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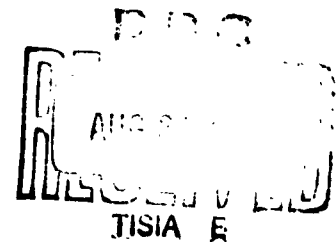
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EIGHTFOLD WAY AND WEAK INTERACTIONS \*

Michel GOURDIN<sup>+</sup>  
Laboratoire de Physique Théorique - Orsay.



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<sup>+</sup> Postal address : Laboratoire de Physique Théorique et Hautes Energies, B.P.12, Orsay (Seine-et-Oise), France.

## SOMMAIRE

Le modèle de l'octet de Ne'oman et Gell-Mann est généralisé de manière à permettre de définir des courants conservés  $|\Delta S| = 1$   $|\Delta I| = \frac{3}{2}$  observés expérimentalement. On montre que la solution nominale de ce problème est une symétrie orthogonale basée sur le groupe  $SO(8)$ , les baryons étant à nouveau dans une représentation irréductible de dimension huit.

Le couplage de Yukawa meson-baryon et les courants baryoniques et mésoniques sont étudiés et classés.

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## INTRODUCTION

The problem of a possible existence of high symmetries in the strong interactions has been extensively studied the last few years. All the various approaches can be characterized, from a mathematical point of view, in a common way : one assumes the existence of a semi-simple Lie algebra,  $g$ , containing, of course, the isospin rotations, the hypercharge and the baryonic number gauge transformations and one constructs a composite model of strong interacting elementary particles from a particular representation of this algebra associated with the spin  $\frac{1}{2}$  baryons. Let us call  $G$  this group realization of  $g$ . In general, the space time properties are assumed to be independent of these new internal symmetries and the complete symmetry group turns out to be the direct product of the inhomogeneous Lorentz group  $P$  by the symmetry group  $G$  [1] .

The general problem of global symmetries has been investigated by Speiser and Tarski [2] for a large class of simple and semi-simple Lie algebras. The more successful models, from a physical point of view are :

- a) The octet model [3][4]
- b) The Sakata model [5][6]
- c) The  $G_2$  model [7][8] .

The octet and the triplet models are the only two physically possible group realizations of the same 9-parameters semi-simple Lie algebra (see the next section). Nevertheless, the properties and the predictions of these models are very different.

The experimental spectrum of masses in a supermultiplet is generally explained by a breakdown of the global symmetry due to the presence in the Lagrangian of interaction, for instance, of a term invariant only in a weaker symmetry [9][10][11]. First order calculations give, in the octet model, surprisingly good results. The electromagnetic interactions are introduced in an analogous way and correspond to another weaker symmetry of the same type of the preceding ones [9][10][11][12][13].

The inclusion of the weak interactions is a more complicated problem. The main difficulty is the experimental evidence of both  $|\Delta S| = 1$ ,  $|\Delta \vec{I}| = \frac{1}{2}$  and  $|\Delta \vec{I}| = \frac{3}{2}$  currents [14] and at least for the models previously quoted, it is not possible to associate the weak interactions with a subgroup of the group of strong interactions if one requires the existence of conserved currents [15][16].

Another way is to postulate the existence of a larger group  $H$  such that the strong interaction group  $G$  is a subgroup of  $H$  and which permits to construct the weak currents experimentally observed.

As pointed out by Behrends and Sirlin [7] a possible solution of such a problem for the  $G_2$  model is the orthogonal group in seven-dimensional space  $SO(7)$ . A mathematical treatment of the inclusion of  $G_2$  in  $SO(7)$  is given in reference [7] and also in [17].

This paper is devoted to a generalization of the octet model in a way permitting to include the weak interactions. It turns out that the more simple solution - but not the only one of course - is an eight-fold way based on the orthogonal group in an eight-dimensional space  $SO(8)$  [18].

Some years ago, Gürsey [19] used this group for weak interactions. The main difference is due to a different non-equivalent place [20] of the isospin and hypercharge operators in the  $D_4$  simple Lie algebra. In both cases, one can construct a unitary group  $SU(3)$  as subgroup of  $SO(8)$  but, in Gürsey's version, the inclusion admits  $G_2$  and  $SO(7)$  as intermediate steps and, really, the Gürsey model appears as a generalization of the  $G_2$  model instead of the octet model. An interesting point of Gürsey's work is the

introduction of space-time properties in the symmetry space; more precisely, the two component baryon spinors of given helicity, are associated with the two spinor representations of the  $D_4$  algebra and the octet of the vector representation describes seven pseudo-scalar and one scalar meson. It follows that, for strong interactions, one cannot find an octet model as a subgroup.

In the first section, the mathematical structure of the inclusion between the algebra of the unitary group and the algebra of the orthogonal group is studied. The reduction under unitary transformation of the irreducible representations of the orthogonal group is examined and it can be shown that the orthogonal  $SO(8)$  group can generalize the octet model but not the triplet model. Moreover, the adjoint representation contains the weak currents and one can try to extend the octet model in this way.

