\frac{g\lambda c}{q_0^3} , \quad q_0 = \tau(1) ,$$

where the variables are the same as in (27) [for references and a derivation see Wehausen and Laitone, 1960, section 32]. By assuming an expansion of $\theta(\gamma)$ in powers of the parameter $\mu' = \mu - 3$,

$$\theta(\gamma) = \sum_{n=1}^{\infty} \theta_n(\gamma)\mu'^n . \tag{30}$$

Nekrasov obtained explicit approximate formulas as well as an existence proof. However, some of his approximate formulas did not agree with those of Levi-Civita and others, and his convergence proof is in any case unpleasantly intricate. Jolas and Kravtchenko [1958; see also Jolas, 1961] have treated the equation differently. Instead of finding recursion formulas for θ_n in terms of $\theta_1, \cdots, \theta_{n-1}$, they assume that one may express θ_n as a Fourier series

$$\theta_n(\gamma) = \sum_p a_{np} \sin p\gamma , \quad \tau(\gamma) = \sum_{n,p} a_{np} \mu'^n \cos p\gamma \tag{31}$$

and substitute directly into (28). This allows a relatively straightforward procedure for computing the a_{np} and for obtaining estimates required for a convergence proof. Their approximate formulas agree with others (for infinite depth, the only case considered) and they are also able to resolve the apparent discrepancies of Nekrasov's results. As the authors suspect, their procedure is adumbrated in the excellent but not widely available treatise of Sretenskii [1936, pp. 185-189].

Recently Beckert [1962 a,b] has presented new proofs for the existence of steady periodic flows in water of infinite and finite depth and over undulating bottoms. He has essentially made use of Nekrasov's integral equation, although it is never written out as such, introduces a transformation in a suitable Banach space and shows that he may apply the Schauder-Leray theory to prove the existence of a fixed point. The results are, of course, included in Krasovskii's. Hyers and Ferling [1957] have shown the uniqueness of the solutions in deep water if the waves are sufficiently close to those of linearized theory.

The last few years have also produced several papers dealing with exact solutions for flows past a submerged disturbance. Moiseev [1957] pointed out that the standard way of approximating such flows by linearized theory draws attention away from other possible solutions. The linearized theory treats the flow past the disturbance as a perturbation of a uniform flow. However, as is evident from Fig. 1, for values of $\nu = c^2/gh$ such that $1 < \nu < \nu_0 \cong 1.6$ a solitary wave is another type of flow approaching a horizontal asymptote upstream which one might use as a basic flow to be perturbed. A formal study of the second perturbation procedure is made in the cited paper and in Moiseev and Ter-Krikorov [1958, 1959].

The question of existence of flows reducing to either a uniform flow or a solitary-wave flow when the disturbance vanishes has been considered in several papers. Ter-Krikorov [1958a] has shown that for $c^2/gh > 1$ there exist flows about a submerged vortex of sufficiently small circulation $\gamma = \Gamma/ch$ which satisfy the exact boundary conditions and which converge to a uniform flow as $\gamma \to 0$. More recently Filippov [1961] has replaced the restriction that γ be small by $\gamma < 2\cot(\pi/2\beta)$, β a constant depending upon the submersion of the vortex. Ter-Krikorov [1958b] has also devised an inverse procedure for proving the existence of similar flows about a submerged hydrofoil. Filippov [1960a] has also shown for sufficiently small γ and for $c^2/gh > 1$ that there exists an exact flow converging to a solitary wave when $\gamma \to 0$. In both cases $c^2/gh - 1$ must be small. Kyner [1962] has proved a similar result, but his disturbance is a small obstacle on the bottom. In a later paper Filippov [1960b] has assumed the existence of a maximum solitary wave with a 120° corner with a submerged vortex beneath the corner and then computed numerically the value of c^2/gh for selected values of the submergence depth and strength γ of the vortex.

The corresponding result for $c^2/gh < 1$ does not seem to have been obtained as yet. In this case the linearized theory predicts a periodic wave downstream [cf. Wehausen and Laitone, 1960, section 20]. From inspection of Fig. 1 one may surmise that such behavior also may persist when $c^2 > gh$, but that the motion then converges to a solitary wave as the amplitude of disturbance decreases.

Carter [1961a] has investigated properties of a steady two-dimensional flow in the neighborhood of an intersection of a free and fixed boundary. Let the intersection be at the origin in both the z-plane and the f-plane, the region $\psi > 0$ corresponds to the fluid, the positive φ-axis to the fixed boundary and the negative one to the free boundary (but any of the other three possible cases is also allowable). Then near $f = 0$ each of the functions $z(f), w(f), \theta(f), \tau(f)$ (in the notation of (27) with C a typical velocity) has an infinite asymptotic expansion in the variables $f^{1/2}$ and $f^{1/2}\log f$, whose term-by-term derivative is also an infinite asymptotic expansion for the derivatives, with each of the expansions converging on a sector including $0 \le \arg f \le \psi$. The expansion for $w(f)$ begins as follows:

$$w(f) = q_0 e^{-i\theta_0}\left[1 + a f^{1/2} + \left(g q_0^{-3}\cos\theta_0 + \tfrac{1}{2}a^2 - i\kappa q_0^{-1}\right) f\right] + o(f) \qquad (32)$$

where $q_0 = |w(0)|$ and θ_0 and κ_0 are the inclination and curvature, respectively, of the fixed boundary at $z = 0$. Carter [1961b] has also proved the existence of certain steady flows corresponding to the flow from an orifice in certain kinds of containers.

It is evident that the successes of the exact water-wave theory have been considerable for steady flows. Furthermore, the methods give promise of further application. However, the situation is quite different for time-dependent flows. Even the existence of a periodic standing wave awaits confirmation. Gouyon [1961] has been able to find explicit expressions for an nth approximation, but he disclaimed a convergence proof. More general time-dependent problems such as initial-value problems seem not to have been broached at all. John [1953] has given a method for constructing time-dependent exact flows, but it does not allow control of the boundary conditions.

It is evident from the complexion of the papers described above that the exact theory of gravity waves is chiefly a province for the mathematical applied mathematician.

Irrotational flows without gravity (jets and cavities). In the particular class of gravity flows considered above a horizontal surface (i.e., one perpendicular to gravity) represented a kind of norm to which one might compare the free surface itself. This situation was achieved, of course, at the expense of deliberately neglecting other types of gravity flow. (Carter's papers are an exception.) In the flows customarily treated in the theory of jets and cavities there is no such simple norm except in certain problems of impact on water where gravity is neglected during a short time interval. However, there are several configurations which may serve as prototypes for these flows. For example, we may take the following: jet issuing from an orifice or striking an object, cavity formed behind a moving body, impact of a body on a water surface, collapse of a vapor cavity in a liquid. The first two can, but need not, be treated as steady flows, the last two are by nature unsteady. In each case one would like to know the pressure distribution on solid boundaries and the geometry of the flow.

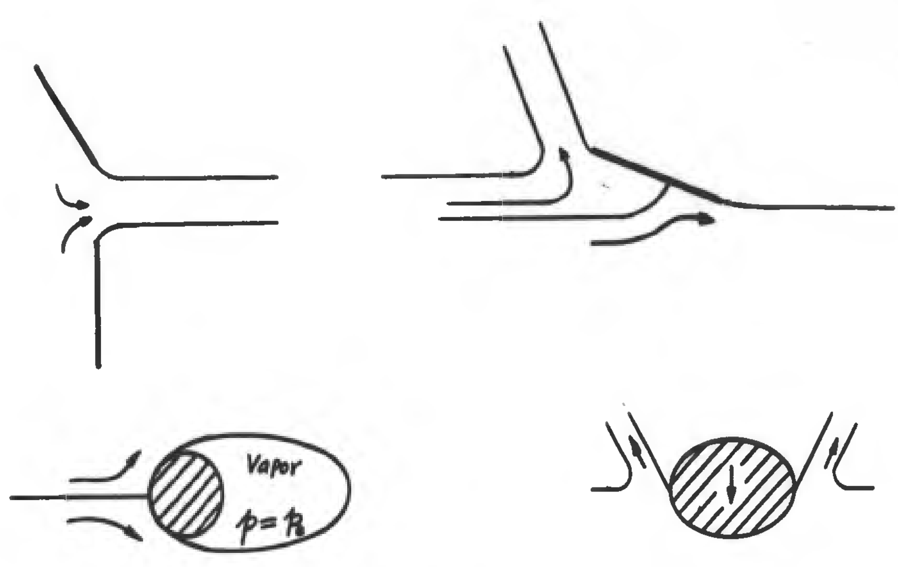


Fig. 2 -

In the jet problems and in the impact problems there are no physical constants associated with pressure, so that an arbitrary ambient pressure can be imposed on the whole flow without changing its geometry. It is usual here simply to take $p=0$ on the free boundaries. The situation is clearly different with the vapor cavity. In fact, if p_0, V_0 are pressure and velocity, respectively, far upstream, then the character of the flow will depend upon the dimensionless number

$$\sigma = (p_0 - p_c) \Big/ \tfrac{1}{2} \rho V_0^2 \quad , \tag{33}$$

the "cavitation number". The other cases correspond to having $\sigma = 0$.

If the surface tension is neglected, one finds immediately from (15) the boundary condition on the cavity wall:

$$\frac{1}{2} V_c^2 + \Phi_t = C(t) , \tag{34a}$$

or, if the motion is steady,

$$\left(\frac{V_c}{V_0}\right) = 1 + \sigma . \tag{34b}$$

In the latter case V_c is evidently constant on the cavity wall. We have spoken here in terms of cavities. However, the same condition is, of course, satisfied on the free surfaces in the other situations, but usually with $\sigma = 0$. Without neglect of surface tension, (34a) and (34b) become

$$\frac{1}{2} V_c^2 + \Phi_t - T'\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = C(t), \quad T' = T/\rho , \tag{35}$$
$$\left(\frac{V_c}{V_0}\right)^2 = 1 + \sigma + \frac{T}{\frac{1}{2}\rho V_0^2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

It is well known [see e.g. Gilbarg, 1960, sect. 4] that, as long as one adheres to the mathematical model above, a steady flow with a finite cavity $(\sigma > 0)$ is not possible. This has lead to the construction of various mathematical models which represent somewhat imperfectly the physical situation. The reentrant jet model (see Fig. 3, left) is perhaps the most satisfactory in that these jets have frequently been observed. In fact, if the jet could somehow be

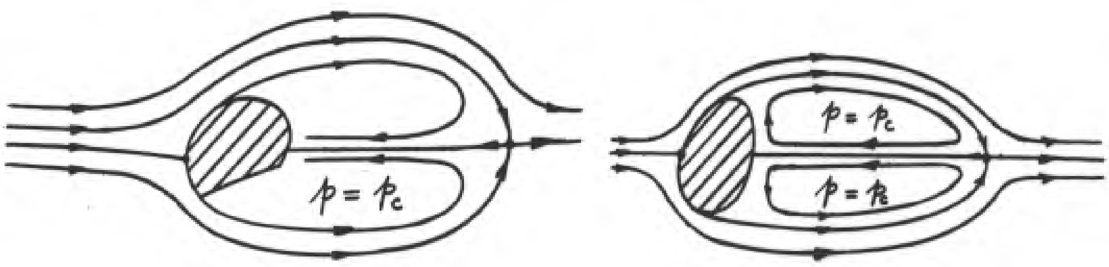


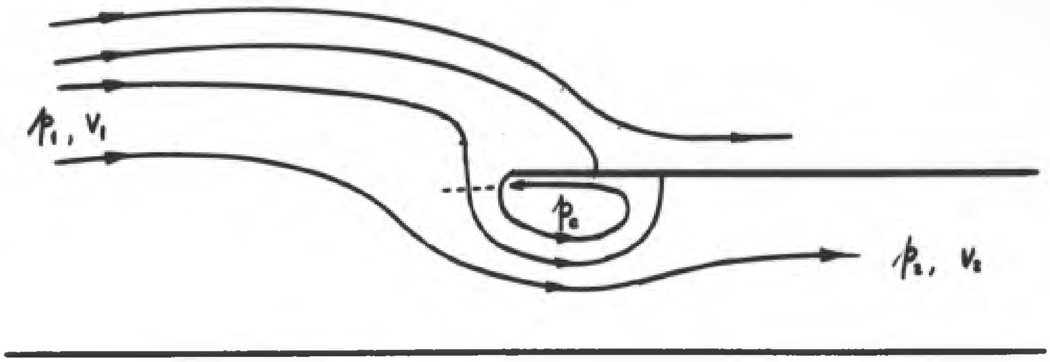
Fig. 3

removed through the body, this would appear to be a satisfactory model. However, that it is, in fact, not completely satisfactory is attested to by the continued appearance of proposed new models. Recent ones are due to Wu [1962] and Lavrent'ev[1962] (see Fig. 3, right).

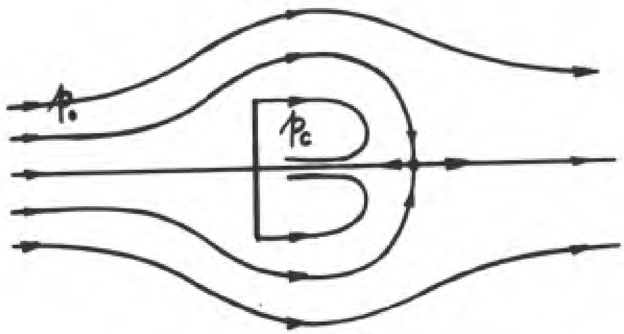
We note parenthetically that in the presence of surface tension or of gravity a finite cavity is not excluded in a steady flow. In view of the second condition, one may also hope to find finite cavities behind accelerated bodies. Yih [1960] has, in fact, constructed examples of both for two-dimensional motion.

The reëntrant-jet model is still not completely defined until the position of detachment of the cavity wall from the body is specified. If the body has a sharp corner, this is usually taken as the position of detachment (bottom part of Fig. 3, left); if the body is everywhere smooth, one generally requires finite curvature of the free surface at the detachment (in which case the curvatures of the free surface and body are equal).

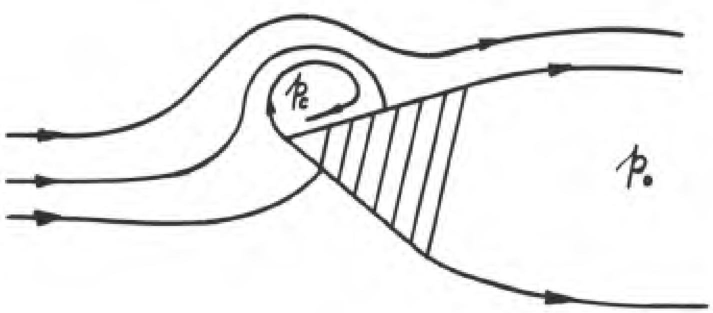
This formulation of the problem is clearly designed to provide unique solutions within the general framework of inviscid-fluid theory and without adding features without physical counterpart (as in the Riabouchinsky model; the Roshko-Eppler model represents wakes rather than cavities). However, it may be criticized in various ways, and one may find in Birkhoff [1960] a discussion of its shortcomings. In fact, this paper is chiefly devoted to a general critique of the present status of the theory of jets and cavities, together with some suggested directions for research.



(a)



(b)



(c)

Fig. 4

Nevertheless, it is this model which has been most actively investigated by applied mathematicians. Once the model has been accepted, the mathematical problem is almost identical with that of steady jets. Methods of solving these problems, insofar as they can be solved, are expounded in Birkhoff and Zarantonello [1957], Gilbarg [1960], and Gurevich [1961].

The exact problems which have received attention since the first two expositions cited above are two-dimensional and are generally modifications of known flows to include finite cavitation numbers. For example, Figures 4a, b, c, caricaturize recent papers of Pykhteev [1960], Kuznetsov [1962], and Cox and Clayden [1958], respectively. Gurevich [1961b] has considered the classical problem of a two-dimensional jet issuing from a vertical wall, taking into account surface tension on the jet. His method is an approximate one so that its discussion belongs properly in a later section; however, we give his main result here: If 2ℓ is the width of the opening and V_0 the final velocity of the jet, then the contraction coefficient is approximately

$$\frac{\pi}{\pi+2}\left\{1+\frac{0.318}{\rho\ell V_0^2}\left[2\ln\frac{0.39\,\rho\ell V_0^2}{T}+6.52\right]\right\}.$$

Salamatov [1959] has considered the three-dimensional jet issuing from a conical container with its tip cut off. Its solution is reduced to that of an integro-differential equation which is then solved numerically by successive approximations; for the case of a cone of half-angle $\pi/4$ he finds a contraction coefficient 0.869. Garabedian [1960] has made a rather general study of contraction coefficients for jets from straight walls. For a circular opening he finds a contraction coefficient 0.58. For some other configurations see Levey [1960] and Wagner [1958].

Fedorov [1958] has developed the theory of flow about a flat-plate glider like that shown in Figure 2 except that he allows detachment to take place on the upper surface of the plate, thus adding another parameter to the problem. The results of a number of computations of free-surface profiles, force coefficients, etc. are shown graphically.

Problems of impact are usually treated with neglect of gravity because the time interval during which the significant events connected with the impact occur is short compared with, say, V/g, where V is the impact velocity. However, the treatment may be subdivided still further as follows. In some formulations one assumes that initially the body is at rest in a steady flow with a cavity, or else at rest on the surface of fluid also at rest. At time $t = 0$ the body acquires impulsively a velocity $\underline{V}$ and angular velocity $\underline{\omega}$. One then asks for the pressure distribution over the body at that moment as a result of the impulsive motion. The results are usually presented as added mass coefficients. The more interesting, but more difficult, impact problem treats the whole time history up to the time when gravity can no longer be neglected.

In the first type of problem one may introduce a separate potential, say φ_1, to describe the fluid motion resulting from the impulse. The associated impulsive pressure is $p_1 = -\rho\varphi_1$; hence the boundary condition for φ_1 on the free surface is $\varphi_1 = 0$ and on the body is, as usual, $\partial\varphi_1/\partial n = \underline{v}\cdot\underline{n} + (\underline{\omega}\times\underline{r})\cdot\underline{n}$. Parkhomovskii [1958, 1959, 1960] has recently treated several two-dimensional problems of this sort. Rogozhin [1959] has considered the inverse problem in which the impulsive pressure along a contour is prescribed and the form of the contour then found. Mossakovskii and Rvachev [1958] compute the added mass for impulsive horizontal motion of a half-submerged sphere and for a sphere half full of liquid.

