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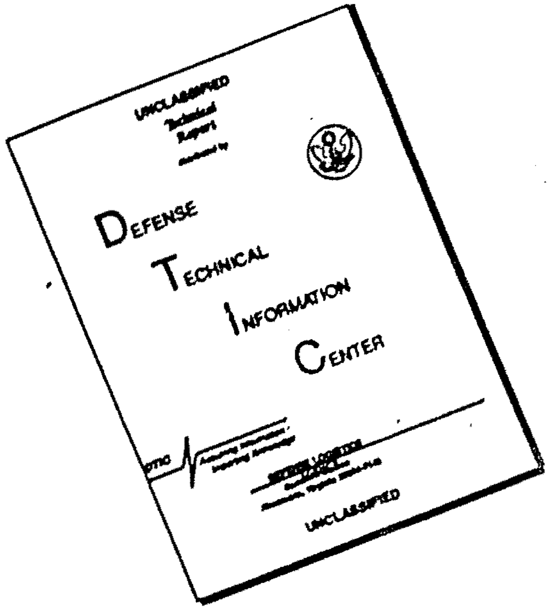

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Reports of the Institute of Geodesy, Photogrammetry and Cartography

Report No. 27

Practical Computation of Gravity at High Altitudes

by

R. A. Hirvonen

and

Helmut Moritz

Prepared for

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts

Contract No. AF 19(628)-2771
Project No. 7600
Task No. 760002

The Ohio State University
Research Foundation
Columbus, Ohio 43212

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W. A. Heiskanen, Director

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FOREWORD

This report was prepared by Dr. R. A. Hirvonen and Dr. Helmut Moritz, Research Associates, of the Institute of Geodesy, Photogrammetry and Cartography of The Ohio State University, under Air Force Contract No. AF 19(628)-2771, OSURF Project No. 1613, under the supervision of Dr. Weikko A. Heiskanen, Director of the Institute. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen Williams and Mr. Bela Szabo, Project and Task scientists.

ABSTRACT

This report presents and compares methods for computing the gravity vector outside the earth. The gravity vector is conveniently split up into a normal part, the vector of normal gravity, and an anomalous part, the vector of gravity disturbance.

Part I gives the theoretical foundations and a practical method for the computation of the vectors of normal gravity and gravitation. The method is adapted for electronic computation. It is illustrated by a numerical example.

Then the formulas for the computation of the gravity disturbance vector from free-air gravity anomalies Δg at the surface of the earth are developed, (a) for the direct method, which uses Δg only, and (b) for the coating method, which requires Δg and the geoid heights N but involves simpler formulas.

The components of the gravity vector are first computed in a local coordinate system and then transformed to the geocentric world system by a spatial rotation. If the gravity vector is required along a rocket trajectory, then also the components along and normal to the trajectory can be computed by a spatial rotation.

Part II is concerned with accuracy studies in order to determine the best practical procedure for the computation of the gravity disturbances.

The standard errors of the components of the gravity disturbances, due to the interpolation of the gravity material, are, approximately, inversely proportional to the elevation and very small for high altitudes, provided there is uniform coverage by gravity stations.

The influence of the distant zones is considered in some detail. This influence decreases very slowly beyond a certain radius, so that it is impractical to go farther than about 30° in the direct method and 20° in the coating method (with an error of about ±5 mgal in both cases), unless the integration is extended over the whole earth, which is necessary for higher accuracy.

There is another method, for which the influence of the distant zones is completely negligible and which furthermore is the simplest. the upward continuation of the surface disturbances. If presupposes.

however, the deflections of the vertical ξ and η for the horizontal components, and Δg and N for the vertical component.

For practical use the following methods are proposed: if only Δg is given, the direct method; if Δg and N are given, the coating method for the horizontal components and the upward continuation for the vertical component; if Δg , ξ , η are given, the upward continuation for all three components.

Finally, a detailed practical computation procedure is described for the practically most important case that Δg and N are given.

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PART I

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R. A. Hirvonen

INTRODUCTION

The purpose of this report is to develop a method of computation of the gravity vector for a great number of points at high altitudes, e.g. for trajectories of moving bodies, when the geocentric coordinates are given. Formulas will be summarized in a form suitable for automatic high speed computers. Numerical examples illustrate the method.

First, the components of the normal gravitation (attraction of a reference ellipsoid) will be computed. Those of the normal gravity can then be obtained by subtraction of the components of the centrifugal force. Finally, the deviations of the actual components from these normal values will be evaluated on the basis of the free air anomalies of the gravity observed on the physical surface of the earth. The accuracy of the method is studied in Part II.

The method is based on the formulas first published in [Hirvonen, 1959], but several practical improvements have been introduced.

1. CONVERSION OF THE COORDINATES

It is supposed that the Cartesian geocentric coordinates x,y,z are given for each point P at which the gravity will be computed. The z-axis is the axis of the earth's rotation, the x-axis has the longitude 0° (Greenwich) and the y-axis has the longitude 90°E. These coordinates must be converted into three other systems.

I. Geocentric coordinates, defined by

$$x = r \cos \Psi \cos \lambda$$

$$\text{(I)} \qquad y = r \cos \Psi \sin \lambda$$

$$z = r \sin \Psi$$

and called

r = radius vector

Ψ = geocentric latitude

λ = geographic longitude

When x,y,z are given, r,Ψ,λ can easily be computed:

$$(1.1) \qquad p^2 = x^2 + y^2$$

$$(1.2) \qquad r^2 = p^2 + z^2$$

$$(1.3) \qquad \cos \lambda = \frac{x}{p}$$

$$(1.4) \qquad \sin \lambda = \frac{y}{p}$$

$$(1.5) \qquad \cos \Psi = \frac{p}{r}$$

$$(1.6) \qquad \sin \Psi = \frac{z}{r}$$

For the computation of the normal gravitation, we shall use the function

(1.7) $F = \cos 2\Psi = \frac{p^2 - z^2}{r^2}$

II. Geographic coordinates, defined by

$$\text{(II)} \quad \begin{aligned} p &= (N + H)\cos\varphi \\ z &= (N + H - e^2N)\sin\varphi \end{aligned}$$

(1.8) $N = \frac{a}{W}$

(1.9) $W^2 = 1 - e^2 \sin^2 \varphi$

$$e^2 = \frac{a^2 - b^2}{a^2}$$

and called

φ = geographic latitude

H = geographic height

The geographic coordinates refer to a reference ellipsoid with equatorial radius a and polar radius b , and they are used for the computation of triangulations, in spite of the fact that the "orientation" of the ellipsoid (or the "data" of the triangulation system) cannot be determined without world-wide geodetic operations. The names suggested here are, however, not yet in a general use.

The normal gravitation is the attraction of the reference ellipsoid which is supposed to have the same mass as the actual earth. The masses inside the rotating ellipsoid are supposed to be distributed in such a way that the combined potential of the attraction and of the centrifugal force is constant at the outer surface and very close to the actual potential of the gravity at the mean sea level.

At higher altitudes, the geographic coordinates are less practical for the exact computation of the geodetic quantities. Especially, we cannot use them for the computation of the normal gravity. However, we should use them for the evaluation of the deviations of the actual gravity from the normal gravity on the basis of the anomalies observed on the ground.

The computation of φ and H from x,y,z is rather complicated. First, we obtain

(1.10) $$H = \frac{p}{\cos \varphi} - N$$

(1.11) $$H = \frac{z}{\sin \varphi} - N + e^2 N$$

The difference of these equations gives

(1.12) $$\tan \varphi = \frac{z}{p} + e^2 \frac{N}{p} \sin \varphi$$

This equation will be solved with respect to φ by successive approximations. The first approximation is usually obtained by $H = 0$ in which case we have

(1.13) $$\tan \varphi_1 = \frac{1}{1-e^2} \cdot \frac{z}{p}$$

For the higher altitudes, it is slightly better to start from $H = e^2 N \approx 43$ km, in which case we obtain

$$(1.14) \quad \tan \varphi_1 = (1 + e^2) \frac{z}{p}$$

The latter case will be used in our present applications.

Any approximate value $\tan \varphi_1$ can be improved as follows. Compute

$$(1.15) \quad \sin^2 \varphi_1 = \frac{\tan^2\varphi_1}{1+\tan^2\varphi_1}$$

and W_1 by the aid of the value thus obtained. Then we have a second approximation

$$(1.16) \quad W_1{}^2 = 1 - e^2 \sin^2 \varphi_1$$

$$(1.17) \quad \delta = e^2 a \frac{\sin\varphi_1}{pW_1}$$

$$(1.18) \quad \tan \varphi_2 = \tan \Psi + \delta$$

If this new approximation is improved in the same way, the change is already much smaller and it can be estimated roughly:

$$(1.19) \quad \tan \varphi_3 - \tan \varphi_2 = \delta \cdot \frac{\cos^2\varphi_1}{\tan\varphi_1} (\tan \varphi_2 - \tan \varphi_1)$$

In the numerical example given in chapter 4 we have H = 129. Even in this extreme case the error of φ_3 is 0".0002 only. That of φ_2 is 0".035 and that of φ_1 is 9".

In our present problem, we shall use φ only for the computation of the disturbances of gravity. Therefore, we may think that the first approximation φ_1 already is quite sufficient for this particular purpose. However, we have included the computation of the correct value into our program because it may be useful for other purposes.

Note that φ is not the direction of the normal gravity at the elevated point P but the direction of the normal of the reference ellipsoid. The former direction can be computed only when the components of the normal gravity are known:

$$\tan \varphi'' = \frac{\gamma_z}{\gamma_p} \tag{1.20}$$

We shall call φ'' the geodetic latitude, bearing in mind the definition: ξ - component of the deflection of the vertical is the difference of the astronomical latitude φ' and geodetic latitude φ'' at the elevated point P .

The difference $\varphi'' - \varphi$ is caused by the curvature of the normal plumb line. In [Hirvonen, 1960], a series is given for the computation of $\varphi'' - \varphi$

$$\begin{aligned} \varphi'' - \varphi = \sin 2\varphi \, (0''.170293 \, H \\ + 0\ .001103 \, H \cos 2\varphi \\ + 0\ .000034 \, H^2) \end{aligned} \tag{1.21}$$

(H in kilometers). In the present problem, however, we cannot use this formula for the computation of φ because H still is unknown.

Therefore, we have to compute φ by successive approximations as described above. When the final approximation has been found, H can be obtained from (1.10).

For latitudes higher than 45°, our program should be based on an alternative set of formulas. Instead of (1.14) through (1.19), compute

$$\text{(1.22)} \quad \cot \varphi_1 = \frac{p}{z(1+e^2)}$$

$$\text{(1.23)} \quad \cos^2 \varphi_1 = \frac{\cot^2\varphi_1}{1+\cot^2\varphi_1}$$

$$\text{(1.24)} \quad \cot \varphi_2 = \frac{p}{z} - \frac{e^2 a}{W_1} \frac{\cos \varphi_1}{z} = \cot \Psi - \delta$$

$$\text{(1.25)} \quad \cot \varphi_3 - \cot \varphi_2 = \delta \cdot \frac{\sin^2\varphi_1}{\cot\varphi_1} (\cot \varphi_2 - \cot \varphi_1)$$

With the final approximation of φ, compute H from (1.11).

III. Elliptic coordinates, defined by

$$p = \frac{c}{\sin \epsilon} \cos \beta$$

$$\text{(III)} \quad z = \frac{c}{\tan \epsilon} \sin \beta$$

$$c = e\ a$$

and called

ϵ = angular eccentricity

β = reduced latitude

At the surface of the reference ellipsoid, ϵ has a constant value

ϵ_o with

(1.26) $\sin \epsilon_o = e$

When p and z are given, we could compute ϵ and β by rigorous formulas

(1.27) $k^2 = r^2 + c^2$

(1.28) $h^4 = k^4 - 4p^2 c^2$

(1.29) $\sin^2 \epsilon = \frac{k^2 - h^2}{2p^2}$

(1.30) $\cos^2 \beta = \frac{2p^2}{k^2 + h^2}$

For practical computations, however, we shall use power series. By the aid of

(1.31) $\varkappa^2 = \frac{c^2}{r^2}$

we can replace the formulas above by

(1.32) $k^2 = r^2(1 + \varkappa^2)$

(1.33) $h^4 = r^4(1 - 2\varkappa^2 \cos 2\Psi + \varkappa^4)$

(1.34) $\sin^2 \epsilon = \varkappa^2[1 - \varkappa^2 \sin^2 \Psi$

$- \varkappa^4 \sin^2 \Psi \cos 2\Psi$

$- \varkappa^6 \sin^2 \Psi (\frac{5}{4} \cos 2\Psi - \frac{1}{4})]$

(1.35) $\cos^2 \beta = \cos^2 \Psi [1 - \varkappa^2 \sin^2 \Psi$

$- \varkappa^4 \sin^2 \Psi \cos 2\Psi]$

Using the abbreviating symbol

(1.7) $F = \cos 2\Psi$

we obtain, after lengthy but easy computations,

$$\begin{aligned} \epsilon = {} & \varkappa + \frac{1}{12}\varkappa^3(3F - 1) \\ (1.36) \qquad & + \frac{1}{160}\varkappa^5(35F^2 - 10F - 13) \\ & + \frac{1}{128}\varkappa^7(33F^3 - 9F^2 - 21F + \frac{19}{7}) \end{aligned}$$

2. THE POTENTIAL OF NORMAL GRAVITATION

The Newtonian attraction of the reference ellipsoid is called the normal gravitation of the earth. The potential of this attraction can be expressed in a closed form:

(2.1) $V = \frac{fM}{c}\epsilon + \frac{\omega^2 a^2}{2q_o} q(\frac{2}{3} - \cos^2\beta)$

where

f : gravitational constant,

M : mass of the earth,

ω : rotation speed of the earth.

The auxiliar variable

(2.2) $q = \frac{1}{2}[\epsilon - 3\cot\epsilon\,(1 - \epsilon\cot\epsilon)]$

is a function of ϵ only. At the surface of the reference ellipsoid q has a constant value q_o.

The closed formula (2.2) of q is not suitable for practical computations, but must be replaced by power series:

$$(2.3) \qquad q = \frac{2}{15} \tan^3 \epsilon - \frac{4}{35} \tan^5 \epsilon + \frac{2}{21} \tan^7 \epsilon - \ldots$$

or

$$(2.4) \qquad q = \frac{2}{15} \sin^3 \epsilon + \frac{3}{35} \sin^5 \epsilon + \frac{5}{84} \sin^7 \epsilon + \ldots$$

Therefore, it is only logical that we use power series throughout for our computations. By the aid of (1.34), (2.4) gives

$$(2.5) \qquad q = \frac{2}{15} \kappa^3 + \frac{1}{10} \kappa^5 \left(F - \frac{1}{7}\right) + \frac{1}{560} \kappa^7 \left(63F^2 - 10F - \frac{59}{3}\right)$$

Now we have to insert (1.35), (1.36) and (2.5) into (2.1). Using another abbreviating symbol for the constant

$$(2.6) \qquad C = \frac{2}{15} \frac{\omega^2 a^2 c}{fMq_o}$$

we can write the result in form

$$(2.7) \qquad \begin{aligned} V = \frac{fM}{c} \kappa \Big\{ 1 &+ \frac{1}{4} \kappa^2 (1 - C)\left(F - \frac{1}{3}\right) \\ &+ \frac{1}{224} \kappa^4 (7 - 10C)\left(7F^2 - 2F - \frac{13}{5}\right) \\ &+ \frac{1}{128} \kappa^6 (3 - 5C)\left(11F^3 - 3F^2 - 7F + \frac{19}{21}\right) \Big\} \end{aligned}$$

3. THE COMPONENTS OF NORMAL GRAVITY

Differentiation of (2.7) and (1.31) with respect to r gives the component of the normal gravitation toward the center of the earth:

$$\Gamma_r = -\frac{\partial V}{\partial r} = \frac{fM}{c^2} \varkappa^2 \{1 + \frac{1}{4} \varkappa^2 (1 - C)(3F - 1)$$

$$(3.1) \qquad + \frac{1}{224} \varkappa^4 (7 - 10C)(35F^2 - 10F - 13)$$

$$+ \frac{1}{384} \varkappa^6 (3 - 5C)(231F^3 - 63F^2 - 147F + 19)\}$$

Differentiation of (2.7) and (1.7) with respect to Ψ and division by r give the component perpendicular to Γ_r:

$$\Gamma_\Psi = -\frac{1}{r} \frac{\partial V}{\partial \Psi} = \frac{fM}{c^2} \varkappa^4 \sin\Psi \cos\Psi \{1 - C$$

$$(3.2) \qquad + \frac{1}{28} \varkappa^2 (7 - 10C)(7F - 1)$$

$$+ \frac{1}{32} \varkappa^4 (3 - 5C)(33F^2 - 6F - 7)\}$$

In order to facilitate the numerical computations, we replace the increasing powers of $\varkappa^2$ by those of

$$(3.3) \qquad G = \frac{fM}{r^2}$$

using convenient units. In other words, we insert:

$$(3.4) \qquad \varkappa^2 = G \cdot \frac{c^2}{fM}$$

For the international ellipsoid with the international normal formula of gravity we obtain, when r is expressed in kilometers and Γ in gal = cm $\sec^{-2}$:

$$(3.5) \qquad G = \frac{3986\ 3290.45}{r^2}$$

$$
\begin{aligned}
\Gamma_r = {} & 1000\ G \\
& - 0.835\,888\ G^2 \\
& + 2.507\,664\ G^2 \cos 2\Psi \\
& - 0.005\,118\ G^3 \\
& - 0.003\,937\ G^3 \cos 2\Psi \\
& + 0.013\,778\ G^3 \cos^2 2\Psi \\
& + 0.000\,007\ G^4 \\
& - 0.000\,054\ G^4 \cos 2\Psi \\
& - 0.000\,023\ G^4 \cos^2 2\Psi \\
& + 0.000\,085\ G^4 \cos^3 2\Psi
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\Gamma_\Psi = \sin\Psi \cos\Psi \{ & 3.343\,551\ G^2 \\
& - 0.003\,149\ G^3 \\
& + 0.022\,045\ G^3 \cos 2\Psi \\
& - 0.000\,031\ G^4 \\
& - 0.000\,026\ G^4 \cos 2\Psi \\
& + 0.000\,145\ G^4 \cos^2 2\Psi \}
\end{aligned} \tag{3.7}
$$

As G is always slightly smaller than 1, it is easy to see that the accuracy of one milligal can be obtained without the terms with G^4.

