TECHNICAL NOTE R-25

A MODEL FOR PREDICTION OF THE ELECTROMAGNETIC INTERACTION WITH THE WAKE OF RE-ENTRY BODIES

Prepared By

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BROWN ENGINEERING COMPANY INC.
HUNTSVILLE, ALABAMA

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ABSTRACT

The laminar wake of a re-entry vehicle is simulated by a system of cylindrical plasma shells. The radar cross-section per unit length is found, taking into account the energy losses due to collisions. Numerical results await completion of a computer program.

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LIST OF SYMBOLS

A_n	Coefficient as defined within
a_n	Arbitrary constant
$\overline{B}$	Magnetic induction vector
B_n	Coefficient as defined within
B_n'	Coefficient as defined within
b_n	Arbitrary constant
c_n	Arbitrary constant
$\overline{D}$	Electric displacement vector
d_n	Arbitrary constant
d_n'	Arbitrary constant
$\overline{E}$	Electric field intensity
$\overline{H}$	Magnetic field intensity
$H_n^{(1)}(\beta r)$	Hankel function of the first kind
$H_n^{(2)}(\beta r)$	Hankel function of the second kind
h	Separation constant
i(superscript)	Incident field
$\overline{J}$	Current density
$J_n(\beta r)$	Bessel Function of the first kind
j	$\sqrt{-1}$
$\overline{K}$	Propagation vector

ℓ	Length of section of cylinder
$N_n(\beta r)$	Bessel function of the second kind
$\overline{N}_r$	Time average of radial component of Poynting Vector
n	Separation constant
$\overline{R}$	Vector from origin to arbitrary observation point on cylinder
R_o	Distance from axis of cylinder to point of reception
R'	Distance from point of surface of integration to point of reception
r	Cylindrical co-ordinate
s(super-script)	Scattered field
x	Rectangular co-ordinate
y	Rectangular co-ordinate
z	Rectangular and cylindrical co-ordinate
β	$\sqrt{K^2 - h^2}$
γ	Polarization angle
δ	Spherical polar angle
ϵ	Permittivity
ϵ_o	Permittivity of free space
ϵ_r	$\epsilon_r = \frac{\epsilon}{\epsilon_o}$ Relative permittivity
ϵ'	Complex permittivity
ζ	Spherical polar angle
θ	Cylindrical co-ordinate

μ	**Permeability**
ν	**Collision frequency**
ρ	**Charge density**
σ	**Conductivity**
σ_r	**Radar cross-section**
ϕ	**Scalar Green's function**
ψ	**Scalar wave function**
ω	**Excitation frequency**
ω_p	**Plasma frequency**
I, II, III	**Superscripts referring to cylindrical shell regions**

INTRODUCTION

It has been observed that a radar echo is received from the wake of a vehicle upon re-entry into the atmosphere, and that this echo represents a considerable enhancement of the total radar return from the re-entry event. The purpose of this paper is to provide a theory which, under certain conditions, will predict the return from such a wake.

The laminar wake is assumed to be of such form that it can be simulated by cylindrical shells concentric about the axis of the vehicle, each shell being a homogeneous, isotropic medium. The media are assumed to be weakly ionized plasmas containing no true charge; i.e., the net charge in any finite volume is zero. A complex permittivity is expressed in terms of plasma, collision and excitation frequencies in order that a symmetrical wave equation may be used to describe the electromagnetic fields. Initially, a plane wave of arbitrary orientation with respect to the wake, is assumed to be incident on the outer shell. Complexity of the equations requires that only normal incidence be considered at this writing. Solutions for the fields in the shells are found, and the resulting scattered wave is found by matching the fields at the various boundaries. Once the scattered field is determined at the surface of the outer shell, the field at an arbitrary observation point may be found by using the Kirchhoff-Huygens principle for vector waves as derived by Stratton (5). The radar cross-section is then calculated.

ANALYSIS

The fields in the various regions must satisfy Maxwell's equations.