In the second section, one constructs explicitly the model and gives a systematic classification of the weak currents with respect to the quantum numbers of the strong interactions.

## II. MATHEMATICAL TREATMENT.

1°) The octet model [3] [4] and the triplet model [6] are constructed on the same Lie algebra  $A_0 \oplus A_2$  [21] but are two distinct group realizations. In the eight-fold way the basic group is a direct product :

$$U(1) \otimes \frac{SU(3)}{Z_3}$$

where  $Z_3$  is the center of  $SU(3)$  [22] and it is well-known that the factor group  $\frac{SU(3)}{Z_3}$  can be entirely generated by the tensorial powers of the eight-dimensional adjoint representation only. Generally, two non-negative numbers,  $\lambda_1$  and  $\lambda_2$ , characterize the representations of  $SU(3)$  and the only possible representations of  $\frac{SU(3)}{Z_3}$  are restricted by the condition :

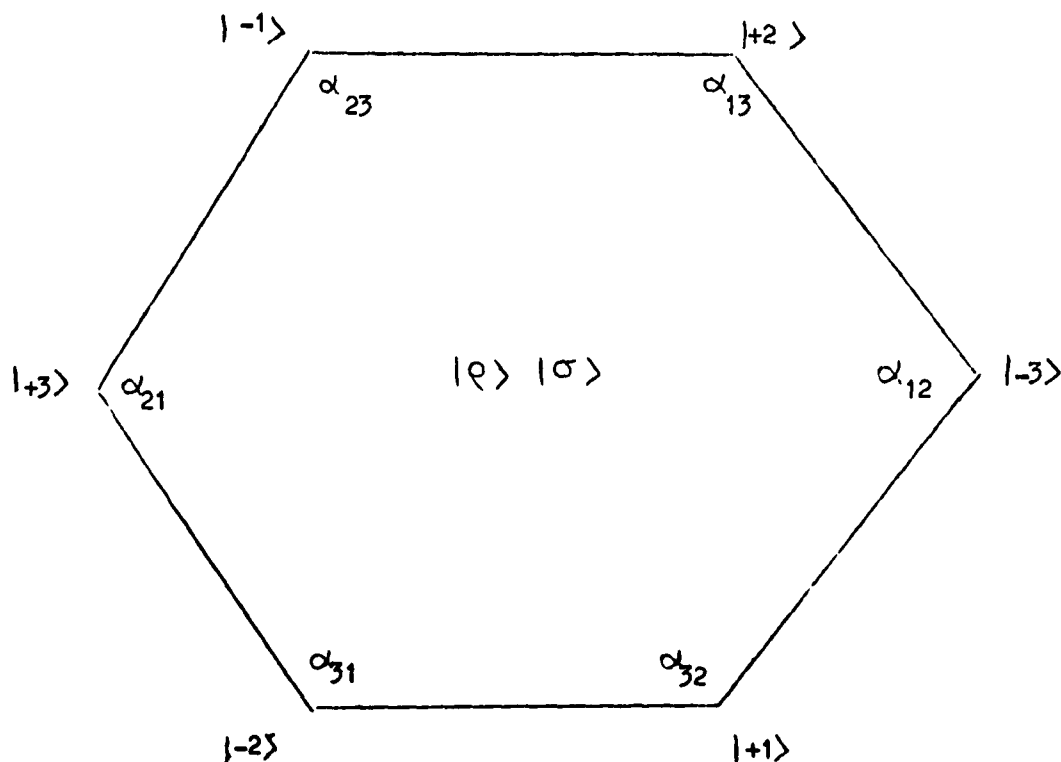
$$\lambda_1 + 2\lambda_2 \equiv 0 \quad (3) \quad .$$

2°) We are now interested in the 8-dimensional adjoint representation of the  $A_2$  Lie algebra. It is well-known that, in associated vector space, one can find a bilinear symmetric form  $C$ , conserved under the transformations contained in  $A_2$  [23]. In an equivalent way, one can define a  $8 \times 8$  symmetric matrix  $C$  such that :

$$C X_\sigma C^{-1} = -X_\sigma^T$$

where the  $X_\sigma$ 's are the adjoint representation of the infinitesimal generators of  $A_2$ .

If now we use the notations as indicated in Fig. 1. for the basis of the adjoint representation  $[11]$  :



**Fig. 1** : Root and Weight Diagram for the adjoint representation.

The vectors  $|j\rangle$  where  $j = \pm 1, \pm 2, \pm 3$  are orthonormalized and associated with the non-zero roots of  $A_2$ . Because of the degenerescence of the root zero, there exists one degree of freedom in the definition of the corresponding eigenvectors and we use  $|\rho\rangle$  and  $|\sigma\rangle$  as two arbitrary orthonormalized weights.