In the other type of impact problem one prescribes the motion of the solid body and satisfies on the free surface the boundary condition

$$\frac{1}{2} v^2 + \Phi_t = 0 .$$

The mathematical problem is known for its difficulty in the general case. However, for wedges or cones striking normally against a plane surface similarity considerations simplify the problem [cf. Gilbarg, 1960, sect. 20]. Two recent papers have made use of the possibilities of similarity in a somewhat different way. Cumberbatch [1960], in order to estimate the force of a wave breaking against an obstacle, considers a wedge of water moving with constant velocity against a plane. He approximates the solution at large distances from the wall and at large distances from the wedge, and matches these in order to find an approximate solution over the whole region. Numerical results are given for wedges of apex angle $22\frac{1}{2}^{\circ}$ and

45°. Borisova, Koryavov and Moiseev [1959] consider either a wedge or a cone of liquid, but suppose it to be struck by another solid wedge or cone, respectively, moving according to a power law: if the y-axis is the axis of symmetry, and the solid body is above, then $v = -ct^{\gamma}$. After an appropriate change of variables, these authors show that, if the shape of the free surface is known, the velocity field can be found from an ordinary differential equation, integrable by quadratures for certain values of γ (e.g., $\gamma = 0$), and thence the force on the entering wedge or cone. Their scheme for finding the shape of the free surface is to assume that it can be approximately represented by a given three-parameter family of functions and then to fix the parameters by conditions at infinity and by conservation of mass. Computed results for wedges and cones penetrating a horizontal surface are in good agreement with earlier results. Application of the procedure to certain other problems of the impact of jets is also discussed.

The calculation of flows associated with water entry of other shapes requires approximations of considerably bolder type. One can find a review of various procedures and a rather extensive bibliography supplementing those in Gilbarg [1960] and Birkhoff and Zarantonello [1957] in a review paper by Chu and Abramson [1961]. Although the qualitative aspects of such flows are fairly well understood, the computation of pressure distribution and free-surface shape will probably continue to rely upon rather unappealing approximation methods until such time as the complete problem can be handled conveniently by digital computers.

There exist other possible topics in unsteady flow which we have not touched upon. Various problems associated with unsteady cavity flow are discussed in Wu [1960]; the paper of Yih [1960] has already been mentioned. Unsteady motion of jets appears to have received very little theoretical attention. Two recent papers on the collapse of cavities by Benjamin [1958] and Hunter [1960] are aimed at elucidating the role of compressibility of the water.

Approximate Solutions

It is quite evident from the preceding discussion of exact solutions that the class of configurations for which either explicit solutions can be obtained or for which the existence of solutions can be proved is extremely small in comparison with those for which one would like solutions, either to explain geophysical phenomena or for use in engineering design. As a result, approximate ways of finding solutions have been developed for which much more extensive classes of problems can be treated. This is not to say that approximate methods of computation have not been used for some of the "exact" problems of the preceding section, for they have. However, in such cases the method of approximation does not effect the formulation of the problem (or at least one hopes not) but is aimed at providing by means of numerical analysis as close a solution as possible to the problem in its exact formulation. The spirit of the approximations to be discussed here is somewhat different: The exact formulation will be replaced by another one for which the solutions can be considered to be approximate solutions to the exact problem only under certain circumstances connected with the geometry of the configuration under study.

The common procedure for generating this type of approximation is that of small perturbations. One starts with some known flow and assumes that the desired one deviates only a little from the known one. The measure of the deviation must, of course, be some dimensionless parameter ε associated with the deviated flow or with the cause of the deviation, and its vanishing with the vanishing of the deviation. It is allowable to change the fundamental variables before introducing the deviation or even to have the parameter ε involved in the transformation.

The procedure is more easily illustrated than described. Since the computations tend to be rather painful in their most general form, we shall carry them through here only in a simple case. More extensive ones may be found in Wehausen and Laitone [1960, sections 10, 27, and 31]. As an example we shall use the problem formulated in eq. (21). First the infinitesimal-wave and then the shallow-water or long-wave approximation will be derived. However, it doesn't seem appropriate here to carry the details very far. In each case the fundamental flow is the shear flow shown in (26), which we shall denote by $u_0(y)$ when given as a function of y.

For the <u>infinitesimal-wave theory</u> one assumes that

$$\begin{aligned} \psi &= \psi_0(y) + \varepsilon\psi_1(x,y) + \varepsilon^2\psi_2(x,y) + \cdots , \\ \eta &= \varepsilon\eta_1(x) + \varepsilon^2\eta_2 + \cdots , \\ Q &= Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \cdots . \end{aligned} \tag{36}$$

Then also, if one assumes $f(\psi)$ analytic,

$$f(\psi) = f(\psi_0) + \varepsilon\psi_1 f'(\psi_0) + \cdots . \tag{37}$$

Substitution in (21) and equating like powers of ε yield successively

$$\varepsilon^0: \ \psi_{0yy}(y) = f(\psi_0), \ \psi_0(0) = 0, \ \psi_0(-h) = -Q_0, \ \psi_{0y}^2(0) = \text{const.},$$

$$\varepsilon^1: \ \Delta\psi_1(x,y) = \psi_1 f'(\psi_0), \ \psi_1(x,-h) = -Q_1, \tag{38}$$

$$\psi_1(x,0) + \psi_{0y}(0)\eta_1(x) = 0, \ \psi_{0y}(0)\psi_{1y}(x,0) + [\psi_{0y}\psi_{0yy} + g]\eta_1 = 0,$$

$$\cdots$$

The stream function $\psi_0(y)$ gives, of course, the fundamental flow which we are perturbing:

$$\int_{-Q_0}^{\psi_0} \left[c^2 + \int_{-Q_0}^{\psi} f(\tau)\,d\tau\right]^{-1/2} d\psi = y + h. \tag{39}$$

Since $\psi_{0y}(y) = u_0(y)$, the equations for ψ_1, may be put into the following forms:

$$\Delta\psi_1(x,y) = \frac{u_0''}{u_0}\psi_1, \ u_0^2(0)\psi_{1y}(x,0) - [g + u_0(0)u_0'(0)]\psi_1(x,0) = 0 \tag{40}$$

$$\psi_1(x,-h) = -Q_1 .$$

The equations for $\psi_2, \psi_3, \cdots$ become more and more complex. Explicit periodic solutions for ψ_1 can be found without difficulty for vorticity distributions of the form $f(\psi) = a\psi + b$ (i.e., u_0''/u_0 = const.) It would have been advantageous in some respects to have applied the expansion scheme to the problem as reformulated in eq. (22). This is, in fact, essentially what Gouyon did in assuming the expansions (25).

We turn next to the other classical approximation procedure, the <u>shallow-water theory</u>. For this one makes a preliminary change of variable which involves the perturbation parameter ε : $\bar{x} = \varepsilon x$, $\bar{y} = y$, $\psi(x,y) = \bar{\psi}(\bar{x},\bar{y})$, $\eta(x) = \bar{\eta}(\bar{x})$.

Then the equations (21) take the following form:

$$\begin{aligned}
&\varepsilon^2 \bar{\psi}_{\bar{x}\bar{x}} + \bar{\psi}_{\bar{y}\bar{y}} = f(\bar{\psi}), \\
&\bar{\psi}(\bar{x}, \bar{\eta}(\bar{x})) = 0, \quad \bar{\psi}(\bar{x}, -h) = -Q, \qquad (41) \\
&\varepsilon^2 \bar{\psi}_{\bar{x}}^2(\bar{x}, \bar{\eta}(\bar{x})) + \bar{\psi}_{\bar{y}}^2 + 2g\bar{\eta}(\bar{x}) = \text{const.}
\end{aligned}$$

We now assume once again series expansions, but in power of ε^2

$$\begin{aligned}
\bar{\psi}(\bar{x},\bar{y}) &= \bar{\psi}_0(\bar{y}) + \varepsilon^2 \bar{\psi}_1(\bar{x},\bar{y}) + \varepsilon^4 \bar{\psi}_2 + \cdots, \\
\bar{\eta}(\bar{x}) &= \varepsilon^2 \bar{\eta}_1(\bar{x}) + \varepsilon^4 \bar{\eta}_2(\bar{x}) + \cdots, \qquad (42) \\
Q &= Q_0 + \varepsilon^2 Q_1 + \varepsilon^4 Q_2 + \cdots, \\
f(\bar{\psi}) &= f(\bar{\psi}_0) + \varepsilon^2 \bar{\psi}_1 f'(\bar{\psi}_0) + \varepsilon^4 \left[\bar{\psi}_2 f'(\bar{\psi}_0) + \tfrac{1}{2} \bar{\psi}_1^2 f''(\bar{\psi}_0)\right] + \cdots.
\end{aligned}$$

After arranging terms according to powers of ε^2, one obtains successively the following equations:

$$\varepsilon^0:\ \bar\psi_{0\zeta\zeta}(\zeta) = f(\bar\psi_0),\ \bar\psi_0(0) = 0,\ \bar\psi_0(-h) = -Q_0,\ \bar\psi_{0\zeta}^2(0) = \text{const.};$$

$$\begin{aligned}\varepsilon^2:\ &\bar\psi_{1\zeta\zeta}(\bar x,\zeta) = \bar\psi_1 f'(\bar\psi_0),\quad \bar\psi_1(\bar x,-h) = -Q_1,\\ &\bar\psi_1(\bar x,0) + \bar\psi_{0\zeta}(0)\bar\eta_1(\bar x) = 0,\ \bar\psi_{0\zeta}(0)\left[\bar\psi_{1\zeta}(\bar x,0) + \bar\eta_1\bar\psi_{0\zeta\zeta}(0)\right] + g\bar\eta_1 = 0;\end{aligned} \tag{43}$$

$$\begin{aligned}\varepsilon^4:\ &\bar\psi_{1\bar x\bar x} + \bar\psi_{2\zeta\zeta} = \bar\psi_2 f'(\bar\psi_0) + \tfrac{1}{2}\bar\psi_1^2 f''(\bar\psi_0),\quad \bar\psi_2(\bar x,-h) = -Q_2,\\ &\bar\psi_2(\bar x,0) + \bar\eta_1\bar\psi_{1\zeta} + \tfrac{1}{2}\bar\eta_1^2\bar\psi_{0\zeta\zeta} + \bar\eta_2\bar\psi_{0\zeta} = 0,\\ &2\bar\psi_{0\zeta}(0)\bar\psi_{2\zeta}(\bar x,0) + \bar\psi_{1\zeta}^2 + 2\left[\bar\psi_{1\zeta}\bar\psi_{0\zeta\zeta} + \bar\psi_{0\zeta}\bar\psi_{1\zeta\zeta}\right]\bar\eta_1 + \left[\bar\psi_{0\zeta}\bar\psi_{0\zeta\zeta\zeta} + \bar\psi_{0\zeta\zeta}^2\right]\bar\eta_1^2\\ &\qquad + 2\left[\bar\psi_{0\zeta}\bar\psi_{0\zeta\zeta} + g\right]\bar\eta_2 = 0;\end{aligned}$$

The solution for $\bar\psi_0$ is the same as (39), as we expect since the fundamental flow is the same in either case. The equations for $\bar\psi_1$ and $\bar\psi_2$ can be made somewhat neater by eliminating $\bar\eta_1$ and $\bar\eta_2$ and by introducing $u_0(\zeta)$ as was done in (40), but this will be postponed until later. As we shall see then, the equations (43) may be used to derive both solitary and cnoidal waves on a shear flow. As in the other approximation method, this one could also, and somewhat more conveniently, be applied to the equation (22), but again at the expense of not producing formulas which are immediately familiar. One advantage of starting with (22) is that one may avoid assuming analyticity of the function $f(\psi)$.

Both approximation schemes lead to solutions which approach a horizontal shear flow as $\varepsilon \to 0$. However, the physical significance of ε is different in the two cases. In the infinitesimal-wave approximation one may associate ε with the maximum slope of the waves, and when $\varepsilon \to 0$, the flow converges to the flow $u = u_0(y), v = 0$, but with $u_0(0) < u_{crit.}$. On the other hand, for the other approximation, as $\varepsilon \to 0$ the flow converges to the flow $u = u_0(y), v = 0$ with $u_0(0) = u_{crit.}$, and ε is associated with the deviation of $u_0(0)$ from $u_{crit.}$.

One might reasonably ask whether one could not obtain other classes of useful approximate solutions by making a different preliminary transformation of variables. Inspection of Fig.1 indicates that this is probably not the case for irrotational flow as long as one is perturbing a uniform flow. Moiseev and Ter-Krikorov [1959, p.39] state that this can be proved. However, there are other possibilities, and one of these has been mentioned earlier: the fundamental flow need not be a horizontal flow, it could also be an exact solitary-wave flow, or, as far as the general procedure is concerned, any other known exact flow.

There are other points concerning the approximation procedures above which deserve mention. One is primarily mathematical. The function $\psi(x, y)$ is defined initially only where there is fluid; yet in (38) and (43) values of $\psi_i(x, 0)$ enter into the formulas, even though for some values of x the point $(x, 0)$ may be outside the fluid. One must evidently assume that the region of definition of ψ can be enlarged; again, use of (22) as a starting point would have avoided this.

The next point concerns the nature of whatever is disturbing the fundamental flow, a matter touched upon at the beginning of this section. The choice of the parameter ε must evidently describe some characteristic quantity of the disturber which is such that when this quantity vanishes the disturbance of the surface vanishes also. For a submerged body this might, for example, be the ratio of some characteristic length to the depth of submersion, or it might be the ratio of some characteristic length in a direction perpendicular to the flow to the greatest one parallel to the flow. Evidently the function describing the flow, say ψ in (21), will have ε as a parameter, i.e. $\psi(x, y, \varepsilon)$. In finding the conditions to be satisfied by various functions $\psi_i(x, y)$ in the expansion in powers of ε, one may anticipate that the boundary condition for ψ_i on the disturbing body will not be the same as that for ψ itself. It is conceivable, of course, that one might also approximate the exact solution by segments of a series $\sum \varepsilon^i \psi_i$ in which the ψ_i satisfy their proper boundary conditions associated with the free surface but with ψ_1 satisfying the same condition as ψ itself on the disturbing body and the remaining ψ_i other conditions conforming appropriately to this initial one. However, there would be no reason to assume in general that the first n terms of this approximation will be more accurate than the same number of terms of the usual one, unless, of course, special circumstances prevail and numerical estimates show this to be the case. Since each configuration requires its own analysis with respect to the proper boundary conditions, further discussion of this point will be generally omitted.

Finally, one is naturally curious as to the status of the series in ε^n. In what sense do they converge, if at all? Since a convergence proof would be also an existence proof for an exact solution, it is evident from the earlier discussion of exact solutions that such proofs do not exist except for a rather small class of configurations. However, in certain cases it has been proved that the infinitesimal-wave approximation converges, as for example in the cited work of Gouyon, and it is not unreasonable to conjecture that this will also turn out to be the case with more complicated boundary configurations. Experimental evidence does not contradict this conjecture. The situation with respect to the long-wave approximation method is not quite similar, for the convergence of the series is at most asymptotic and it is even possible that taking more terms in the series may make the result less rather than more accurate [cf. Ivanilov, Moiseev and Ter-Krikorov, 1958].

In the following we shall consider first some recent results in the first-order infinitesimal-wave approximation, often called linearized theory, next, higher-order version of this approximation ("Stokes theory"), and finally, the long-wave and other approximations. As one might expect, the linearized theory is by far the richest in the variety of problems which it can successfully solve, in fact, so much so that we can hardly do more than roughly classify them.

First-order ("linearized") theory.

The number of papers of substantial content which have appeared in recent years and which employ linearized theory are so numerous that it seems out of the question to try to summarize them individually or to give any mathematical details. Rather, we shall attempt to group them into classes of related papers, hoping that the titles are sufficiently descriptive beyond that, and only occasionally point out a noteworthy advance. Many of the papers overlap and some rediscover results already published. We shall make no attempt to single these out, for to do so would involve documenting the statement and this again would lead us into bypaths.

There has been a noticeable shift in the types of problems which are receiving emphasis. For example, there has been a large increase in the number of papers devoted to the study of sloshing of fluids in both fixed and moving tanks, to the investigation of stability of motion of a body containing a liquid with a free surface, and to stability of the free surface itself; in some cases the whole configuration is rotating. The motivation for these studies is obvious, and there undoubtedly exists a large body of additional literature in report form. Geophysical problems have led to a renewed interest in internal waves in stratified fluids and to waves in rotating basins. Studies of ship waves have moved from resistance in calm water to the more difficult problem of motion and resistance in waves or else to other complications like stratified fluids or underlying shear flows; high-speed computers have made possible numerical computation of both older and more recent results. In the theory of cavitation flow about hydrofoils Tulin's linearization of the problem transformed what was chiefly an academic problem into an important tool for designers and at the same time stimulated a considerable amount of further research.

The waves which one observes in the laboratory are not, of course, familiar with methods or degree of approximation. However, since many experiments are carried out in order to determine whether or not the measurements are in agreement with some theoretical computations, it seems reasonable to include these papers together with those whose computations have the same basis.

For the simplest gravity-wave problems, if one assumes the flow irrotational, the linearized boundary condition for the free surface is easily derived from (10) and (15):

$$\Phi_{tt}(x,0,z,t) + g\,\Phi_y + T'\Phi_{yyy} = -\frac{1}{\rho}\,p_{At}(x,z,t), \tag{44}$$

where p_A is a given atmospheric pressure, and $T' = T/\rho$. The free surface itself, $y = \eta(x,z,t)$, may be obtained from

$$\eta_t(x,z,t) = \Phi_y(x,0,z,t) \tag{45}$$

or from the differential equation

$$T'(\eta_{xx} + \eta_{zz}) - g\eta = \Phi_t(x,0,z,t) + \frac{1}{\rho}\,p_A(x,z,t),$$

which is evidently the simpler form if surface tension may be neglected. In these formulas the coordinates are inertial and the fluid at rest at infinity. If there is a uniform flow of velocity c in the direction Ox with a superposed wave, it follows immediately from (40) that

$$c^2 \psi_y(x,0) - g\psi = 0, \qquad \eta(x) = -\frac{c}{g}\psi_y(x,0) \tag{46}$$

or, in terms of φ when the flow is irrotational,

$$c^2 \varphi_{xx}(x,0) + g\varphi_y = 0, \qquad \eta = -\frac{c}{g}\varphi_x(x,0). \tag{46'}$$

The situation for flows referred to moving coordinates requires some discussion of cases. If the absolute velocity field is irrotational in the fixed coordinate system, it remains so in a moving coordinate system, whereas the relative velocity $\underline{V}'$ may not be since [see (6)]

$$\operatorname{curl} \underline{V} = \operatorname{curl} \underline{V}' + 2\,\underline{\omega} \tag{47}$$

It is not possible to discuss here all situations likely to be met with. However, several modifications will be discussed below in connection with special problems.