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{1}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial D}{\partial t} \tag{2}$$

$$\nabla \cdot \bar{D} = \rho \tag{3}$$

$$\nabla \cdot \bar{B} = 0 \tag{4}$$

The constitutive equations are:

$$\bar{D} = \epsilon \bar{E} \tag{5}$$

$$\bar{B} = \mu \bar{H} \tag{6}$$

$$\bar{J} = \sigma \bar{E} \tag{7}$$

Since we have assumed constant permittivity, charge-free regions,

$$\nabla \cdot \bar{E} = 0 \tag{8}$$

Assuming an exp. ($j\omega t$) time dependence for the field vector, (2) may be written

$$\nabla \times \bar{H} = (\sigma + j\omega\epsilon)\bar{E} = j\omega\epsilon_0\left(\epsilon_r + \frac{\sigma}{j\omega\epsilon_0}\right)\bar{E}$$

where

$$\epsilon_r = \frac{\epsilon}{\epsilon_o}$$

Let

$$\epsilon' = \epsilon_o \left(\epsilon_r + \frac{\sigma}{j\omega\epsilon_o}\right)$$

where ϵ' is the complex permittivity of the region. It may be expressed as

$$\epsilon' = \left[1 - \frac{\omega_p^2}{\nu^2 + \omega^2} + j\,\frac{\omega_p^2\, \nu/\omega}{\nu^2 + \omega^2}\right]\epsilon_o \tag{9}$$

where ω_p is the plasma frequency, ν is the collision frequency and ω, the excitation frequency (5). Then

$$\nabla \times \bar{H} = j\omega\epsilon' \bar{E} \tag{10}$$

From (1), (6), and (10), we write

$$\nabla \times \nabla \times \bar{E} = -j\omega\mu\, \nabla \times \bar{H} = \omega^2 \epsilon' \mu \bar{E}$$

$$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = \omega^2 \epsilon' \mu \bar{E}$$

Letting $\omega^2 \epsilon' \mu = K^2$ from (8), we may write

$$\nabla^2 \bar{E} + K^2 \bar{E} = 0 \tag{11}$$

Using the same method, it may be shown that

$$\nabla^2 \bar{H} + K^2 \bar{H} = 0 \tag{12}$$

With the assumed time dependence, the square root of K^2 must be extracted such that the sign of the imaginary part is negative. The Laplacian of a vector is defined to be the sum of the Laplacian's of the rectangular components of the vector. Therefore, either (11) or (12) may be written

$$\sum_{\alpha} e_{\alpha} (\nabla^2 \psi_{\alpha} + K^2 \psi_{\alpha}) = 0$$

where α = x, y, z and $\bar{e}_{\alpha}$ is a unit vector. Due to the orthogonal properties of the components, each scalar component must satisfy the equation independently so that

$$\nabla^2 \psi_{\alpha} + K^2 \psi_{\alpha} = 0 \tag{13}$$

In a transformation of co-ordinates from rectangular to cylindrical, the z component is unaffected. Therefore, a solution of (13) is a solution for the z component of the E or H field in cylindrical co-ordinates. In cylindrical co-ordinates (13) becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} + K^2 \psi = 0 \tag{14}$$

The variables may be separated so that

$$\psi = Q(\theta)\, R(r)\, T(z)$$

is a solution of (14). The equation is then satisfied for

$$Q(\theta) = e^{-jn\theta} \qquad n = 0 \pm 1, \pm 2, \ldots . \tag{15}$$

$$R(r) = z_n\left(r\sqrt{K^2-h^2}\right) \tag{16}$$

$$T(z) = e^{-jhz} \tag{17}$$

where $Z_n(r\sqrt{K^2 - h^2})$ is a solution of the Bessel equation and $\underline{h}$ is an arbitrary constant.

The following are particular solutions of the Bessel equation, and their important characteristics (2). The symbol β will represent $\sqrt{K^2 - h^2}$

I.

$$J_n(\beta r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(n+m)!} \left(\frac{\beta r}{2}\right)^{n+2m} \tag{18}$$

If $\underline{n}$ is replaced by $-\underline{n}$ in (16), the equation is unaltered and $J_{-n}(\beta r)$ is also a solution. However, for integral $\underline{n}$, the two solutions are not linearly independent. $J_n(\beta r)$ and $J_{-n}(\beta r)$ are known as Bessel functions of the first kind. They are the only solutions of (16), which remain finite for $r = 0$.

II.

$$N_n(\beta r) = \frac{1}{\sin n\pi}\left[J_n(\beta r)\cos n\pi \quad J_n(\beta r)\right]$$

This is known as a Neumann function or Bessel function of the second kind. For integral $\underline{n}$, this function becomes indeterminate but may be evaluated by L'Hospital's rule.