In this basis, the matrix  $C$  has the simple form

$$C = \sum_{j = \pm 1, \pm 2, \pm 3} |j\rangle \langle -j| + |\rho\rangle \langle \rho| + |\sigma\rangle \langle \sigma|$$

and satisfies evidently :

$$C^T = C \quad C C^T = I .$$

3°) The linear transformations leaving the bilinear form  $C$  of the previous section invariant can be generally represented by the set of all  $8 \times 8$  orthogonal matrices which satisfy

$$O^T C O = C$$

The orthogonal  $O(8)$  and special orthogonal  $SO(8)$  groups are naturally introduced in this way and we are now interested by the properties of the simple Lie algebra  $D_4$  of these orthogonal groups.

The 8-dimensional space is the vector representation space of the  $D_4$  algebra and it is convenient, as usual, to define the following basis for the  $8 \times 8$  matrices [24] :

$$[\xi_{\alpha\beta}]_{\gamma\delta} = c_{\alpha\gamma} c_{\beta\delta}$$

A simple non normalized form of the vector representation of the  $D_4$  algebra is then given, as for orthogonal groups, by

$$Z_{\alpha\beta} = \xi_{\alpha\beta} - \xi_{\beta\alpha}$$

The unimodular unitary transformations associated with  $A_2$  appear, in this way, as a particular subset of the orthogonal transformations of  $D_4$ . We then have the inclusion relation for the Lie algebra :

$$A_2 \subset D_4$$

and our aim is now to give the precise position of  $\Lambda_2$  in  $D_4$  by writing the infinitesimal generators of  $\Lambda_2$  as linear combinations of those of  $D_4$ . We use for this the explicit 8-dimensional representation of the Lie algebras on the basis previously introduced. The result is given in Table 1 :

$$\begin{aligned}
 \sqrt{6} \quad E_{12} &= Z_{21} + \sqrt{2} \quad Z_{-3r} \\
 \sqrt{6} \quad E_{21} &= Z_{-1-2} - \sqrt{2} \quad Z_{3r} \\
 \sqrt{6} \quad E_{23} &= Z_{32} - \frac{1}{\sqrt{2}} \left[ Z_{-1r} + \sqrt{3} \quad Z_{-1s} \right] \\
 \sqrt{6} \quad E_{32} &= Z_{-2-3} + \frac{1}{\sqrt{2}} \left[ Z_{1r} + \sqrt{3} \quad Z_{1s} \right] \\
 \sqrt{6} \quad E_{31} &= Z_{13} - \frac{1}{\sqrt{2}} \left[ Z_{-2r} - \sqrt{3} \quad Z_{-2s} \right] \\
 \sqrt{6} \quad E_{13} &= Z_{-3-1} + \frac{1}{\sqrt{2}} \left[ Z_{2r} - \sqrt{3} \quad Z_{2s} \right] \\
 \sqrt{6} \quad H_1 &= Z_{2-2} - Z_{3-3} \\
 \sqrt{6} \quad H_2 &= Z_{3-3} - Z_{1-1} \\
 \sqrt{6} \quad H_3 &= Z_{1-1} - Z_{2-2}
 \end{aligned}$$

Table 1.

The state vectors  $|r\rangle$  and  $|s\rangle$  correspond to a particular definition of  $|\rho\rangle$  and  $|\sigma\rangle$  on the basis of a physical choice explained in the next section.

One can immediately verify on the previous expressions that the set of eight linearly independent generators  $E_{ij}$ ,  $H_k$ , generates a sub-algebra of  $D_4$  of type  $\Lambda_2$  by using the commutation rules of the  $D_4$  algebra deduced for instance, from the matrix representation of the  $Z_{\alpha/\beta}$ .

4°) The irreducible representations of the  $D_4$  algebra are defined by four non-negative integers  $\mu_1, \mu_2, \mu_3, \mu_4$  from the four fundamental representations [25] which can be classified in the following way :

- a)  $D(1,0,0,0)$  8-dimensional spinor representation  $8_{sp}$
- b)  $D(0,1,0,0)$  8-dimensional spinor representation  $8_{sp'}$
- c)  $D(0,0,1,0)$  8-dimensional vector representation  $8_v$
- d)  $D(0,0,0,1)$  28-dimensional adjoint representation  $28_R$

The three 8-dimensional representations are inequivalent, of course, if one considers all orthogonal transformations. If now, we restrict ourselves to the transformations contained in the  $A_2$  sub-algebra, they are irreducible and equivalent. At the same time, the adjoint representation is reducible according to :

$$28_R \rightarrow 8 \oplus 10 \oplus \overline{10} .$$

Such a result can easily be obtained by looking<sup>at</sup> the projection of the four-dimensional weight diagrams of  $D_4$  on a convenient plane. The study of the characters gives an algebraic method of demonstration - see appendix.