The components of gravitation in directions of the Cartesian coordinate axes are here chosen to be positive towards the origin. They can be computed by formulas

$$\Gamma_p = \Gamma_r \cos\Psi - \Gamma_\Psi \sin\Psi \tag{3.8}$$

$$\Gamma_x = \Gamma_p \cos\lambda \tag{3.9}$$

(3.10) $\Gamma_y = \Gamma_p \sin \lambda$

(3.11) $\Gamma_z = \Gamma_r \sin \Psi + \Gamma_\Psi \cos \Psi$

The components of the normal gravity are obtained by subtraction of those of the centrifugal force:

(3.12) $\gamma_p = \Gamma_p - \omega^2 p$

(3.13) $\gamma_x = \Gamma_x - \omega^2 x$

(3.14) $\gamma_y = \Gamma_y - \omega^2 y$

(3.15) $\gamma_z = \Gamma_z$

The total gravitation is

(3.16) $\Gamma = \sqrt{\Gamma_r^2 + \Gamma_\Psi^2}$

and the total gravity

(3.17) $\gamma = \sqrt{\gamma_p^2 + \gamma_z^2}$

The direction of the former is

(3.18) $\Phi = \arc \tan \frac{\Gamma_z}{\Gamma_p}$

that of the latter was given in (1.20).

4. SUMMARY OF FORMULAS FOR NORMAL GRAVITY

The formulas given above will be summarized here in form of a program for automatic computers. First, the constants are listed which pertain to the international ellipsoid and to the international normal formula of gravity. Then the computations are described in detail, using a numerical example as illustration.

Input			Constants
(1)			1,0000 0000
(2)	e^2	=	0,0067 2267
(3)	$1 + e^2$	=	1,0067 2267
(4)	a	=	6378,3880
(5)	e^2a	=	42,8798
(6)	f M	=	3986 3290
(7)			1000,0000
(8)			0,83589
(9)			2,50766
(10)			0.00512
(11)			0,00394
(12)			0,01378
(13)			3,34355
(14)			0,00315
(15)			0,02204
(16)	ω^2	=	0,000 531 749

Input	Variables	Example
(17)	Identification number of P	——
(18)	x	521,0517
(19)	y	4957.4762
(20)	z	4169.1994

Program of Computations

(21)	p^2	(18) · (18) + (19) · (19)	2484 8065
(22)	z^2	(20) · (20)	1738 2224
(23)	r^2	(21) + (22)	4223 0289
(24)	cos 2Ψ	[(21) - (22)] ÷ (23)	0.1767 8877
(25)	p	√(21)	4984.7833
(26)	r	√(23)	6498.4836
(27)	cos Ψ	(25) ÷ (26)	0.7670 6869
(28)	sin Ψ	(20) ÷ (26)	0.6415 6496
(29)	tan Ψ	(20) ÷ (25)	0.8363 8528
(30)	cos λ	(18) ÷ (25)	0.1045 2846
(31)	sin λ	(19) ÷ (25)	0.9945 2191
(32)	cot λ	(18) ÷ (19)	0.1051 0423
(33)	cosΨ cosλ	(27) · (30)	0.0802
(34)	sinΨ cosλ	(28) · (30)	0.0670
(35)	cosΨ sinλ	(27) · (31)	0.7629
(36)	sinΨ sinλ	(28) · (31)	0.6381
(37)	λ	arc cot (32)	84°00! 0000
(38)	tan φ_1	(3) · (29)	0.8420 0802
(39)	$\tan^2 \varphi_1$	(38) · (38)	0.7089 7751
(40)	$\sin^2 \varphi_1$	(39) ÷ [(1) + (39)]	0.4148 5479
(41)	W^2	(1) - (2) · (40)	0.9972 1107
(42)	sin φ_1	√(40)	0.6440 9222
(43)	W	√(41)	0.9986 0456
(44)		(5) · (42) ÷ (25)	0.0055 4057
(45)	δ	(44) ÷ (43)	0.0055 4831

(46)	tan φ_2	(29) + (45)	0.8419 3359
(47)		(38) - (46)	0.0000 7443
(48)		(38) · [(1) + (39)]	1.4390
(49)		(47) · (45) ÷ (48)	0.0000 0029
(50)	tan φ	(46) - (49)	0.8419 3330
(51)	$\tan^2 \varphi$	(50) · (50)	0.7088 5168
(52)	sec φ	$\sqrt{(1) + (51)}$	1.3072 3054
(53)	cos φ	(1) ÷ (52)	0.7649 7601
(54)	sin φ	(50) · (53)	0.6440 5878
(55)	φ	arc tan (50)	40°05'.7085
(56)		(25) · (52)	6516.2610
(57)		(4) ÷ (43)	6387.3011
(58)	H	(56) - (57)	128.96
(59)	G	(6) ÷ (23)	0.9439 5022
(60)	G^2	(59) · (59)	0.8910 42
(61)	G^2F	(60) · (24)	0.1575 26
(62)	G^3	(60) · (59)	0.8411
(63)	G^3F	(62) · (24)	0.1487
(64)	G^3F^2	(63) · (24)	0.0263
(65)		(7) · (59)	943.950 22
(66)		(8) · (60)	0.744 81
(67)		(9) · (61)	0.395 02
(68)		(10) · (62)	0.004 31
(69)		(11) · (63)	0.000 59
(70)		(12) · (64)	0.000 36

Input		Variables	Example
(71)	Γ_r	(65)-(66)+(67)-(68)-(69)+(70)	943.595 90
(72)		(13) · (60)	2.979 24
(73)		(14) · (62)	0.002 65
(74)		(15) · (63)	0.003 28
(75)		(72) - (73) + (74)	2.979 87
(76)	Γ_ψ	(75) · (27) · (28)	1.466 47
(77)	Γ_p	(71) · (27) - (76) · (28)	722.862 04
(78)	$\Gamma_z = \gamma_z$	(71) · (28) + (76) · (27)	606.502 95
(79)	Γ_x	(77) · (30)	75.559 66
(80)	Γ_y	(77) · (31)	718.902 14
(81)	γ_p	(77) - (16) · (25)	720.211 39
(82)	γ_x	(79) - (16) · (18)	75.282 59
(83)	γ_y	(80) - (16) · (19)	716.266 01
(84)	tan Φ	(78) ÷ (77)	0.8390 3002
(85)	tan φ''	(78) ÷ (81)	0.8421 1796
(86)	Γ^2	(71) · (71) + (76) · (76)	890375.37
(87)	γ^2	(78) · (78) + (81) · (81)	886550.27
(88)	Γ	$\sqrt{(86)}$	943.59704
(89)	γ	$\sqrt{(87)}$	941.56799
(90)	Φ	arc tan (84)	39°59'.8596
(91)	φ''	arc tan (85)	40°06'.0800

Output

(17), (55), (37), (58), (53), (79), (80), (78)	in full
(33), (34), (31), (35), (36), (30), (28), (27)	four digit values

5. GRAVITY DISTURBANCES

If the free air anomalies Δg are known everywhere on the physical surface S of the earth, the potential disturbance T at any point on or above S can be computed. The computation is very complicated, especially if the point is very close to high and steep mountains. In most cases, however, the generalized formula of Stokes gives a sufficient approximation.

We shall use following notations:

P : fixed point at which T is wanted;

M : "moving" point at S ;

r : distance from the centre of the earth;

ψ : angle between radii r_P and r_M

α = azimuth from P to M ;

(5.1) $t = \frac{r_M}{r_P}$;

(5.2) $D^2 = 1 - 2t \cos \psi + t^2$;

(5.3) $d\sigma = \sin \psi \, d\psi \, d\alpha = \cos \varphi \, d\varphi \, d\lambda$

is the element of solid angle, situated at point M .

The generalized formula of Stokes reads

(5.4) $$T_P = \frac{r_P}{4\pi} \int \Delta g_M \, t^2 \left\{\frac{2}{D} + 1 - 3D - t \cos \psi \left(5 + 3 \ln \frac{D + 1 - t \cos \psi}{2}\right)\right\} d\sigma$$

The integration must be carried out over the entire surface S . To a very good approximation,

$$(5.5) \quad r_M = r_P - H$$

where H is the height of P above S . Usually, r_M can even be replaced by a constant R , the mean radius of the earth.

Especially, if H = 0 or t = 1, we have

$$(5.6) \quad D = 2 \sin \frac{\psi}{2}$$

and

$$(5.7) \quad T_S = \frac{r_S}{4\pi} \int \Delta g_M \{ \operatorname{cosec} \frac{\psi}{2} + 1 - 6 \sin \frac{\psi}{2}$$

$$- \cos \psi [5 + 3 \ \ln (\sin \frac{\psi}{2} + \sin^2 \frac{\psi}{2})]\} d\sigma$$

The quantity

$$(5.8) \quad \zeta = \frac{T}{\gamma}$$

is called the <u>height anomaly</u> and

$$(5.9) \quad \zeta_S = \frac{T_S}{\gamma_S}$$

is approximately the elevation of the geoid above the reference ellipsoid.

Because ζ_S is one of the most important quantities of geodesy, we may often assume that the values of it have already been computed.

Then we can compute the "coating" function of Helmert:

$$(5.10) \quad \mu = \Delta g + \frac{3}{2} \frac{\zeta_S}{r_S}$$

This function has the advantage that the long formula (5.4) for the computation of T at higher altitudes can be replaced by a shorter one:

$$(5.11) \quad T_P = \frac{r^2 M}{2\pi r_P} \int \frac{\mu}{D} \, d\sigma$$

In our present problem, we have to compute the components of the gravity disturbance:

$$(5.12) \quad \delta_n = - \frac{\partial T}{\partial r}$$

$$(5.13) \quad \delta_m = - \frac{\cos \alpha}{r} \frac{\partial T}{\partial \Psi}$$

$$(5.14) \quad \delta_\ell = - \frac{\sin \alpha}{r} \frac{\partial T}{\partial \Psi}$$

If we take T from the long formula (5.4), the result can be written in form

$$(5.15) \quad \delta_n = \frac{1}{2\pi} \int \Delta g \, F_1 \, d\sigma$$

$$(5.16) \quad \delta_m = \frac{1}{2\pi} \int \Delta g \, F_2 \cos \alpha \, d\sigma$$

$$(5.17) \quad \delta_\ell = \frac{1}{2\pi} \int \Delta g \, F_2 \sin \alpha \, d\sigma$$

where

$$(5.18) \quad F_1 = \frac{1}{2} t^2 \left\{ \frac{1-t^2}{D^3} + \frac{4}{D} + 1 - 6D - t \cos \Psi \left(13 + 6 \ln \frac{D+1-t \cos \Psi}{2}\right) \right\}$$

$$(5.19) \quad F_2 = t^3 \sin \Psi \left\{ \frac{1}{D^3} + \frac{3}{D} - 4 + \frac{3}{2} \left(\frac{D-1+t \cos \Psi}{D \sin^2 \Psi} - \ln \frac{D+1-t \cos \Psi}{2} \right) \right\}$$

If we use the short formula (5.11), we obtain

$$(5.20)\quad \delta_n = \frac{1}{2\pi} \int \mu \, f_1 d\sigma$$

$$(5.21)\quad \delta_m = \frac{1}{2\pi} \int \mu \, f_2 \cos \alpha \, d\sigma$$

$$(5.22)\quad \delta_\ell = \frac{1}{2\pi} \int \mu \, f_2 \sin \alpha \, d\sigma$$

where

$$(5.23)\quad f_1 = \frac{t^2}{D^3} (1 - t \cos \Psi)$$

$$(5.24)\quad f_2 = \frac{t^3}{D^3} \sin \Psi$$

Table 5.1 shows some values of F_1 and f_1, Table 5.2 those of F_2 and f_2. We see that for small values of Ψ, the functions F and f are almost equal.

In practical computations, we must first find the values of Ψ and α. If the earth is considered as a sphere, we have

$$(5.25)\quad \cos \Psi = \sin \varphi_P \sin \varphi_M + \cos \varphi_P \cos \varphi_M \cos (\lambda_M - \lambda_P)$$

$$(5.26)\quad \sin \Psi \cos \alpha = \cos \varphi_P \sin \varphi_M - \sin \varphi_P \cos \varphi_M \cos (\lambda_M - \lambda_P)$$

$$(5.27)\quad \sin \Psi \sin \alpha = \cos \varphi_M \sin (\lambda_M - \lambda_P)$$

For small values of Ψ, we can use the approximate formulas:

$$(5.28)\quad m = \varphi_M - \varphi_P$$

Table 5.1

Ψ	H = 2 km	H = 8	H = 32	H = 128	H = 512
			F_1		
0'	1016xxxx	636xxx	400xx	2559	174
1	4009xxx	588xxx	398xx	2558	174
3	3953xx	3522xx	3824x	2552	174
10	1321x	4002x	2594x	2481	173
30	710	2062	5085	1980	171
1°	182	356	947	1120	162
3	45	51	76	148	105
10	12.1	12.2	12.8	14.9	18.8
30	- 3.7	- 3.7	- 3.6	- 3.4	- 2.7
60	- 3.3	- 3.3	- 3.3	- 3.1	- 2.4
90	- 2.3	- 2.3	- 2.3	- 2.2	- 1.8
120	0.4	0.4	0.4	0.4	0.3
150	3.0	3.0	3.0	2.8	2.3
180	4.1	4.1	4.0	3.8	3.1

f_1

Ψ	H = 2 km	H = 8	H = 32	H = 128	H = 512
0'	1016xxxx	635xxx	397xx	2480	155
1	4005xxx	587xxx	395xx	2480	155
3	3937xx	3512xx	3795x	2472	155
10	1267x	3956x	2568x	2403	154
30	528	1883	4928	1908	152
1°	88	262	858	1061	144
3	12	18	44	118	90
10	2.9	3.1	3.8	6.3	12.2
30	1.0	1.0	1.0	1.1	1.3
60	0.5	0.5	0.5	0.5	0.5
90	0.4	0.4	0.4	0.3	0.3
120	0.3	0.3	0.3	0.3	0.3
150	0.3	0.3	0.3	0.3	0.2
180	0.2	0.2	0.2	0.2	0.2

Table 5.2

Ψ	H = 2 km	H = 8	H = 32	H = 128	H = 512
			F_2		
0'	0	0	0	0	0
1	3714xxx	1362xx	237x	36	1
3	1095xxx	2445xx	662x	109	2
10	1166xx	916xx	1494x	353	6
30	1327x	1286x	859x	842	18
1°	3366	3333	2957	936	33
3	396	394	385	309	62
10	43.5	43.4	42.9	40.5	28.0
30	7.5	7.5	7.5	7.1	5.9
60	1.2	1.2	1.2	1.2	1.1
90	- 1.4	- 1.4	- 1.3	- 1.3	- 1.0
120	- 2.2	- 2.2	- 2.1	- 2.0	- 1.7
150	- 1.5	- 1.5	- 1.5	- 1.4	- 1.2
180	0	0	0	0	0
			f_2		
0'	0	0	0	0	0
1	3713xxx	1360xx	229x	36	1
3	1093xxx	2442xx	660x	107	2
10	1160xx	914xx	1487x	348	6
30	1310x	1271x	850x	827	16
1°	3280	3252	2894	915	31
3	364	363	357	289	72
10	32.7	32.7	32.5	31.2	22.3
30	3.6	3.6	3.6	3.5	3.1
60	0.9	0.9	0.9	0.8	0.8
90	0.4	0.4	0.4	0.3	0.3
120	0.3	0.3	0.3	0.3	0.3
150	0.3	0.3	0.3	0.3	0.2
180	0.2	0.2	0.2	0.2	0.2

(5.29) $\ell = (\lambda_M - \lambda_P) \cos \varphi_P$

(5.30) $n = \frac{H}{R} + \frac{1}{2}(m^2 + \ell^2)$

(5.31) $\Delta^2 = n^2 + m^2 + \ell^2$

(5.32) $f_1 = \frac{n}{\Delta^3}$

(5.33) $f_2 \cos \alpha = \frac{m}{\Delta^3}$

(5.34) $f_2 \sin \alpha = \frac{\ell}{\Delta^3}$

These formulas can be used only when $\Psi < 20°$. On the other hand, the effect of zones $\Psi > 20°$ is often negligible. Otherwise, this effect may be computed for a few points only and interpolated for the other points. When the components δ_n, δ_m, δ_ℓ have been computed, we must convert them to the system x,y,z by the matrix multiplication:

(5.35) $$\begin{pmatrix} \delta_x \\ \delta_y \\ \delta_z \end{pmatrix} = \begin{pmatrix} -\cos\varphi\cos\lambda & -\sin\varphi\cos\lambda & -\sin\lambda \\ -\cos\varphi\sin\lambda & -\sin\varphi\sin\lambda & \cos\lambda \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \delta_n \\ \delta_m \\ \delta_\ell \end{pmatrix}$$

The components

δ_A along the trajectory

δ_R horizontally to the right hand side

δ_D perpendicularly (not vertically) down

can be obtained as follows. Take three consecutive points P_{i-1}, P_i and P_{i+1} of the trajectory. Compute:

$$x' = x_{i+1} - x_{i-1}$$

$$(5.36) \qquad y' = y_{i+1} - y_{i-1}$$

$$z' = z_{i+1} - z_{i-1}$$

$$x'' = \frac{(x'x_i + y'y_i)\, z_i - z'p_i^2}{r'p_i r_i}$$

$$(5.37) \qquad y'' = \frac{x'y_i - y'x_i}{r'p_i}$$

$$z'' = \frac{x'x_i + y'y_i + z'z_i}{r'p_i}$$

$$(5.38) \qquad p'' = \sqrt{x''^2 + y''^2}$$

$$\begin{pmatrix} \delta_A \\ \delta_R \\ \delta_D \end{pmatrix} = \begin{pmatrix} -z'' & -x'' & y'' \\ 0 & \frac{y''}{p''} & \frac{x''}{p''} \\ p'' & -\frac{x''z''}{p''} & \frac{y''z''}{p''} \end{pmatrix} \begin{pmatrix} \delta_n \\ \delta_m \\ \delta_\ell \end{pmatrix}$$

Size of Blocks. Around the point P define four regions, each bounded by latitudes φ_1, φ_2 and longitudes λ_1, λ_2.