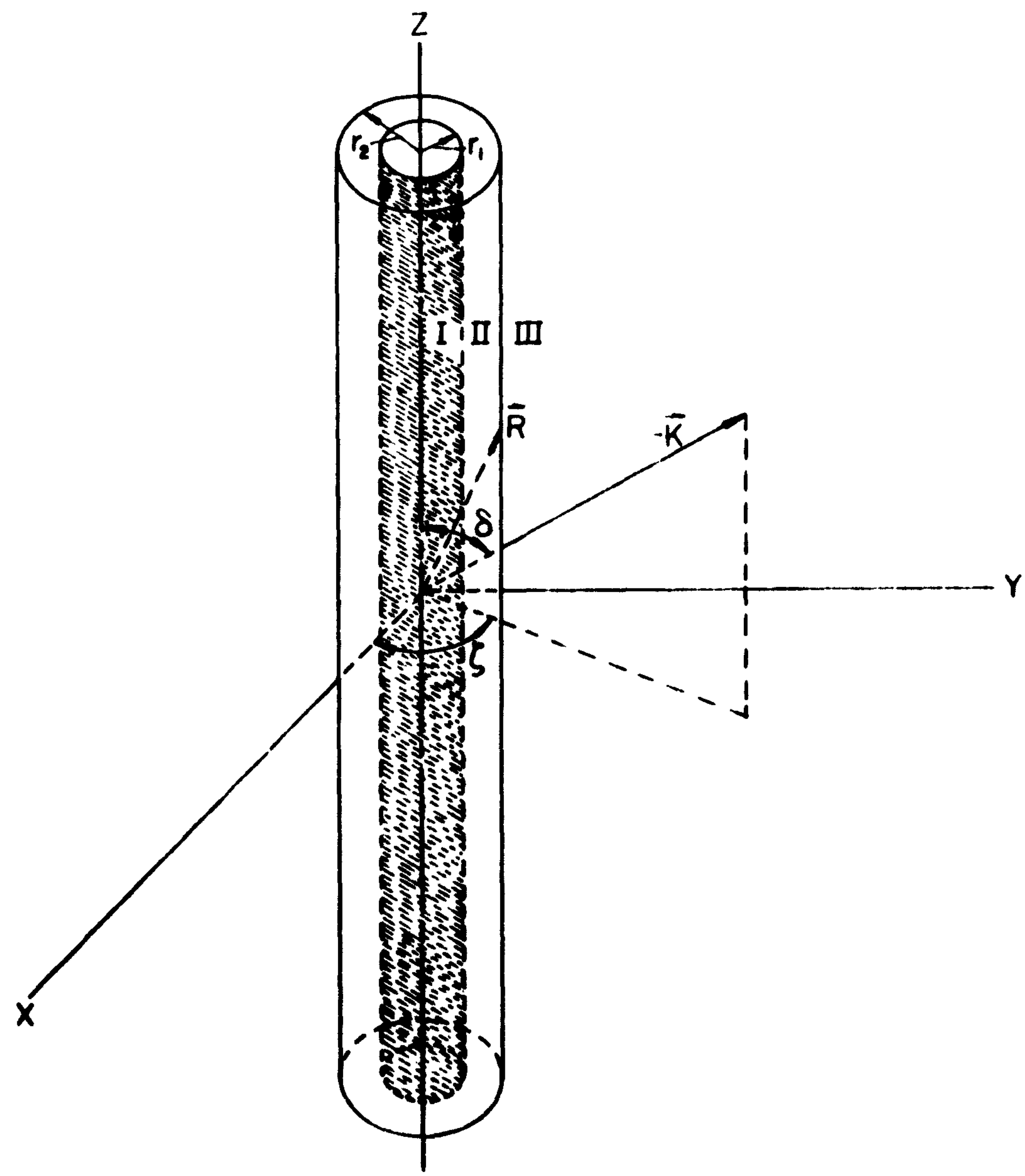

Figure 1

Orientation of Incident Wave on Cylindrical Shell

The Bessel functions of the first and second kind converge for all values of the argument, real or complex, except as noted at the origin. However, for large arguments the convergence is slow and asymptotic forms must be used.

III.

$$H_n^{(1)}(\beta r) = J_n(\beta r) + j N_n(\beta r) \tag{19}$$

$$H_n^{(2)}(\beta r) = J_n(\beta r) - j N_n(\beta r) \tag{20}$$

These functions are known as Hankel functions of the first and second kind respectively. These functions vanish at infinity. With the assumed time dependence exp ($j\omega t$), $H_n^{(1)}(\beta r)$ and $H_n^{(2)}(\beta r)$ describe waves traveling radially inward and outward respectively.