Let us call as  $D_4^*$  the universal covering group of the Lie algebra  $D_4$ . The center of  $D_4^*$  possesses a very simple structure as shown in Table 2.:

|       | $8_{sp}$ | $8_{sp'}$ | $8_v$ | $28_R$ |
|-------|----------|-----------|-------|--------|
| I     | 1        | 1         | 1     | 1      |
| $R_1$ | 1        | -1        | -1    | 1      |
| $R_2$ | -1       | 1         | -1    | 1      |
| $R_3$ | -1       | -1        | 1     | 1      |

Table 2. : Center of  $D_4^*$  and the fundamental representations.

One immediately verifies that this center is isomorphic to a direct product  $Z_2 \otimes Z_2$  [26] and the connected groups associated to the  $D_4$  algebra can be easily classified in the following way :

a)  $D_4^*$  where  $\mu_1, \mu_2, \mu_3, \mu_4$  are independent;

b)  $\Delta_4 \frac{D_4^*}{Z_2}$  : one can realize three isomorphic groups of this type by

considering the tensorial powers of one of the 8-dimensional representations :

$$\left\{ \begin{array}{l} \alpha) \Delta_{4sp} \text{ from } 8_{sp} \text{ with } \mu_2 + \mu_3 = 0 \quad (2) \\ \beta) \Delta_{4sp'} \text{ from } 8_{sp'} \text{ with } \mu_1 + \mu_3 = 0 \quad (2) \\ \gamma) \Delta_{4v} \text{ SO}(8) \text{ from } 8_v \text{ with } \mu_1 + \mu_2 = 0 \quad (2) \end{array} \right.$$

c)  $\frac{D_4^*}{Z_2 \otimes Z_2}$  : generated by the tensorial powers of the adjoint representation and characterized by  $\mu_1, \mu_2, \mu_3$  of the same parity e.g. the representations entering in all  $\Delta_4$  groups.

We are now in a position to give the possible inclusion of the connected groups of the Lie algebra  $A_2$  with respect to the connected groups of the Lie algebra  $D_4$  in our scheme and the main results are the following :

a)  $\frac{SU(3)}{Z_3}$  is a subgroup of  $\Delta_4$

b)  $SU(3)$  cannot be a subgroup in any case.

It follows that our approach cannot generalize any symmetry group constructed from  $SU(3)$  or from  $U(3)$  as the Sakata model.

### III. EIGHT-FOLD WAY.

1°) Let us go back to physics. In such a scheme, it is natural to associate the baryons with the weights of the vector representation in the usual way :

$$\begin{aligned}
 |1\rangle &= |\Xi^0\rangle & |2\rangle &= |p\rangle & |3\rangle &= |\Sigma^+\rangle \\
 |-1\rangle &= |n\rangle & |-2\rangle &= |\Xi^-\rangle & |-3\rangle &= |\Sigma^+\rangle \\
 |r\rangle &= |\Sigma^0\rangle & |s\rangle &= |\Lambda^0\rangle
 \end{aligned}$$

Table 3.

The isospin operators are then given by :

$$\left\{ \begin{aligned}
 I^+ &= Z_{21} + \sqrt{2} Z_{-22} \\
 I^- &= Z_{-1-2} - \sqrt{2} Z_{32} \\
 I_0 &= \frac{1}{2}(Z_{1-1} + Z_{2-2}) - Z_{3-3}
 \end{aligned} \right.$$

and the hypercharge and charge operators by

$$\left\{ \begin{aligned}
 Y &= Z_{2-2} - Z_{1-1} \\
 Q &= Z_{2-2} - Z_{3-3}
 \end{aligned} \right.$$

2°) The product of representations in  $D_4$  can be written as [27] :

$$8_{\alpha} \otimes 8_{\alpha} = 1 \oplus 28 \oplus 35_{\alpha}$$

If one uses a coupling of the Yukawa type between baryons and mesons, the more simple place for the mesons is the adjoint representation. One predicts two 28-dimensional multiplets with  $J = 0$  and  $J = 1$  [28] .

We now consider the baryon-meson resonances. From the product of representations in  $D_4$  :

$$8_{\alpha} \otimes 28 = 8_{\alpha} \oplus 56_{\alpha} \oplus 160_{\alpha}$$

we have essentially the possibility of 8-dimensional and 56-dimensional multiplets. The  $J = \frac{1}{2}^+$  octet can be repeated into a  $J = \frac{5}{2}^+$  octet. Because of the isospin  $\frac{3}{2}$  of the first  $\pi$ -nucleon resonance, one predicts a 56-dimensional  $J = \frac{3}{2}$  multiplet and perhaps another 56-dimensional  $J = \frac{7}{2}$  [29] .

3°) With baryons and antibaryons in the  $8_V$  representation and the mesons in the  $28_R$  representation, the meson baryon coupling can be written as :

$$\left[ z_{kl} \right]_{\alpha\beta} (\bar{\psi}^{\alpha} \gamma_5 \psi^{\beta}) M^{kl}$$

From the point of view of strong interactions, it is convenient to classify the weight  $M^{kl}$  of the adjoint representation according to :

- { a) the values of isospin and hypercharge
- { b) the irreducible representations in the reduction by  $\Lambda_2$ .