<u>Methods</u>. It is probably fair to state that by now the methods for solving most boundary-value problems with linearized boundary conditions are fairly standard. The

solutions are generally obtained by using Fourier or Laplace transforms or expansion in eigenfunctions, by reflection methods, or by use of Green's functions (singularity distributions). Since the Green's functions themselves are usually found by one of the other two methods, the techniques fundamentally reduce to the first two. The various approaches are illustrated in chapter D of Wehausen and Laitone [1960]. Recent papers on Green's functions are by Jinnaka [1960] and Kostyukov [1959b]. MacCamy [1959] deals with a difficulty in asymptotic expansions of solutions for certain water-wave and other problems.

General properties of linearized irrotational waves: One of the most striking properties of waves of the type under discussion is their dispersiveness, that is, free harmonic waves of different wave lengths travel with different velocities, a property shared also by the exact periodic solutions discussed earlier. It is eash to derive this property from, say, (44) with p_A = const.. Suppose that there is also a horizontal bottom at $y = -h$. Then one may easily verify that

$$\Phi = A \cosh k(y+h) \sin(kx - \sigma t + \tau) \tag{48}$$

is a solution of $\Delta\Phi = 0$ and satisfies the boundary conditions, provided that

$$\sigma^2 = (gk + T'k^3) \tanh kh. \tag{49}$$

The phase velocity of the wave is

$$C = \frac{\sigma}{k} = \sqrt{\left(\frac{g}{k} + T'k\right) \tanh kh} \quad . \tag{50}$$

One of the interesting consequences of wave dispersion and the linearized approximation is that an initial configuration of the surface, say some sort of hump, will not preserve its form. Rather, the various harmonic components in its Fourier analysis will each move with its appropriate velocity, so that after a sufficiently long time all of these components will have sorted themselves out and will be individually identifiable. This is easily observed after throwing a stone into a pond. Another phenomenon now presents itself. If in the sorted-out waves an observer fixes his attention upon one wave length and moves along with a crest, he will find that he is moving with the velocity appropriate to waves of that length. However, he will also find that he has soon moved away from the waves of that length and into a faster moving group of waves, his own velocity having in the meantime increased as he followed the crest. If he had wished to remain with the original group of waves, he would have had to travel with a different velocity, the so-called <u>group velocity</u>, defined by

$$U_G = \frac{d\sigma}{dk} \quad . \tag{51}$$

It is not surprising to find that the energy associated with the various wave lengths also travels with the group velocity, keeping pace with the waves themselves.

The definitions above are for plane flow, but are easily generalized: If $\underline{k} = (k_1, k_2)$ is the wave-number vector and $\sigma = \sigma(k_1, k_2)$, which has the form $\sigma(|k|)$ for linearized water waves, then phase and group velocity are, respectively,

$$\underline{c} = \frac{\sigma}{|k|}\frac{\underline{k}}{|k|}\ , \quad U_G = \operatorname{grad}\sigma = \sigma'(|k|)\frac{\underline{k}}{|k|}\ . \tag{51'}$$

The discovery of these properties goes back at least to Kelvin. However, the matter continues to attract attention and to be the subject of published papers, some with the aim of elucidating the phenomenon, others with application of the notions to particular situations. The subject has been recently dealt with in the following: Aris [1958], Becker [1961], Lighthill [1960], Longuet-Higgins and Stewart [1960,1961], Ursell [1960], Warren [1961], Wehausen and Laitone [1960, sect.15], Whitham [1960, 1961].

Waves from ships and moving bodies. The investigation of ship waves is made considerably more complicated if one treats the more general situation of a ship moving on an undulating sea and includes the effect of the sea upon the ship as well as of the ship upon the sea. In this setting it follows naturally from the assumption of infinitesimal waves that the ship motions will be small about a reference system moving with the ship unless a resonance occurs. If this coordinate system is denoted by $Oxyz$ and moves with velocity $c(t)$ in the Ox direction with respect to the original fixed coordinate system, then the free-surface boundary condition for the velocity potential for the absolute velocity field referred to $Oxyz$ is

$$\Phi_{tt}(x,0,z,t) - 2c\Phi_{tx} + c^2\Phi_{xx} - \dot{c}\Phi_x + g\Phi_y = 0, \tag{52}$$

as follows easily from a perturbation expansion applied to (15b) and (9') for $F = y - \eta(x - \int^t c(\tau)d\tau, z, t)$. If c is constant and the motion has reached a steady state, one obtains from (52) the more usual boundary condition [cf. (46')]

$$c^2\Phi_{xx}(x,0,z) + g\Phi_y = 0. \tag{53}$$

Either one of these boundary conditions must be accompanied by a further one concerning the nature of the motion far ahead of the body. For (53) this takes the simple form that $\sqrt{R}\Phi_x \to 0$ as $x \to \infty$, where $R^2 = x^2 + y^2$; for (52) it will be either this or a requirement that the waves be purely outgoing, depending upon the circumstances.

There is still another aspect to the linearization problem for moving ships: oncoming small waves cause small ship motions, but the forward motion of the ship must also cause only small waves. The latter may be accomplished by having a deeply submerged ship, i.e. a submarine, and in this case proper boundary conditions on the body are the exact boundary conditions. However, as has been mentioned earlier, this goal may also be attained by having the body present to the flow only a small cross-section area relative to its total area.

This may be achieved by having the body thin in some direction perpendicular to the direction of motion, and leads to both "thin-ship" and "thin-hydrofoil" theory. (The theory of thin cavitating hydrofoils will be considered later.) In the first of these, for example, if the equation of the ship hull is given by $y = \pm f(x, z)$ for (x, z) in S, the linearized boundary condition for Φ for motion in calm water is

$$\Phi_z(x, y, \pm 0, t) = \mp \frac{C(t)}{2\pi} f_x(x, z), \quad (x, z) \text{ in } S. \tag{54}$$

Here S is the area bounded by the profile of the ship in its equilibrium position and by the undisturbed water surface. We shall not formulate the more difficult problem of a "thin" ship moving with a given thrust through waves. This was first formulated in a systematic fashion by Peters and Stoker [1958] who expanded in terms of a single parameter. Newman [1961a], using two parameters, one describing the incoming waves and the other the thinness of the ship, was also able to carry through a consistent theory which in several ways is more satisfactory but which retains terms which would be of second order in the Peters-Stoker development.

The ratio of cross-section area to total area will also be small for a "slender" body. A systematic development of this idea for ships has recently geen carried through by Vossers [1962] and further developed by Maruo [1962] and Ursell [1962].

A classical problem not specifically connected with the form of a ship, but still connected with ship waves, is that of the wave pattern behind a pressure point moving along the free surface. This has been investigated again recently by Ursell [1960] with the aim of elucidating more precisely the behavior of the surface near the path directly behind the point and near the critical lines forming an angle $\arcsin \frac{1}{3}$ with it. Becker [1961] also considers this problem. The remaining papers listed below deal specifically with either moving ships, moving hydrofoils, or similar boundary-value problems.

Ships: Becker [1961], Bessho [1959, 1960, 1961], Constantine [1960], Cummins [1962], Eggers [1960, 1962], Grim [1960, 1962], Guilloton [1960, 1962], Haskind [1957, 1959], Hudimac [1961], Kaplan [1957, 1959], Karp, Kotik and Lurye (1900], Kirsch [1962], Kolberg [1959], Kostyukov [1959], Kotik and Mangulis [1962], Kurlovich [1961], Maruo [1957, 1959, 1960, 1962], Newman [1959, 1961], Newman and Poole [1962], Perzhnyanko [1960], Plesset and Wu [1960], Sabuncu [1961], Sharma [1962], Sizov [1961], Sretenskii [1959], Timman and Newman [1962], Timman and Vossers [1960], Ursell [1960, 1962], Vossers [1962a,b], Warren [1961], Webster and Wehausen [1962], Wehausen [1961], Yeh and Martinek [1962].

Hydrofoils (non-cavitating): Breslin [1961], Isay [1961], Krappinger [1959], Moiseev [1958], Nishiyama [1962], Panchenkov [1960, 1961, 1962], Shebalov [1962].

Steady flow past obstructions: Abdylaev [1960], Binnie [1960], Crapper [1959, 1960].

<u>Forced harmonic waves</u>. As the title suggests such waves result from some action upon the fluid, the motion of a body or other solid boundary or a fluctuating pressure applied to a portion of the surface. The situation which is chiefly of interest is that of harmonic motion. If the motion has proceeded for such a long time that initial transients may be neglected, one may assume

$$\Phi(x,y,z,t) = \varphi_1(x,y,z)\cos\sigma t + \varphi_2(x,y,z)\sin\sigma t = \mathcal{R}e\,\varphi e^{-i\sigma t}, \qquad (55)$$
$$\varphi = \varphi_1 + i\varphi_2 .$$

Then the boundary condition (44) (with $T' = 0$) becomes

$$-\sigma^2\varphi(x,0,z) + g\,\varphi_y = \begin{cases} 0 & \text{if } p_A = 0, \\ -i\sigma p(x,z) & \text{if } p_A = \mathcal{R}e\, p e^{-i\sigma t},\ p = p_1 + i p_2 . \end{cases} \qquad (56)$$

The first case is for waves generated by a moving body, the second for those generated by a fluctuating pressure distribution. In either case a "radiation condition" must be added, i.e. a requirement that the waves be outgoing; without this the solution will not be unique. For waves generated within a region of bounded extent and propagating into an unbounded three-dimensional region the radiation condition is given by

$$\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial\varphi}{\partial R} - i\frac{\sigma^2}{g}\varphi\right) = 0, \quad R^2 = x^2 + z^2. \qquad (57)$$

The appropriate radiation condition is not known for all boundary configurations.

If the waves are generated by an oscillating body, the motions must, of course, be small in order that the linearized free surface condition may be used. As a consequence the boundary condition to be satisfied on the surface of the oscillating body need be satisfied only on its mean position, S, i.e.

$$\frac{\partial \varphi}{\partial n} = V_n \quad \text{on } S, \tag{58}$$

where V_n is the normal component of the velocity of the body at a given point of its surface.

Several of the papers listed below have as one of their aims the comparison of experimentally measured with computed values. We mention in particular those of Paulling, Porter, Tasai, Ursell and Yu and their coauthors. It is encouraging to be able to report that under appropriate circumstances the agreement is quite satisfactory and that irrotational flow of an inviscid fluid provides a good mathematical model for the phenomena of this section.

Oscillating solid boundaries: Barakat [1962], Cherkasov [1961], Grim [1960], Haskind [1957, 1959a], Kaplan [1957], Kaplan and Hu [1960], Kravtchenko [1954], MacCamy [1961a], Moiseev [1959c], Newman [1961], Paulling and Porter [1962], Paulling and Richardson [1962], Porter [1960], Sparenberg [1960], Stelson and Murtha [1957], Tasai [1959, 1960, 1961[, Ursell, Dean and Yu [1960], Yu and Ursell [1961].

Oscillating pressure distributions: Cherkesov [1959,1962], Voit [1959 a, c].

Other: Kravtchenko and McNown [1955].

<u>Diffraction of waves</u>. The diffraction problem is quite similar to the radiation problem considered above. Here a part of the velocity potential is given, namely, that of an "incoming wave", say Φ_I. Some fixed solid boundary is present, say S, and one wishes to find a velocity potential $\Phi = \Phi_I + \Phi_D$ such that

$$\Phi_n = 0 \quad \text{or} \quad \Phi_{Dn} = -\Phi_{In} \quad \text{on } S, \tag{59}$$

where Φ_D satisfies a radiation condition, e.g. (57) under the same conditions. It is usually assumed that $\Phi_I = \Re e\, \varphi_I e^{-i\sigma t}$, so that also $\Phi_D = \Re e\, \varphi_D e^{-i\sigma t}$. Evidently (59) is contained in (58) if one allows a flexible boundary motion. It is worth noting that any solution of a two-dimensional sound-diffraction problem provides at the same time a solution of a water-wave problem in which the obstacle is a vertical cylinder with cross-section the same as that of the diffracting body in the sound-diffraction problem [see Wehausen and Laitone, 1960, p.544].

One may interpret the two-dimensional diffraction problem as a wave train moving down a canal with an obstruction, and being partly reflected and partly transmitted. Let the depth before the obstruction be uniformly h_1 and behind it h_2 and let the amplitudes of the incoming, reflected and transmitted waves be A_I, A_R and A_T respectively. If one defines reflection and transmission coefficients by $R = A_R/A_I$ and $T = A_T/A_I$, respectively, then it is a consequence of conservation of energy that

$$R^2 + T^2 \frac{\cosh^2 m_1 h_1}{\cosh m_2 h_2} = 1 , \tag{60}$$

where $\sigma = [g m_i \tanh m_i h_i]^{1/2}$ is the frequency of the wave motion. R and T are functions of σ^2/gL, where L is some length associated with the obstruction or the transition from depth h_1 to depth h_2. Information concerning them for various configurations is of obvious significance for problems in coastal engineering.

Diffraction, two dimensions: Grim [1960], Haskind [1959a], Ogilvie [1960], Takano [1900], Ursell [1961], Dean and Ursell [1959].

Diffraction, three dimensions: Burger [1959], Haskind [1957], Keller and Magiros [1961], Levine [1963], MacCamy [1961b], Newman [1962], Sretenkii [1959], Voit [1959b, 1961].

<u>Beaches.</u> Let the equation of the ocean bottom be $y = f(x) < 0$, $f(0) = 0$ (thus excluding irregularity and effectively making the problem two-dimensional), let the incoming waves at infinity be given by

$$y = A \cos(k_1 x + k_2 z + \sigma t), \tag{61}$$

and let Φ_I be the corresponding velocity potential. In the customary case $f(x) = -x \tan \gamma$, the condition $k_1^2 + k_2^2 = \sigma^4/g^2$ must be satisfied. However, if $f(x) \to -h$ as $x \to \infty$, the condition is $k_1^2 + k_2^2 = m_0^2$, where $\sigma^2 = g m_0 \tanh m_0 h$. In either case one may set

$$\Phi(x, y, z, t) = \varphi(x, y) \cos k_2 z \, e^{-i\sigma t}, \quad \varphi = \varphi_1 + i\varphi_2, \tag{62}$$

where φ must satisfy

$$\varphi_{xx} + \varphi_{yy} - k_2^2 \varphi = 0, \quad k_2^2 < m_0^2, \tag{63}$$

and also, of course ,

$$\varphi_y(x,0) - \frac{\sigma^2}{g}\varphi = 0 , \qquad \varphi_n(x, f(x)) = 0 . \tag{64}$$

There can obviously be no transmitted wave as in the case of the canal obstruction discussed just above. On the other hand, if the incoming wave is completely reflected, the incident and reflected wave combine to make a standing wave. If there is to be no reflected wave, the energy in the incoming wave must be somehow absorbed at the beach. Inviscid-fluid theory does not provide any physical mechanism for explaining the absorption, but does provide a mathematical one for accomplishing it, namely, a singularity at the intersection of the beach and the water surface. It is possible to construct solutions with a logarithmic singularity and such solutions are unique for the straight sloping beach as has been shown by Lehman and Lewy [1961].

If instead of considering a progressing wave like (61), one considers a wave moving parallel to the beach with an associated velocity potential

$$\Phi(x,y,z,t) = \varphi(x,y)\, e^{i(k_2 z - \sigma t)} , \tag{65}$$

then (63) and (64) must again be satisfied, but no longer with the restriction $k_2^2 < m_0^2$. Let the beach again be a uniformly sloping one making angle γ with the horizontal. If σ is fixed, there do not exist bounded solutions for all values of k_2, but only for certain values. Let $\gamma_n = \pi/2(2n+1)$. Then for $\gamma_1 < \gamma \leq \gamma_0 = \frac{1}{2}\pi$ there is only one value $k^{(1)} = \sigma^2/g \sin\gamma$;

for $\gamma_2 < \gamma \le \gamma_1$ there is an additional value $k^{(2)} = \sigma^2/g \sin 3\gamma$, etc. When γ does not take on one of the values γ_n, the waves decrease exponentially as x increases; for $\gamma = \gamma_n$ the wave remains finite as $x \to \infty$. For $\gamma \neq \gamma_n$ these are called "edge waves". When k_2 is not one of these values, Roseau [1958,1959] has shown that there still exist solutions which decay as $x \to \infty$ but which have singularities at $x = 0$ which are logarithmic except for $\gamma = \pi/2n$, $n = 1, 2, \cdots$. Lehman and Lewy have shown that the solutions which are either bounded or have logarithmic singularities are unique and cannot coexist for a given value of k.

Waves on beaches: Van Dantzig [1958,1959], Kajiura [1958], Lauwerier [1959, 1960, 1961], Lehman [1960], Lehman and Lewy [1961], LeMéhauté [1961], Reid [1958], Roseau [1958, 1959], B. N. Rumyantsev [1960], Stavrovskii [1959], Williams [1961].

<u>Waves in a fixed basin</u>. Here the boundary-value problem is to satisfy (44), usually with T' and p_A set equal to zero, and $\partial\Phi/\partial n = 0$ on the basin walls. A case of special interest is that of standing harmonic waves, when (44) becomes (52). There is now a solution only for a discrete set of σ's, $\sigma_1, \sigma_2, \cdots$, although to a given σ_i there may correspond several φ's. Many of the recent papers on the subject are concerned with explicit calculation of these characteristic frequencies and solutions for special shapes. Two of the papers listed below, those of Kirillov and Miles, treat an initial-value problem which is a modification of the usual one in that the fluid is being drained from the bottom at the same time that it is sloshing. The draining has a damping effect upon the motion.

Waves in a fixed basin: Abramson and Ransleben [1961], Budiansky [1960], Lawrence, Wang and Reddy [1958], Mikishev and Dorozhkin [1961], Miles [1962a], Petrov [1961], Riley and Tremblath [1961].

Waves in a moving basin. It is necessary to distinguish between several cases in formulating the boundary-value problem. One obviously necessary distinction is between basins undergoing a prescribed motion and those whose motion is to be determined as a part of the problem. Furthermore, if the basin moves only translationally with respect to the fixed axes, and if the absolute motion is irrotational, the motion of the fluid relative to the basin is also irrotational, which greatly simplifies the problem. On the other hand, if an angular velocity also occurs, then the relative motion becomes rotational, as is evident from (47). Finally, if there is an acceleration in a direction perpendicular to the direction of gravity, the equilibrium surface will no longer be parallel to $z = 0$ and some further modification of the theory so far expounded becomes necessary. Moiseev [1953] treats the general theory insofar as the fluid motion can still be linearized. We mention below a few easily described cases.