Region A

$$\varphi_2 - \varphi_1 = 3^\circ$$
$$\lambda_2 - \lambda_1 = 4^\circ$$
$$|\varphi - \varphi_P| < 1^\circ$$
$$|\lambda - \lambda_P| < 1^\circ\ 30'$$

Region B

outside A

$$\varphi_2 - \varphi_1 = 7^\circ$$
$$\lambda_2 - \lambda_1 = 9^\circ$$
$$|\varphi - \varphi_P| < 3^\circ$$
$$|\lambda - \lambda_P| < 4^\circ$$

Region C: $\varphi_2 - \varphi_1 = 25^\circ$

outside B $\lambda_2 - \lambda_1 = 30^\circ$

$|\varphi - \varphi_P| < 10$

$|\lambda - \lambda_P| < 12^\circ\ 30'$

$\varphi_1, \varphi_2, \lambda_1, \lambda_2$ are all divisible by 5°.

Region D outside C

For region A , pick up the mean anomaly cards $5' \times 5'$ ($36 \times 48 = 1728$ cards), for region B , $20' \times 20'$ cards ($21 \times 27 - 9 \times 12 = 459$ cards), for region C , $1^\circ \times 1^\circ$ cards ($25 \times 30 - 7 \times 9 = 687$ cards) and for region D , $5^\circ \times 5^\circ$ cards. Region D will be used for a few points only.

The influence of zone D on δ_x, δ_y, δ_z can also be computed directly by means of the Cartesian coordinates x,y,z. If these coordinates x,y,z are computed for the centers of the $5^\circ \times 5^\circ$ squares, then the integration can be carried out by the formulas

(5.40)
$$\Delta\delta_x = \frac{R^2}{2\pi} \int_D \mu \cdot \frac{x_M - x_P}{\Delta^3}\, d\sigma$$

$$\Delta\delta_y = \frac{R^2}{2\pi} \int_D \mu \cdot \frac{y_M - y_P}{\Delta^3}\, d\sigma$$

$$\Delta\delta_z = \frac{R^2}{2\pi} \int_D \mu \cdot \frac{z_M - z_P}{\Delta^3}\, d\sigma$$

where

(5.41)
$$\Delta^2 = (x_M - x_P)^2 + (y_M - y_P)^2 + (z_M - z_P)^2$$

and $\Delta\delta_x$, $\Delta\delta_y$, $\Delta\delta_z$ are the contribution of the zone D to δ_x, δ_y, δ_z.

PART II

by

H. Moritz

6. INFLUENCE OF AN INACCURATE GRAVITY MATERIAL

For simplicity, the error propagation will be studied for the coating method.

Since, by eqs. (5.20) - (5.24), the components of the gravity disturbance are given by

$$\delta_n = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \mu \cdot \frac{t^2}{D^3} (1 - t \cos \Psi) \sin \Psi \, d\Psi \, d\alpha,$$

$$\begin{Bmatrix} \delta_m \\ \delta_\ell \end{Bmatrix} = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \mu \cdot \frac{t^3}{D^3} \sin^2 \Psi \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} d\Psi \, d\alpha,$$

we find for their standard errors m_n, m_m, m_ℓ:

$$m_n^2 = \frac{1}{4\pi^2} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \int_{\alpha'=0}^{2\pi} \int_{\Psi'=0}^{\pi} \sigma(\Psi,\alpha,\Psi',\alpha') \cdot \frac{t^4}{D^3 D'^3} (1 - t \cos \Psi) \cdot$$

$$\cdot (1 - t \cos \Psi') \sin \Psi \sin \Psi' \, d\Psi \, d\alpha \, d\Psi' \, d\alpha',$$

(6.1)

$$\begin{Bmatrix} m_m^2 \\ m_\ell^2 \end{Bmatrix} = \frac{1}{4\pi^2} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \sigma(\Psi,\alpha,\Psi',\alpha') \cdot \frac{t^6}{D^3 D'^3} \sin^2 \Psi \sin^2 \Psi' \cdot$$

$$\cdot \begin{Bmatrix} \cos \alpha \cos \alpha' \\ \sin \alpha \sin \alpha' \end{Bmatrix} d\Psi \, d\alpha \, d\Psi' \, d\alpha',$$

where $\sigma(\Psi,\alpha,\Psi',\alpha')$ is the error covariance function, or error function, of μ [Moritz, 1962a, 1963].

We shall assume uniform coverage of the whole earth by gravity stations so that σ approximately depends on the relative position of points $P(\Psi,\alpha)$ and $P'(\Psi',\alpha')$ only. Furthermore we assume that the error function has a sharp maximum at $P' = P$ and drops off rapidly to zero with increasing distance PP'.

Then we can simplify the integrations considerably by approximately replacing $\sigma(\Psi,\alpha,\Psi',\alpha')$ by

$$(6.2) \qquad \frac{S}{\sin\Psi'} \cdot \delta(\Psi' - \Psi)\, \delta(\alpha' - \alpha)$$

where $\delta(\Psi' - \Psi)$, $\delta(\alpha' - \alpha)$ are Dirac's delta functions and S is a constant given by

$$(6.3) \qquad S = \int_{\alpha'=0}^{2\pi} \int_{\Psi'=0}^{\pi} \sigma(\Psi,\alpha,\Psi',\alpha') \sin\Psi'\, d\Psi'\, d\alpha' .$$

Since according to a property of the delta function

$$\int f(x')\, \delta(x' - x)\, dx' = f(x),$$

the integrations with respect to Ψ', α' can be performed immediately and we get

$$m_n^2 = \frac{S}{4\pi^2} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \frac{t^4}{D^6} (1 - t\cos\Psi)^2 \sin\Psi\, d\Psi\, d\alpha,$$

$$\begin{Bmatrix} m_m^2 \\ m_\ell^2 \end{Bmatrix} = \frac{S}{4\pi^2} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \frac{t^6}{D^6} \sin^3\Psi \begin{Bmatrix} \cos^2\alpha \\ \sin^2\alpha \end{Bmatrix} d\Psi\, d\alpha,$$

and finally,

$$(6.4)\quad \begin{aligned} m_n^2 &= \frac{S}{2\pi}\left[\frac{t^4(t^2+1)}{2(t^2-1)^2} + \frac{t^4}{t^2-1} - \frac{t^3}{4}\,\ell n\left|\frac{t-1}{t+1}\right|\right], \\ m_m^2 = m_\ell^2 &= \frac{S}{2\pi}\left[\frac{t^4(t^2+1)}{4(t^2-1)^2} + \frac{t^3}{8}\,\ell n\left|\frac{t-1}{t+1}\right|\right]. \end{aligned}$$

Since

$$t = \frac{R}{R+H} = 1 - \frac{H}{R} + \dots$$

(R:mean radius of the earth) we can develop in series, and neglecting higher powers of $\frac{H}{R}$ we get

$$(6.5)\quad m_n^2 = \frac{SR^2}{8\pi H^2}, \quad m_m^2 = m_\ell^2 = \frac{SR^2}{16\pi H^2}.$$

(For H = 500 km the error of these approximations is smaller than 7%.)

<u>Validity of the Simplified Integration</u>

Now we have to investigate in more detail the validity of replacing the error function σ by a product of Delta functions. For simplicity we limit ourselves to the plane approximation. In this case, eqs. (5.20) - (5.22) are simplified to

$$(6.6)\quad \begin{aligned} \delta_n &= \frac{H}{2\pi}\iint_{-\infty}^{\infty} \frac{\mu(x,y)}{(H^2+x^2+y^2)^{3/2}}\,dx\,dy, \\ \begin{Bmatrix}\delta_m\\ \delta_\ell\end{Bmatrix} &= \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{\mu(x,y)}{(H^2+x^2+y^2)^{3/2}}\begin{Bmatrix}x\\ y\end{Bmatrix}dx\,dy, \end{aligned}$$

where the xy-plane is horizontal and the z-axis contains point P.

These equations have the general form

$$(6.7) \qquad \iint_{-\infty}^{\infty} \mu(x,y)\, f(x,y)\, dx\, dy$$

and the mean square error is given by

$$(6.8) \qquad m^2 = \iiiint_{-\infty}^{\infty} \sigma(x,y,x',y')\, f(x,y)\, f(x',y')\, dx\, dy\, dx'\, dy' \ .$$

As before we assume the error function $\sigma(x,y,x',y')$ to have appreciable magnitude only for $(x',y') \doteq (x,y)$ and to be almost zero elsewhere. Then, only the points $(x',y') \doteq (x,y)$ will contribute significantly to the integral and if we put

$$x' = x + \xi \ , \quad y' = y + \eta \ ,$$

then only small values ξ, η need to be considered. If the usual conditions of continuity and differentiability are satisfied, $f(x',y')$ can be developed in a Taylor series with respect to ξ, η:

$$f(x',y') = f(x,y) + \xi f_x + \eta f_y +$$

$$+ \frac{1}{2} \xi^2 f_{xx} + \xi\eta f_{xy} + \frac{1}{2} \eta^2 f_{yy} + \ . \ . \ .$$

where f_x, f_{xx}, ... denote partial derivatives at the point (x,y) and

$$\xi = x' - x \ , \quad \eta = y' - y \ .$$

Inserting this in (6.8) we can separate the integrations over x, y and over x', y', getting

$$m^2 = \iint_{-\infty}^{\infty} \sigma(x,y,x',y')\, dx'\, dy' \cdot \iint_{-\infty}^{\infty} [f(x,y)]^2\, dx\, dy +$$

$$+ \iint_{-\infty}^{\infty} (x'-x)\, \sigma(x,y,x',y')\, dx'\, dy' \cdot \iint_{-\infty}^{\infty} ff_x\, dx\, dy +$$

$$+ \iint_{-\infty}^{\infty} (y'-y)\, \sigma(x,y,x',y')\, dx'\, dy' \cdot \iint_{-\infty}^{\infty} ff_y\, dx\, dy +$$

$$+ \frac{1}{2} \iint_{-\infty}^{\infty} (x'-x)^2\, \sigma(x,y,x',y')\, dx'\, dy' \cdot \iint_{-\infty}^{\infty} ff_{xx}\, dx\, dy +$$

$$+ \iint_{-\infty}^{\infty} (x'-x)(y'-y)\, \sigma(x,y,x',y')\, dx'\, dy' \cdot \iint_{-\infty}^{\infty} ff_{xy}\, dx\, dy +$$

$$+ \frac{1}{2} \iint_{-\infty}^{\infty} (y'-y)^2\, \sigma(x,y,x',y')\, dx'\, dy' \cdot \iint_{-\infty}^{\infty} ff_{yy}\, dx\, dy \; \ldots ,$$

provided $\sigma(x,y,x',y')$ depends on $x-x'$, $y-y'$ only, i.e., we have the same accuracy everywhere. Furthermore, if σ is a symmetrical function of $x'-x$ and $y'-y$ which is a very natural assumption, then the integrals containing $x'-x$ and $y'-y$ linearly will vanish and there remains

$$m^2 = S_0 \iint_{-\infty}^{\infty} [f(x,y)]^2\, dx\, dy + \tag{6.9}$$
$$+ \frac{1}{2} S_1 \iint_{-\infty}^{\infty} ff_{xx}\, dx\, dy + \frac{1}{2} S_2 \iint_{-\infty}^{\infty} ff_{yy}\, dx\, dy \; \ldots$$

where

$$S_0 = \iint_{-\infty}^{\infty} \sigma\, dx'\, dy', \quad S_1 = \iint_{-\infty}^{\infty} (x'-x)^2\, \sigma\, dx'\, dy', \tag{6.10}$$
$$S_2 = \iint_{-\infty}^{\infty} (y'-y)^2 \sigma\, dx'\, dy' .$$

If we would have replaced, as before, the error function σ by a product of delta functions,

$$S_o \cdot \delta(x'-x)\ \delta(y'-y)\ ,$$

we would have got

$$m^2 = S_o \iint_{-\infty}^{\infty} [f(x,y)]^2\ dx\ dy\ ,$$

which is the first and principal term of (6.9).

Thus the use of delta functions is justified. Transferring these results from the plane to the spherical case offers no difficulties. S_o, the integral of the error function over the plane (6.10), is replaced by S, the integral over the unit sphere (6.3); we have the relation

$$(6.11) \qquad S = \frac{S_o}{R^2}$$

where the scaling factor R^2 represents the transition from the terrestrial sphere (or its tangential plane) to the unit sphere.

The error of the delta function method is the effect of S_1 and S_2 in (6.9).

In order to evaluate the quantities S, S_o, S_1, S_2 we have to make some assumption concerning the error function σ . If this function has the character assumed above - everywhere the same form, sharp maximum at $x' = x$, $y' = y$, vanishing farther away from this point - then, within the limit of the obtainable accuracy, we can usually take the well-known function

$$(6.12) \qquad \sigma(x,y,x',y') = \sigma_o e^{-c^2[(x'-x)^2 + (y'-y)^2]} \, .$$

(6.10) can then be evaluated immediately to give

$$(6.13) \qquad S_o = \frac{\pi\sigma_o}{c^2} \, , \quad S_1 = S_2 = \frac{\pi\sigma_o}{2c^4}$$

and

$$(6.14) \qquad S = \frac{\pi\sigma_o}{c^2R^2} \, .$$

Now we can return to eqs. (6.6). First we consider δ_n. Here

$$f(x,y) = \frac{H}{2\pi} \frac{1}{(H^2+x^2+y^2)^{3/2}} \, .$$

Differentiating twice, inserting in (6.9) and integrating yields

$$m_n^2 = \frac{S_o}{8\pi H^2} - \frac{3(S_1+S_2)}{64\pi H^4} \, .$$

The first term is clearly the same as the first equation in (6.5). Inserting (6.13) we finally get

$$(6.15) \qquad m_n^2 = \frac{\sigma_o}{8c^2H^2} \left(1 - \frac{3}{8c^2H^2} + \ldots\right)$$

and in a similar way we find

$$(6.16) \qquad m_m^2 = m_\ell^2 = \frac{\sigma_o}{16c^2H^2} \left(1 - \frac{3}{8c^2H^2} + \ldots\right) \, .$$

The correction term in the parenthesis is practically zero for elevations $H \gg c^{-1}$, but in view of the practical purpose of these accuracy formulas, it can already be neglected for $H = c^{-1}$.

Discussion

To sum up the formulas we have for the standard errors of δ_n, δ_m, δ_ℓ :

$$(6.17) \qquad m_n = \frac{k_1}{H} , \quad m_m = m_\ell = \frac{k_2}{H}$$

where

$$(6.18) \qquad k_1 = \sqrt{\frac{SR^2}{8\pi}} , \quad k_2 = \sqrt{\frac{SR^2}{16\pi}}$$

and

$$S = \frac{\pi\sigma_0}{R^2c^2} .$$

These formulas are valid for elevations $H > c^{-1}$. For smaller elevations they cannot be applied (for $H = 0$ they would yield ∞ , which is obviously wrong).