The various calculations are, of course, identical to those of the decomposition of the antisymmetric part of the product of two adjoint representations of  $\Lambda_2$ . The results are given in Tables 4, 5 and 6.

| Y  | I             | $I_3$          | $D^8(1,1)$ representation of $A_2$                                                           |         |
|----|---------------|----------------|----------------------------------------------------------------------------------------------|---------|
| 1  | $\frac{1}{2}$ | $\frac{1}{2}$  | $\frac{1}{\sqrt{3}} \left[ M^{-3-1} + \frac{1}{\sqrt{2}} (M^{2r} - \sqrt{3} M^{2s}) \right]$ | $K^+$   |
|    |               | $-\frac{1}{2}$ | $\frac{1}{\sqrt{3}} \left[ M^{32} - \frac{1}{\sqrt{2}} (M^{-1r} + \sqrt{3} M^{-1s}) \right]$ | $K^0$   |
| 0  | 1             | 1              | $\frac{1}{\sqrt{3}} \left[ M^{21} + \sqrt{2} M^{-3r} \right]$                                | $\pi^+$ |
|    |               | 0              | $\frac{1}{\sqrt{6}} \left[ M^{1-1} + M^{2-2} - 2 M^{3-3} \right]$                            | $\pi^0$ |
|    |               | -1             | $\frac{1}{\sqrt{3}} \left[ M^{-1-2} - \sqrt{2} M^{3r} \right]$                               | $\pi^-$ |
|    | 0             | 0              | $\frac{1}{\sqrt{2}} \left[ M^{1-1} - M^{2-2} \right]$                                        | $\eta$  |
| -1 | $\frac{1}{2}$ | $\frac{1}{2}$  | $\frac{1}{\sqrt{3}} \left[ M^{-2-3} + \frac{1}{\sqrt{2}} (M^{1r} + \sqrt{3} M^{1s}) \right]$ | $K^0$   |
|    |               | $-\frac{1}{2}$ | $\frac{1}{\sqrt{3}} \left[ M^{13} - \frac{1}{\sqrt{2}} (M^{-2r} - \sqrt{3} M^{-2s}) \right]$ | $K^-$   |

Table 4.

| Y  | I             | $I_3$          | $D^{10} (3,0)$ representation of $A_2$                                                       |
|----|---------------|----------------|----------------------------------------------------------------------------------------------|
| 1  | $\frac{3}{2}$ | $\frac{3}{2}$  | $M^{2-3}$                                                                                    |
|    |               | $\frac{1}{2}$  | $\frac{1}{\sqrt{3}} \left[ M^{3-1} - \sqrt{2} M^{2r} \right]$                                |
|    |               | $-\frac{1}{2}$ | $\frac{1}{\sqrt{3}} \left[ M^{32} + \sqrt{2} M^{-1r} \right]$                                |
|    |               | $-\frac{3}{2}$ | $M^{3-1}$                                                                                    |
| 0  | 1             | 1              | $\frac{1}{\sqrt{3}} \left[ M^{21} - \frac{1}{\sqrt{2}} (M^{-3r} + \sqrt{3} M^{-3s}) \right]$ |
|    |               | 0              | $\frac{1}{\sqrt{6}} \left[ M^{1-1} + M^{2-2} - M^{3-3} - \sqrt{3} M^{rs} \right]$            |
|    |               | -1             | $\frac{1}{\sqrt{3}} \left[ M^{-1-2} + \frac{1}{\sqrt{2}} (M^{3r} - \sqrt{3} M^{3s}) \right]$ |
| -1 | $\frac{1}{2}$ | $\frac{1}{2}$  | $\frac{1}{\sqrt{3}} \left[ M^{-2-3} + \frac{1}{\sqrt{2}} (M^{1r} - \sqrt{3} M^{1s}) \right]$ |
|    |               | $-\frac{1}{2}$ | $\frac{1}{\sqrt{3}} \left[ M^{13} - \frac{1}{\sqrt{2}} (M^{-2r} + \sqrt{3} M^{-2s}) \right]$ |
| -2 | 0             | 0              | $M^{-21}$                                                                                    |

Table 5.