If the basin motion is purely translational and the fluid motion irrotational, then it is evident from (6b) that the equation for the relative motion is the same as (5') if one replaces $\underline{F} = -\operatorname{grad} U$ by $\underline{F} = -\dot{\underline{v}}_0 = -\operatorname{grad}\{U + \dot{\underline{v}}_0 \cdot \underline{r}'\}$. With this modified potential function for the external force one may still use the integral (15a) with the moving coordinate system. In particular, if the basin is undergoing a motion of only small amplitude, described by the motion of the origin $(x_0(t), y_0(t), z_0(t))$ of the coordinate system fixed in the basin, then the free-surface boundary conditions for the velocity potential of the relative motion are given by

$$\begin{aligned} &\Phi'_t(x', 0, z', t) + [g + \ddot{y}_0(t)]\eta(x', z', t) + \ddot{x}_0 x' + \ddot{z}_0 z' = 0, \\ &\Phi'_{y'}(x', 0, z', t) - \eta_t(x', z', t) = 0, \end{aligned} \tag{66}$$

and on the basin walls by

$$\Phi'_{n'} = 0 .$$

Note that $\ddot{y}_0$ does not need to be small in order to use these boundary conditions. If the basin is also undergoing small angular motions, one may avoid the difficulty with rotational flow by considering the velocity of the fluid relative to a coordinate system with origin (x_0, y_0, z_0) and axes parallel to the fixed axes. The difficulties are not greatly increased, but we shall not discuss them here. However, later on, in connection with rotational flow, we shall return to waves in rotating basins.

Waves in movable basins: Bogoryad [1962], Borisova [1962], Bublik and Merkulov [1960], Cooper [1960], Kito [1958, 1959, 1960, 1961, 1962], Lukovskii [1961], Mal'tsev [1962], Miles [1958], Moiseev [1958, 1959b,c, 1962], Shved [1959], Troesch [1960], Yoshioka [1962].

Initial-value problems. The classical initial-value problem is the Cauchy-Poisson problem in which the initial position and velocity of the free surface is given and the subsequent motion sought. This problem is still studied in various generalizations. The motion ensuing when a solid boundary is set into motion or a pressure suddenly applied is frequently studied in connection with technical and geophysical applications. The motion following an underwater explosion presents a similar problem. These types are all represented in the following list.

Chen [1961], Kisler [1960], Kranzer and Keller [1959], Miles [1962], Ross [1961], B. N. Rumyantsev [1960], T. Ya. Sekerzh-Zeńkovich [1959a], Ya. I. Sekerzh-Zeǹkovich [1959a], Sen [1959, 1960, 1963], Sretenskii [1959, 1960], Sretenskii and Stavrovskii [1961].

Waves in stratified fluids. If one perturbs a state of rest of two inviscid fluids of different densities situated in a gravitational field, the linearized interfacial boundary conditions can be easily derived from (9) and (16):

$$\eta_t(x,z,t) = \Phi_y^{(1)}(x,0,z,t) = \Phi_y^{(2)}(x,0,z,t), \tag{67}$$

$$\rho^{(2)}\Phi_t^{(2)}(x,0,z,t) - \rho^{(1)}\Phi_t^{(1)} + [\rho^{(2)} - \rho^{(1)}]g\eta + T[\eta_{xx} + \eta_{zz}] = 0.$$

It follows that the two velocity potentials $\Phi^{(1)}$ and $\Phi^{(2)}$ must satisfy the juncture conditions

$$\begin{aligned} &\Phi_y^{(1)}(x,0,z,t) = \Phi_y^{(2)}, \\ &\rho^{(2)}[\Phi_{tt}^{(2)}(x,0,z,t) + g\Phi_y^{(2)}] = \rho^{(1)}[\Phi_{tt}^{(1)} + g\Phi_y^{(1)}] + T\Phi_{yyy}^{(1)}. \end{aligned} \tag{68}$$

Here the equilibrium position of the interface has been taken as $y=0$. However, it could equally well be at any other value of y, and if there are to be a number of different layers of fluid, one might find it convenient to take $y=0$ as a free surface for fluid 1, $y=-h_1$ as the interface between fluid 1 and fluid 2, $y=-h_2$ as the interface between fluid 2 and fluid 3, etc. At each interface there will be a different T, i.e. $T^{(01)}$ at the free surface, $T^{(12)}$ at the interface at $y=-h_1$, etc. The juncture conditions (68) must be satisfied at each $y=-h_i$; with appropriate modification of the notation.

Some properties of periodic motions in stratified fluids together with further references may be found in Wehausen and Laitone (1960, §148 and §25]. The theory of such waves is developed as a part of the general theory of waves in fluids with variable density, including discontinuities, in Yih [1960a] and Yanowitch [1962]. (We shall return to this more general case later when we consider rotational waves.) One interesting property is the following. If only two fluids are present, each of which either fills a whole half-space or is bounded by a rigid horizontal plane, the equation relating frequency σ to wave number k is quite similar in structure to that for a free surface. However, if the upper fluid is bounded by a free surface, or if there are three fluids with the top and bottom ones bounded by rigid walls, the motion has, so to speak, an additional degree of freedom. As a result the equation which determines σ^2/gk is quadratic, so that two frequencies are associated with one wave number. As the number of interfaces increases, the order of the equation determining σ^2/gk increases. For a progressive wave a different velocity of propagation $\sigma_i = \sigma_i/k$ is associated with each value of σ_i, so that more than one wave length may be propagating at a given speed.

The phenomenon mentioned above is closely connected with that of so-called "dead-water" resistance of ships, in which a ship moving through a layer of fresh water lying on top of salt water experiences an abnormally high wave resistance at rather small speeds. The matter has been completely elucidated only recently by an extension of the thin-ship theory [see eq.(50)] to this situation, and indeed by several persons independently [Hudimac, Sabuncu, Sretenskii, Uspenskii]. Havelock in unpublished work has carried through the analogous computations for a moving pressure distribution. In the thin-ship theory the essential preliminary step is to find the Green's function, a somewhat more complicated computation than for a homogeneous fluid. Sabuncu [1962] has computed resistance curves for a source-sink body moving just above the interface which provide striking theoretical confirmation of the existence of the "dead-water" effect. This is also confirmed by Havelock's calculation.

Papers on stratified fluids: Eckart [1960, 1961], Esch [1962], Hudimac [1961], Long [1959], Sabuncu [1961, 1962], Samuseva [1961], T. Ya. Sekerzh-Zen'kovich [1959a,c], Sretenskii [1959], Uspenskii [1959], Voit [1959 a, c], Warren [1960], Yanowitch [1962], Yih [1960a].

Other boundary conditions. Certain geophysical problems lead to the consideration of other interfaces than that between two liquids. For example, consideration of the propagation of waves in an ice sheet overlying fluid leads one to consider a thin elastic plate in contact on one side with a fluid. Seismological problems might lead one to consider a layer of fluid over an elastic half-space. Such problems introduce new equations for the second medium and new dynamical boundary conditions. Situations of this sort have not been very extensively investigated, but occasional papers appear, of which two follow: Kheisin [1962], Papadopoulos [1960].

Rotational waves in inviscid fluids. Although the assumption of irrotational flow yields a reasonably good approximation to observed phenomena in many situations, there are some in which such an assumption would be clearly contrary either to observed behavior or to the nature of the problem. River flows, for example, are known not to be uniform flows in general and the theory of free-surface waves propagating on such a flow should be developed in such a way as to take account of the underlying shear flow.

For the case of steady two-dimensional flows the linearized boundary-value problem has been formulated in (40). It is mentioned there that if u_0''/u_0 = a constant, say a, where $u_0(y)$ is the underlying shear flow, then it is a simple matter to find periodic solutions. We give the solution for wave number k in order to see the effect of this special class of shear flows upon the waves: Let $u_0(0) = c > 0$ and $u_0'(0) = d$. Then

$$\psi_1 = A\cos(kx+\alpha)\sinh\sqrt{k^2+2a}\,(y+h),$$

$$\eta_1 = -Ac^{-1}\sinh\sqrt{k^2+2a}\,h\cos(kx+\alpha), \tag{69}$$

$$\frac{c^2}{(g+cd)h} = \frac{\tanh\sqrt{k^2+2a}\,h}{\sqrt{k^2+2a}\,h}.$$

If $a<0$, corresponding to a velocity profile

$$u_0 = c \cos\sqrt{-a}\, y + \frac{d}{\sqrt{-a}} \sin\sqrt{-a}\, y, \tag{70}$$

there are no waves of wave number less than $\sqrt{-2a}$. In the case of a linear profile ($a=0$, $u_0 = c + dy$), one may show without difficulty that if $\bar{c}$ is the mean velocity and if $d>0$, then $\bar{c} > c$ for a given k [Biesel, 1950]. Further investigations along this line may be found in Hunt [1955] and Tsao [1959,1960]. Kolberg [1959, 1961] has carried through an analysis of the steady motion of thin ships and of pressure distributions for shear flows. In the next section we shall discuss a special problem which also leads to flows with vorticity.

<u>Waves in rotating basins</u>. In an earlier section we also dealt with waves in moving basins. However, there the absolute motion was assumed irrotational and in equations (53) the equilibrium position of the fluid such that the free surface is $y=0$. In connection with geophysical problems, but in recent times more especially rocket problems, there has been a considerable interest in waves on a fluid which is at rest with respect to a rotating basin containing it. The fluid motion is then necessarily rotational.

In order to see the nature of the boundary-value problem consider a fluid inside a container rotating about the vertical y-axis with angular velocity ω in the positive direction (from Oz to Ox). Let $R^2 = x^2 + z^2$. Then the equilibrium position of the free surface is given by $R^2 = R_0^2 + 2\gamma y$, where $\gamma = g/\omega^2$. Let the deviation from this surface be measured along the normal external to the fluid and be denoted by $\zeta(R,\theta,t)$, where θ is an angular coordinate measured in the positive direction from Ox. R and θ are then parameters describing the surface, which in cylindrical coordinates ρ, θ, y is then given by the equations

$$\begin{aligned} \rho &= \rho(R,\theta,t) = R\left[1-\zeta(R,\theta,t)/S\right], \\ y &= \eta(R,\theta,t) = (R^2-R_0^2)/2\gamma + \gamma\zeta/S, \end{aligned} \tag{71}$$

where $S^2 = R^2+\gamma^2$. Let U_R, U_θ, U_y, respectively, be the relative velocity components in the directions R, θ, y. If these are small the equations (6b) may be approximated by the linearized equations

$$\begin{aligned} \frac{\partial U_R}{\partial t} + 2\omega U_\theta &= -\frac{\partial H}{\partial t}, \\ \frac{\partial U_\theta}{\partial t} - 2\omega U_R &= -\frac{1}{R}\frac{\partial H}{\partial \theta}, \\ \frac{\partial U_y}{\partial t} &= -\frac{\partial H}{\partial y}, \quad H = \frac{p}{\rho} + gy - \frac{1}{2}\omega^2 R^2, \end{aligned} \tag{72}$$

to which must be added the continuity equation

$$\frac{\partial}{\partial R}(RU_R) + \frac{\partial}{\partial \theta}U_\theta + R\frac{\partial}{\partial y}U_y = 0.$$

The kinematic and dynamic boundary conditions yield the following equations after linearization:

$$\begin{aligned} -\frac{R}{S}U_R + \frac{\gamma}{S}U_y &= \frac{\partial \zeta}{\partial t}, \\ \frac{\partial U_R}{\partial t} - \frac{R}{\gamma}\frac{\partial U_y}{\partial t} + 2\omega U_\theta &= \frac{\omega^2 R}{S}\zeta + \omega^2 S\frac{\partial \zeta}{\partial R}, \\ \frac{\partial U_\theta}{\partial t} - 2\omega U_R &= -\frac{\omega^2 S}{R}\frac{\partial \zeta}{\partial \theta}, \end{aligned} \tag{73}$$

all to be satisfied on the equilibrium position.

If $\omega = 0$ the boundary conditions reduce, as they should, to

$$\frac{\partial \zeta}{\partial t} = u_y, \quad \frac{\partial u_R}{\partial t} = g\frac{\partial \zeta}{\partial R}, \quad \frac{\partial u_\theta}{\partial t} = -\frac{g}{R}\frac{\partial \zeta}{\partial \theta}, \tag{74}$$

to be satisfied on $y = 0$. If ω is so large that one may neglect the gravitational force, the boundary conditions become

$$\frac{\partial \zeta}{\partial t} = -u_R, \quad \frac{\partial u_\theta}{\partial t} - 2\omega u_R = -\omega^2\frac{\partial \zeta}{\partial \theta}, \quad \frac{\partial u_y}{\partial t} = -\omega^2\frac{\partial \zeta}{\partial y}, \tag{75}$$

to be satisfied on $R = R_0$ (here one must take ζ as a function of y instead of R).

These equations do not cover all the situations which may arise under this heading. They are evidently more complex than the linearized equations which have been discussed above. However, in spite of this, useful solutions have been obtained in special cases. One problem, in particular, which has attracted considerable attention is the stability of a spinning top containing liquid with a free surface.

Waves in rotating basins: Chaikovskaya [1961], Kostandyan [1960,1961], Malashenko [1960], Miles [1959], Miles and Troesch [1960], Narimanov [1957], Nigam and Nigam [1962], Phillips [1960], V. V. Rumyantsev [1959, 1960], Saint-Guily [1962], T. Ya. Sekerzh-Zen'kovich [1959], Stewartson [1959], Tsao [1960], Uspenskiǐ [1959], Voǐtsekhovskiǐ and Koval′ [1960].

<u>Flows with variable density</u>. We have considered earlier the situation in which a number of fluids of different densities flow in strata with the motion being irrotational within each stratum. A natural generalization is to suppose that the density is continuously or piecewise continuously distributed. For a steady two-dimensional flow one may derive linearized equations from (18'). However, let us instead consider the more general case of time-dependent three-dimensional flows. We shall suppose that when the fluid is undisturbed the density is given by $\rho = \rho_0(y)$. and that in the disturbed state

$$\underline{v} = \varepsilon \underline{v}_1 + \varepsilon^2 \underline{v}_2 + \cdots, \tag{76}$$

$$p = p_0 + \varepsilon p_1 + \cdots,$$

$$\rho = \rho_0(y) + \varepsilon \rho_1(x, y, z, t) + \cdots .$$

After substituting into equations (21), (3) and (5), one may then derive the linearized equations

$$\rho_{1t} + v_1 \rho_0' = 0, \quad u_{1x} + v_{1y} + w_{1z} = 0, \quad u_{1tz} = w_{1tx}, \tag{77}$$

$$[\rho_0 u_{1tt}]_y = \rho_0 v_{1ttx} - g\rho_0' v_{1x}, \quad [\rho_0 w_{1tt}]_y = \rho_0 v_{1ttz} - g\rho_0' v_{1z} .$$

If there are discontinuities in $\rho_0(y)$, then at such an interface the following juncture conditions [cf. (168)] must be satisfied:

$$v^{(2)} = v^{(1)}, \quad [\rho_0(u_{tt} + g v_x)] = -T(v_{xxx} + v_{zzx}), \tag{78}$$

$$[\rho_0(w_{tt} + g v_z)] = -T(v_{xxz} + v_{zzz}),$$

where $[f] = f^{(2)} - f^{(1)}$. If there is a free surface, the boundary condition is obtained by setting $\rho_0^{(2)} = 0$ in the second and third equations, and discarding the first.

In the discussion of finitely stratified fluids it was remarked that several wave lengths could travel with the same velocity, the number depending upon the number of interfaces. One might hence expect to find an infinite sequence of wave lengths for a given velocity in a continuously stratified fluid with $\rho_0'(\eta) \neq 0$. However, this is not the case; there are only a finite number. On the other hand, for a given wave length there is an infinite sequence of associated velocities c_n with $c_n \to 0$ as $n \to \infty$. For the proof of these statements and for the derivation of a number of other properties of internal waves one may consult the recent paper of Yanowitch [1962]. Yih [1960a], in addition to considering some general properties of internal waves, has considered two boundary-value problems in heterogeneous fluids, the plane wave-maker and the stability of fluid in a vertically oscillated container. The flow around a three-dimensional obstacle is considered by Drazin [1961]. The method of analysis does not properly put this paper into the category of approximation being discussed above. However, it should be mentioned, for it develops systematic methods of approximation; references to other relevant literature may also be found here.

Papers on waves in heterogeneous fluids: Alterman [1961], Benton [1953], Drazin [1961], Eckart [1960], Heyna and Groen [1958], Howard [1961], Long [1959], Miles [1961], Ter-Krikorov [1962], Yanowitch [1962], Yih [1959, 1960a].

<u>Infinitesimal waves in viscous fluids</u>. If the motion is a perturbation of a state of rest, the appropriate linearized equations are easily derived from (5), (7) and (8). If, as above, superscript 2 refers to the upper and 1 to the lower fluid, the equations are

$$\frac{\partial \underline{V}^{(i)}}{\partial t} = -\frac{1}{\rho^{(i)}}\, grad\left(p^{(i)} + \rho^{(i)} g y\right) + \nu^{(i)} \Delta \underline{V}^{(i)}, \quad div\ \underline{V}^{(i)} = 0. \tag{79}$$

The juncture conditions, to be satisfied at the undisturbed interface, are the following

$$\underline{V}^{(1)} = \underline{V}^{(2)} = \eta_t, \quad \mu^{(1)}\left(u_y^{(1)} + v_x^{(1)}\right) = \mu^{(2)}\left(u_y^{(2)} + v_x^{(2)}\right), \quad \mu^{(1)}\left(w_y^{(1)} + v_z^{(1)}\right) = \mu^{(2)}\left(w_y^{(2)} + v_z^{(2)}\right),$$
$$p^{(2)} - p^{(1)} + \left(\rho^{(2)} - \rho^{(1)}\right) g \eta - 2\left(\mu^{(2)} v_y^{(2)} - \mu^{(1)} v_y^{(1)}\right) = T\left(\eta_{xx} + \eta_{zz}\right). \tag{80}$$

If the fluids are bounded above or below by horizontal planes, then on such a plane there is a boundary condition $\underline{V} = 0$.