The error covariance function σ is supposed to have the form (6.12):

$$(6.19) \qquad \sigma(x,y,x',y') = \sigma_0 e^{-c^2 s^2}$$

where s is the distance of the points (x,y) and (x',y'). According to the derivation of the above formulas, σ is the error function of μ , the density of the coating. Since $\mu = \Delta g + \frac{3\gamma}{2R} N$, it contains the influence of errors in Δg and in N . The errors of N have a small influence but they diminish much less with increasing distance and can hardly be represented in the form (6.19). So, in order to apply our formulas, we must be able to neglect the errors of N. That this is possible can be seen in the following way.

We would have avoided the trouble concerning the errors of N, had we used the direct formulas (5.15) - (5.19) instead of using the coating method (5.20) - (5.24). These formulas are, however, very difficult to handle for our purpose. On the other hand, for smaller values of ψ, where the effect of errors in μ is largest, F and f are closely equal so that, for this particular purpose, we might replace μ in (5.20) - (5.24) by Δg. Then we can take σ in (6.1) and in the subsequent developments to be the error covariance function of the gravity anomalies Δg only. From this we can conclude that the error of (6.17) due to neglecting the inaccuracy of N must be small.

Thus, we can consider σ in (6.19) to be the error function for the interpolation of gravity anomalies. Then,

$$\sigma_0 = m^2$$

is the square of the average standard error of interpolation, m. In Table 6.1 some numerical values of m and c are given, together with the constants k_1 and k_2 defined above.

Table 6.1

Constants k_1, k_2 for average station distances s

s km	m mgal	c^{-1} km	k_1 mgal.km	k_2 mgal.km
10	1	13	2.5	1.8
50	9	25	45	32
100	14	45	126	89
200	16	90	287	203

To get the standard errors of $\delta_n^{\cdot}$ and of $\delta_m^{\cdot}$, $\delta_\ell^{\cdot}$, the above values for k_1 and k_2 have to be divided by the elevation H in kilometers.

For the computation of the dependence of m and c on the average station distances, an idealized gravity station net consisting of equilateral triangles with side s has been assumed. The computation was done in a way described in [Moritz, 1962b, Appendix], using numerical values for the covariance function of gravity anomalies given in [Kaula, 1959] combined with the standard interpolation errors of [Molodenskii et. al., 1962, p. 172]. Since the method used is not quite rigorous, the results of Table 2 must be considered somewhat preliminary.

Error Correlation of δ_n, δ_m, δ_ℓ

For the error covariances of δ_n, δ_m, δ_ℓ in two points P, P' we have found [Moritz, 1963] :

$$\sigma_{n,PP'} = \frac{\sigma_o}{8c^2H^2}\left[1 - \frac{3}{8}\frac{s^2}{H^2} + \dots\right],$$

$$(6.20) \quad \sigma_{m,PP'} = \frac{\sigma_o}{16c^2H^2}\left[1 - \frac{3}{16}\frac{s^2}{H^2}(1 + 2\cos^2\alpha) + \dots\right],$$

$$\sigma_{\ell,PP'} = \frac{\sigma_o}{16c^2H^2}\left[1 - \frac{3}{16}\frac{s^2}{H^2}(1 + 2\sin^2\alpha) + \dots\right].$$

P, P' are assumed to have the same elevation H, s is the distance and α is the azimuth from P to P'. These formulas, being the first terms of power series, are valid for small distances s only. For $P = P'$, by (6.5) and (6.14)

$$(6.21) \quad \sigma_{n,PP} = m_n^2, \quad \sigma_{m,PP} = m_m^2, \quad \sigma_{\ell,PP} = m_\ell^2 ;$$

the error covariances become variances, i.e., the squares of the standard errors of δ_n, δ_m, δ_ℓ .

Dividing the covariances (6.20) by the variances (6.21), we get the correlation coefficients

$$\rho_{n,PP'} = 1 - \frac{3}{8}\frac{s^2}{H^2} + \dots ,$$

$$(6.22) \quad \rho_{m,PP'} = 1 - \frac{3}{16}\frac{s^2}{H^2}(1 + 2\cos^2\alpha) + \dots ,$$

$$\rho_{\ell,PP'} = 1 - \frac{3}{16}\frac{s^2}{H^2}(1 + 2\sin^2\alpha) + \dots .$$

$\rho = \pm 1$ means maximum correlation, i.e., complete functional dependence; $\rho = 0$ means independence (more strictly speaking, complete lack of correlation). For a definite s, the correlation coefficients (6.22) will be the closer to 1, the greater the altitude H is. Thus, with increasing altitude, the standard errors of δ_n, δ_m, δ_ℓ decrease, but the correlation increases. Since the accuracy is characterized by the standard errors and the error correlation, this is significant, too.

7. INFLUENCE OF THE REMOTE ZONES

The purpose of the estimation of the influence of the distant zones is to decide how far the integration must be extended if certain accuracy requirements are prescribed.

Since the effect of the distant zones is almost independent of the elevation H, we may, for simplicity, put $H=0$, i.e., we consider P to be situated directly on the geoid. Then, since

$$\delta_n = \Delta g + \frac{2\gamma}{R} N, \quad \delta_m = \gamma\xi, \quad \delta_\ell = \gamma\eta ,$$

we can, for the direct method, use one of the evaluations of the effect of distant zones on the components ξ, η of the deflection of the vertical and on N (e.g., [Kaula, 1957]). For our purpose it is best to use the formulas of [Molodenskii et al., 1962] which can also be easily adapted to the coating method.

Direct Method (using Δg). The disturbing potential is

$$(7.1.a) \qquad T = \gamma N = \frac{R}{4\pi} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \Delta g \, S(\cos \Psi) \sin \Psi \, d\Psi \, d\alpha$$

where $S(\cos \Psi)$ is Stokes' function

$$(7.2.a) \qquad S(\cos \Psi) = \frac{1}{\sin \frac{\Psi}{2}} - 3 \cos \Psi \, \ln \left(\sin \frac{\Psi}{2} + \sin^2 \frac{\Psi}{2}\right) -$$

$$- 6 \sin \frac{\Psi}{2} + 1 - 5 \cos \Psi .$$

If we extend the integration with respect to Ψ only up to a spherical radius $\Psi_o < \pi$, we commit an error.

$$(7.3.a) \qquad \Delta N = \frac{R}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\Psi=\Psi_o}^{\pi} \Delta g \, S(\cos \Psi) \sin \Psi \, d\Psi \, d\alpha$$

which can be also written

$$(7.4.a) \qquad \Delta N = \frac{R}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \Delta g \, \bar{S}(\cos \Psi) \sin \Psi \, d\Psi \, d\alpha$$

where

$$(7.5.a) \qquad \bar{S}(\cos \Psi) = \begin{cases} 0 \text{ if } \Psi < \Psi_o \\ S(\cos \Psi) \text{ if } \Psi \geqq \Psi_o \end{cases} .$$

Expand $\bar{S}(\cos \Psi)$ in a series of Legendre's polynomials:

$$(7.6.a) \quad \bar{S}(\cos\Psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n P_n(\cos\Psi)$$

where the coefficients Q_n depend on Ψ_o. Then, by multiplying both sides by $P_n(\cos\Psi)\sin\Psi$ and integrating from 0 to π we find

$$Q_n = \int_0^{\pi} \bar{S}(\cos\Psi) P_n(\cos\Psi) \sin\Psi \, d\Psi = \int_{\Psi_o}^{\pi} S(\cos\Psi) P_n(\cos\Psi)\sin\Psi \, d\Psi .$$

By substituting

$$(7.7.a) \quad z = \sin\frac{\Psi}{2}, \quad t = \sin\frac{\Psi_o}{2}$$

we get

$$(7.8.a) \quad Q_n = -4 \int_1^t P_n(1 - 2z^2) \, S(1 - 2z^2) \, z \, dz .$$

Performing the integration yields (we need the Q_n only for $n \geqq 2$):

$$Q_2 = 2 - 4t + 5t^2 + 14t^3 - \frac{53}{2}t^4 - 30t^5 + 47t^6 + 18t^7 - \frac{51}{2}t^8 + (6t^2 - 24t^4 + 36t^6 - 18t^8) \ln t(1+t),$$

$$(7.9.a) \quad Q_3 = 1 - 4t + 5t^2 + 22t^3 - 46t^4 - \frac{372}{5}t^5 + 136t^6 + 104t^7 - 166t^8 - 48t^9 + \frac{352}{5}t^{10} + (6t^2 - 42t^4 + 108t^6 - 120t^8 + 48t^{10}) \ln t(1+t),$$

$$Q_4 = \frac{2}{3} - 4t + 5t^2 + \frac{98}{3}t^3 - 72t^4 - 156t^5 + 320t^6 + 360t^7 - 645t^8 - \frac{1120}{3}t^9 + 602t^{10} + 140t^{11} - 210t^{12} + (6t^2 - 66t^4 + 260t^6 - 480t^8 + 420t^{10} - 140t^{12}) \ln t(1+t).$$

Higher order Q_n (up to 8th order) may be found in [Molodenskii et al., 1962, p. 149].

By (7.4.a) and (7.6.a) we get

$$\Delta N = \frac{R}{2\gamma} \sum_{n=2}^{\infty} Q_n \Delta g_n \qquad (7.10.a)$$

if

$$\Delta g = \sum_{n=2}^{n} \Delta g_n$$

is the development of Δg in Laplace's spherical harmonics (Δg_o and Δg_1 are missing, as usual).

The coating method can be treated in an exactly analogous way.

Coating Method (using μ). The disturbing potential is

$$T = \gamma N = \frac{R}{4\pi} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \mu \, M (\cos \Psi) \sin \Psi \, d\Psi \, d\alpha \qquad (7.1.b)$$

where

$$M(\cos \Psi) = \frac{1}{\sin \frac{\Psi}{2}} . \qquad (7.2.b)$$

If we extend the integration with respect to Ψ only up to a spherical radius $\Psi_o < \pi$, we commit an error

$$\Delta N = \frac{R}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\Psi=\Psi_o}^{\pi} \mu \, M (\cos \Psi) \sin \Psi \, d\Psi \, d\alpha \qquad (7.3.b)$$

which can be also written

$$(7.4.b) \qquad \Delta N = \frac{R}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \mu \, \bar{M} \, (\cos \Psi) \sin \Psi \, d\Psi \, d\alpha$$

where

$$(7.5.b) \qquad \bar{M}(\cos \psi) = \begin{cases} 0 & \text{if } \Psi < \Psi_o \\ M \, (\cos \Psi) = \dfrac{1}{\sin \frac{\Psi}{2}} & \text{if } \Psi \geq \Psi_o \end{cases} .$$

Expand $\bar{M}(\cos \Psi)$ in a series of Legendre's polynomials:

$$(7.6.b) \qquad \bar{M}(\cos \Psi) = \sum_{n=0}^{\infty} (n-1) \, q_n \, P_n \, (\cos \Psi)$$

where the coefficients q_n depend on Ψ_o. Then, by multiplying both sides by $P_n(\cos \Psi) \sin \psi$ and integrating from 0 to π we find

$$q_n = \frac{2n+1}{2(n-1)} \int_0^{\pi} \bar{M} \, (\cos \Psi) \, P_n \, (\cos \Psi) \sin \Psi \, d\Psi =$$

$$= \frac{2n+1}{2(n-1)} \int_{\Psi_o}^{\pi} \frac{1}{\sin\frac{\Psi}{2}} \, P_n \, (\cos \Psi) \sin \Psi \, d\Psi \, .$$

By substituting

$$(7.7.a) \qquad z = \sin \frac{\Psi}{2}, \quad t = \sin \frac{\Psi_o}{2}$$

we get

$$(7.8.b) \qquad q_n = \frac{2(2n+1)}{n-1} \int_t^1 P_n (1-2z^2) \, dz.$$

Performing the integration yields (we need the q_n only for $n \geq 2$):

$$q_2 = \frac{2}{1}(1 - 5t + 10t^3 - 6t^5),$$

$$q_3 = \frac{2}{2}(1 - 7t + 28t^3 - 42t^5 + 20t^7),$$

$$q_4 = \frac{2}{3}(1 - 9t + 60t^3 - 162t^5 + 180t^7 - 70t^9),$$

$$q_5 = \frac{2}{4}(1 - 11t + 110t^3 - 462t^5 + 880t^7 - 770t^9 + 252t^{11}),$$

(7.9.b) $$q_6 = \frac{2}{5}(1 - 13t + 182t^3 - 1092t^5 + 3120t^7 - 4550t^9 + 3276t^{11} - 924t^{13}),$$

$$q_7 = \frac{2}{6}(1 - 15t + 280t^3 - 2268t^5 + 9000t^7 - 19250t^9 + 22680t^{11} - 13860t^{13} + 3432t^{15}),$$

$$q_8 = \frac{2}{7}(1 - 17t + 408t^3 - 4284t^5 + 22440t^7 - 65450t^9 + 111384t^{11} - 109956t^{13} + 58344t^{15} - 12870t^{17}).$$

By (7.4.b) and (7.6.b) we get

$$\Delta N = \frac{R}{\gamma} \sum_{n=2}^{\infty} \frac{n-1}{2n+1} \; q_n \mu_n$$

if

$$\mu = \sum_{n=2}^{n} \mu_n$$

if the development of μ in Laplace's spherical harmonics (μ_o and μ_1 are missing together with Δg_o and Δg_1).

If

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n$$

then

$$N = \frac{R}{\gamma} \sum_{n=2}^{\infty} \frac{\Delta g_n}{n-1} .$$

Therefore,

$$\mu = \Delta g + \frac{3\gamma}{2R} N = \sum_{n=2}^{\infty} \frac{2n+1}{2(n-1)} \Delta g_n$$

so that

$$\mu_n = \frac{2n+1}{2(n-1)} \Delta g_n .$$

Inserting this in the formula for N we finally obtain

$$(7.10.b) \quad \Delta N = \frac{R}{2\gamma} \sum_{n=2}^{\infty} q_n \Delta g_n ,$$

which is analogous to (7.10.a).

Mean Square Effect on N, ξ, η. Squaring (7.10.a) and averaging over the whole earth we get

$$\overline{\Delta N^2} = \frac{R^2}{4\gamma^2} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} Q_n Q_{n'} \cdot \overline{\Delta g_n \Delta g_{n'}} .$$

Since the integral, over the sphere, of the product of two Laplace harmonics of different degree is zero (Orthogonality !), only the terms where $n' = n$ remain and we obtain

$$(7.11.a) \quad \overline{\Delta N}^2 = \frac{R^2}{4\gamma^2} \sum_{n=2}^{\infty} Q_n^2 \overline{\Delta g}_n^2$$

where $\overline{\Delta N}$ is the root mean square influence on N of the zones beyond a spherical radius ψ_o , and $\overline{\Delta g}_n$ is the root mean square average of the Laplace spherical harmonic Δg_n .

The derivation of the RMS influence on the components ξ, η of the deflection of the vertical is rather involved. Since it is omitted in [Molodenskii et. al., 1962], it may be of interest to give it here.

By (7.10.a),

$$\Delta\xi = \frac{1}{R}\frac{\partial N}{\partial\varphi} = \frac{1}{2\gamma}\sum_{n=2}^{\infty} Q_n \frac{\partial\Delta g_n}{\partial\varphi} ,$$

(7.12)

$$\Delta\eta = \frac{1}{R\cos\varphi}\frac{\partial N}{\partial\lambda} = \frac{1}{2\gamma\cos\varphi}\sum_{n=2}^{\infty} Q_n \frac{\partial\Delta g_n}{\partial\lambda}$$

for the direct method. (For the coating method replace Q_n by q_n.)

Hence,

$$\overline{\Delta\xi^2} = \frac{1}{4\gamma^2}\sum_{n=2}^{\infty}\sum_{n'=2}^{\infty} Q_n Q_{n'} \overline{\frac{\partial g_n}{\partial\varphi}\frac{\partial g_{n'}}{\partial\varphi}} ,$$

$$\overline{\Delta\eta^2} = \frac{1}{4\gamma^2}\sum_{n=2}^{\infty}\sum_{n'=2}^{\infty} Q_n Q_{n'} \overline{\frac{1}{\cos^2\varphi}\frac{\partial g_n}{\partial\lambda}\frac{\partial g_{n'}}{\partial\lambda}} ,$$

so that

$$(7.13)\quad \overline{\Delta\xi^2} + \overline{\Delta\eta^2} = \frac{1}{4\gamma^2}\sum_{n=2}^{\infty}\sum_{n'=2}^{\infty} Q_n Q_{n'} \overline{\frac{\partial g_n}{\partial\varphi}\frac{\partial g_{n'}}{\partial\varphi} + \frac{1}{\cos^2\varphi}\frac{\partial g_n}{\partial\lambda}\frac{\partial g_{n'}}{\partial\lambda}} .$$

(The bar denotes averaging over the sphere.)