| Y  | I             | $I_3$          | $D^{10}(0,3)$ representation of $A_2$                                                       |
|----|---------------|----------------|---------------------------------------------------------------------------------------------|
| 2  | 0             | 0              | $M^{12}$                                                                                    |
| 1  | $\frac{1}{2}$ | $\frac{1}{2}$  | $\frac{1}{\sqrt{3}} \left[ M^{3-1} + \frac{1}{\sqrt{2}} (M^{2r} + \sqrt{3} M^{2s}) \right]$ |
|    |               | $-\frac{1}{2}$ | $\frac{1}{\sqrt{3}} \left[ M^{32} - \frac{1}{\sqrt{2}} (M^{1r} - \sqrt{3} M^{1s}) \right]$  |
| 0  | 1             | 1              | $\frac{1}{\sqrt{3}} \left[ M^{21} - \frac{1}{\sqrt{2}} (M^{3r} - \sqrt{3} M^{3s}) \right]$  |
|    |               | 0              | $\frac{1}{\sqrt{6}} \left[ M^{1-1} + M^{2-2} + M^{3-3} + \sqrt{3} M^{rs} \right]$           |
|    |               | -1             | $\frac{1}{\sqrt{3}} \left[ M^{1-2} + \frac{1}{\sqrt{2}} (M^{3r} + \sqrt{3} M^{3s}) \right]$ |
| -1 | $\frac{3}{2}$ | $\frac{3}{2}$  | $M^{1-3}$                                                                                   |
|    |               | $\frac{1}{2}$  | $\frac{1}{\sqrt{3}} \left[ M^{2-3} - \sqrt{2} M^{1r} \right]$                               |
|    |               | $-\frac{1}{2}$ | $\frac{1}{\sqrt{3}} \left[ M^{13} + \sqrt{2} M^{2r} \right]$                                |
|    |               | $-\frac{3}{2}$ | $M^{3-2}$                                                                                   |

Table 6.

4°) In our scheme, the "conserved" currents belong to the 28-dimensional adjoint representations. For instance, the baryonic part of the vector currents can be written as :

$$J_{\mu}^B [k\ell] = [z_{k\ell}]_{\alpha\beta} (\bar{\Psi}^{\alpha} \gamma_{\mu} \Psi^{\beta})$$

For the mesonic part, one uses the adjoint representation of the  $D_4$  algebra - e.g. the structure constants- following :

$$J_{\mu}^M [k\ell] = c [k\ell] [m n] [p q] M^{[m n]} \partial_{\mu} M^{[p q]}$$

It is useful, for practical applications, to classify the various currents in the same way as the weights of the adjoint representation. The results are obtained from Tables 4, 5 and 6, by the simple substitution  $M^{k\ell} \Rightarrow J_{k\ell}$ .

5°) If now we assume the existence of terms of the structure current  $\times$  current, responsible of the non-leptonic interactions, the weak lagrangian has a variance, under  $D_4$ , given by the product of representations :

$$D^{28}(0001) \otimes D^{28}(0001) = D^1(0000) \oplus D^{28}(0001) \oplus D^{35}(2000) \oplus D^{35}(0200) \oplus D^{35}(0020) \\ \oplus D^{300}(0002) \oplus D^{350}(1110)$$

The more natural assumption seems to associate the weak lagrangian with the 28 place. It follows then the two interesting properties :

- As required by  $K_{\ell 3}$  decay experiments [14] one obtains both  $|\Delta S| = 1, |\Delta \vec{I}| = \frac{1}{2}$  and  $\frac{3}{2}$  terms in the lagrangian,
- the  $|\Delta S| = 2$  currents are associated with  $|\Delta \vec{I}| = 0$  and not excluded by experiments, whereas the  $|\Delta S| = 2, |\Delta \vec{I}| = 1$  transitions are forbidden at first by the  $K_1^0 K_2^0$  mass difference result.

### CONCLUSIONS

In this first paper, we have studied, from a mathematical point of view, a possible generalization of the octet model which permits to include the weak interactions in a natural way. The orthogonal  $SO(8)$  symmetry is not, of course, the unique solution of such a problem but it is the more economical one satisfying the two minimal requirements :

- $$\left\{ \begin{array}{l} \text{a) } \frac{SU(3)}{\mathbb{Z}_3} \text{ is a subgroup,} \\ \text{b) both } |AS| = 1, |\vec{\Delta I}| = \frac{1}{2} \text{ and } \frac{3}{2} \text{ currents appear in the adjoint representations.} \end{array} \right.$$

It is clear, from the result of Section II, that  $\Delta_{4sp}$ ,  $\Delta_{4sp'}$  and  $\Delta_{4v}$  are three isomorphic but non equivalent solutions. The equivalence however holds for the strong interactions. More precisely, the 8-dimensional representations of the infinitesimal generators are not equivalent for the total  $D_4$  algebra but only for the  $A_2$  sub-algebra. For instance, the coupling between mesons and baryons is identical in the three cases for the  $J = 0^-$  pseudoscalar mesons associated with the  $\underline{8}$  part of the reduction of the  $28_R$  adjoint representation of  $D_4$  by the  $A_2$  subalgebra (Table 4). It follows equally that the weak currents associated with Table 4. are the same in the three situations whereas a difference must appear for the  $|\vec{\Delta I}| = \frac{3}{2}$  current

The possibility to include the leptons and the vector bosons has not been considered in this paper. It seems that such a tentative is very difficult and actually at the present time, unsuccessful in any model. Nevertheless, it is a necessary problem to solve in a general scheme in order to know the variance of the complete weak lagrangian with respect to the group operations - if such a group exists, of course.