Consider a set of three concentric cylinders, the outer cylinder bounded by infinity (Figure 1). The wave functions for the three regions may now be determined. Region I contains the origin and thus $Z_n(\beta^{I} r)$ must be a Bessel function of the first kind.

$$\psi^{I} = a_n e^{-jn\theta} J_n(\beta^{I} r) e^{-jhz} \tag{21}$$

Region II will contain waves transmitted from Region III and waves reflected from the surface at Region I. Therefore, a combination of Hankel functions of the first and second kind is appropriate.

$$\psi^{II} = \left[b_n \cdot H_n^{(1)}(\beta^{II} r) + C_n H_n^{(2)}(\beta^{II} r) \quad e^{-jn\theta} e^{-jhz} \right] \tag{22}$$

Region III contains an incident wave yet to be specified and a scattered wave described by a Hankel function of the second kind since the wave is traveling radially outward and must vanish at infinity.

$$\psi^{III} = \psi^{i} + d_n' e^{-jn\theta} H_n^{(2)}(\beta^{III} r) e^{-jhz}$$

All wave functions will be understood to have a time dependence exp. $(j\omega t)$.

Now consider a plane wave incident upon the boundary between regions II and III. The wave is described by E_0 exp. $-j(\overline{K} \cdot \overline{R} - \omega t)$ where $\overline{K}$ is the propagation vector and $\overline{R}$ the vector to a point of observation (Figure 1). E_0 is the amplitude of the wave. The propagation vector may be written in the rectangular components

$$K_x = K \sin\delta \cos\zeta \qquad \text{(a)}$$

$$K_y = K \sin\delta \sin\zeta \qquad \text{(b)} \qquad (23)$$

$$K_z = K \cos\delta \qquad \text{(c)}$$

so that E_0 exp. $(-j\overline{K} \cdot \overline{R})$ is a solution of (13).

Now

$$e^{-j\bar{K}\cdot\bar{R}} = e^{-jK\sin\delta(x\cos\zeta + y\sin\zeta)} e^{-jKz\cos\delta}$$

$$= e^{-jKr\sin\delta\cos(\theta-\zeta)}$$

where r, θ, and z are cylindrical co-ordinates. This may be expanded into cylindrical wave functions (1) such that

$$\psi_n^{i} = E_0 j^{n} J_n(Kr\sin\delta) e^{-jn\theta} e^{-jKz\cos\delta}$$

The wave function for region III may now be written as the sum of the incident and the scattered wave.

$$\psi^{III} = E_0 j^{-n} e^{-jn\theta} J_n(Kr\sin\delta)\, e^{-jKz\cos\delta}$$

$$+ d_n' H_n^{(2)}(\beta^{III} r)\, e^{-jn\theta} e^{-jhz} \qquad (24)$$

From (21), (22), and (24), the separation constant h may be inferred to be the z component of the propagation vector and since this is a rectangular component,

$$h = K_z = K\cos\delta$$

Then

$$\beta = \sqrt{K^2 - h^2} = K\sin\delta \qquad (25)$$

This solution for h satisfies boundary conditions to be imposed on the fields, but at the same time it limits the aspect angle such that $\delta > 0$ for if δ is allowed to become zero, ψ^{II} and ψ^{III} are not finite.

A more general solution for h must describe the propagation of energy along the z axis of the cylinders. As in the case of circular waveguides, various modes of propagation are possible and an infinite number of solutions of the propagation constant exist (4). If a plane wave is incident normal to the surface of the cylinder, no energy is propagated along the z axis, and (25) is an exact solution for h. The remaining discussion will be confined to this case.

The incident field may now be considered to be a superposition of two fields, the first having an E vector parallel to the z axis with its associated H field normal to z and the second having an E vector normal to z with H parallel to the axis. Then ψ is a solution for E in the first case, and a solution for H in the second. The remaining components may then be found from Maxwell's equations.

Since ψ describes the z component of the fields and tangential components must be continuous at a boundary, then the wave functions may be equated at their respective boundaries.

Let the incident field have amplitude E_{zo} and be polarized parallel to the z axis. Then

$$\psi^{i} = E_{zo} \sum_{n=-\infty}^{\infty} j^{-n} e^{-jn\theta} J_n(\beta^{III} r)$$

where

$$\beta^{III} = K^{III} = \sqrt{\omega^2 \varepsilon_o \mu_o} = \frac{\omega}{c}$$

for

$$h = K \cos \delta = 0$$

From (1),

$$H_\theta = \frac{1}{j\omega\mu} \frac{\partial E_z}{\partial r}$$

Now equate the z components of E and the θ components of H at the boundaries represented by r_1 and r_2.