It is immediately obvious that in the presence of horizontal walls one cannot convert the problem formulated above into a steady flow by referring the variables to moving axes. The difficulties are, of course, well known. The flows discussed earlier in which a given steady non-uniform velocity profile is perturbed are attempts to treat this problem theoretically within the framework of inviscid-fluid theory. However, there does exist one situation in which one need not retreat to an inviscid fluid. One of the well known exact solutions of the Navier-Stokes equations is for a steady flow of constant depth down an inclined plane. Let the free surface be at the plane $y = 0$ and be subject to a constant shearing stress τ, let the bottom be at $y=-h$, and let the x-axis make an acute angle α with the horizontal (so that in eq.(5) $\underline{F} = g(\sin\alpha, -\cos\alpha, 0)$. Then this solution is

$$u = \frac{1}{2}\frac{g}{\nu}(h^2-y^2)\sin\alpha + \frac{\tau}{\mu}y, \quad v=0, \quad p = -\rho g y \cos\alpha. \tag{81}$$

With this as the fundamental flow, one may now look for flows which deviate only a little from it by expressing them as perturbation series in the manner already described. The resulting linearized theory has been studied by several persons with the intent of finding any regions of wave numbers in which the flow is unstable. In this category are the following papers: Benjamin [1957, 1961], Binnie [1957, 1959], Bushmanov [1960], Ivanilov [1960], Lyubutskaya [1961], Maurin and Sorokin [1962], Miles [1960] (here $\alpha = 0$), Yih [1954]. Similar stability problems with two fluids are considered by Feldman [1957], Graebel [1960], and Zaitsev [1960]. In several of these papers the method of analysis belongs more properly to the theory of long waves; they will be mentioned again.

We recall one property of standing waves in a viscous fluid [see Wehausen and Laitone, 1960, p. 642]. For a fluid of infinite depth with a free surface an oscillatory motion at the surface can exist only for waves for which

$$\Lambda = \frac{1}{k}\sqrt{\frac{g}{\nu^2} + \frac{T'}{\nu^2}k^2} > 0.835; \tag{82}$$

The other wave lengths are critically damped. If the depth is finite, the region of critically damped waves depends upon the depth. Oborotov [1960] has made some numerical calculations of the boundary curve in the $(\Lambda, h(g/\nu^2)^{1/3})$-plane. The curve begins to separate from the asymptote $\Lambda = 0.835$ at about $h(g/\nu^2)^{1/3} = 4$, reaches a minimum of about 2.3 for Λ between 2 and 3 and then slowly increases, so that sufficiently long waves are also critically damped. However, it is not the sort of phenomenon one expects to observe easily; moreover, for water the limit 0.835 corresponds to k of the order of 10^6. The theoretically computed damping coefficients for oscillatory waves appear to be in good agreement with the experiments of Hughes and Stewart [1961], but not with those of Grosch, Ward and Lukasik [1960] or of Eagleson [1962].

Other papers on small waves in viscous fluids: Hunt [1961], Moiseev [1961]. A paper by Ivanilov [1961] will be discussed later.

<u>Stability of free surfaces</u>. The general problem of stability of an interface between two fluids under various flow conditions belongs in another chapter, and in any case is too extensive to discuss here. Certain problems are, however, intimately connected with the presence of a gravity field. Some of these have been mentioned in passing in the foregoing paragraphs. Two others of interest are the generation of gravity waves by the flow of one fluid over another, as air over water, and the motion of an interface being accelerated in the direction of its normal.

The first problem is obviously of great importance in understanding how the wind initially generates water waves and to this end was first studied by Helmholtz and Kelvin [see Wehausen and Laitone, 1960, pp. 649-650 for the classical theory and references]. Their treatment assumed a uniform flow over still water, thus making the surface a sheet of concentrated vorticity. The predicted minimum wind velocity for wave formation was not in agreement with observations. The problem has been reconsidered in recent years in a series of papers by Miles [1957, 1959, 1962] and one by Benjamin [1959]. The assumption of a more realistic wind profile over the water leads to a more difficult mathematical problem, but the resulting theory yields predictions which are not in disagreement with experiment. This work may be considered as one of the substantial advances in this subject. A more physically oriented exposition of Miles' theory is given by Lighthill [1962]. Longuet-Higgins [1962a] reviews the present status of the theory of wave generation and ocean wave spectra and includes a description of recent observations. A related stability problem is treated by Esch [1962].

The recent interest in the stability of interfaces subject to normal acceleration appears to have begun with a paper of G. I. Taylor [1950] in which it is pointed out that an acceleration normal to the interface (or free surface) acts in the same way as a gravitational field. This enables him to extend certain classical results and formulate the following rule: if two fluids with an interface with no surface tension are being accelerated in a direction from the less to the more dense fluid, the interface is unstable, otherwise stable. (Such instability is now often called "Taylor instability".) An analogous statement can easily be formulated for the interface between two fluids flowing as a result of a pressure gradient imposed normal to the interface, as in pipe flow. A brief discussion of the problem and some references will be found in Wehausen and Laitone [1960, pp.647-648]. Further more recent investigations may be found in the following papers: Bankoff [1959], Carrier and Chang [1959], Case [1960], Chang [1959], Emmons, Chang and Watson [1960], Hunt [1961], Reid [1961], Richtmyer [1960]. The further development of such instabilities in certain restricted situations has been investigated in several papers by Saffman and Taylor [1958, 1959], and by Taylor [1961], but their treatment does not fit into the present context. Birkhoff [1962] examines in a critical way both stability problems.

Linearized flows without gravity. Under appropriate conditions several of the problems considered earlier in the discussion of exact solutions can be solved approximately by linearization. A particularly fruitful development has been Tulin's extension of the thin-wing linearization to cavitating hydrofoils. As in the thin-wing theory, the angle of attack, camber and thickness ratio are all supposed to be small, so that the boundary conditions on the wing and cavity are to be satisfied on a segment in the direction of the undisturbed flow. We omit the formal discussion of the perturbation expansion. Let $\underline{V}_0 = (u_0, 0)$ be the velocity at infinity, $v_c > u_0$ the (constant) speed on the cavity wall and (u, v) the velocity. We define the perturbation velocity (u_1, v_1) by

$$u = v_c(1 + u_1), \qquad v = v_c v_1 \, . \tag{83}$$

The following figures show a partially cavitated and a fully cavitated two-dimensional hydrofoil and the associated segments, which will be used to formulate the linearized boundary-value problem for steady flow.

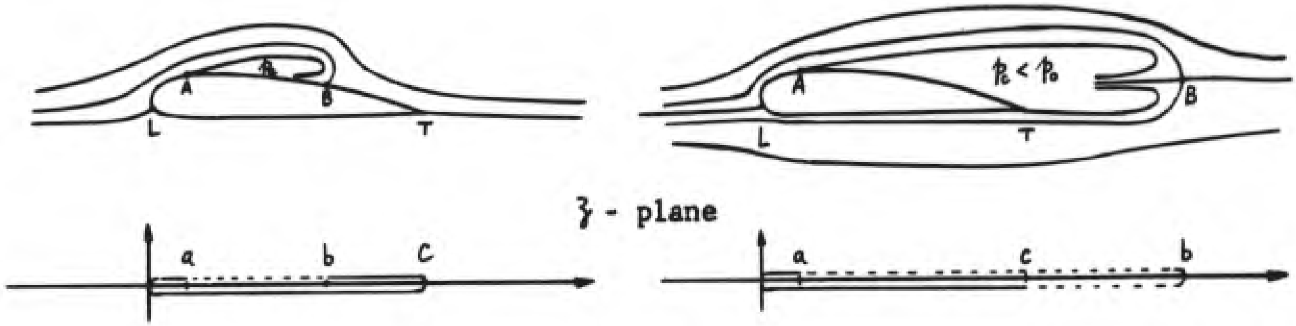


Figure 5

Let the upper surface of the hydrofoil be described by $y = m(x)$ and the lower by $y = n(x)$. Then the complex perturbation velocity $w_1(\zeta) = u_1 - i v_1$ must be analytic everywhere except on the segment representing the wing plus cavity, and satisfy at infinity the conditions

$$\lim_{\zeta \to \infty} w_1 = -1 + \frac{u_0}{v_c} = -1 + (1+\sigma)^{-1/2}, \quad \oint_C w_1(\zeta)\, d\zeta = 0 \tag{84}$$

for any contour containing the segment. The last condition results from the closure of the cavity. (If $p_c = p_0$, i.e. $\sigma = 0$, the cavity is infinite in length so that the segment extends along the whole positive axis and the last condition must be dropped.) The z-plane is cut along the segment and w_1 must satisfy the following boundary conditions on the two sides of the cut:

Partial cavitation: $0 < x < a \qquad \mathrm{Im}\, w_1(x+i0) = -m'(x)$, (85)

$a < x < b \qquad \mathrm{Re}\, w_1(x+i0) = 0$,

$$b < x < c \qquad \operatorname{Im} w_1(x+i0) = -m'(x),$$
$$0 < x < c \qquad \operatorname{Im} w_1(x-i0) = -n'(x);$$

full cavitation:

$$\begin{aligned} &0<x<a \qquad && \operatorname{Im} w_1(x+i0) = -m'(x), \\ &a<x<b && \operatorname{Re} w_1(x+i0) = 0, \\ &0<x<c && \operatorname{Im} w_1(x-i0) = -n'(x), \\ &c<x<b && \operatorname{Re} w_1(x-i0) = 0. \end{aligned} \tag{86}$$

There are still juncture conditions to be satisfied: For both cases there must be continuity of slope and curvature at A:

$$\operatorname{Im} w_1(a+0+i0) = -m'(a), \quad \operatorname{Im} w_1'(a+0+i0) = -m''(a).$$

For the fully cavitated hydrofoil there must be continuity of slope at T: $\operatorname{Im} w_1(c+0-i0) = -n'(c)$. For the partially cavitated hydrofoil the Kutta-Joukowsky condition must hold at T, i.e. w_1 must be finite at $\zeta = c$. It is known from thin-wing theory, and a moment's consideration will convince one of it, that the linearized approximation must break down near a stagnation point. Here both L and B are stagnation points, and at the corresponding values $\zeta = 0$ and $\zeta = b$ one may expect to find singularities. The values of a and b must be determined as part of the solution.

Finally, the integral condition in (84) deserves some further mention, for the models of the assumed exact flow show a loss of fluid through the reëntrant jet. This loss is, however, of the second order in any parameter describing the deviation from a horizontal segment, so that for the first-order theory the integral condition in (84) is consistent. A second-order theory would have to take this loss into account. (The only such theory available [Chen, 1962] has been worked out only for $\sigma = 0$, so that this problem doesn't occur.) If one starts from a Riabouchinsky model, say with a small vertical plate at the cavity end of the same vertical extent as the hydrofoil, one avoids this difficulty, but at the expense of introducing an artificial model. However, it leads to the same linearized boundary-value problem. Guerst [1959a] has, in fact, shown that for a vertical flat plate an appropriate expansion of the exact solutions for each of the two models has the same leading term for each and also the same as given by linearized theory. This is, of course, more than one usually achieves in linearized theory, where the perturbation expansion is only formal. The closure condition has been relaxed by Fabula [1962], but his open cavity corresponds to a different exact model.

As far as the author is aware, there are no published solutions for a lifting hydrofoil of the boundary-value problem as formulated above. However, various modifications have been studied. Fabula [1962] has considered the fully cavitated case, but has fixed beforehand the value of a (corresponding to venting at this point). Tulin [1958] has given a solution for a fully cavitating symmetric strut without prior determination of the point of detachment. Geurst [1959b] treated the case of partial cavitation, but with fixed $a = 0$. In addition, the following papers are all concerned with some aspect of two-dimensional steady flow about cavitating hydrofoils: Cohen and DiPrima [1958], Cohen, Sutherland and Tu [1957], Fabula [1961], Ivanov [1960], Oba [1961] (one should consult Wu's review in Math. Rev. 26, #962), Parkin [1959], Sutherland and Cohen [1958], Yamaoka and Mimura [1960]. The report by Parkin is a general survey of the linearized theory and covers all but the most recent work and also includes new material.

The boundary-value problem formulated above is for steady two-dimensional flow. There are two obvious directions in which to extend it: to unsteady flow and to three-dimensional flow. We shall not consider here the various ways of approaching either extension as a linearized theory. Unsteady cavity flow about two-dimensional hydrofoils has been considered by Cumberbatch [1961b], Geurst [1960], Timman [1958], and Wu [1958], and is also covered in Parkin's survey [1959] cited above. Three-dimensional steady flow is investigated in Cumberbatch [1961a], Cumberbatch and Wu [1961], Kermeen [1961], and Tulin [1958, 1959]. Experimental investigation of various aspects of cavity flow may be found in the following papers: Johnson [1958], Kermeen [1961], Koval and Novikov [1960], Meijer [1959], Silberman and Song [1961]. The Second Symposium on Naval Hydrodynamics (Washington, D.C., 1958) was partly devoted to topics in cavitation flow and most of the referenced papers from this source are, at least in part, of an expository nature, giving descriptions of the present status of different aspects of the subject and of current problems.

The boundary conditions for impact of a body upon a water surface can also be linearized. Here an appropriate physical requirement is that the striking body be either wedge-like or cone-like with a small apex angle, i.e. either a thin or a slender body. The boundary condition on the free surface reduces to $\Phi = 0$ and on the body itself may be replaced by the usual thin-body or slender-body boundary conditions. This linearized problem has been considered in recent years by Sagomonyan [1952, 1956, 1959], Bagdoev [1955, 1957, 1958], and the two jointly [1955]. Many of these results for the slender-body theory are given by Bagdoev in a book [1961]. This theory has also been recently developed independently by Mackie [1962] and Moran [1960, 1961, 1962]. If the impacting body is not thin or slender, the results of a linearized computation, i.e. of replacing the surface condition by $\Phi = 0$, can be interpreted as an approximate answer for a short time interval after impact. Several papers dealing with impact problems in this way were mentioned at the end of the section on exact solutions. Kudryavtseva [1960] has performed experiments on impact parallel to the water surface which indicate that this is a poor approximation for this case.

Another type of problem in which the boundary condition $\Phi = 0$ may be used is that of high-frequency oscillation, for this approximates the boundary condition (56) for small g/σ^2. Papers of Landweber and Macagno [1957, 1959, 1960] and of Macagno and Macagno [1961] are in this category. The papers of Sabaneev [1960] are similar in nature, but not restricted to periodic motion.

Higher-order theory of infinitesimal waves.

As has been stated earlier, much effort has been devoted in recent times to various higher-order calculations. In some of the relevant papers the purpose is simply to improve the accuracy of the first-order theory. In others the higher-order theory brings out facts of physical behavior which necessarily escape the first-order theory. Perhaps the most systematic effort in developing higher-order theory has come from Grenoble under the leadership of Kravtchenko and Santon. This group has attempted to give explicit formulas for nth-order approximations, not to assume irrotationality at the outset, and in a general way to provide theoretical information which can be used for analyzing experiment data, so that one may try to decide whether, or under what conditions, waves generated in the laboratory are irrotational, how well theory conforms with observation, to what extent linearized boundary conditions provide usable results, etc. Summaries of earlier parts of this work may be found in Kravtchenko and Santon [1955, 1957] and Kravtchenko [1956]. The approach is essentially that of the mathematician. The other noticeably fruitful approach has its origins in attempts to explain various oceanographic phenomena. The mathematics is not more elementary because of this, but the choice of problem and the criteria of success are different. Somewhat less attention will be given to these papers, for they have appeared in readily accessible journals and are not likely to be overlooked.

<u>Progressive periodic waves: rotational</u>. One method of organizing the computations is shown in equations (36) - (38). One must, of course, add the condition of periodicity. Also, an approximate distribution of vorticity must be decided upon at the start. The linearized equations have already been given in (69) and some consequences of the presence of vorticity mentioned. The necessary formulas for the second- and third-order theories have been derived for arbitrary vorticity distributions by Daubert [1958], and for a linear shear flow by Sun Tsao [1959]. We also recall that Gouyon [1958] in connection with his existence proof for steady periodic rotational waves derived explicit formulas for the higher-order approximations. However, the usefulness of both Daubert's and Gouyon's calculations is somewhat restricted by the assumption that the vorticity distribution is of the same order as the wave amplitude [cf.(24)]. This precludes their application to river-like flows. On the other hand, this is a proper assumption if one is interested in deriving formulas for comparison with observations of laboratory waves in order to decide whether, and how much, the wave motion departs from irrotationality. Daubert shows that under the mentioned condition the influence of vorticity can appear at the earliest in the second-order corrections. This makes detection of such effects dependent upon very precise measurements.

A theory of three-dimensional (i.e., doubly modulated) waves progressing in one direction and not necessarily irrotational has been developed in a series of notes by Gaillard [1957, 1958, 1962].

Inspection of (87) shows that even in the absence of gravity there may exist an infinitesimal periodic wave in a linear shear flow. This possibility has been examined by Sun Tsao [1960].

<u>Progressive periodic waves: irrotational</u>. For infinite depth explicit computation of coefficients in the perturbation expansion had been carried to the fifth order long ago. For irrotational waves in water of finite depth these computations have been made to the same order by several persons, in particular De [1955], Chappelear [1961] and Skjelbreia [1962]. The latter has also produced a set of tables which may be used in computing the properties of such waves. Fig. 6 shows A/h plotted against c^2/gh for various values of h/λ for both the third-order and the fifth-order approximation (these are often called "Stokes waves"). The first-order theory would, of course, give vertical lines starting at the same points on the c^2/gh-axis. It is evident that for $h/\lambda < 0.15$ the behavior of the curves is what one hopes for, i.e., the fifth-order approximation shows a small correction to the third-order, which becomes more noticeable as A/h increases. Furthermore, the fifth-order approximation passes almost through the points which have been computed by Chappelear [1959] for waves of greatest height, following Michell's original proposal. Since this computation follows a different method of approximation, it gives grounds for believing that carrying the approximation higher than the fifth order would yield only insignificant changes, and that the perturbation series is convergent up to the Michell waves with 120° corners at their crests. The situation is, however,

quite different for $h/\lambda = 0.10$ and 0.05. Here the fifth-order curves deviate widely from the third-order ones, the curve for 0.10 does not come near the terminal point computed for Michell's wave, and the curve for 0.05 begins to deviate already at $A/h = 0.05$ and soon thereafter begins to behave pathologically. In connection with this behavior we recall the sketch in Fig. 1 of the nature of the domain in which Struik's convergence proof is valid. It is limited to $c^2/gh < 1$ and the boundary curve of the domain ends at $c^2/gh = 1$, $A/h = 0$. The curve is, of course, only qualitative, and, even if it did show a known domain of convergence, would not mean that the series could not converge in a larger domain. However, the numerical evidence apparently indicates that this method of approximation breaks down for $c^2/gh \geqslant 1$ and presumably also for some values $c^2/gh < 1$ when A/h is large enough.

Fig. 6 is also useful in showing the regions in which linearized theory may be used with reasonable accuracy.

The other curves in Fig. 6 were computed by other methods of approximation, and will be discussed later. It seems appropriate to remark here, however, that if the fifth-order Stokes approximation had not been computed, one would feel constrained to comment upon how well the two main methods of approximation agree over a wide range of values. The two do, in fact, seem to agree, but only in a region corresponding to the overlap of the regions of validity of the two theories as indicated in Fig. 1.