Δg_n can be written as

$$(7.14)\quad \Delta g_n = \sum_{m=0}^{n} (a_{nm}\cos m\lambda + b_{nm}\sin m\lambda)\, P_n^m(\sin\varphi)$$

where a_{nm}, b_{nm} are constant coefficients and φ, λ are geographical coordinates. Now we can differentiate with respect to φ and λ:

$$\frac{\partial \Delta g_n}{\partial \varphi} = \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) \frac{dP_n^m}{d\varphi} ,$$

(7.15)

$$\frac{\partial \Delta g_n}{\partial \lambda} = \sum_{m=0}^{n} (mb_{nm} \cos m\lambda - ma_{nm} \sin m\lambda) P_n^m .$$

Hence we get, taking $n' \geqq n$,

$$4\pi \overline{\frac{\partial \Delta g_n}{\partial \varphi} \frac{\partial \Delta g_{n'}}{\partial \varphi}} = \int_{\lambda=0}^{2\pi} \int_{\varphi=-\pi/2}^{\pi/2} \frac{\partial \Delta g_n}{\partial \varphi} \frac{\partial \Delta g_{n'}}{\partial \varphi} \cos \varphi \, d\varphi \, d\lambda =$$

(7.16)

$$= 2\pi \, a_{no} a_{n'o} \int_{-\pi/2}^{\pi/2} \frac{dP_n^o}{d\varphi} \frac{dP_{n'}^o}{d\varphi} \cos \varphi \, d\varphi +$$

$$+ \pi \sum_{m=1}^{n} (a_{nm} a_{n'm} + b_{nm} b_{n'm}) \int_{-\pi/2}^{\pi/2} \frac{dP_n^m}{d\varphi} \frac{dP_{n'}^m}{d\varphi} \cos \varphi \, d\varphi .$$

Here we have used the well-known orthogonality relations

$$\int_0^{2\pi} \cos p\lambda \cos q\lambda \, d\lambda = \begin{cases} 0, & p \neq q \\ \pi, & p = q \neq 0 \\ 2\pi, & p = q = 0 \end{cases} ,$$

(7.17)

$$\int_0^{2\pi} \cos p\lambda \sin q\lambda \, d\lambda = 0 ,$$

$$\int_0^{2\pi} \sin p\lambda \sin q\lambda \, d\lambda = \begin{cases} 0, & p \neq q \\ \pi, & p = q \neq 0 \\ 0, & p = q = 0 \end{cases} .$$

Similarly

$$4\pi \overline{\frac{1}{\cos^2\varphi} \frac{\partial\Delta g_n}{\partial\lambda} \frac{\partial\Delta g_{n'}}{\partial\lambda}} = \int_{\lambda=0}^{2\pi} \int_{\varphi=-\pi/2}^{\pi/2} \frac{1}{\cos^2\varphi} \frac{\partial\Delta g_n}{\partial\lambda} \frac{\partial\Delta g_{n'}}{\partial\lambda} \cos\varphi \, d\varphi \, d\lambda =$$

(7.18)
$$= \pi \sum_{m=1}^{n} m^2 (a_{nm} a_{n'm} + b_{nm} b_{n'm}) \int_{-\pi/2}^{\pi/2} \frac{P_n^m P_{n'}^m}{\cos^2\varphi} \cos\varphi \, d\varphi \; .$$

Putting

$$\sin\varphi = x \; , \quad \cos\varphi \, d\varphi = dx$$

we find by formulas of [Molodenskii et.al., 1962, pp. 165-6]

$$\int_{-\pi/2}^{\pi/2} \frac{dP_n^m}{d\varphi} \frac{dP_{n'}^m}{d\varphi} \cos\varphi \, d\varphi = \int_{-1}^{1} \frac{dP_n^m}{d\varphi} \frac{dP_{n'}^m}{d\varphi} dx =$$

$$= \begin{cases} \dfrac{2n(n+1)}{2n+1} \cdot \dfrac{(n+m)!}{(n-m)!} - m^2 C_n^m, & \text{if } n' = n, \\ -\, m^2 C_n^m, & \text{if } n' - n \text{ is even, } n' > n, \\ 0 \; , & \text{if } n' - n \text{ is odd;} \end{cases}$$

$$\int_{-\pi/2}^{\pi/2} \frac{P_n^m P_{n'}^m}{\cos^2\varphi} \cos\varphi \, d\varphi = \int_{-1}^{1} \frac{P_n^m P_{n'}^m}{1-x^2} dx =$$

$$= \begin{cases} C_n^m, & \text{if } n' = n, \\ C_n^m, & \text{if } n' - n \text{ is even, } n' > n, \\ 0 \; , & \text{if } n' - n \text{ is odd,} \end{cases}$$

where

$$C_n^m = 2(2n - 1)\frac{(n+m-2)!}{(n-m)!} + 2(2n-5)\frac{(n+m-4)!}{(n-m-2)!} + \ldots .$$

(Concerning C_n^m, there is a misprint on p. 165, op. cit.: " + C_n^m" should read " = C_n^m".)

Hence we find, for $n' \neq n$:

$$4\pi\,\overline{\frac{\partial\Delta g_n}{\partial\varphi}\frac{\partial\Delta g_{n'}}{\partial\varphi}} = \begin{cases} \pi \sum_{m=1}^{n} (a_{nm}a_{n'm} + b_{nm}b_{n'm})\cdot(-m^2 C_n^m), & n'-n \text{ even}, \\ 0, & n' - n \text{ odd}, \end{cases}$$

$$4\pi\,\overline{\frac{1}{\cos^2\varphi}\frac{\partial\Delta g_n}{\partial\lambda}\frac{\partial\Delta g_{n'}}{\partial\lambda}} = \begin{cases} \pi \sum_{m=1}^{n} (a_{nm}a_{n'm} + b_{nm}b_{n'm})\cdot m^2 C_n^m, & n'-n \text{ even}, \\ 0, & n' - n \text{ odd} \end{cases}$$

so that

$$\overline{\frac{\partial\Delta g_n}{\partial\varphi}\frac{\partial\Delta g_{n'}}{\partial\varphi} + \frac{1}{\cos^2\varphi}\frac{\partial\Delta g_n}{\partial\lambda}\frac{\partial\Delta g_{n'}}{\partial\lambda}} = 0 \tag{7.19}$$

for $n' \neq n$.

For $n' = n$ we find in the same way

$$\overline{\left(\frac{\partial\Delta g_n}{\partial\varphi}\right)^2 + \frac{1}{\cos^2\varphi}\left(\frac{\partial\Delta g_n}{\partial\lambda}\right)^2} = \frac{n(n+1)}{2n+1}a_{no}^2 + \frac{1}{2}\frac{n(n+1)}{2n+1}\sum_{m=1}^{n}\frac{(n+m)!}{(n-m)!}(a_{nm}^2 + b_{nm}^2).$$

Forming the average of Δg_n^2 itself we find easily

$$\overline{\Delta g_n^2} = \frac{1}{4\pi} \int_{\lambda=0}^{2\pi} \int_{\varphi=-\pi/2}^{\pi/2} \Delta g_n^2 \cos\varphi \, d\varphi \, d\lambda =$$

$$(7.20) \qquad = \frac{1}{2n+1} a_{no}^2 + \frac{1}{2(2n+1)} \sum_{m=1}^{n} \frac{(n+m)!}{(n-m)!} (a_{nm}^2 + b_{nm}^2)$$

so that

$$(7.21) \qquad \overline{\left(\frac{\partial \Delta g_n}{\partial \varphi}\right)^2 + \frac{1}{\cos^2\varphi}\left(\frac{\partial \Delta g_n}{\partial \lambda}\right)^2} = n(n+1)\,\overline{\Delta g_n^2} \ .$$

Taking (7.19) and (7.21) into account, we find by (7.13)

$$\overline{\Delta\xi^2} + \overline{\Delta\eta^2} = \frac{1}{4\gamma^2} \sum_{n=2}^{\infty} n(n+1)\, Q_n^2\, \overline{\Delta g_n^2} \ .$$

For reasons of symmetry, $\overline{\Delta\xi}$ and $\overline{\Delta\eta}$ can be taken to be equal, and we finally get

$$(7.22.a) \qquad \overline{\Delta\xi^2} = \overline{\Delta\eta^2} = \frac{1}{8\gamma^2} \sum_{n=2}^{\infty} n(n+1)\, Q_n^2\, \overline{\Delta g_n^2}$$

for the direct method.

For the <u>coating method</u> we must replace Q_n by q_n, obtaining

$$(7.11.b) \qquad \overline{\Delta N^2} = \frac{R^2}{4\gamma^2} \sum_{n=2}^{\infty} q_n^2\, \overline{\Delta g_n^2} \ ,$$

$$(7.22.b) \qquad \overline{\Delta\xi^2} = \overline{\Delta\eta^2} = \frac{1}{8\gamma^2} \sum_{n=2}^{\infty} n(n+1)\, q_n^2\, \overline{\Delta g_n^2} \ .$$

Relationship between $\overline{\Delta g_n}$ and the Covariance Function. Comparing eq. (7.20) with eq. (33) of [Kaula, 1959] and taking the relationship between conventional and fully normalized spherical harmonics into account, we find that

$$\overline{\Delta g_n^2} = c_n, \tag{7.23}$$

the coefficient of n-th degree in the development of the covariance function of gravity anomalies, C(s), in a series of Legendre's polynomials:

$$C(s) = \sum_{n=2}^{\infty} c_n P_n(\cos s). \tag{7.24}$$

Mean Square Effect on δ_N, δ_M, δ_L. In order to avoid confusion, we shall change our notation of the components of the gravity disturbance vector, now writing as subscripts, capital letters N, M, L instead of n, m, ℓ.

Then,

$$\delta_N = \Delta g + \frac{2\gamma}{R} N, \quad \delta_M = \gamma\xi, \quad \delta_L = \gamma\eta. \tag{7.25a}$$

Denoting the RMS effect of the zones beyond Ψ_o on these components by $\overline{\Delta\delta_N}$, $\overline{\Delta\delta_M}$, $\overline{\Delta\delta_L}$, respectively, we find from (7.11.a,b) and (7.22.a,b), taking (7.23) into account, the following formulas:

Direct Method

$$\overline{\Delta\delta_N^2} = \sum_{n=2}^{\infty} Q_n^2 c_n ,$$

(7.26.a)

$$\overline{\Delta\delta_M^2} = \overline{\Delta\delta_L^2} = \frac{1}{8} \sum_{n=2}^{\infty} n(n+1)\, Q_n^2 c_n .$$

If we use the coating μ , then

(7.25.b) $\delta_N = \mu + \frac{\gamma}{2R}N, \quad \delta_M = \gamma\xi, \quad \delta_L = \gamma\eta,$

and we have instead:

Coating Method

$$\overline{\Delta\delta_N^2} = \frac{1}{16} \sum_{n=2}^{\infty} q_n^2 c_n ,$$

(7.26.b)

$$\overline{\Delta\delta_M^2} = \overline{\Delta\delta_L^2} = \frac{1}{8} \sum_{n=2}^{\infty} n(n+1)\, q_n^2 c_n .$$

Numerical Results. First we give tables for the Q_n (Table 7.1.a, taken from [Molodenskii et.al., 1962]), and for q_n (Table 7.1.b, computed by eqs. (7.9.b)).

Table 7.1.a

Coefficients Q_n for the direct method as functions of radius Ψ_o

Ψ_o	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8
0°	+2.000	+1.000	+0.667	+0.500	+0.400	+0.333	+0.286
5°	+1.801	+0.802	+0.470	+0.304	+0.206	+0.141	+0.095
8.99°(1000km)	+1.634	+0.639	+0.312	+0.152	+0.061	+0.004	-0.032
10°	+1.593	+0.596	+0.274	+0.118	+0.030	-0.024	-0.056
13.49°(1500km)	+1.457	+0.472	+0.159	+0.015	-0.058	-0.095	-0.110
17.99°(2000km)	+1.306	+0.339	+0.049	-0.070	-0.116	-0.126	-0.118
20°	+1.247	+0.290	+0.011	-0.094	-0.127	-0.125	-0.105
25°	+1.133	+0.204	-0.044	-0.116	-0.120	-0.093	-0.057
30°	+1.062	+0.161	-0.058	-0.103	-0.087	-0.050	-0.013
40°	+1.021	+0.144	-0.051	-0.080	-0.057	-0.024	+0.002
50°	+1.044	+0.122	-0.094	-0.116	-0.067	-0.006	+0.033
60°	+1.040	+0.024	-0.188	-0.134	-0.009	+0.069	+0.063
70°	+0.958	-0.130	-0.240	-0.053	+0.095	+0.081	-0.015
80°	+0.807	-0.260	-0.187	+0.073	+0.111	-0.016	-0.075
90°	+0.636	-0.306	-0.068	+0.128	+0.023	-0.076	-0.011
100°	+0.504	-0.274	+0.024	+0.090	-0.046	-0.034	+0.041
110°	+0.441	-0.222	+0.049	+0.038	-0.043	+0.008	+0.020
120°	+0.431	-0.208	+0.047	+0.028	-0.035	+0.010	+0.010
130°	+0.430	-0.237	+0.078	+0.019	-0.050	+0.034	-0.003
140°	+0.389	-0.263	+0.140	-0.039	-0.025	+0.050	-0.045
150°	+0.291	-0.234	+0.171	-0.107	+0.050	-0.006	-0.023
160°	+0.159	-0.145	+0.127	-0.107	+0.086	-0.064	+0.044
170°	+0.045	-0.044	+0.043	-0.041	+0.039	-0.037	+0.035
180°	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 7.1.b

Coefficients q_n for the coating method as functions of radius Ψ_o

Ψ_o	q_2	q_3	q_4	q_5	q_6	q_7	q_8
0°	+2.000	+1.000	+0.667	+0.500	+0.400	+0.333	+0.286
5°	+1.565	+0.697	+0.408	+0.265	+0.179	+0.123	+0.083
8.99° (1000km)	+1.226	+0.465	+0.215	+0.095	+0.026	-0.016	-0.042
10°	+1.142	+0.408	+0.170	+0.056	-0.007	-0.044	-0.066
13.49° (1500km)	+0.858	+0.222	+0.024	-0.062	-0.102	-0.119	-0.121
17.99° (2000km)	+0.512	+0.009	-0.128	-0.170	-0.173	-0.156	-0.129
20°	+0.366	-0.075	-0.182	-0.202	-0.185	-0.152	-0.113
25°	+0.033	-0.251	-0.275	-0.233	-0.169	-0.101	-0.040
30°	-0.255	-0.374	-0.309	-0.206	-0.103	-0.018	+0.040
40°	-0.676	-0.460	-0.228	-0.045	+0.066	+0.103	+0.085
50°	-0.878	-0.363	-0.037	+0.116	+0.127	+0.057	-0.021
60°	-0.875	-0.156	+0.138	+0.153	+0.040	-0.058	-0.071
70°	-0.707	+0.070	+0.206	+0.061	-0.074	-0.073	+0.007
80°	-0.433	+0.235	+0.149	-0.064	-0.090	+0.014	+0.060
90°	-0.121	+0.293	+0.018	-0.119	-0.007	+0.068	+0.003
100°	+0.165	+0.241	-0.101	-0.070	+0.070	+0.020	-0.050
110°	+0.376	+0.115	-0.147	+0.026	+0.060	-0.048	-0.011
120°	+0.484	-0.028	-0.105	+0.086	-0.012	-0.039	+0.040
130°	+0.488	-0.137	-0.014	+0.068	-0.058	+0.020	+0.015
140°	+0.406	-0.178	+0.065	+0.002	-0.035	+0.041	-0.029
150°	+0.275	-0.153	+0.092	-0.050	+0.018	+0.004	-0.016
160°	+0.138	-0.088	+0.066	-0.051	+0.038	-0.027	+0.018
170°	+0.037	-0.025	+0.021	-0.019	+0.017	-0.015	+0.014
180°	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Now we are going to evaluate formulas (7.26.a,b) for the direct and the coating method, for different sets of c_n. First we use the maximum estimates for c_n of [Kaula, 1959, p. 2419]:

$$c_2 = 15, \; c_3 = 43, \; c_4 = 30, \; c_5 = c_6 = c_7 = c_8 = 25 \text{ mgal}^2.$$

The results are given in Table 7.2. The summation has first been extended to n = 8; higher order c_n have been neglected. In order to see the convergence of the series (7.26.a,b) we have also performed the summation up to n = 5 only.

Table 7.2.