$$\sum_n E_{zo}\, j^{-n} J_n(\beta^{III} r_2) + d_n' H_n^{(2)}(\beta^{III} r_2)$$

$$= \sum_n b_n H_n^{(1)}(\beta^{II} r_2) + C_n H_n^{(2)}(\beta^{II} r_2) \qquad (26)$$

$$\frac{1}{j\omega\mu} \sum_n E_{zo}\, j^{-n} \left\{ \frac{d}{dr}\left[J_n(\beta^{III} r_2)\right] + d_n' \frac{d}{dr}\left[H_n^{(2)}(\beta^{III} r_2)\right]\right\}$$

$$= \frac{1}{j\omega\mu} \sum_n \left\{ b_n \frac{d}{dr}\left[H_n^{(1)}(\beta^{II} r_2)\right] + C_n \frac{d}{dr}\left[H_n^{(2)}(\beta^{II} r_2)\right]\right\} \qquad (27)$$

$$\sum_n \left[b_n H_n^{(1)}(\beta^{II} r_1) + C_n H_n^{(2)}(\beta^{II} r_1)\right] = \sum_n a_n J_n(\beta^{I} r_1) \qquad (28)$$

$$\frac{1}{j\omega\mu} \sum_n \left\{ b_n \frac{d}{dr}\left[H_n^{(1)}(\beta^{II} r_1)\right] + C_n \frac{d}{dr}\left[H_n^{(2)}(\beta^{II} r_1)\right]\right\} = \frac{1}{j\omega\mu} \sum_n a_n \frac{d}{dr}\left[J_n(\beta^{I} r_1)\right] \qquad (29)$$

From (28) and (29)

$$C_n = -A_n b_n$$

where

$$A_n = \frac{H_n^{(1)}(\beta^{II} r_1) \frac{d}{dr}\left[J_n(\beta^{I} r_1)\right] - J_n(\beta^{I} r_1) \frac{d}{dr}\left[H_n^{(1)}(\beta^{II} r_1)\right]}{H_n^{(2)}(\beta^{II} r_1) \frac{d}{dr}\left[J_n(\beta^{I} r_1)\right] - J_n(\beta^{I} r_1) \frac{d}{dr}\left[H_n^{(2)}(\beta^{II} r_1)\right]} \qquad (30)$$

Substituting for C_n in (26) and (27),

$$E_{zo}\, j^{-n} J_n(\beta^{III} r_2) + d_n' H_n^{(2)}(\beta^{III} r_2) = b_n \left[H_n^{(1)}(\beta^{II} r_2) - A_n H_n^{(2)}(\beta^{II} r_2)\right]$$

$$E_{zo}\, j^{-n} \frac{d}{dr}\left[J_n(\beta^{III} r_2)\right] + d_n' \frac{d}{dr}\left[H_n^{(2)}(\beta^{III} r_2)\right]$$

$$= b_n \left\{ \frac{d}{dr}\left[H_n^{(1)}(\beta^{II} r_2)\right] - A_n \frac{d}{dr}\left[H_n^{(2)}(\beta^{II} r_2)\right]\right\}$$

then

$$d_n' = \frac{j^{-n} E_{zo} \left\{ B_n \frac{d}{dr} \left[J_n(\beta^{III} r_2) \right] - B_n' J_n(\beta^{III} r_2) \right\}}{B_n' H_n^{(2)}(\beta^{III} r_2) - B_n \frac{d}{dr} \left[H_n^{(2)}(\beta^{III} r_2) \right]} \qquad (31)$$

where

$$B_n = H_n^{(1)}(\beta^{II} r_2) - A_n H_n^{(2)}(\beta^{II} r_2) \qquad (32)$$

$$B_n' = \frac{d}{dr}\left[H_n^{(1)}(\beta^{II} r_2) \right] - A_n \frac{d}{dr}\left[H_n^{(2)}(\beta^{II} r_2) \right] \qquad (33)$$

It is more convenient to express the derivative with respect to r of the Bessel functions as the derivative with respect to the argument.

Thus,

$$\frac{d}{dr} Z_n(\beta r) = \beta \frac{d}{d(\beta r)} Z_n(\beta r)$$

Henceforth, $\frac{d}{d(\beta r)}$ will be written $\frac{d}{d(\beta r)} Z_n(\beta r) = Z_n'(\beta r)$