Mattson [1962] has attempted to explore experimentally the region in Fig. 6 just to the left of $c^2/gh = 1$. In his experiments he has used waves of period 2 sec., three mean depths, $h = 0.10$, 0.15, 0.20 cm., and then varied the amplitude from small values up to breaking. The data were subjected to an elaborate analysis described by Marcou [1957] remove any reflected waves from the end of the wave tank and also any higher harmonics introduced by the wave-maker. For the values $h/\lambda = 0.0739, 0.0635, 0.0514$, he finds maximum attainable values of $A/h = 0.217, 0.205, 0.165$, respectively. Even if one takes into consideration that other methods of data analysis might produce somewhat different results, it is evident that the highest values of A/h which were realized experimentally were far short of those indicated on Figures 1 or 6 as theoretically possible. Also, as far as one can judge from the data, breaking occurred in the neighborhood of $c^2/gh = 1$. It should be added that Mattsson has started with the premise that periodic steady waves will exist only within the region where the Stokes method of approximation can be proved to converge and that his experiments are an attempt to find a part of the boundary of this region by direct observation. It seems to the author that this premise is incorrect, although this fact does not detract from the value of the experiments or the possible significance of the apparent impossibility of generating a wave of supercritical velocity. Finally, one should note that, although Mattsson's experiment data do not seem to confirm the theoretical limits, there do exist data for progressive waves reported by Danel [1952] which appear to confirm quite well down to $h/\lambda = 0.05$ the limits indicated on Fig. 6.

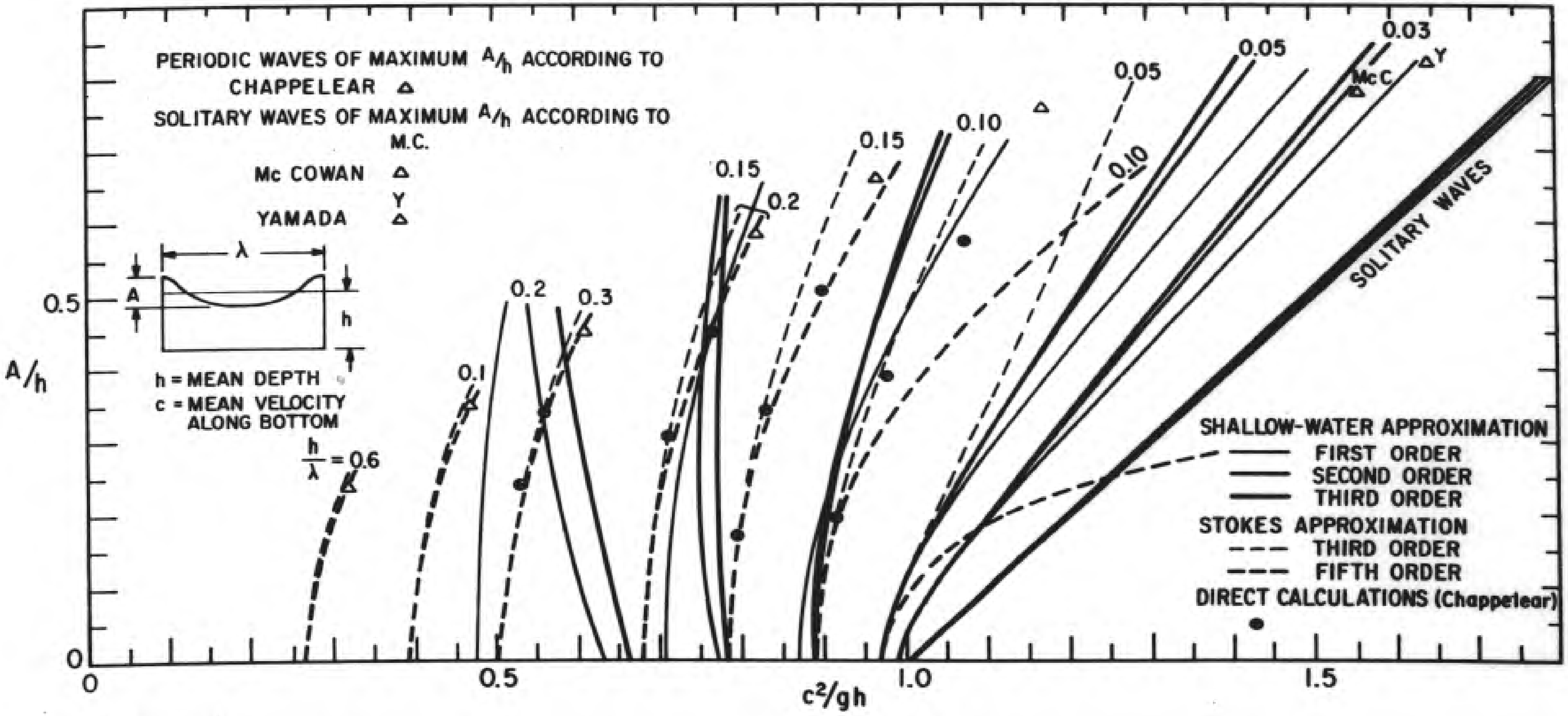


Figure 6, p. 30a

<u>Progressive irrotational waves with surface tension</u>. The theory of combined gravity-capillary waves has attracted both theoretical and experimental attention in recent years, and especially to an old result of Wilton [1915]. Consider the solution (48) and for simplicity let $h=\infty$. It then follows easily from (50) that for each $c>(4gT')^{1/4}$ there are two different wave numbers $k(c)$ and $\hat{k}(c)$ with the same phase velocity, a well known property of these waves. If $\hat{k}(c)=n\,k(c)$, $n=2,3,\cdots$, the two associated wave systems will be in phase, with $\lambda=n\hat{\lambda}$. This will, in fact, happen for a denumerable sequence of $c^{(n)}$, as one can easily convince onself from the graph of c. It turns out for $h=\infty$ that the associated $k^{(n)}$ and $\hat{k}^{(n)}$ are $(g/nT')^{1/2}$ and $(ng/T')^{1/2}$ and that $c^{(n)2}=(n+1)(gT'/n)^{1/2}$. For $h<\infty$, the $k^{(n)}$ are determined by the equation

$$\frac{gh^2}{nT'}=\frac{n\tanh nkh-\tanh kh}{n\tanh kh-\tanh nkh} \tag{88}$$

and do not exist if $h<\sqrt{3T'/g}$ (about $\frac{1}{2}$ cm for water).

It is a result of this analysis that the proper linearized solution associated with one of these resonant velocities $c^{(n)}$ is not (48) but instead

$$\Phi^{(1)}=A\cosh k^{(n)}(y+h)\sin(k^{(n)}x-\sigma^{(n)}t)+B\cosh\hat{k}^{(n)}(y+h)\sin(\hat{k}^{(n)}x-\hat{\sigma}^{(n)}t). \tag{89}$$

If this fact had not been observed in constructing the linearized solution, it would in any case be forced upon one's attention in finding the second-order term, where $k^{(2)}$ would occur as a singularity. For example, for $h=\infty$ one finds

$$\varphi = A c_0 e^{ky} \sin k(x - c_0 t) + \frac{3}{2} A^2 k c_0 \frac{k^2 T'}{g - 2k^2 T'} e^{2ky} \sin 2k(x - c_0 t),$$

$$\eta = A \cos k(x - c_0 t) - \frac{1}{2} A k \frac{g + k^2 T'}{g - 2k^2 T'} \cos 2k(x - c_0 t), \tag{90}$$

where the constant A has been chosen so as to be related to the wave amplitude and $c_0 = (g k^{-1} + T' k)^{1/2}$ is the first term in the expansion $c = c_0 + Ak c_1 + (Ak)^2 c_2 + \cdots$ [cf. (36)]. Here it follows from the analysis that $c_1 = 0$. Evidently (90) cannot be used for $k = k^{(2)}$ and, as stated above, one must start anew with the complete linearized solution for this wave number. Use of this solution as a starting point brings out two surprising facts. The coefficient of the term in $\sin 2k^{(2)}(x - c_0^{(2)} t)$ is not independent of that of the other term, and c_1 is no longer zero. One finds the following expressions for η and c; the corresponding ones for Φ will not be given.

$$\eta = A \cos k^{(2)}(x - ct) \pm \frac{1}{2} A \cos 2k^{(2)}(x - ct)$$
$$\mp \frac{3}{2} A^2 k^{(2)} \cos 3k^{(2)}(x - ct) - \frac{3}{4} A^2 k^{(2)} \cos 4k^{(2)}(x - ct), \tag{91}$$

$$c = c_0^{(2)} \left(1 \pm \frac{1}{4} A k^{(2)}\right).$$

A parasitical solution of the form (90) with k replaced by $2k^{(2)}$ should properly not be added to (91), for its velocity will be $c_0^{(2)}$ instead of c.

Further details concerning this phenomenon may be found in Wehausen and Laitone [1960, § 27] and especially in Sekerzh-Zenkovich [1956] and Pierson and Fite [1961]. The latter authors add a necessary refinement to the analysis. If (90) is incorrect at $k = k^{(2)}$ it is also evidently unusable in the near neighborhood of this wave number, for otherwise there would be a qualitative difference between the waves at $k^{(2)}$ and those arbitrarily near to $k^{(2)}$. In order to investigate the behavior in the neighborhood of the wave numbers $k^{(n)}$ Pierson and Fite make use of an additional perturbation of the wave number near these resonant values. Their procedure yields a behavior near $k^{(2)}$, the only case computed, which is consistent with that at $k^{(2)}$. For recent experimental investigations of these waves we refer to Schooley [1958, 1960].

Longuet-Higgins [1963] has studied capillary-gravity waves under circumstances where the type of perturbation analysis used above is not applicable. In order to explain the occurrence of capillary waves ahead of the crests of very steep waves, he considers the capillary waves as a perturbation of a steep gravity wave. A particular choice of the latter is made on the basis of an approximation method due to Davies [see Wehausen and Laitone, section 34γ], a choice which can hardly affect the conclusions in any important way. In the boundary conditions for the perturbing flow the gravitational terms are then discarded. However, the effect of gravity is felt through the properties of the underlying flow. The method successfully predicts the capillary waves on the forward face of the crest. Further refinements are carried through in order to find out how the steepness of the capillary waves varies and how much energy is absorbed by them.

<u>Progressive waves: viscosity</u>. Longuet-Higgins [1953] developed a second-order theory of waves in viscous fluids in which boundary-layer type approximations were used near the free surface and the bottom. It was a result of this theory that the

mass transport velocity just above the bottom boundary layer should be

$$\frac{5}{4}\,\frac{A^2\sigma k}{\sinh^2 kh}$$

and that the gradient of this velocity at the surface should be

$$4A^2\sigma k^2 \coth kh .$$

The Stokes theory had predicted coefficients $\frac{1}{2}$ and 2, respectively, for these two quantities, and further a qualitatively different profile in the intermediate region. The coefficients themselves are sufficiently different so that one can hope to discriminate between them in an experiment. The first one has been confirmed by several persons. Some confirmation of the second one has been reported by Longuet-Higgins [1960]. In this paper Longuet-Higgins also points out an error in a second-order computation of Harrison's [see Wehausen and Laitone, 1960, section 27γ] for progressive waves in a viscous fluid.

<u>Standing waves</u>. A standing wave or clapotis occurs when the motion of each particle is periodic in time and when furthermore the curves of intersection of the free surface with the plane of the mean water level also move periodically with time. For two-dimensional motion over a horizontal plane it suffices to require that the motion be periodic in both space and time and be symmetric about some vertical plane, say x = o. Although the latter definition can be extended to include doubly modulated standing waves over a horizontal bottom, it is not extendable to all situations, for example, standing ring waves.

It has been noted earlier that the existence of a standing wave satisfying the exact boundary conditions has not been proved even for the simplest situations. However, approximations of the first few orders have been computed by several persons. Two procedures have been used. In one it is assumed from the outset that the motion is irrotational and the velocity potential is assumed to have the form of a power series in a perturbation parameter ε and a Fourier series in the time. The further boundary conditions and geometrical requirements then determine the specific terms in the series. The method is illustrated in Wehausen and Laitone [1960, Sect. 27β] for two-dimensional motion in infinite depth. Aside from the references given there, it has also been applied by Tadjbakhsh and Keller [1960] (two-dimensional motion, finite depth, third order), Verma and Keller [1962] (three-dimensional motion, finite depth, second order), Mack [1962, 1963] (axisymmetric three-dimensional motion, finite depth, third order), and Hunt [1961] (two-dimensional motion, interfacial waves, fourth order). A general discussion of the procedure for standing waves in a basin has been given by Moiseev [1958].

The other procedure makes use of Lagrangian coordinates and may be carried through without assuming initially irrotationality of the motion. However, one of the interesting consequences of this method is that the motion is, in fact, irrotational at each order of approximation. This approach to standing waves has been developed especially by Sekerzh-Zenkovich since 1947. He has computed to the third order two-dimensional waves in finite depth [1951], and has recently [1959] published a detailed account of results announced in 1952 for doubly modulated three-dimensional waves in infinite depth, computed to the second order; two-dimensional interfacial standing waves are computed to the third order [1961]; both zonal [1959a] and sectorial [1959b] waves on the surface of a sphere are computed to the second order. Lagrangian coordinates are also used in the investigations of Chabert d'Hières [1960] and Gouyon [1961]. For a finitely deep fluid Gouyon has succeeded in finding explicit expressions for the approximation of nth order.

Since Lagrangian coordinates have not been used in any of the earlier developments, it may be instructive to sketch the computation for two-dimensional motion. Let each particle be identified by its coordinates (a, b) when the fluid is at rest. The fluid motion may then be described by giving $x(a,b,t)$, $y(a,b,t)$. The equations satisfied by x, y and p are

$$\frac{\partial(x,y)}{\partial(a,b)} = 1,$$
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} = -\frac{\partial}{\partial a}\left(\frac{p}{\rho} + gy\right), \tag{92}$$
$$\frac{\partial^2 x}{\partial t}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} = -\frac{\partial}{\partial b}\left(\frac{p}{\rho} + gy\right).$$

Eliminating p gives

$$\frac{\partial}{\partial t}\left\{\frac{\partial(x,x_t)}{\partial(a,b)} + \frac{\partial(y,y_t)}{\partial(a,b)}\right\} = 0,$$

or, after integrating

$$\frac{\partial(x,x_t)}{\partial(a,b)} + \frac{\partial(y,y_t)}{\partial(a,b)} = \zeta(a,b), \tag{93}$$

where ζ is the vorticity. On the free surface and bottom x and y must satisfy, respectively,

$$x_{tt}(a,0,t)x_a + (g + y_{tt})y_a = 0, \quad y(a,-h,t) = -h. \tag{94}$$

Periodicity in time and space requires

$$x(a, b, t+T) = x(a, b, t), \quad y(a, b, t+T) = y(a, b, t),$$
$$x(a+\lambda, b, t) = \lambda + x(a, b, t), \quad y(a+\lambda, b, t) = y(a, b, t). \tag{95}$$

Finally, in order to ensure that the motion behaves like a standing wave and to fix the phase we require, respectively,

$$x(0, b, t) = 0, \qquad y_t(0, 0, 0) = 0. \tag{96}$$

The perturbation expansion analogous to (36) now takes the following form:

$$x(a, b, t) = a + \sum_1^\infty \varepsilon^n \xi_n(a, b, t), \quad y(a, b, t) = b + \sum_1^\infty \varepsilon^n \eta_n(a, b, t),$$
$$\zeta(a, b, t) = \sum_1^\infty \varepsilon^n \zeta_n(a, b, t), \quad T^2 = \sum_0^\infty \varepsilon^n T_n^2. \tag{97}$$

Substitution in the equations and boundary conditions yields equations of the following form:

$$\frac{\partial \xi_n}{\partial a} + \frac{\partial \eta_n}{\partial b} = P_n, \qquad \frac{\partial^2 \xi}{\partial b \partial t} - \frac{\partial^2 \eta_n}{\partial a \partial t} = \zeta_n + Q_n,$$
$$g\frac{\partial \eta_n}{\partial a} + \frac{\partial^2 \xi_n}{\partial t^2} = R_n \text{ on } b = 0, \quad \xi_n(0, b, t) = \eta_n(a, -h, t) = \frac{\partial \eta_n}{\partial t}(0, 0, 0) = 0, \tag{98}$$

where $P_1 = Q_1 = R_1 = 0$ and P_n, Q_n, R_n for $n > 1$ depend in a relatively simple way upon the solutions ξ_n, η_n, T_p for $1 \le p \le n-1$. As mentioned above, Gouyon was able to solve the equations above and give explicit expressions for ξ_n, η_n and T_n and to show that $\zeta_n = 0$, a conclusion established earlier by Chabert d'Hière and by Sekerzh-Zenkovich.

Among the physical consequences which can be drawn from the computations are the following: (1) There are no fixed nodes. (2) The surface is never flat. (3) For $h/\lambda < 0.1684$ the period T is less than that predicted by linearized theory and decreases with increasing wave steepness whereas for $h/\lambda > 0.1684$ the reverse holds. This is quite different from progressive periodic waves where the period is less than that for linearized theory and decreases with increasing steepness or h/λ. This fact was explicitly pointed out by Chabert d'Hière and Tadjbakhsh and Keller but is also implicit in the formulas given by Sekerzh-Zenkovich. The phenomenon has been investigated experimentally by Fultz [1962], who finds a reversal in behavior of T near (~ 0.14) the predicted value of h/λ. It is stated by Moiseev [1958] that (1) and (2) hold in any cylindrical basin, conclusions confirmed by the various cited papers dealing with three-dimensional standing waves. The third conclusion is also found in the three-dimensional problems treated by Verma and Keller [1962] and Mack [1962].

If surface tension is included in the boundary condition at the free surface a resonance phenomenon analogous to that discussed earlier for progressive waves occurs. The matter is discussed in Wehausen and Laitone [1960, sec. 25β) and formulas for $h = \infty$ are carried to the second order. Recently Concus [1962] has provided third-order computations for $h < \infty$, but without special investigation of the resonant frequencies. The reversal in behavior of the period as a function of h/λ discussed above for pure gravity waves occurs but is followed at a still smaller value of h/λ corresponding to the first resonant frequency by a further reversal. The relevant curves are plotted by Concus.

<u>Composite waves</u>. A natural next step after an investigation of waves, either progressive or standing, of a given wave length is to examine the effect of retaining higher-order terms in the computation when two or more wave lengths are present. Aside from pure curiosity there have been at least two direct motivations. In a wave channel or in a towing tank for ship models the wave maker is almost inevitably so constructed that, when second-order terms are kept, it generates not only components like $\sin 2(kx-\sigma t)$ travelling with the velocity of the primary component $\sin(kx-\sigma t)$ but also components like $\sin(k'x-2\sigma t)$ which travel with different velocities. Hence an experimental investigation aimed at a comparison of second-order theory with laboratory measurements will not be properly carried out unless the second-order theory is appropriate to the circumstances. The other direct motivation has come from oceanography. In order to explain how it is possible for energy to pass from one part of an ocean-wave spectrum to another, it is necessary to go beyond the linearized theory. The problem has been studied by several persons, and will be returned to below.