An other way to generalize the strong interactions unitary models is to find a large group  $G$  where the previous condition a) is replaced by the more general one : a')  $SU(3)$  is a subgroup of  $G$ .

In mathematical terms, the unitary rotations associated with the center  $Z_3$  of  $SU(3)$  must be represented in a non-trivial way in  $G$ . The simple Lie group of lower rank solution of such a problem is the unimodular unitary group  $SU(6)$ . The 35-dimensional adjoint representation of  $A_5$  reduces, under the  $A_2$  subalgebra, into  $8 \oplus 27$ . It follows that condition b) is satisfied but at the same time one obtains  $|\Delta S| = 2, |\vec{\Delta I}| = 1$  weak currents and this feature is certainly unfavourable.

We acknowledge the help of Professor Michel and of Dr. Lascoux for illuminating discussions on the group theoretical aspect of this problem.

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## APPENDIX

Characters for SU(3) and SO(8).

The characters for an irreducible representation can be calculated from the knowledge of the weights  $\underline{m}$  of multiplicity  $\gamma_{\underline{m}}$ . The formula given by Weyl, is the following [25] :

$$\chi(\underline{\phi}) = \sum_{\underline{m}} \gamma_{\underline{m}} \exp i (\underline{m}, \underline{\phi})$$

where the scalar product, in the exponential, is performed in the weight space.

1°) Characters for SU(3). In order to preserve the symmetry of the algebra, exhibited on the root diagram (see Fig. 1.), it is convenient to use triangular coordinates of sum zero in the 2-dimensional weight space [25] [11]. We then obtain, for the fundamental representation :

$$\chi_3(\varphi) = \exp \frac{1}{3} (2\varphi_1 - \varphi_2 - \varphi_3) + \exp(-\varphi_1 + 2\varphi_2 - \varphi_3) + \exp \frac{1}{3} (-\varphi_1 - \varphi_2 + 2\varphi_3)$$

After a change of variables :

$$\varphi_2 - \varphi_1 = \phi_3 \quad \varphi_1 - \varphi_3 = \phi_2 \quad \varphi_3 - \varphi_2 = \phi_1 ,$$

one can use also for the  $\phi$  vector triangular coordinates :

$$\phi_1 + \phi_2 + \phi_3 = 0$$

and the character formula becomes :

$$\chi_3(\phi) = \exp \frac{i}{3} (\phi_2 - \phi_3) + \exp \frac{i}{3} (\phi_3 - \phi_1) + \exp \frac{i}{3} (\phi_1 - \phi_2)$$

One easily deduces for the 8, 10,  $\overline{10}$  and 27 representations of  $\frac{SU(3)}{Z_3}$ , the following expressions :

$$\chi_8(\phi) = 2 [1 + \cos \phi_1 + \cos \phi_2 + \cos \phi_3]$$

$$\chi_{10}(\phi) = 1 + 2 [\cos \phi_1 + \cos \phi_2 + \cos \phi_3] + \exp i(\phi_2 - \phi_3) + \exp i(\phi_3 - \phi_1) + \exp i(\phi_1 - \phi_2)$$

$$\chi_{\overline{10}}(\phi) = 1 + 2 [\cos \phi_1 + \cos \phi_2 + \cos \phi_3] + \exp i(\phi_3 - \phi_2) + \exp i(\phi_1 - \phi_3) + \exp i(\phi_2 - \phi_1)$$

$$\begin{aligned} \chi_{27}(\phi) = & 3 + 2 [\cos 2\phi_1 + \cos 2\phi_2 + \cos 2\phi_3] + 2 [\cos \phi_1 + \cos \phi_2 + \cos \phi_3] + \\ & + 4 [\cos \phi_1 \cos \phi_2 + \cos \phi_2 \cos \phi_3 + \cos \phi_3 \cos \phi_1] \end{aligned}$$

2°) Characters for  $D_4^*$ . All weights of the fundamental representations are simple and deducible from the highest weight by the operations of the Weyl group. The weight space is four-dimensional, with basis  $e_j$ ,  $j = 1, 2, 3, 4$ . We immediately know the

characters for the fundamental representations :

a)  $8_{sp}$  Highest weight  $\frac{1}{2} (e_1 + e_2 + e_3 + e_4) :$

$$\chi_{8_{sp}}(\phi) = 2 \left[ \cos \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4}{2} + \cos \frac{\phi_1 + \phi_2 - \phi_3 - \phi_4}{2} + \cos \frac{\phi_1 - \phi_2 - \phi_3 + \phi_4}{2} + \cos \frac{\phi_1 - \phi_2 + \phi_3 - \phi_4}{2} \right]$$