Now

$$d_n = \frac{\beta^{III} B_n J_n'(\beta^{III} r_2) - B_n' J_n(\beta^{III} r_2)}{B_n' H_n^{(2)}(\beta^{III} r_2) - \beta^{III} B_n H_n^{(2)\prime}(\beta^{III} r_2)}$$

where

$$B_n = H_n^{(1)}(\beta^{II} r_2) - A_n H_n^{(2)}(\beta^{II} r_2) \qquad (35)$$

$$B_n' = \beta^{II}\left[H_n^{(1)\prime}(\beta^{II} r_2) - A_n H_n^{(2)\prime}(\beta^{II} r_2) \right] \qquad (36)$$

$$A_n = \frac{\beta^{I} H_n^{(1)}(\beta^{II} r_1)\, J_n'(\beta^{I} r_1) - \beta^{II} J_n(\beta^{I} r_1)\, H_n^{(1)\prime}(\beta^{II} r_1)}{\beta^{I} H_n^{(2)}(\beta^{II} r_1)\, J_n'(\beta^{I} r_1) - \beta^{II} J_n(\beta^{I} r_1)\, H_n^{(2)\prime}(\beta^{II} r_1)} \qquad (37)$$

The fields in region III for the parallel polarization case are now completely defined.

$$\bar{E} = \bar{e}_z E_{zo} \sum_{n=-\infty}^{\infty} j^{-n} e^{jn\theta} \left[J_n(\beta^{III} r_2) + d_n\, H_n^{(2)}(\beta^{III} r_2) \right] \qquad (38)$$

$$\bar{H} = -\bar{e}_\theta \sqrt{\frac{\epsilon_o}{\mu_o}}\, E_{zo} \sum_{n=-\infty}^{\infty} j^{-n} e^{jn\theta} \left[J_n'(\beta^{III} r_2) + d_n\, H_n^{(2)}(\beta^{III} r_2) \right] \qquad (39)$$

where $\bar{e}_z$ and $\bar{e}_\theta$ are unit vectors.

The Kirchhoff-Huygens principle states that if the value of a scalar field quantity is known at every point on any closed surface surrounding a source-free region, each unit of surface can be considered as a radiating source, and the total field at any interior point is given by integrating the contributions of all the individual elements over the surface. The extension of this principle to cover vector waves is given by Stratton (5) in section 8.14. We consider the field at an interior point (P) of a volume bounded by the cylinder and infinity.

The electric field at P is given by

$$\bar{E} = \frac{1}{4\pi} \int_S j\omega\mu (\bar{n} \times \bar{H})\phi - (\bar{n} \times \bar{E}) \times \nabla\phi - (\bar{n} \cdot \bar{E}) \nabla\phi \; da \qquad (40)$$

where $\bar{n}$ is a unit vector normal to the surface and the field quantities in the integrand are those just inside the surface enclosing the volume. Since the fields vanish at infinity, the quantities in the integrand are defined by equations (38) and (39) if $\underline{r}$ is replaced by $\underline{r}_2$. Phi is the Green's function for free space defined by:

$$\phi = \frac{e^{-jKR'}}{R'}$$

where R' is the distance from a point on the surface to the point P.

For large R', relative to the wake length the gradient may be written

$$\nabla\phi = -\bar{n}_o \frac{jKe^{-jKR'}}{R'}$$

and the 1/R' attenuation factor may be approximated by $1/R_o$, where R_o is the normal distance from the cylinder axis to the point P. However, a more exact value is necessary for the phase factor and this will be

$$R' = R_o + r_2 \cos\theta$$

The R_o in the phase factor may be neglected since it is constant. The Green's function and its gradient are now defined by

$$\phi = \frac{1}{R_o} e^{-jKr_2\cos\theta}$$

$$\nabla\phi = -\bar{n}_o jK \frac{e^{-jKr_2\cos\theta}}{R_o}$$

The geometry is illustrated in Figure 2. Replacing the vector quantities by the appropriate scalars, equation (40) may now be written:

$$E_z = \frac{1}{4\pi R_0}\int_0^{2\pi}\int_0^{\ell} j\omega\mu\, e^{jKr_2\cos\theta} H_\theta + jKe^{-jKr_2\cos\theta}\cos\theta\, E_z\ r_2\, d\theta\, dz$$

where ℓ is the length of the cylinder.

$$E_z = \frac{r_2 K \ell}{4\pi R_0} E_{z0}\sum_n j^{-n}\left[J_n'(\beta^{III} r_2) + d_n H_n^{(2)\prime}(\beta^{III} r_2)\right]\left[\int_0^{2\pi} e^{-jn\theta} e^{-jKr_2\cos\theta}\, d\theta\right]$$

$$+\ j\ \frac{r_2 K \ell}{4\pi R_0} E_{z0}\sum_n j^{-n}\left[J_n(\beta^{III} r_2) + d_n H_n^{(2)}(\beta^{III} r_2)\right]\left[\int_0^{2\pi} e^{-jn\theta} e^{-jKr_2\cos\theta}\left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right) d\theta\right] \tag{41}$$