RMS influence of the zones beyond a radius Ψ_o on δ_N, δ_M, δ_L, based on the maximum estimates for the degree variances of [Kaula, 1959]

Ψ_o	Summation up to 8th order				Summation up to 5th order			
	Direct		Coating		Direct		Coating	
	$\overline{\Delta\delta}_N$	$\overline{\Delta\delta}_M = \overline{\Delta\delta}_L$	$\overline{\Delta\delta}_N$	$\overline{\Delta\delta}_M = \overline{\Delta\delta}_L$	$\overline{\Delta\delta}_N$	$\overline{\Delta\delta}_M = \overline{\Delta\delta}_L$	$\overline{\Delta\delta}_N$	$\overline{\Delta\delta}_M = \overline{\Delta\delta}_L$
	mgal		mgal		mgal		mgal	
5°	9.4	10.6	2.0	9.3	9.3	10.1	2.0	8.8
8.99° (1000km)	7.8	8.1	1.4	6.0	7.8	8.1	1.4	5.9
10°	7.5	7.7	1.3	5.4	7.5	7.6	1.3	5.3
13.49° (1500km)	6.6	6.7	1.0	4.3	6.5	6.3	0.9	3.4
17.99° (2000km)	5.7	5.9	0.7	4.3	5.6	5.2	0.6	2.6
20°	5.4	5.6	0.6	4.4	5.2	4.9	0.5	2.8
25°	4.7	4.8	0.7	4.5	4.7	4.3	0.6	3.8
30°	4.4	4.1	0.8	4.7	4.3	4.0	0.8	4.6
40°	4.2	3.8	1.1	5.2	4.1	3.7	1.0	4.8
50°	4.2	4.0	1.1	4.6	4.2	3.9	1.0	4.3
60°	4.3	4.3	0.9	4.0	4.2	4.1	0.9	3.7
70°	4.1	4.3	0.8	3.3	4.0	4.0	0.8	3.1
80°	3.8	4.2	0.6	3.1	3.7	3.8	0.6	2.8
90°	3.3	3.7	0.5	2.8	3.3	3.5	0.5	2.6
100°	2.7	3.0	0.5	2.6	2.7	2.9	0.5	2.3
110°	2.3	2.4	0.5	2.2	2.3	2.4	0.5	2.0
120°	2.2	2.3	0.5	2.2	2.2	2.3	0.5	2.0
130°	2.4	2.6	0.5	2.2	2.3	2.5	0.5	2.1
140°	2.4	3.0	0.5	2.2	2.4	2.8	0.5	2.0
150°	2.2	2.8	0.4	1.8	2.2	2.8	0.4	1.8
160°	1.6	2.4	0.2	1.3	1.4	2.0	0.2	1.1
170°	0.6	1.1	0.1	0.5	0.4	0.7	0.1	0.4
180°	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Fig. 7.1 shows a graphical representation of the first part of Table 7.2.

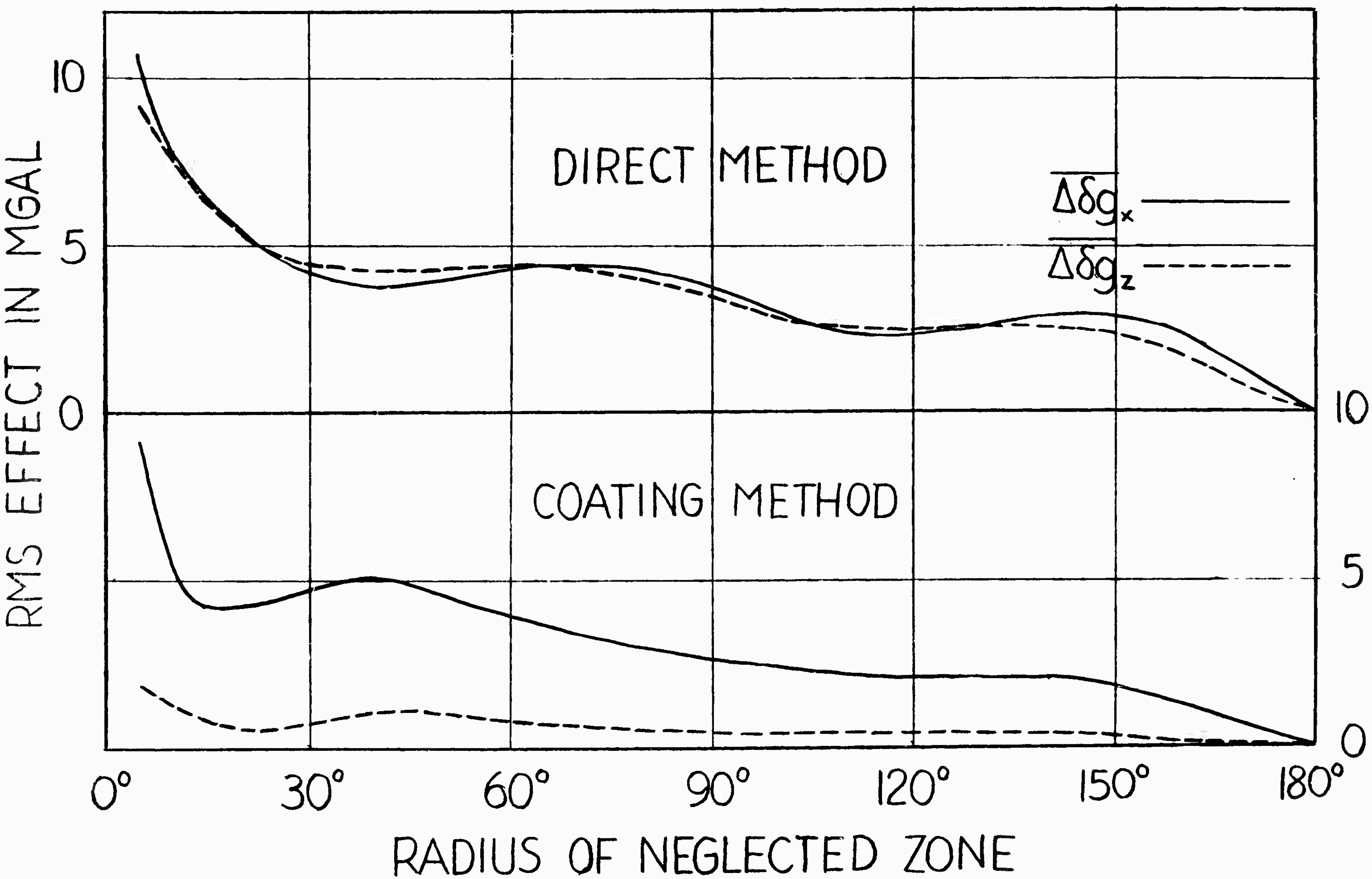

Figure 7.1

Graphical representation of the values of Table 7.2 for summation up to 8th order.

One would expect a rather monotonous decrease of the curves from 0° to 180°. The non-monotonousness is especially apparent in the coating method. It is probably due mainly to neglecting terms of higher than 8th order: it is still stronger if we use only terms up to 5th order as Table 7.2. shows. It seems, however, to be hardly worth-while to go up higher than to 8th order, in view of the increasing complexity of the formulas for Q_n and q_n and the uncertainty of the numerical material for c_n. Since the higher order terms can be expected to influence mainly the dips, it is justifiable to "bridge" them empirically in order to get a smooth monotonous function.

More recent values for the c_n can be obtained from [Kaula, 1961] and from [Uotila, 1962]. Both give a development of the gravity anomaly field in fully normalized spherical harmonics. Writing (7.20) in fully normalized harmonic coefficients we get [Kaula, 1959, eq. (33)]

$$c_n = \overline{\Delta g_n^2} = \sum_{m=0}^{n} (a_{nm}^2 + b_{nm}^2).$$

By this equation, the c_n can be computed.

From [Uotila, 1962, Table 5] we get (flattening 1/297):

$$c_2 = 22, \quad c_3 = 42, \quad c_4 = 32.$$

Only values up to 4th order are given. The almost perfect agreement with the c_n of [Kaula, 1959] is accidental.

From [Kaula, 1961] we get for the flattening 1/298.24:

$$c_2 = 0.7, \quad c_3 = 19, \quad c_4 = 11, \quad c_5 = 6, \quad c_6 = 11, \quad c_7 = c_8 = 5;$$

if we use the flattening 1/297, then σ_2 is changed to $\sigma_2 = 17$. The influence of the distant zones, according to these values, is given in Table 7.3 and in Fig. 7.2. We see that the effect of a wrong flattening is much stronger in the direct than in the coating method. This can be expected since, as Tables 7.1.a and 7.1.b show, q_2 is smaller and changes sign more often than Q_2.

Table 7.3.

RMS influence of the zones beyond a radius Ψ_o on δ_N, δ_M, δ_L, for two different flattenings, based on a spherical harmonics development obtained by a combined adjustment of gravimetric, astro-geodetic, and satellite data [Kaula, 1961] (summation up to 8th order)

	flattening 1/298.24				flattening 1/297			
	Direct		Coating		Direct		Coating	
Ψ_o	$\overline{\Delta\delta_N}$	$\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$	$\overline{\Delta\delta_N}$	$\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$	$\overline{\Delta\delta_N}$	$\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$	$\overline{\Delta\delta_N}$	$\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$
	mgal		mgal		mgal		mgal	
5°	4.3	5.7	0.9	5.0	8.5	8.6	1.9	7.4
8.99° (1000km)	3.3	4.1	0.6	3.0	7.5	7.1	1.4	5.3
10°	3.1	3.8	0.5	2.6	7.2	6.8	1.3	4.8
13.49° (1500km)	2.5	3.1	0.3	1.9	6.5	6.0	0.9	3.6
17.99° (2000km)	2.0	2.5	0.3	2.2	5.7	5.3	0.6	2.8
20°	1.8	2.4	0.3	2.4	5.4	5.0	0.5	2.7
25°	1.4	1.9	0.4	2.7	4.8	4.4	0.4	2.7
30°	1.2	1.5	0.5	2.9	4.5	4.1	0.6	3.1
40°	1.1	1.3	0.6	3.0	4.3	3.8	0.9	3.8
50°	1.2	1.4	0.5	2.4	4.4	4.0	1.0	3.9
60°	1.2	1.5	0.3	1.6	4.4	4.0	0.9	3.5
70°	1.4	1.9	0.3	1.5	4.2	3.9	0.8	2.9
80°	1.6	2.1	0.3	1.8	3.6	3.6	0.5	2.3
90°	1.6	1.9	0.3	1.7	3.0	3.0	0.4	1.8
100°	1.3	1.7	0.3	1.6	2.4	2.4	0.3	1.7
110°	1.1	1.3	0.2	1.2	2.1	2.0	0.4	1.8
120°	1.0	1.2	0.2	0.9	2.0	2.0	0.5	1.9
130°	1.2	1.4	0.2	1.0	2.1	2.1	0.5	2.0
140°	1.3	1.7	0.2	1.1	2.0	2.2	0.5	1.8
150°	1.2	1.7	0.2	1.0	1.7	2.0	0.3	1.4
160°	0.9	1.4	0.1	0.7	1.1	1.5	0.2	0.9
170°	0.3	0.6	0.0	0.3	0.4	0.6	0.1	0.3
180°	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

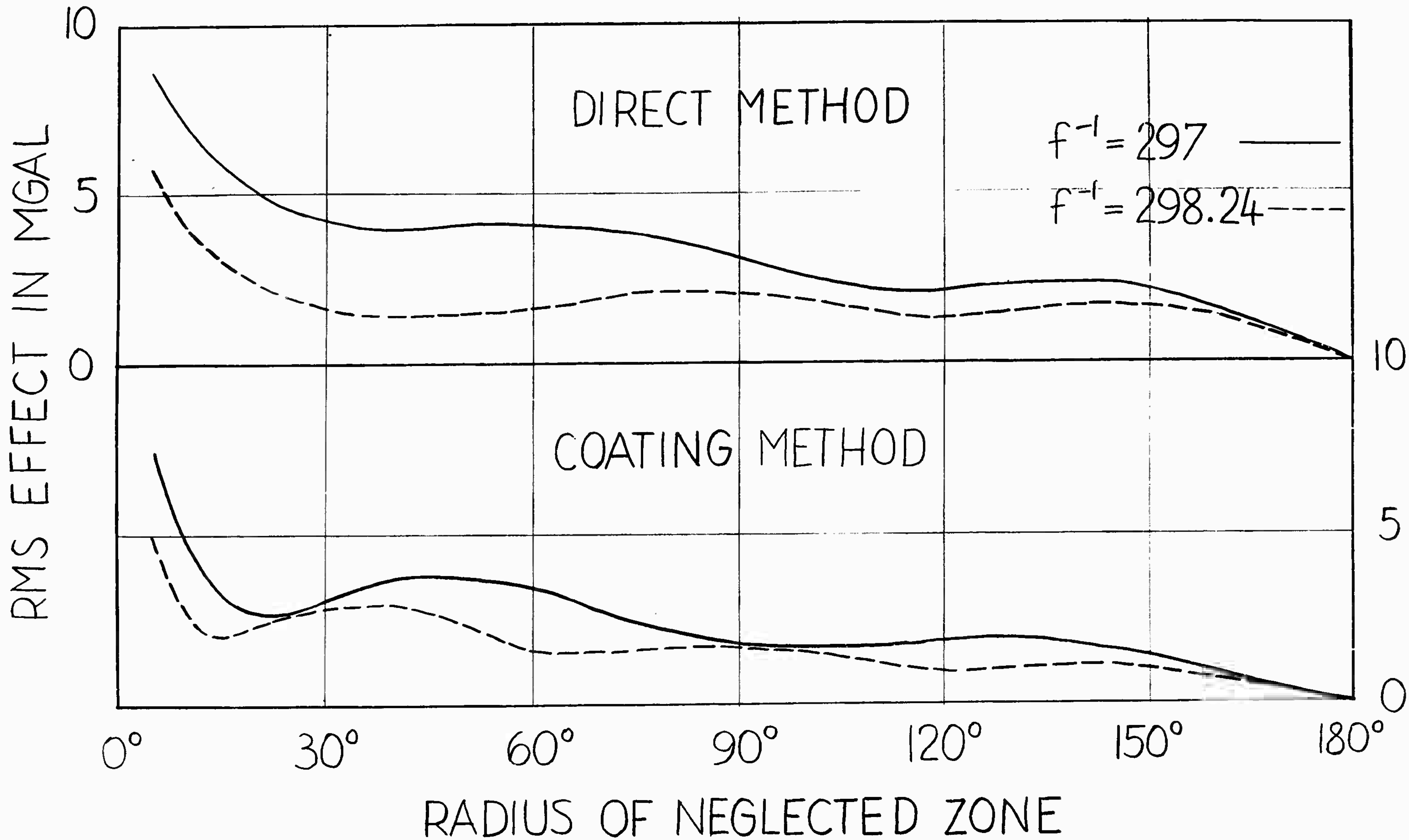

Figure 7.2

Graphical representation of $\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$ of Table 7.3.

Finally we consider the spherical harmonics expansion of Zhongolovich quoted and used in [Molodenskii et al., 1962, pp. 145-6]. Here,

$$c_2 = 61, \quad c_3 = 96, \quad c_4 = 12, \quad c_5 = 8, \quad c_6 = 14, \quad c_7 = 5, \quad c_8 = 8 \text{ mgal}$$

(computed by $c_n = \overline{\Delta g_n^2}$). With these coefficients we find values considerably higher than those given above (Table 7.4). They are, however, probably too high since the development of Zhongolovich is already out of date (it was computed in 1952).

Table 7.4

RMS influence of the zones beyond a radius Ψ_o on δ_N, δ_M, δ_L, based on the spherical harmonics development of Zhongolovich, 1952 (summation up to 8th order).

Ψ_o	Direct $\overline{\Delta\delta_N}$ mgal	Direct $\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$ mgal	Coating $\overline{\Delta\delta_N}$ mgal	Coating $\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$ mgal
5°	16.2	16.0	3.5	13.9
8.99° (1000km)	14.2	13.6	2.6	10.1
10°	13.8	13.0	2.4	9.2
13.49° (1500km)	12.3	11.5	1.8	6.6
17.99° (2000km)	10.7	9.9	1.0	4.2
20°	10.2	9.3	0.8	3.7
25°	9.1	8.2	0.7	4.0
30°	8.4	7.5	1.1	5.3
40°	8.1	7.2	1.7	7.4
50°	8.2	7.3	1.9	7.5
60°	8.2	7.2	1.8	6.4
70°	7.6	6.9	1.4	5.0
80°	6.8	6.5	1.0	4.3
90°	5.8	5.7	0.8	3.7
100°	4.8	4.8	0.7	3.3
110°	4.1	4.0	0.8	3.4
120°	4.0	3.9	1.0	3.4
130°	4.1	4.1	1.0	3.7
140°	4.0	4.2	0.9	3.5
150°	3.3	3.6	0.7	2.7
160°	2.0	2.4	0.4	1.5
170°	0.6	0.8	0.1	0.5
180°	0.0	0.0	0.0	0.0

Conclusion. On the whole, the values of Fig. 1a,b, based on [Kaula, 1959] and confirmed by [Uotila, 1962], might be the most realistic estimates, apart from the dips. We should like to propose the smoothed values of Table 7.5.

Table 7.5

RMS influence of the zones beyond a radius Ψ_o on δ_N, δ_M, δ_L; proposed values.

	Direct		Coating	
Ψ_o	$\overline{\Delta\delta_N}$	$\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$	$\overline{\Delta\delta_N}$	$\overline{\Delta\delta_M} = \overline{\Delta\delta_L}$
	mgal		mgal	
8.99° (1000km)	8	8	2	6
13.49° (1500km)	7	7	2	5
17.99° (2000km)	6	6	2	5
20°	6	6	2	5
25°	5	5	2	5
30°	5	5	1	5
45°	5	5	1	5
60°	4	5	1	4
90°	3	4	1	3
120°	2	3	1	2
150°	2	3	0	2

These values are referring to a flattening in the neighborhood of 1/297. If we use a better flattening, then these values should be somewhat less.