The most extensive recent study of two-dimensional composite waves has been made by Daubert [1960], who has carried his computations to the third order in Lagrangian coordinates. He does not assume irrotationality, but in the notation of (92) does assume that $\zeta = \zeta(b)$, which is appropriate to steady periodic waves but an extra assumption here. In the first order Daubert's expressions reduce to a superposition of n sinusoidal waves.

As one might expect, the third-order expressions are extremely complicated, involving various sum and difference waves which can be generated from the fundamentals taken two and three at a time. One conclusion is striking. Although the total motion cannot be steady in any coordinate system, it can be expressed as the superposition of a finite number of periodic motions each of which is steady in some moving coordinate system. Daubert calls these components pseudo-steady waves ("pseudo-houles"). The velocities of propagation of these pseudo-steady waves is not the same as that of an ordinary steady wave of the same length. This result of Daubert's appears to contradict the results of Phillips and others described below. This is a consequence of the fact that Daubert has deliberately excluded the very solutions which would have given Phillips' conclusions, presumably because they could be valid only for a very limited time interval (in Phillips' formulation), whereas he was seeking solutions which held for all values of t.

The property of composite two-dimensional waves mentioned above might seem to exclude the possibility that transfer of energy from one frequency to another could occur in a theoretical model of this sort. However, as stated above, this conclusion is a result of Daubert's method of analysis. When the problem is treated, say, as an initial-value problem, then for certain appropriately related frequencies a transfer of energy from one or several of them to a different one may take place. For this to occur the waves need not be propagating in one direction. In order for this transfer to show up the theoretical calculations must be carried through the third order. The fundamental facts were first discovered by Phillips [1960]. The phenomenon has since been studied by Longuet-Higgins [1962], Longuet-Higgins and Phillips [1962], Hasselmann [1962, 1963] and Benney [1962]. The latter's treatment of the problem overcomes certain objections to the earlier ones. The result may be stated as follows. Let the equation of the surface reduce in the first order to

$$\eta(x,z,t) = \sum_{i=1}^{4} A_i \sin(k_{i1}x + k_{i2}z - \sigma_i t), \tag{99}$$

where, for deep water $\sigma_i^2 = g[k_{i1}^2 + k_{i2}^2]^{1/2} = g|\underline{k}_i|$. Then if the wave-number vectors $\underline{k}_i$ and the frequencies σ_i are related in any of the following ways,

$$\underline{k}_1 \pm \underline{k}_2 \pm \underline{k}_3 \pm \underline{k}_4 = 0, \quad \sigma_1 \pm \sigma_2 \pm \sigma_3 \pm \sigma_4 = 0, \tag{100}$$

with the same sequence of signs in each, there will occur "resonant interactions" with the result that the energy distribution among the components will vary with time. In particular, if one component is absent at $t = 0$, energy can be transferred from the other modes so that the missing component will later appear among the first-order components. The situation which has been most studied is that in which $\underline{k}_4 = 2\underline{k}_1 - \underline{k}_3$ and $\sigma_4 = 2\sigma_1 - \sigma_3$. For two-dimensional motion the interchange of energy is between waves of wave numbers $\underline{k}_0$, $\frac{9}{4}\underline{k}_0$, and $-\frac{1}{4}k_0$. The implications of these resonant interactions for the development in time of ocean-wave spectra have been especially explored by Hasselmann in the several cited papers. It is perhaps appropriate to remark that the degree of complexity of the details of these investigations is such that one must be strongly motivated to wish to examine them.

Related to the investigations described above, but aimed at explaining other phenomena, are a series of papers by Longuet-Higgins and Stewart [1960, 1961, 1962], Hughes and Stewart [1961], G. I. Taylor [1962], and Whitham [1962]. The phenomena in question are the interaction between waves and currents of various types, especially non-uniform currents. The non-uniform current may, in fact, be provided by another wave. Consider a composite wave consisting in the linearized theory of two components with wave numbers and amplitudes k_1, $k_2 >> k_1$ and A_1, A_2 respectively. Even though the numbers $A_1 k_1$ and $A_2 k_2$ are each small enough so that linearized theory would appear allowable, A_1/A_2 may be so large that the usual linearization assumption

$$\Phi(x,\eta,z,t) \cong \Phi(x,0,z,t)$$

may not really be compatible with the resolution

$$\Phi = \Phi_1 + \Phi_2 ,$$

where Φ_i is the potential for the wave of wave number k_i. In fact, not only $A_1 k_1$ and $A_2 k_2$ but also $A_1 k_2$ should be small in order to apply linearized theory. The investigation of this and similar problems has led the authors mentioned above to reconsider in a fundamental way various other aspects of wave propagation connected with the transport of mass, momentum, and energy when the theory is carried through the second order and when currents are present. A superficial summary of certain new concepts introduced by these authors could only be misleading.

Other initial- and boundary-value problems. Although most problems of this nature might seem sufficiently forbidding in their linearized formulation that one would not wish to become involved in higher-order approximations, nevertheless several of them do not lead to new mathematical difficulties, but primarily to algebraic complexity. This is not to say, of course, that the extension is simply a matter of bookkeeping and not of great interest. As we have seen in the preceding sections, certain phenomena can only be explained by going to higher orders.

One of the classical initial-value problems of water-wave theory is the Cauchy-Poisson problem in which the initial surface profile and velocity are the given data. Sretenskii [1961] has developed the higher-order theory for the special case in which the motion is two-dimensional and the fluid bounded by two vertical walls. Further, his initial profile is taken flat. Lagrangian coordinates are used in a somewhat similar way to that in the standing-wave problem discussed above. In fact, the same notation may be kept. Assume from the start that $\zeta(a,b)=0$ so that there exists $\varphi(a,b)$ with $\xi_t(a,b,0)=\varphi_a$, $\eta_t(a,b,0)=\varphi_b$. To the first two perturbation expansions (97) one now adds

$$\varphi(a,b) = \sum_{n=1}^{\infty} \varepsilon^n A_n e^{nb} \cos na , \qquad H(a,b,t) = \frac{p}{\rho} + gy = \sum_{n=1}^{\infty} \varepsilon^n H_n , \tag{101}$$

where the initial and free-surface conditions require

$$\xi_i(a,b,0) = \eta_i(a,b,0) = 0, \qquad \xi_{it}(a,b,0) = -nA_n e^{nb} \sin na, \tag{102}$$

$$\eta_{it}(a,b,0) = nA_n e^{nb} \cos na, \qquad H_n(a,0,t) = g\,\eta_n(a,0,t).$$

Sretenskii's procedure for solving this is to introduce a sequence of auxiliary variables $w_i = \sigma_i t$, $i = 1, 2, \cdots$, and to assume the expansions

$$\frac{\partial \xi}{\partial t} = \sum \frac{\partial \xi}{\partial w_i} \sigma_i , \quad \frac{\partial^2 \xi}{\partial t^2} = \sum \frac{\partial^2 \xi}{\partial w_i \partial w_j} \sigma_i \sigma_j , \tag{103}$$

where $\sigma_i = \sigma_{i0} + \sigma_{i1} \varepsilon + \cdots$. Substitution of the various series in (92) enables him to determine successively the σ_{ij} and $\xi_i(a, b, t)$, etc.

The theory of waves generated in a tank of rectangular cross-section by a wave maker in the form of a deforming wall at one end has been carried to the second order by Fontanet [1961]. The solution of the boundary-value problem for the second-order terms is not more difficult in principle than for the first order, but, of course, considerably more complicated. Fontanet's investigation is very complete, including a study of the singularities at the top and bottom of the wave maker and numerical computations for an oscillating-bulkhead wave-maker.

A related problem is that of the waves in a sheltered harbor excited through an inlet to the open sea. This problem has been considered in a simplified model by Gaillard [1960] and Takano [1959]. The model consists of a rectangular basin of finite depth with a narrower rectangular inlet centered on one side and with a wave-maker at one end. In the second-order theory seiches may occur which are of much longer period than the lowest resonant periods predicted by linearized theory. Such long-period seiches have been observed in harbors and often result in the breaking of mooring lines. Certain general aspects of this problem have also been discussed by Moiseev [1958].

Another problem of this genre is the <u>propagation of waves over an obstacle.</u> In order to simplify the problem let us suppose that waves of frequency σ are propagating down a canal of rectangular section in which there is an obstacle across the canal in the form of a rectangular solid. Thus the waves move first in water of depth h , then of depth $h_0 < h$ for a distance ℓ and then again of depth h. Even in its linearized formulation the problem is not easy, although a formal solution has been given by Takano [1960]. However, certain phenomena are observed both in the laboratory and in the ocean which cannot possibly be explained by linearized theory. In particular, behind the obstacle the transmitted wave decomposes into a sum of waves of frequencies $\sigma, 2\sigma, \cdots$, each propagating with its proper velocity according to the linearized theory. The number of higher harmonics increases with decrease of h_0 for fixed ℓ, and the phenomenon is not observed if either h_0/λ or λ/ℓ is too large. It is not difficult to explain qualitatively what happens. As the waves propagate from the deeper to the shallower water the amount of transmitted energy per wave length requires such a large value of A_0/h_0 that the linearized theory is a poor approximation for the new value of h_0/λ_0 which is considerably less than h/λ. It is helpful in this respect to refer to Fig. 1. Further, the action of the fluid before the obstacle upon that over it may be likened to that of a wave-maker; in particular, the matching of solutions at the edge will correspond to satisfying boundary conditions on the wave maker. If the action is intense enough to require higher-order approximations, one may expect wave components of frequency $2\sigma, 3\sigma, \cdots$, each travelling at its own proper velocity. At the lower end of the obstacle these waves again act as wave makers for the water below. However, even though linearized theory may be adequate for this region, waves of frequency $2\sigma, 3\sigma, \ldots,$ as well as σ may now be present. A start on an analytic treatment has been made by Takano [1959], who considers the second-order theory for $\ell = \infty$; some experiments have been carried out by Jolas [1961, 1962].

The second-order theory of ship motion in waves has also been broached, although the general tendency is to try to avoid taking this step and to resolve certain difficulties by means of other linearization methods such as the slender-body approximation. The problem of higher-order approximation is discussed in a general way by Lugovskii [1961], but the only problem he solves is to find the velocity potential to the second-order for the plane flow generated by a submerged line source of strength $\varepsilon \gamma_1 \cos \sigma t + \varepsilon^2 \gamma_2 \cos 2\sigma t$. In a certain sense this problem has, in fact, been discussed by Newman [1959, 1961]. He expands his velocity potential in a perturbation series with two parameters, one α measuring the steepness of the incoming waves, the other β the thinness of the ship. Not only the terms associated with α and β are retained but also those associated with $\alpha\beta$. The procedure was successful in that diffracted waves and waves caused by the ship's own motion, absent in the pure linearized theory, are present here. Agreement between measured and computed values has also been reasonably good. One interesting physical phenomenon which appears when the terms in $\alpha\beta$ are retained is a resonance at the value $\sigma U/g = 1/4$ for deep water, i.e. when $C_g - U = U$ where C_g is the group velocity. For example, for a ship of length L moving head-on into oncoming deep-water waves of wave number k there will be a singularity in the heaving and pitching response when the Froude number

$$\frac{u}{\sqrt{gL}} = \frac{-1+\sqrt{2}}{2}\,\frac{1}{\sqrt{kL}} ,$$

corresponding to an effective frequency $\sigma_e = \sigma(k) + uk = g/4u$. This phenomenon has often been observed in model tests, and also occurs in other contexts [cf. Nigam and Nigam, 1962]. With respect to the second-orderness of Newman's analysis one should note that only the first power of α is retained, so that only the linearized free-surface boundary condition is used.

Instability. We discuss briefly two classical stability problems in which higher-order approximations have been carried out. The computations for Taylor instability (see p. 68) have been carried to the third order by Emmons, Chang and Watson [1960]. The most striking prediction is that the region of stable wave numbers according to the linearized theory (i.e. $k^2 > (\rho_2 - \rho_1) g/T$) is also unstable in the third-order theory in the sense that the surface oscillates with an amplitude increasing with time, but only linearly. Experiments reported in the paper seem to confirm this behavior and also the initial rates of growth according to the linearized theory. A fifth-order computation with neglect of surface tension has been made by Gillis and Goldstine [1957].

The other problem is that of the stability of the surface of a fluid in a vertically oscillated container. The linearized theory has been studied by several persons and is given in Wehausen and Laitone [1960, pp. 651-653]. As mentioned earlier, Yih [1960a] has extended the linearized analysis to stratified fluids. Skalak and Yarymovych [1962] have carried the computations to the third order for a rectangular basin. One consequence of the more exact theory is to bound the amplitudes of the waves for unstable frequencies. An experimental determination of the maximum amplitude as a function of frequency in the neighborhood of the $\frac{1}{2}$-subharmonic, the only one which could be excited, showed good agreement with the theoretical predictions.

Cavitational flows. Tulin's linearized theory of flow past thin hydrofoils with cavities (see p. 70 ff.) has been extended by Chen [1962] to include second-order terms. The extension is limited to zero cavitation number and is carried through for two cases: a hydrofoil with small angle of attack and detachment at the nose as well as tail; a symmetric wedge with a sharp cut-off at the after end. Numerical computations were carried through for a circular-arc hydrofoil and for a straight-sided wedge. The agreement between exact and second-order theory is still remarkably good at angles of attack or wedge angles where the linearized theory is in error by substantial percentages.

The shallow-water theory.

The problems treated by the shallow-water approximation can be classified in a manner similar to that used for the infinitesimal-wave theory, i.e., first by the order of the approximation and then by various other characteristics. It will be evident from the topics discussed below that this method of finding an approximate solution does not allow one to deal easily with a wide variety of geometrical configurations. This is essentially a consequence of the non-linearity of the equations.

Since the derivation of the fundamental equations (43) for this approximation was carried through, by way of illustration, only for a steady continuous two-dimensional flow with vorticity we give here the more complete equations but for irrotational motion. Their derivation may be found in Wehausen and Laitone [1960, sect. 10β]. With a notation analogous to that in (42), the equations for motion in three dimensions which correspond to the first set in (43) are

$$
\begin{aligned}
&u_{0t} + u_0 u_{0x} + w_0 u_{0z} + g\eta_{0x} = 0\,, \\
&w_{0t} + u_0 w_{0x} + w_0 w_{0z} + g\eta_{0z} = 0\,, \\
&\eta_{0t} + [u_0(\eta_0 + h)]_x + [w_0(\eta_0 + h)]_z = 0\,,
\end{aligned}
\tag{104}
$$

where $y = -h(x,z)$ is the equation of the bottom. As is well known, these equations are the same as those satisfied by two-dimensional isentropic irrotational motion of a perfect gas if one assumes h constant, identifies the gas density with $\eta + h$ and takes $\gamma = 2$ in the adiabatic equation of state.

By assuming the velocity and its gradient and the surface and its slope sufficiently small one may further approximate (104) by the linearized equations

$$
\begin{aligned}
&u_{0t} + g\eta_{0x} = 0, \quad w_{0t} + g\eta_{0z} = 0, \\
&\eta_{0tt} - g(h\eta_{0x})_x - g(h\eta_{0z})_z = 0\,.
\end{aligned}
\tag{105}
$$

These are the equations usually used in the theory of seiches and tides in situations in which the curvature of the earth and the Coriolis force may be neglected. Since the derivation of (105) from (104) is based upon neglections of the same sort used in the infinitesimal-wave theory, one might guess that it would be possible to derive (105) directly from those equations by bringing into play the effect of shallowness of the water. This has, in fact, been done by Shinbrot [1962] under some further assumptions. Included are estimates of the error in using (105) instead of the infinitesimal-wave equations and an explicit solution showing among other things that η varies as $h^{-1/4}$ [see also Longuet-Higgins, 1956].

The equations analogous to (104) for the next higher approximation will not be given here, but may be found in Wehausen and Laitone [1960, sect. 10β]. References for still higher orders of approximation under special conditions will be given later.

We now turn to the consideration of special aspects of the shallow-water theory which have received attention during recent years.

The linearized theory. Since the most important application of the linearized theory is to oceanographic problems, the equations (105) are inadequate because the rotation of the earth is omitted. We shall not introduce the usual equations, but note that terms similar to those containing ω in (72) must be included in the first two equations of (105). Generally one is looking for a time-periodic solution to the equations with boundary conditions corresponding to some configuration of geophysical interest. Among such recent work is that of Chaikovskaya [1961], Crease [1956, 1958], Fjeldstad [1958], T. Ya. Sekerzh-Zenkovich [1959b, c] and Voit [1961]. Sretenskii and Stavrovskii [1961] have recently considered an initial-value problem by linearized theory, the tsunami caused by a sudden motion of the bottom near a straight vertical cliff. Computations of the wave height along the cliff are given for various times after the waves reach the shore when the bottom disturbance is confined to a narrow rectangle parallel to the shore. (We note that for this problem equations (105) have to be extended to include motion of the bottom.) T. Ya. Sekerzh-Zenkovich [1959a] has also considered a Cauchy-Poisson initial-value problem. There are many more papers of specific oceanographic interest; the few mentioned above were selected because they correspond to configurations which may be, and frequently are, also studied by the usual infinitesimal-wave theory. Some problems of mathematical interest are described by van Dantzig [1959] and by Lauwerier [1960, 1961]. Sretenskii [1960] has recently provided a more comprehensive survey of recent accomplishments in the theory of tidal waves and an extensive bibliography.

The non-linear theory. The application of the nonlinear first-order equations (104) to the solution of various problems of steady and unsteady one-dimensional flow and of steady two-dimensional flow when h is constant is described in Wehausen and Laitone [1960, sect. 30] and in other places, e.g., Khristianovich [1938] and Stoker [1957]. Some peculiarities of the solutions should be mentioned. In certain initial-value problems the solution becomes discontinuous after a finite interval of time even though the initial data are continuous. Thereafter the solution is discontinuous. Also, in certain steady-flow problems no continuous solution exists. The discontinuities are usually called "bores" or "hydraulic jumps".

It is evident that the equations (104) must be supplemented by conditions to be satisfied at a discontinuity. Let B be the curve in the $(x,\mathfrak{z})$-plane along which it occurs and V_n its velocity at each point in the direction of the normal there.