The integrals in (41) are the integral representations of Bessel functions.

$$2\pi j^{-n} J_n(Kr_2) = \int_0^{2\pi} e^{-jn\theta} e^{-jKr_2\cos\theta}\, d\theta \tag{42}$$

$$2\pi\left[j^{-(n-1)} J_{n-1}(Kr_2) + j^{-(n+1)} J_{n+1}(Kr_2)\right] = \int_0^{2\pi} e^{-jKr_2\cos\theta}\left(e^{-j(n-1)\theta} + e^{-j(n+1)\theta}\right) d\theta \tag{43}$$

Substituting (42) and (43), (41) becomes

$$E_z = \frac{r_2 \ell}{2R_0} E_{z0}\ \frac{\omega}{c}\sum_n (-1)^n d_n\left[J_n(\beta^{III} r_2) H_n^{(2)\prime}(\beta^{III} r_2) - H_n^{(2)}(\beta^{III} r_2) J_n'(\beta^{III} r_2)\right]$$

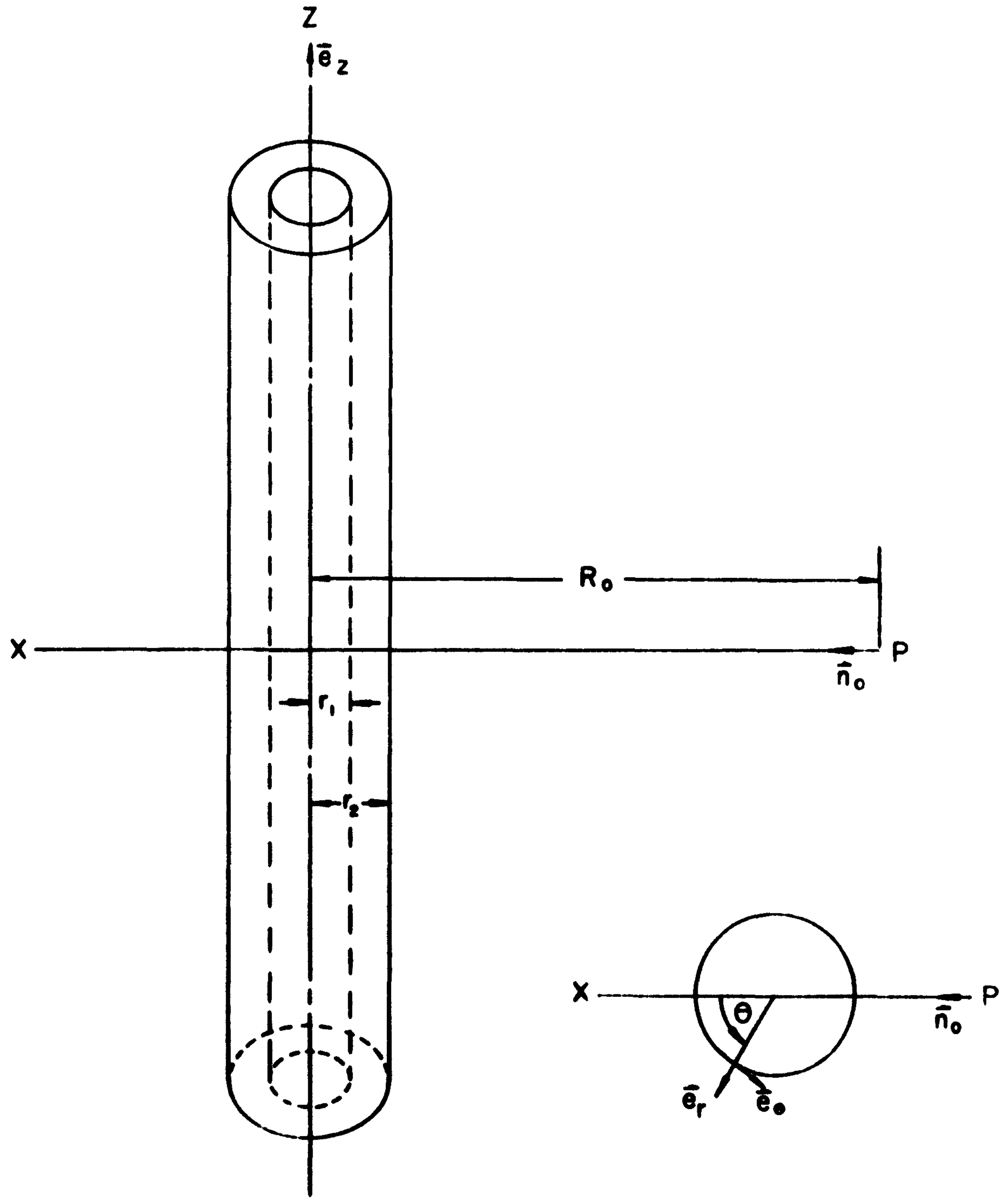

Figure 2

Cross-Section Geometry

The function in brackets is the Wronskian of the Bessel functions (2) and is given by