b)  $8_{sp'}$  Highest weight  $\frac{1}{2} (e_1 + e_2 + e_3 - e_4) :$

$$\chi_{8_{sp'}}(\phi) = 2 \left[ \cos \frac{\phi_1 + \phi_2 + \phi_3 - \phi_4}{2} + \cos \frac{\phi_1 + \phi_2 - \phi_3 + \phi_4}{2} + \cos \frac{\phi_1 - \phi_2 - \phi_3 - \phi_4}{2} + \cos \frac{\phi_1 - \phi_2 + \phi_3 + \phi_4}{2} \right]$$

c)  $8_v$  Highest weight  $e_1 :$

$$\chi_{8_v}(\phi) = 2 \left[ \cos \phi_1 + \cos \phi_2 + \cos \phi_3 + \cos \phi_4 \right]$$

d)  $28_R$  Highest weight  $e_1 + e_2$

$$\chi_{28_R}(\phi) = 4 \left[ 1 + \cos \phi_1 \cos \phi_2 + \cos \phi_2 \cos \phi_3 + \cos \phi_3 \cos \phi_1 + \cos \phi_1 \cos \phi_4 + \cos \phi_2 \cos \phi_4 + \cos \phi_3 \cos \phi_4 \right]$$

For completeness, we also give the characters for the  $35_V$  and  $56_V$  representations :

$$\chi_{35_V}(\phi) = 3 + 4 \sum_{i < j} \cos \phi_i \cos \phi_j + 2 \sum_i \cos 2\phi_i$$

$$\chi_{56_V}(\phi) = 6 \sum_i \cos \phi_i + \sum_{\ell} \cos(\pm \phi_i \pm \phi_j \pm \phi_k)$$

where  $(i j k \ell)$  is a permutation  $(1,2,3,4)$ .

3°) Reduction. The inclusion  $A_2 \subset D_4$  one considers, is realized by the projection of the four-dimensional weight diagrams on the two-dimensional plane defined by :

$$\begin{aligned} \phi_4 &= 0 \\ \phi_1 + \phi_2 + \phi_3 &= 0 \end{aligned}$$

The first equation assures the equivalence of the two spinor representations and, consequently, of all  $N_{sp}$  and  $N_{sp'}$  representations. From the second equation, one deduces the equivalence between the three 8-dimensional representations of  $D_4^*$  and their irreducibility. The following results are then straightforward :

$$8_{sp} \Rightarrow 8 \quad 8_{sp'} \Rightarrow 8 \quad 8_V \Rightarrow 8$$

$$\left\{ \begin{aligned} 28_R &\Rightarrow 8 \oplus 10 \oplus \overline{10} \\ 35_\alpha &\Rightarrow 8 \oplus 27 \\ 56_\alpha &\Rightarrow 1 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 \end{aligned} \right.$$

FOOTNOTES AND REFERENCES

- [1] For the problem of the extensions of the Poincaré group, see for instance, L. Michel, notes of lectures given at the Istanbul Summer School, July 1962.
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- [18] Recently, the inclusion has been noted by Y. Ne'eman, Phys. Lett. 4, 81 (1963).
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- [20] We call "inequivalent places" positions one cannot relate by means of an inner automorphism.
- [21] In Cartan's notation, the simple Lie algebra  $A_n (n \geq 1)$  is realized, for instance by its universal covering group, the unimodular unitary group  $SU(n+1)$ . One extends the notation for  $n = 0$ , to the one-dimensional Lie algebra, for which the more interesting physical realization is the one-dimensional unitary group  $U(1)$  commonly called the gauge group of the first kind.
- [22] In a general way,  $Z_n$  is a finite abelian cyclic group isomorphic to the set of the algebraic  $n^{\text{th}}$  roots of the unity. It follows naturally that  $Z_n$  is a subgroup of  $U(1)$ .
- [23] This form is associated with the one-dimensional representation figuring in the direct product of two adjoint representations :

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27$$

- [24] P.M. Cohn, Lie groups. Cambridge 1957.
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- [27] The index  $\alpha$  distinguishes three inequivalent representations of  $D_4^*$ , entering in only one of the  $\Delta_4$  groups.
- [28] In general, an irreducible representation of  $\Delta_4$  is reducible under the strong interaction transformations and one can mix the parities into a  $\Delta_4$  multiplet but not into the subrepresentations irreducible in  $A_2$ . For instance, we know that :

$$28_R \Rightarrow 8 \oplus 10 \oplus \overline{10}$$

and the well-known  $J = 0^-$  and  $J = 1^-$  octets can take place in the 8 part of the 28 representation. For the 10 and  $\overline{10}$  part, it is possible to put  $J = 0^+$  and  $J = 1^+$  mesons.

- [29] The  $56_\alpha$  representation reduces, under  $A_2$ , according to :

$$56 \Rightarrow 1 \oplus 8 + 10 \oplus \overline{10} \oplus 27.$$

The  $J = \frac{3}{2}^+$  multiplet can take place in the 10 part and the  $J = \frac{3}{2}^-$  octet (?) in the 8 part.