Let $\underline{V} = (u, w)$ be the particle velocity, $v_n = \underline{V} \cdot \underline{n}$ and $\xi_n = v_n - V_n$. According to the usual notation, we let $[f] = f_2 - f_1$, when f_2 and f_1 are the limiting values of f on the positive and negative sides of β respectively. It then follows from conservation of mass and momentum that

$$[\rho(\eta+h)\xi_n] = 0, \quad \rho(\eta+h)\xi_n[\underline{V}] - \tfrac{1}{2}\rho g[(\eta+h)^2]\,\underline{n} = 0. \tag{106}$$

By various manipulations other relations may be derived from these, e.g.,

$$[v^2] = [V_n^2], \quad [\xi_n^2] + \tfrac{1}{2} g[(\eta+h)^2]\left\{\frac{1}{\eta_1+h} + \frac{1}{\eta_2+h}\right\} = 0, \tag{107}$$

$$\xi_{n1}\xi_{n2} = \tfrac{1}{2} g(\eta_1+h)(\eta_2+h),$$

all analogues of similar shock conditions in a compressible fluid. It turns out that conservation of mechanical energy is not compatible with a discontinuous solution. It is then assumed that energy must be lost at a bore or jump; this gives the inequality

$$\begin{aligned}\rho(\eta+h)\xi_n[\tfrac{1}{2}v^2] &+ \tfrac{1}{2}\rho g(\eta+h)\xi_n[\eta+h] + \tfrac{1}{2}\rho g[(\eta+h)^2 V_n] \\ &= \rho(\eta+h)\xi_n[\tfrac{1}{2}\xi_n^2 + g(\eta+h)] = -\tfrac{1}{4} g\rho(\eta+h)\xi_n \frac{[(\eta+h)^3]}{(\eta_1+h)(\eta_2+h)} \le 0. \end{aligned} \tag{108}$$

This inequality plays a role similar to that of the entropy-increase condition in a shock wave. In particular, if $\underline{n}$ has been selected so that $\xi_n > 0$, then

$$\eta_2 + h > \eta_1 + h, \tag{109}$$

i.e. relative to the bore the particles must move through it in the direction from lower to higher water levels. It now follows easily from the first equation in (106) that $\xi_{n2} < \xi_{n1}$ and from the last one in (107) that

$$g(\eta_1+h) < \xi_{n1}\,\xi_{n2} < g(\eta_2+h),$$

so that

$$\xi_{n1}^2 > g(\eta_1+h) \equiv c_1^2 \,, \qquad \xi_{n2}^2 < g(\eta_2+h) \equiv c_2^2 \,. \tag{110}$$

In the usual terminology the relative normal velocity must be supercritical ahead and subcritical behind the bore or jump.

For a discussion of the usual methods of solving the equations (104) and of using the jump conditions we refer to the sources cited above. Several recent investigations consider situations not covered in those sources. Most deal with waves on beaches. Carrier and Greenspan [1958] have constructed a time-periodic solution for an incoming and reflected wave on a plane sloping beach such that the motion remains continuous. They have also constructed continuous solutions to the initial-value problem where $u(x,0) = 0$ and $\eta(x,0)$ belongs to a particular class of functions. On the other hand, Greenspan [1958] has shown that, if a disturbance moving into still water has a positive amplitude and nonzero slope at its front, it will develop a discontinuity before reaching the shore. The remaining papers assume the bore already present and moving into still water. The paper of Keller, Levine and Whitham [1960] is partly aimed at making a numerical check of an approximation proposed by Whitham [1958] which has wider applicability. Among the conclusions reached that are of significance for the problem at hand are the facts that the bore height approaches zero and its speed a finite positive limit as it approaches the shore, in particular, that $[\eta]$ varies as $h^{1/2}$. (We recall that the linearized "acoustic" theory had given $\eta \propto h^{-1/4}$.) Ho and Meyer [1962] and Shen and Meyer [1963] have considered the behavior at the shore in considerably greater detail. The first authors confirm the result above and show further the existence of a singularity in the acceleration at the shore. In the second paper of the last two authors the behavior of the flow after the shore is reached is studied and among other things the existence of a thin sheet of water moving up the beach with constant deceleration is deduced. Dressler [1958] has considered the dam-break problem in a sloping channel.

Cnoidal and solitary waves and higher approximations.

For higher-order approximations only two-dimensional steady flows have as yet been considered, and hence equations (43) will be adequate for most of the following. In order to find $\bar{\psi}_1$, one must make use not only of the equations associated with ε^2 but also with ε^4. First we note that after eliminating $\bar{\eta}_1$ and $\bar{\eta}_2$ in (43) and evaluating f and its derivatives in terms of u_0 the boundary conditions at $y=0$ may be written as follows:

$$u_0^2(0)\bar{\psi}_{1\bar{y}}(\bar{x},0) - (u_0u_0' + g)\bar{\psi}_1(\bar{x},0) = 0,$$

$$u_0^2(0)\bar{\psi}_{2\bar{y}}(\bar{x},0) - (u_0u_0' + g)\bar{\psi}_2(\bar{x},0) = u_0\bar{\psi}_{1\bar{y}\bar{y}}\bar{\psi}_1 - \frac{1}{2}u_0\bar{\psi}_{1\bar{y}}^2 \tag{111}$$

$$- \left\{\frac{1}{u_0}(u_0u_0' + g) - u_0'\right\}\bar{\psi}_{1\bar{y}}\bar{\psi}_1 - \frac{1}{2}\frac{1}{u_0}\left\{u_0u_0'' + u_0'^2 - \frac{u_0'}{u_0}(u_0u' + g)\right\}\bar{\psi}_1^2.$$

Consider now the simple case where $u_0'' = 0$, so that $f(\bar{\psi}_0) = u_0'(\bar{y})$ and $f'(\bar{\psi}_0) = 0$. Then from $\bar{\psi}_{1\bar{y}\bar{y}} = 0$ and $\bar{\psi}_1(\bar{x}, -h) = -Q_1$ we find $\bar{\psi}_1 = a(\bar{x})(\bar{y}+h) - Q_1$. Substitution in (111) yields

$$a(\bar{x})\left\{u_0^2(0) - (u_0 u_0' + g)h\right\} = -(u_0 u_0' + g)\,Q_1.$$

Either $a(\bar{x})$ = const., giving a trivial solution, or else $Q_1 = 0$. The latter implies

$$u_0^2(0) = (u_0(0)u_0'(0) + g)h \quad \text{and} \quad \bar{\eta}_1(\bar{x}) = -h\,a(\bar{x})/u_0(0). \qquad (112)$$

To proceed further we need the equations $\bar{\psi}_{1\bar{x}\bar{x}} + \bar{\psi}_{2\bar{y}\bar{y}} = 0$ and $\bar{\psi}_2(\bar{x}, -h) = -Q$. These yield

$$\bar{\psi}_2 = -\frac{1}{6}\,a''(\bar{x})(\bar{y}+h)^3 + b(\bar{x})(\bar{y}+h) - Q_2.$$

After substitution in (111) and some simple reductions one finds easily

$$-\frac{1}{3}\,u_0^2(0)h^2 a'' + \frac{1}{2}\,u_0\left[2 - \frac{h u_0'}{u_0} + \left(1 - \frac{h u_0'}{u_0}\right)^2\right]a^2 + \frac{u_0^2}{h}\,Q_2 = 0, \qquad (113)$$

a differential equation for $a(\bar{x})$. The usual case of irrotational flow is obtained by setting $u_0' = 0$. Then (113) becomes

$$a'' - \frac{9}{2}\,\frac{1}{h^2\sqrt{gh}}\,a^2 - \frac{3}{h^3}\,Q_2 = 0. \qquad (114)$$

It follows from (114) and also from (113) if $u_0(0) > 0$ that one must have $Q_2 < 0$ for a physically significant solution, for otherwise one would have $\bar{\eta}'' < 0$ everywhere. Let us recast (114) as an equation for $\bar{\eta}_1$:

$$\bar{\eta}_1'' + \frac{9}{2}\frac{1}{h^3}\bar{\eta}_1^2 + \frac{3}{u_0 h^2} Q_2 = 0 . \tag{115}$$

From the integral

$$\bar{\eta}_1'^2 + 3\frac{1}{h^3}\bar{\eta}_1^3 + \frac{6}{u_0 h^2} Q_2 \bar{\eta}_1 + C = 0 \tag{116}$$

one can easily deduce that for a given value of Q_2 there will be a one-parameter family of solutions if one fixes a crest at $x = 0$. Then $C = -6h^{-3}(-2hQ_2/u_0)^{3/2}$ corresponds to the solitary wave and larger values of C to periodic cnoidal waves. The solitary wave has the largest amplitude for a fixed value of Q_2. The mean depth is no longer given by h and is least for the solitary wave.

There is no need to repeat here the well known properties of these waves. Rather, we wish to point out that (113) has exactly the same structure as (114) and that to this order of approximation the properties of solitary and cnoidal waves on a stream with linearly varying velocity distribution can be computed with little extra trouble. This possibility was apparently first pointed out by Burns [1953], who was more concerned, however, with the linearized theory with vorticity. Since then Hunt [1955], Moiseev [1960], Ter-Krikorov [1961], and Benjamin [1962] have treated the problem more intensively. Moiseev still restricts the vorticity distribution so that the underlying flow must be either linear or quadratic. However, both Ter-Krikorov and Benjamin allow arbitrary differentiable functions of ψ but discuss only the solitary wave. Moiseev and Benjamin start from the equations (22) instead of (21) as we have done; Ter-Krikorov introduces a different variable to replace x. It has already been mentioned that Ter-Krikorov proves also the existence of solitary waves under appropriate restrictions. His approximate expression for the surface is

$$\eta = A \operatorname{sech}^2 \beta x,$$

$$A = h^2 \frac{c^{*2} - gh}{gh\, c^{*4} \int_0^h u_0^{-4}(y)\,dy}, \qquad c^{*2} = \left[\frac{1}{h}\int_0^h u_0^{-2}(y)\,dy\right]^{-1}, \tag{117}$$

$$\beta^2 = \frac{A}{4h} \frac{c^{*2} \int_0^h u_0^{-4}(y)\,dy}{\int_0^h u_0^2(y)\left[\int_0^y u_0^{-2}(t)\,dt\right]^2 dy},$$

where $u_0(y)$ is the velocity distribution as $x \to \pm\infty$, h is the depth there, and one must have $c^{*2} > gh$ for there to be a solution.

For cnoidal waves in irrotational flow Wiegel [1960] has made numerical calculations of several important properties. These are presented as graphs in which various combinations of A/h, $c/\sqrt{gh}$, $\lambda^2 A/h^3$, $T\sqrt{g/h}$ (T = period) are plotted. A useful collection of formulas and results of experiments are also included.

From the form of (43) and (111) it is already clear that approximations of still higher orders will take the same mathematical form, but with the combinations of known lower-order functions becoming more and more complex. Such higher-order computations have, in fact, been made. Laitone [1960, 1962] has computed η_2 and the associated quantities and

Chappelear [1962] has carried the computations one step further. His formulas allow computation of the same graphs shown in the paper of Wiegel cited above. Such computations should, however, be approached with caution. The series are asymptotic and there is no guarantee that more accurate numerical results will be obtained by retaining more terms. However, since their computation was relatively easy with an IBM 7090, a certain amount of optimism seemed in order. The results of first-, second-, and third-order computations of A/h against c^2/gh for fixed values of h/λ were computed and are shown in Figure 6. As one can see, there does seem to be a tendency for the curves to "converge" for $h/\lambda < 0.1$. Also, the agreement with the Stokes-wave approximations is reasonable in the region where both methods may be presumed to be valid. However, as was mentioned earlier, the agreement would have appeared even better if the fifth-order Stokes approximation had not been made. A further order of approximation for either might produce even greater discrepancies for the larger values of A/h.

There is a further development in the theory of solitary and cnoidal waves which will only be mentioned but which is not less interesting on that account. When a layer of fluid of density ρ_1 depth d_1 and with a free upper surface overlays another of density ρ_2 and depth d_2 resting on a solid bottom, there exist two critical speeds C_1 and C_2 instead of the single one $\sqrt{gh}$ when $\rho_1 = \rho_2$. One would anticipate that solitary and cnoidal waves might exist at both interface and free surface near each value. This is confirmed by Peters and Stoker (1960] who study especially the behavior of the solitary wave. They then further consider the situation where the density varies like an exponential and establish the existence of an infinite sequence of critical speeds converging to zero, each one having solutions possible in its neighborhood which behave like solitary and cnoidal waves. Long [1956] had previously studied solitary waves in two-fluid systems when the upper boundary was rigid.

Long waves with viscosity. Although the assumption of inviscid fluid is compatible with the shallow-water approximation in many physical situations, this ceases to be the case in some types of open-channel flow. In some cases the effect of viscosity can be accounted for artificially in an inviscid fluid by assuming an underlying velocity profile based upon observation. This has been considered above. However, it is not always satisfactory. For example, in flows down an inclined channel of slope more than 3° and Reynolds number $(Q/\nu = g h^3 \sin\alpha / 3\nu^2$; cf. (81)) between 75 and 420 Mayer [1959] has observed steady laminar periodic flows which are not properly described by cnoidal-wave theory as developed above (see also Binnie [1959]). Furthermore, the flow described in (81) has been investigated theoretically by both Benjamin [1957] and Ivanilov [1960], to determine its stability with respect to disturbances of long wave length. They found that the flow was unstable to waves of wave number k if

$$\frac{2}{5}\frac{g h^3}{\nu^2}\sin^2\alpha - \cos\alpha \geq (kh)^2 \frac{T'}{g h^2}\,, \tag{118}$$

or

$$\frac{Q}{\nu} \geq \frac{5}{6}\cot\alpha + \frac{5}{6}(kh)^2\frac{T'}{g h^2}\csc\alpha\,.$$

In view of these facts it appears reasonable to try to develop a shallow-water approximation of the steady-state Navier-Stokes equations and to look for periodic solutions. This has been done by Ivanilov [1961] who follows a procedure similar to that used in deriving (43). For small amplitude and small Reynolds number he finds a sinusoidal solution which is of larger discharge rate than the parallel flow (81) of the same mean depth and which is stable to long-wave disturbances whereas the flow (81) is unstable. In addition, he finds cnoidal-wave solutions. However, both these solutions have the property that there is only one free parameter, say, the wave length, when Q/ν and α have been fixed. In the inviscid-fluid theory there remain two free parameters. A solution for large Reynolds number is also given.

Other approximation methods.

Although the two classical approximation methods have been very fruitful, they are obviously limited in application by the assumptions inherent in their formulations. Other methods have been proposed which are effective for restricted categories of problems. Some are purely numerical treatments of the exact equations and boundary conditions. Others are methods of approximation analogous to the two already discussed. Several have been mentioned earlier in other contexts.

With one exception the other methods discussed below are directed at finding more precisely the profiles and properties of periodic progressive waves and solitary waves. The earliest attempts in this direction were Michell's and McCowan's procedures for finding, respectively, the periodic and solitary wave of greatest amplitude and Havelock's modification of Michell's method to find waves of any amplitude-length ratio in deep water. These are discussed in Wehausen and Laitone [1960, sect. 33].

Michell carried out numerical calculations only for infinitely deep water, but indicated how the analysis could be done for finite depth. Chappelear [1959] has made numerical calculations. Points obtained by interpolating his results are shown on Figure 6. Since the accuracy of his calculations began to decrease for h/λ below 0.10, the point for $h/\lambda = 0.05$ has been omitted. For $h/\lambda \leq 0.15$ they conform well with the curves from the Stokes-wave approximation of fifth order. Points corresponding to McCowan's and Yamada's [1957] calculation of the highest solitary wave are also shown. Filippov [1961] has recently computed the highest solitary wave over a submerged vortex. His computation for zero vortex strength appears to confirm McCowan's. Longuet-Higgins [1963] has recently pointed out an interesting fact concerning the motion near the crest of highest progressive waves, namely, that the acceleration of the particles is $\frac{1}{2} g$ upwards. For standing waves a criterion for the highest wave has been taken to be the attainment of an acceleration of g downwards.

Yamada [1958] has approached the computation of periodic waves in finite depth by a method which embraces Stokes waves, cnoidal and solitary waves and which could conceivably be the basis for an existence proof as universal as Krasovskii's. The waves are defined by the two parameters $Q/c\lambda$ and the ratio of the velocity at the crest to that at the trough. His computation method is based upon an iteration procedure. For his particular formulation of the problem one should consult the original paper. Another recent procedure not tied to either small slopes or long waves has been proposed by Bagin [1962].

Chappelear [1961] has proposed a direct calculation of periodic waves in finite depth which uses the exact boundary conditions. The general idea is to express $u(x,y)$, $v(x,y)$ and $\eta(x)$ as finite Fourier series satisfying all except the two free-surface conditions. The coefficients are then determined so as to satisfy best according to a least-squares criterion the two remaining conditions. In five cases waves computed according to this method satisfied the exact boundary conditions better than when computed by Stokes theory of the fifth order. Figure 6 shows several points computed according to this scheme; the numerical values were provided by Dr. Leon Borgman.

For certain flows of importance in hydraulic engineering, for example, the flow under a sluice gate, neither of the two standard approximation methods is suitable. Marchi [1953] has proposed a method which appears to be satisfactorily accurate for practical use. A mapping associated with the angle of the gate takes the image of the free streamline in the hodograph plane into a curve resembling a quarter of an ellipse. By assuming that this curve is an ellipse, Marchi is able to find the flow. A disadvantage of the scheme is that it does not carry with it any prescription for improvement. The method has also been used by Benjamin [1956] and Melkonyan [1957]. Procedures for dealing with other free surface flows with gravity have been given by Duisheev [1958], Dumitrescu [1956, 1960], Gurevich and Pykhteev [1960], and Woronetz [1953]. Others are described by Gilbarg [1960], Birkhoff and Zarantonello [1957], and Gurevich [1961].

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The papers listed below have been classified in three large groups: gravity waves, jets and cavities, and capillarity, bubbles, drops and thin sheets. The classes are not mutually exclusive, so that there may occasionally be some confusion as to the proper one in which to look for a particular paper. The subdivision was made in this way because it seemed to correspond to one already established by convention and would thus facilitate comparison with other bibliographies. There is only a very small overlap with the bibliographies in either Gilbarg's Jets and Cavities [1960], or Wehausen and Laitone's Surface Waves [1960]. Almost all the papers in the first two categories are mentioned in the text. Much of the material in the third category was not discussed.

Forms of reference and journal abbreviations are standard. Titles are given in the original language except for Russian and other not widely read languages. It may be assumed that any paper in a journal from the USSR is written in Russian. For others the language is indicated. No attempt has been made to give references to translations. However, many Russian journals are now translated cover to cover.

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