$$J_n(\beta^{III} r_2)\, H_n^{(2)\prime}(\beta^{III} r_2) - H_n^{(2)}(\beta^{III} r_2)\, J_n'(\beta^{III} r_2) = \frac{-2j}{\pi \beta^{III} r_2}$$

Therefore,

$$E_z = \frac{-j\ell}{\pi R_0} E_{z0} \sum_n (-1)^n d_n \tag{44}$$

When the incident field is polarized transverse to the z axis, the same method is used to calculate the magnetic field which is given by:

$$H_z = \frac{-j\ell}{\pi R_0} H_{z0} \sum_n (-1)^n e_n \tag{45}$$

where

$$e_n = \frac{\beta^{II} E_n J_n'(\beta^{III} r_2) - E_n' J_n(\beta^{III} r_2)}{E_n' H_n^{(2)}(\beta^{III} r_2) - \beta^{II} E_n H_n^{(2)\prime}(\beta^{III} r_2)} \tag{46}$$

$$E_n = H_n^{(1)}(\beta^{III} r_2) - D_n H_n^{(2)}(\beta^{III} r_2) \tag{47}$$

$$E_n' = \beta^{III} H_n^{(1)\prime}(\beta^{II} r_2) - D_n H_n^{(2)\prime}(\beta^{II} r_2) \tag{48}$$

$$D_n = \frac{\beta^{II} H_n^{(1)}(\beta^{II} r_1)\, J_n'(\beta^{I} r_1) - \beta^{I} H_n^{(1)\prime}(\beta^{II} r_1)\, J_n(\beta^{I} r_1)}{\beta^{II} H_n^{(2)}(\beta^{II} r_1)\, J_n'(\beta^{I} r_1) - \beta^{I} H_n^{(2)\prime}(\beta^{II} r_1)\, J_n(\beta^{I} r_1)} \tag{49}$$

The H field is then found at P by

$$\bar{H} = \frac{1}{4\pi} \int_S \left[-j\omega\epsilon (\bar{n} \times E)\phi - (\bar{n} \times \bar{H}) \times \nabla\phi - (\bar{n} \cdot H)\nabla\phi \right] da$$

where all quantities are as previously defined.

For arbitrary polarization, the amplitudes of the incident fields E_{zo} and H_{zo} are related by the polarization angle.

The radar cross-section σ_r is defined by

$$\sigma_r = 4\pi R^2 \left| \frac{\bar{N}^R}{\bar{N}^T} \right| \tag{50}$$

where the N's are time averaged, received and transmitted Poynting vectors and R is the distance to the target, and equal to R_o. Since N is proportional to the square of the fields, we may write

$$\sigma_r = 4\pi R^2 \left| \frac{E^R}{E^T} \right|^2 = 4\pi R^2 \left| \frac{H^R}{H^T} \right|^2$$

For parallel polarization (50) becomes

$$\sigma_r = \frac{4\ell^2}{\pi} \left| \sum_n (-1)^n d_n \right|^2$$

For transverse polarization (50) becomes

$$\sigma_r = \frac{4\ell^2}{\pi}\left|\sum_n (-1)^n e_n\right|^2$$

The solution to equations (51) and (52) has been programmed for the IBM 7090 computer. The initial input data are determined as accurately as possible from flow field analysis (7). These data are then automatically incremented to cover all values assumed to exist in a laminar wake. The following method is used to obtain the data.

1. An arbitrary value, near unity, is assumed for the ratio, ω/ω_p in the inner cylinder.

2. The largest radius for which condition one holds is determined from flow field calculations and becomes r_1.

3. The radius r_2 is determined from flow field calculations such that ω/ω_p is equal to two at the outer boundary.

4. The complex permittivities for regions one and two are averages of the values found between the boundaries.

5. The complex arguments of the Bessel functions are incremented systematically by incrementing the modulus and the argument.

An attempt is being made to keep the increments small enough to observe resonances and yet vary the parameters sufficiently to cover all possible situations. Results of these computations will be reported at a later date.

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