We see that for $\Psi_o > 20°$ or $30°$ the influence of the distant zones decreases very slowly. It is, therefore, impractical to extend the integration much farther than 20° (coating method) or 30° (direct method), unless it is extended over the whole earth.

The effect of the distant zones on δ_M and δ_L is somewhat less in the coating than in the direct method. The effect on δ_N is much less. By using another method which will be described in the next section, however, the influence of the distant zones on δ_N can be made still smaller.

8. PRACTICAL COMPUTATION OF THE GRAVITY DISTURBANCES

We shall describe now a method for the practical computation of gravity disturbances. Since the geoid undulations N have been computed for large parts of the world--the accuracy is as good as the existing gravity material--we can assume them to be known. Therefore, the coating method can be applied which has certain advantages over the direct method: simplicity, less influence of the distant zones, etc.

Upward Continuation Method. For the vertical component δ_n there is, however, a method which is still better than the coating method.

Compute the vertical component at ground (or at the geoid) by

(8.1) $$\delta_n^{\circ} = \Delta g + \frac{2\gamma}{R} N .$$

Since $r\delta_n = -r \frac{\partial T}{\partial r}$ is a harmonic function, Poisson's integral for harmonic functions can be applied to give

$$r\delta_n = \frac{r^2 - R^2}{4\pi R} \iint_{\sigma} \frac{R\delta_n^{\circ}}{D_s^3} \cdot R^2 d\sigma$$

where

$$r = R + H ,$$

$$D_s = \sqrt{R^2 + r^2 - 2Rr \cos \Psi} .$$

Setting as before

$$t = \frac{R}{r} = \frac{R}{R + H} ,$$

$$D = \sqrt{1 - 2t \cos \Psi + t^2}$$

we have

$$D_s = rD .$$

Hence

(8.2) $$\delta_n = \frac{t^2(1-t^2)}{4\pi} \iint_{\sigma} \frac{\delta_n^{\circ}}{D^3} d\sigma$$

where δ_n° is given by (8.1).

This formula (8.2) is called "upward continuation integral"; it can also be used for the upward continuation of the gravity anomalies Δg , since $r\Delta g$ is also a harmonic function. Here, however, it is used for the upward continuation of the gravity disturbances.

If we compare (8.2) to the corresponding formula for the coating method,

$$(8.3) \qquad \delta_n = \frac{1}{2\pi} \iint_\sigma \mu \cdot \frac{t^2}{D^3} (1 - t \cos \Psi)\, d\sigma$$

where

$$\mu = \Delta g + \frac{3\gamma}{2R} N ,$$

then we find that the integrand in (8.3) decreases, with increasing distance D, like $1/D$, whereas in the upward continuation integral (8.2) it decreases much faster, like $1/D^3$ .. It can, therefore, be expected that the influence of the distant zones in the upward continuation method is much smaller than in the coating method.

Indeed, if we put H = 0, then $\delta_n = \delta_n^\circ$, i.e., the gravity disturbance is equal to the point value in P itself and it does not at all depend on the other values and on the distant zones. Thus, for H = 0 the influence of the distant zones is rigorously zero in the upward continuation method whereas, as we have seen in section 7, this is not at all the case in the coating method (neither in the direct method, as a matter of fact). From this we can conclude that, even for higher elevations, only the nearest surroundings of P will be of some influence.

Therefore, we can replace the terrestrial sphere by its tangential plane at P. If in this plane we assume a rectangular coordinate system x,y (origin P; the x-axis pointing northward, the y-axis pointing eastward), we have instead of (8.2) the plane formula

$$(8.4) \qquad \delta_n = \frac{H}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta_n^\circ}{(H^2+x^2+y^2)^{3/2}} \, dx \, dy \; .$$

In order to estimate the influence of the distant zones beyond a certain distance s_o from P we introduce polar coordinates s, α by

$$x = s \cos \alpha,$$

$$y = s \sin \alpha,$$

getting

$$\delta_n = \frac{H}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=0}^{\infty} \frac{\delta_n^\circ}{(H^2+s^2)^{3/2}} \, s \, ds \, d\alpha \; .$$

The influence $\Delta\delta_n$ of the zones beyond the distance s_o is given by

$$\Delta\delta_n = \frac{H}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=s_o}^{\infty} \frac{\delta_n^\circ}{(H^2+s^2)^{3/2}} \, s \, ds \, d\alpha \; .$$

Since for large s, $\sqrt{H^2 + s^2} \doteq s$, we can simplify this to

$$\Delta\delta_n = H \int_{s_o}^{\infty} \frac{\overline{\delta_{n,s}^\circ}}{s^2} \, ds$$

where $\overline{\delta_{n,s}^\circ}$ is the average of δ_n° over the circle with radius s. If M is the upper limit of the absolute amount of $\overline{\delta_{n,s}^\circ}$,

$$\overline{\delta_{n,s}^\circ} < M \; ,$$

we get

$$\Delta\delta_n < HM \int_{s_o}^{\infty} \frac{ds}{s^2} = \frac{H}{s_o} M .$$

To get some kind of average effect $\overline{\Delta\delta_n}$ rather than this maximum estimate, we may replace M by the average disturbance $\overline{M}$ outside the radius s_o which is certainly close to 0:

$$(8.5) \qquad \overline{\Delta\delta_n} = \frac{H}{s_o} \overline{M} .$$

From this we see immediately that, to get the same error for different elevations, s_o must be proportional to H. If, e.g., $s_o = 10$ H, then

$$\overline{\Delta\delta_n} = 0.1 \, \overline{M} .$$

Since $\overline{M}$ will hardly exceed 10 mgal, $\overline{\Delta\delta_n}$, in this case, will be smaller than 1 mgal. So, it should, in general, be sufficient to go as far as 10 times the elevation. For larger H, conditions are even more favorable since, the larger s_o is, the smaller can $\overline{M}$ be expected to be.

Plane formulas analogous to (8.4) also hold for the horizontal components δ_m and δ_ℓ:

$$(8.6) \qquad \begin{Bmatrix} \delta_m \\ \delta_\ell \end{Bmatrix} = \frac{H}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(H^2+x^2+y^2)^{3/2}} \begin{Bmatrix} \delta_m^{\,o} \\ \delta_\ell^{\,o} \end{Bmatrix} dx\, dy .$$

But, since $\delta_m^{\,o} = \gamma\xi$, $\delta_\ell^{\,o} = \gamma\eta$, the use of these formulas would require the knowledge of the components ξ, η of the deflection of the vertical in the entire neighborhood of P. If we know ξ and η, then these formulas can indeed be applied with advantage. (To compute ξ and η by differentiating the N-field will not, in general, be accurate enough.) In the following, however, we shall regard only Δg and N as known, so that δ_m and δ_ℓ must be computed by the coating method.

Formulas and Approximations. For easy reference we put the relevant formulas together:

$$(8.7.a)\quad \delta_n = \frac{t^2(1-t^2)}{4\pi} \iint_\sigma \frac{\delta_n^\circ}{D^3}\, d\sigma ,$$

$$(8.7.b)\quad \begin{Bmatrix} \delta_m \\ \delta_\ell \end{Bmatrix} = \frac{t^3}{2\pi} \iint_\sigma \frac{\mu}{D^3} \sin\Psi \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} d\sigma$$

where

$$t = \frac{R}{R+H} , \quad D = \sqrt{1 - 2t\cos\Psi + t^2} , \quad d\sigma = \sin\Psi\, d\Psi\, d\alpha;$$

$$\delta_n^\circ = \Delta g + \frac{2\gamma}{R} N ,$$

$$\mu = \Delta g + \frac{3\gamma}{2R} N .$$

At least for the nearest neighborhood of P, the spherical formulas can be approximated by plane ones:

$$(8.8.a)\quad \delta_n = \frac{H}{2\pi} \iint \frac{\delta_n^\circ}{D_o^3}\, dx\, dy ,$$

$$(8.8.b)\quad \begin{Bmatrix} \delta_m \\ \delta_\ell \end{Bmatrix} = \frac{1}{2\pi} \iint \frac{\mu}{D_o^3} \begin{Bmatrix} x \\ y \end{Bmatrix} dx\, dy$$

where

$$(8.9) \qquad D_o = \sqrt{H^2 + x^2 + y^2} \, .$$

They follow from (8.7.a,b) by a series development with respect to $\Psi = \frac{s}{R}$ and to $\frac{H}{R}$ if we put $s \cos \alpha = x$, $s \sin \alpha = y$.

In order to get an estimate on the validity of these approximations, we develop $1/D^3$ or, which is equivalent,

$$\frac{1}{D_s^3} = \frac{1}{(R+H)^3 D^3} = \frac{1}{(R^2+r^2-2Rr \cos \Psi)^{3/2}}$$

in this manner, finding

$$(8.10) \qquad \frac{1}{D_s^3} = \frac{1}{D_o^{\,3}} \left(1 - \frac{3}{2} \, \frac{H}{R} \, \frac{s^2}{D_o^{\,2}} + \frac{1}{8} \, \frac{s^4}{R^2 D_o^{\,2}} + \ldots \right).$$

Here $1/D_s^3$ is spherical and $1/D_o^{\,3}$ is the equivalent plane quantity (8.9); and $s^2 = x^2 + y^2$.

The first small quantity in the bracket is

$$\frac{3}{2} \, \frac{H}{R} \, \frac{s^2}{D_o^{\,2}} \approx \frac{3H}{2R} \, ,$$

independent of the distances. If H = 63.7 km, this is 0.015 or 1.5% , if H = 637 km it is 15% . Since the higher up we go the smaller the gravity disturbances are, this can be neglected anyway. The second term is

$$\frac{s^4}{8R^2 D_o^{\,2}} \approx \frac{s^2}{8R^2} \, .$$

For s = 1000 km this is 0.003 or 0.3% ; for s = 2000 km it is 0.01 or 1% . This can be neglected, too; the more so as $1/D_o^{\,3}$ itself becomes smaller with increasing distance.

We see that up to 20° distance from P we can, in general, use the plane formulas (8.8.a,b) instead of the spherical ones (8.7.a.b). In particular this holds practically always for the upward continuation integral (8.8.a), except for very high elevations (> 250 km, say). It is also useful in the case that in the coating formulas (8.7.b) we extend the integration only up to 20° (as we have seen, it is impractical to go much farther unless the integration is extended over the <u>whole</u> earth, which of course is necessary for highest accuracy).

Now we have to show how the x,y in (8.8.a,b) are to be computed. The simplest way is

$$x = R(\varphi - \varphi_P) ,\quad y = R \cos \varphi_P (\lambda - \lambda_P) ; \tag{8.11}$$

another way,

$$x = R(\varphi - \varphi_P) ,\quad y = R \cos \varphi (\lambda - \lambda_P) . \tag{8.12}$$

The x's are the same in both; the y's differ in that in (8.11) we have the factor $\cos \varphi_P$ and in (8.12), the factor $\cos \varphi$. In our latitudes there is, for large $|\varphi - \varphi_P|$, a big difference between both formulas. Take $\varphi_P = 40^\circ$, $|\varphi - \varphi_P| = 20^\circ$, i.e., $\varphi = 20^\circ$ or 60°. Then

$$\cos 20^\circ = 0.940 ,$$

$$\cos 40^\circ = 0.766 ,$$

$$\cos 60^\circ = 0.500 .$$

The differences between $\cos \varphi$ and $\cos \varphi_p$ are 22% and 35%!

It is easily seen that (8.12) is preferable for larger distances; only if we use this formula rather than (8.11) is the reasoning following (8.10) applicable. The disadvantage of (8.12) is that the $5' \times 5'$, $1^\circ \times 1^\circ$, etc., blocks are not rectangles in the system x,y but trapezoids. In (8.11) these blocks are represented by rectangles.

Evaluation of the Integrals. In practice, the integrals must be evaluated by sums. The formulas (8.7.a,b) and 8.8.a,b) are of the type

$$(8.13.a) \quad \delta = \iint \mu \cdot f(\varphi,\lambda)\, dq ,$$

$$(8.13.b) \quad \delta = \iint \mu \cdot f(x,y)\, dq.$$

In δ_n we have δ_n° instead of μ; Ψ,α,D in (8.7.a,b) are, of course, functions of the geographical coordinates φ, λ; dq is the element of area, defined by

$$dq = R^2 d\sigma = R^2 \cos\varphi \; d\varphi\, d\lambda ,$$

$$dq = dx\, dy,$$

respectively.

The preceding integrals can be approximated by sums:

$$(8.14) \quad \delta = \Sigma\, \mu_k\, c_k$$

where

$$(8.15.a) \quad c_k = f(\varphi_k, \lambda_k) \cdot q_k,$$

or

$$(8.15.b) \quad c_k = f(x_k, y_k) \cdot q_k,$$

where q_k is the area of a certain block B_k ($5' \times 5'$ or $1° \times 1°$, say), and φ_k, λ_k or x_k, y_k are the coordinates of the center of the block B_k.

Especially for low elevations and for the nearest neighborhood of P it is better to compute c_k in the integrated form.

We limit ourselves to rectangular coordinates x, y. Then (Fig.8.1) we can compute c_k by

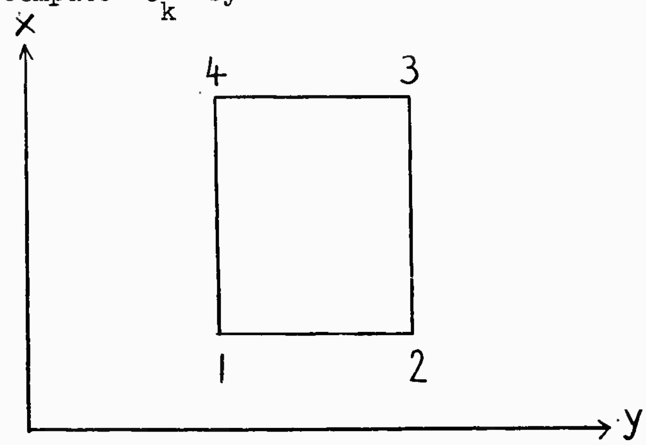

Figure 8.1

(8.16) $c_k = F(1) - F(2) + F(3) - F(4)$

where

(8.17) $F(x,y) = \iint f(x,y)\, dx\, dy$

is the indefinite double integral of f(x,y) so that

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

and, e.g., F(1) is the value of F(x,y) in point 1.

As a matter of fact, this is equivalent to computing c_k by

$$c_k = \iint_{B_k} f(x,y)\, dx\, dy \tag{8.18.a}$$

rather than by

$$c_k = f(x_k,y_k) \cdot q_k \ . \tag{8.18.b}$$

If f(x,y) varies little in the block B_k considered, then (8.18.a) and (8.18.b) are practically equal. If not, then (8.18.a), i.e., (8.16), is to be preferred: if μ is constant throughout the block, then (8.18.a) yields the correct result whereas (8.18.b) does not. (8.18.b) is in this case, therefore, subject to a systematical error.

Let us now return to formulas (8.8.a,b). For (8.8.a),

$$f_n(x,y) = \frac{H}{2\pi D_o^{\,3}} = \frac{H}{2\pi(x^2+y^2+H^2)^{3/2}} \ ;$$

for (8.8.b),

$$f_m(x,y) = \frac{x}{2\pi D_o^{\,3}} = \frac{x}{2\pi(x^2+y^2+H^2)^{3/2}} \ ,$$

$$f_\ell(x,y) = \frac{y}{2\pi D_o^{\,3}} = \frac{y}{2\pi(x^2+y^2+H^2)^{3/2}} \ .$$

Integration by (8.17) yields

$$F_n(x,y) = \frac{1}{2\pi} \tan^{(-1)} \frac{xy}{HD_o} \ , \tag{8.19.a}$$

$$(8.19.b) \quad F_m(x,y) = -\frac{1}{2\pi} \ln (y + D_o) ,$$

$$(8.19.c) \quad F_\ell(x,y) = -\frac{1}{2\pi} \ln (x + D_o) .$$

Then, the coefficients c_k for δ_n, δ_m, δ_ℓ can be computed by (8.16).

In order to be able to apply (8.16), the figure 1234 must be an exact rectangle. Therefore, x,y must in this case be computed by (8.11) instead of (8.12). Therefore, we are limited to the nearest surroundings of P where (8.11) and (8.12) are practically equal, say to a rectangle $3^\circ \times 4^\circ$ in the center of which P is situated (Region A, p. 25). Outside this rectangle, c_k can very well be computed by the simpler formula (8.18.b).

For the computation of the contribution of the innermost zones it is necessary to use as small blocks as possible. The smallest size is usually $5' \times 5'$. Even so, it might be necessary to take the deviation of the gravity anomalies from the mean anomaly of the respective block into account.

For this purpose we consider the gravity anomaly to be a linear function throughout the block. Then,

$$(8.20) \quad \Delta g = a_o + a_1 x + a_2 y .$$

Inserting this in (8.8.a) and (8.8.b) and integrating over one block B_k only we get

$$\Delta\delta_n = H(a_o J_o + a_1 J_1 + a_2 J_2) ,$$

$$\Delta\delta_m = a_o J_1 + a_1 J_{11} + a_2 J_{12} ,$$

$$\Delta\delta_\ell = a_o J_2 + a_1 J_{12} + a_2 J_{22} ,$$

where $\Delta\delta_n$, $\Delta\delta_m$, $\Delta\delta_\ell$ are the contributions of the block B_k to δ_n, δ_m, δ_ℓ and

(8.21) $$J = F(1) - F(2) + F(3) - F(4)$$

where

(8.22)

$$F_o(x,y) = \frac{1}{2\pi} \iint \frac{dx\,dy}{D_o^3} = \frac{1}{2\pi H} \tan^{(-1)} \frac{xy}{HD_o} = \frac{1}{H} F_n(x,y) ,$$

$$F_1(x,y) = \frac{1}{2\pi} \iint \frac{x\,dx\,dy}{D_o^3} = -\frac{1}{2\pi} \ln(y+D_o) = F_m(x,y) ,$$

$$F_2(x,y) = \frac{1}{2\pi} \iint \frac{y\,dx\,dy}{D_o^3} = -\frac{1}{2\pi} \ln(x+D_o) = F_\ell(x,y) ,$$

$$F_{11}(x,y) = \frac{1}{2\pi} \iint \frac{x^2\,dx\,dy}{D_o^3} = \frac{y}{2\pi} \ln(x+D_o) - \frac{H}{2\pi} \tan^{(-1)} \frac{xy}{HD_o} ,$$

$$F_{12}(x,y) = \frac{1}{2\pi} \iint \frac{xy\,dx\,dy}{D_o^3} = -\frac{D_o}{2\pi} ,$$

$$F_{22}(x,y) = \frac{1}{2\pi} \iint \frac{y^2\,dx\,dy}{D_o^3} = \frac{x}{2\pi} \ln(y+D_o) - \frac{H}{2\pi} \tan^{(-1)} \frac{xy}{HD_o} .$$

Consider now the contribution of the innermost block. Let the sides of the block be 2a, 2b, so that the points 1, 2, 3, 4 have the coordinates:

1(-a, -b) 2(-a, +b)

3(+a, +b) 4(+a, -b).

Then,

$$J_1 = J_2 = J_{12} = 0$$

and

(8.23.a) $\Delta\delta_n = a_o H J_o$,

(8.23.b) $\Delta\delta_m = a_1 J_{11}$,

(8.23.c) $\Delta\delta_\ell = a_2 J_{22}$,

where

$$J_o = \frac{2}{\pi H} \tan^{(-1)} \frac{ab}{HD_{o,1}} ,$$

(8.24)

$$J_{11} = \frac{b}{\pi} \ell n \frac{D_{o,1} + a}{D_{o,1} - a} - \frac{2H}{\pi} \tan^{(-1)} \frac{ab}{HD_{o,1}} ;$$

$$J_{22} = \frac{a}{\pi} \ell n \frac{D_{o,1} + b}{D_{o,1} - b} - \frac{2H}{\pi} \tan^{(-1)} \frac{ab}{HD_{o,1}} ;$$

$$D_{o,1} = \sqrt{a^2 + b^2 + H^2} .$$

We see that for the central block $\Delta\delta_n$ is the same as if we would use formula (8.16). For $\Delta\delta_m$ and $\Delta\delta_n$ (8.16) yields zero whereas the correct values are given by (8.23.b,c).

In order to estimate numerically the contribution of the innermost compartment, $\Delta\delta_m$ and $\Delta\delta_\ell$, we take it as a square with sides s , so that

$$a = b = s ; \quad D_{o,1} = \sqrt{2s^2 + H^2} .$$

Since, then, $J_{11} = J_{22}$ it is sufficient to consider $\Delta\delta_m$ only.

We have

$$(8.25) \qquad J_{11} = \frac{s}{\pi} \ln \frac{D_{o,1} + s}{D_{o,1} - s} - \frac{2H}{\pi} \tan^{(-1)} \frac{s^2}{HD_{o,1}} \quad ;$$

we take

$$s = 4 \text{ km}$$

which corresponds to the average size of $5' \times 5'$ blocks. Table 8.1 gives the values of J_{11} for several elevations H.

Table 8.1

Factor J_{11} for $s = 4$ km as a function of elevation H

H km	J_{11} km
0	2.24
1	1.41
2	0.87
5	0.23
10	0.04
20	0.01
50	0.00
100	0.00

If we have a north-south gradient $a_1 = 1$ mgal per 1 km, then J_{11} is numerically equal to the effect $\Delta\delta_m$ in mgal; e.g., for $H = 0$ we have

$$\Delta\delta_m = 2.2 \text{ mgal} .$$

Since also gradients of 2 or even 3 mgal/km are possible, the effect of the gradient on the horizontal components $\Delta\delta_m$ and $\Delta\delta_\ell$ can be considerable for lower elevations, $H < 5$ km. For $H > 5$ km, the effect of the gradient is negligible.

We have also to investigate the influence of the eight blocks adjacent to the central one. Since the effect is a maximum for $H = 0$, we consider this case only. We again take the block to be a square, but with arbitrary sides. We find that the effect of the innermost block is

(8.26) $\Delta\delta_m = 0.561\ a_1 s$

and the combined effect of the eight surrounding blocks is only

(8.27) $\Delta'\delta_m = -\ 0.071\ a_1 s$.

If the gradient $a_1 = 1$ mgal/km and $s = 4$ km ($5' \times 5'$ blocks), then

$$\Delta'\delta_m = -0.3 \text{ mgal} .$$

These values hold for $H = 0$, but even here the effect of the gradient in the surrounding blocks will be negligible in most cases, even if it must be taken into account for the central block (for $H < 5$ km).

These considerations are equally valid for the component δ_ℓ ; we have only to replace a_1 by the east-west gradient a_2 . For the vertical component δ_n , the influence of the gradients a_1 and a_2 will always be negligible.

An easy way of computing the gradients a_1 and a_2 which can also be used for high-speed computations is (Fig. 8.2):

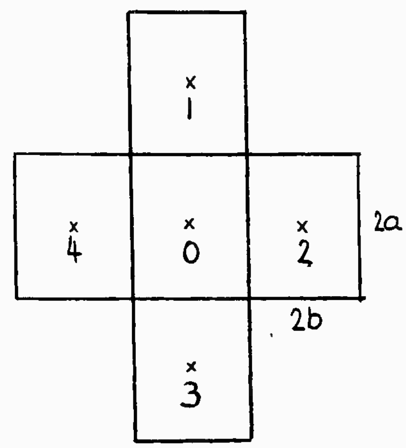

Figure 8.2
Station and Substations

(8.28) $$a_1 = \frac{\Delta g_1 - \Delta g_3}{4a} , \quad a_2 = \frac{\Delta g_2 - \Delta g_4}{4b} .$$

Δg_1 through Δg_4 are the gravity anomalies in the centers of the adjacent blocks 1 to 4, which we call substations. These point anomalies can also be replaced by the mean anomalies of the respective blocks, provided the gravity anomalies are sufficiently linear (this is presupposed in (8.28)).

Station and Substations. For reasons of symmetry the station P at which δ_n, δ_m, δ_ℓ are computed must be at the center of the central block. If we, however, use given $5' \times 5'$ mean anomalies for which the division into blocks is fixed a priori, it is best to select 5 substations (centers of adjacent blocks) 0, 1, 2, 3, 4 (Fig. 8.2) and compute the gravity disturbance in those three substations which are the corners of a triangle containing P in its interior (0, 1, 4 in Fig. 8.2). The value for P is then interpolated between these three substations. This procedure is particularly necessary for elevations smaller than about 70 km.

Summary of Formulas and Outline of Computation. In the following we shall give a computational method valid for any elevation from 0 to several hundred kilometers. It is, therefore, suitable for the computation of the gravity vector along rocket trajectories.

According to p.25 we use the following regions

A	$\varphi_2 - \varphi_1 = 3^\circ$, $\lambda_2 - \lambda_1 = 4^\circ$	$5' \times 5'$ blocks,
B	$\varphi_2 - \varphi_1 = 7^\circ$, $\lambda_2 - \lambda_1 = 9^\circ$	outside A, $20' \times 20'$ blocks,
C	$\varphi_2 - \varphi_1 = 25^\circ$, $\lambda_2 - \lambda_1 = 30^\circ$	outside B, $1^\circ \times 1^\circ$ blocks,
D	outside C, $5^\circ \times 5^\circ$ blocks.	

1. For each block compute

$$\mu = \Delta g + 0.231\ N\ , \qquad \delta_n^{\circ} = \Delta g + 0.308\ N\ ;$$

Δg: mean free-air anomalies, N: geoid undulations; Δg, μ, δ_n° in mgal, N in meters.

2. For each station P select 3 substations, according to Fig. 8.2. One substation, O, is the center of the block in which P lies, the other two are centers of adjacent blocks so that the three substations form the corners of a triangle containing P . The elevation of each substation is equal to the elevation H of P .

The gravity disturbances are computed at each substation and are interpolated for P .

In the following, only the substations will be considered; they will be denoted by Q .

3. Now we consider one definite substation Q only.

Zone A.

Compute for all grid points (= corners of blocks):

$$x = R(\varphi - \varphi_Q)\ ,$$

$$y = R\cos\varphi_Q\,(\lambda - \lambda_Q)\ ;$$

$$D_o = \sqrt{x^2 + y^2 + H^2}\ ;$$

$$F_n = \frac{1}{2\pi} \tan^{(-1)} \frac{xy}{HD_o} ,$$

$$F_m = - \frac{1}{2\pi} \ln (y + D_o) ,$$

$$F_\ell = - \frac{1}{2\pi} \ln (x + D_o) .$$

Compute for each block

$$c_n = F_n(1) - F_n(2) + F_n(3) - F_n(4) ,$$

$$c_m = F_m(1) - F_m(2) + F_m(3) - F_m(4) ,$$

$$c_\ell = F_\ell(1) - F_\ell(2) + F_\ell(3) - F_\ell(4) .$$

Be sure that the grid points 1, 2, 3, 4 are arranged as in Fig. 8.1.

For elevations $H > 40$ km the coefficients c can also be computed as in zone B.

Zone B

Compute for all block centers

$$x = R(\varphi - \varphi_Q) ,$$

$$y = R \cos \varphi(\lambda - \lambda_Q) ;$$

$$D_o = \sqrt{x^2 + y^2 + H^2} ;$$

$$c_n = \frac{H}{2\pi D_o^3} \cdot q ,$$

$$c_m = \frac{x}{2\pi D_o^3} \cdot q ,$$

$$c_\ell = \frac{y}{2\pi D_o^3} \cdot q ;$$

q is the area of the block.

Zone C can be treated either like Zone B or like Zone D.

Zone D

If many points are to be computed then the following computations need not be performed for all points. In this case it is sufficient to compute the effect of Zone D for a few points only and interpolate between these.

Compute :

$$\Psi = \cos^{(-1)}[\sin \varphi_Q \sin \varphi + \cos \varphi_Q \cos \varphi \cos (\lambda - \lambda_Q)] ,$$

$$\alpha = \mathrm{ctn}^{(-1)}\left[\frac{\cos\varphi_Q \sin \varphi - \sin \varphi_Q \cos \varphi \cos (\lambda - \lambda_Q)}{\cos \varphi \sin (\lambda - \lambda_Q)}\right] ;$$

$$t = \frac{R}{R + H} ,$$

$$D = \sqrt{1 - 2t \cos \Psi + t^2} ;$$

$$c_n = \begin{cases} 0 , & H \leq 200 \text{ km} \\ \frac{t^2(1-t^2)}{4\pi D^3} \cdot q , & H > 200 \text{ km} \end{cases} ,$$

$$c_m = \frac{t^3}{2\pi D^3} \sin \Psi \cos \alpha \cdot q ,$$

$$c_\ell = \frac{t^3}{2\pi D^3} \sin \Psi \sin \alpha \cdot q .$$

4. For the substation Q considered compute by summation over all blocks i :

$$\delta_n = \sum_i c_{n,i}\, \delta^{o}_{n,i},$$

$$\delta_m = \sum_i c_{m,i}\, \mu_i,$$

$$\delta_\ell = \sum_i c_{\ell,i}\, \mu_i .$$

For elevations < 5 km we have to add, to δ_m and to δ_ℓ, the terms $\Delta\delta_m$ and $\Delta\delta_\ell$, respectively:

$$\Delta\delta_m = a_1 \cdot \left[\frac{b}{\pi} \ell n \frac{D_1+a}{D_1-a} - \frac{2H}{\pi} \tan^{(-1)} \frac{ab}{HD_1} \right],$$

$$\Delta\delta_\ell = a_2 \cdot \left[\frac{a}{\pi} \ell n \frac{D_1+b}{D_1-b} - \frac{2H}{\pi} \tan^{(-1)} \frac{ab}{HD_1} \right],$$

where 2a,2b are the sides of the central 5' × 5' block and

$$D_1 = \sqrt{a^2 + b^2 + H^2};$$

$$a_1 = \frac{\Delta g_1 - \Delta g_3}{4a}, \quad a_2 = \frac{\Delta g_2 - \Delta g_4}{4b} .$$

These formulas for a_1 and a_2 are valid for the central substation 0 (Fig. 8.2); for the other substations they are to be modified in an evident way.

5. Now denoting by x,y,z the coordinates in the geocentric system defined on p. 2 (z-axis = axis of rotation of the earth) compute the <u>components of the gravity disturbance in this system</u> by

$$\delta_x = - \cos\varphi \cos\lambda \cdot \delta_n - \sin\varphi \cos\lambda \cdot \delta_m - \sin\lambda \cdot \delta_\ell ,$$

$$\delta_y = - \cos\varphi \sin\lambda \cdot \delta_n - \sin\varphi \sin\lambda \cdot \delta_m + \cos\lambda \cdot \delta_\ell ,$$

$$\delta_z = - \sin\varphi \cdot \delta_n + \cos\varphi \cdot \delta_m .$$

These components δ_x, δ_y, δ_z are positive away from the earth's center; φ, λ refer to the substation Q .

6. This procedure must be repeated for all substations and the values of δ_x, δ_y, δ_z for the station P are interpolated.

For elevations $H > 70$ km, it is in general sufficient to use the central substation 0 only and to put the values at P equal to the values at 0 .

7. Computation of the components of the gravity vector:

$$g_x = \gamma_x - \delta_x ,$$

$$g_y = \gamma_y - \delta_y ,$$

$$g_z = \gamma_z - \delta_z .$$

g_x, g_y, g_z; γ_x, γ_y, γ_z are positive toward the earth's center.

References

Hirvonen, R. A., Gravity anomalies and deflections of the vertical above sea level, Trans. Am. Geophys. Union, 33 (6), 801-809, 1952

___________, The new theory of the gravimetric geodesy II, Report 7, Inst. Geod. Phot. Cart., Ohio State Univ., 1959

___________, New theory of the gravimetric geodesy, Ann. Acad. Sci. Fenn., A, III. Geolog.-Geograph., 56, 1960

Kaula, W. M., Accuracy of gravimetrically computed deflections of the vertical, Trans. Am. Geophys. Union, 38 (3), 297-305, 1957

___________, Statistical and harmonic analysis of gravity. J. Geophys. Research, 64 (12), 2401-2421, 1959

___________, A geoid and world geodetic system based on a combination of gravimetric, astro-geodetic, and satellite data, J. Geophys. Research, 66(6), 1799-1812, 1961

Molodenskii, M. S., Eremeev, V. F., and Yurkina, M. I., Methods for study of the external gravitational field and figure of the earth, translated from the Russian by The Israel Program for Scientific Translations, Jerusalem, 1962

Moritz, H., Studies on the accuracy of the computation of gravity at high elevations. Report 21, Inst. Geod. Phot. Cart., Ohio State Univ., 1962a

______________, Interpolation and prediction of gravity and their accuracy, Report 24, Inst. Geod. Phot. Cart., Ohio State Univ., 1962b

______________, On the theory of errors of gravimetric geodesy, Report 26, Inst. Geod. Phot. Cart., Ohio State Univ., 1963

Orlin, H., The three components of the external anomalous gravity field. J. Geophys. Research, 64(12), 2393-2399, 1959

Uotila, U. A., Investigations on the gravity field and shape of the earth, Report 6, Inst. Geod. Phot. Cart., Ohio State Univ., 1959

______________, Corrections to gravity formula from direct observations and anomalies expressed in lower degree spherical harmonics, Report 23, Inst. Geod. Phot. Cart., Ohio State Univ., 1962

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Methods for computing the gravity vector outside the earth are presented and compared. For the purpose of computation, gravity is split up into normal gravity and gravity disturbance. The report first gives a practical method for computing normal gravity. For gravity disturbances, different methods are described and compared by means of accuracy studies (on influence of distant zones, etc.); finally a practical computation procedure is given.

1. Gravity
2. Geodetic Computations
3. Error Analysis

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Task 760002
II. Contract AF 19 (628)2771
III. Ohio State Univ., Columbus, Ohio
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