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DYNAMICS OF A ROCKET WITH AN AXIS OF SYMMETRY

by

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ABSTRACT. A fundamental analysis of the dynamics of a short-range rocket-propelled vehicle with an axis of symmetry is presented. The treatment is rigorous and comprehensive, stressing the interpretation and application of physical principles to this problem.

The number of initial assumptions has been kept small. The final equations are quite general, and are carried through a series of approximations and restrictions.

An index is included.

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AN ACTIVITY OF THE BUREAU OF NAVAL WEAPONS

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FOREWORD

A significant proportion of any modern ballistics effort consists of the mathematical simulation of the weapon and its behavior on a digital computer. If this phase of the work is to be successful, it is necessary that the equations of motion be both well understood and documented in a way that points out the origin of all significant terms.

These equations have been programmed for and run on the IBM 7090 in connection with the ASROC (Antisubmarine Rocket) development program carried out by this department.

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INTRODUCTION

This report presents a fundamental analysis of the ballistics of a short-range rocket-propelled vehicle with an axis of symmetry.

The U. S. Naval Ordnance Test Station carried out a significant part of the ballistics work undertaken in connection with the ASROC (Antisubmarine Rocket) development program. It became necessary to know what physical assumptions are implicit in the equations used for trajectory simulation, and how these assumptions affect the form of the finished equations.

This report represents an effort to meet these requirments; a rather complete list of assumptions relating to non-aerodynamic phenomena is presented near the beginning, and the equations are developed in a way that points out where and how these assumptions are invoked.

Such an objective entails a rigorous and somewhat comprehensive derivation of the motion equations from first principles. The report therefore proceeds along the lines of a textbook. Special care was taken in the preparation of the sections on Thrust and Dynamic Principles. The beginner may find these sections helpful to his understanding of rocket ballistics in terms of the postulates of classical mechanics.

Most of the non-aerodynamic phenomena involved in the ballistics of this kind of missile are explored in detail. The aerodynamic forces and moments, however, are expressed as components of certain vector integrals; they are not evaluated as explicit functions of the state variables.

DYNAMIC PRINCIPLES

Two fundamental principles of classical mechanics may be stated as follows:

1. The total external force on a mass system equals the rate of change of its linear momentum, with respect to an inertial system.

2. The total external moment applied to a system about its center of mass is equal to the rate of change of angular momentum, relative to an inertial frame. The angular momentum is taken with respect to the center of mass (c.m.).

In this section, these principles are stated in detail, and their application to systems that lose mass is discussed briefly.

By a physical force we mean a force generated by some physical agent, such as an electromagnetic, gravitational, or impulsive force. Coriolis "forces," for example, do not fall into this category. An inertial frame is defined as a rectangular Cartesian coordinate system which has the property that every particle referenced to it moves with constant velocity whenever the total physical force on the particle is zero. Newton's second law asserts that the acceleration $\vec{a}$ relative to any inertial frame of a particle of mass m, acted upon by a physical force $\vec{F}$, is given by $\vec{a} = (1/m)\,\vec{F}$.

This postulate is readily extended to systems of particles. Let S be a system of n particles having masses $m_1, \ldots, m_n$. Let $\vec{F}_{ij}$ denote the physical force exerted on particle j by particle i. Then the total physical force on particle j is

$$\vec{F}_j + \sum_{i=1}^{n} \vec{F}_{ij},$$

where $\vec{F}_j$ is the force exerted on the j^{th} particle by agents external to the system. If $\vec{r}_i$ is the vector from the center of mass of S to the i^{th} particle, then

$$\sum_{i=1}^{n} \vec{F}_i \quad \text{and} \quad \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i$$

are, respectively, the total external force on S and the total external moment applied to S about its center of mass. It can be easily demonstrated that by the principle of action-reaction, both the total internal force

$$\sum_{i,\,j=1}^{n} \vec{F}_{ij}$$

and the total internal moment

$$\sum_{i,\ j=1}^{n} \vec{r}_j \times \vec{F}_{ij}$$

are zero.

Principles 1 and 2 follow from these considerations (see, for example, Goldstein; Classical Mechanics, Chapter 1). Let $\vec{v}_i$ be the velocity of m_i relative to an inertial frame, and let m_i, $\vec{F}_i$, $\vec{r}_i$ be as defined above.

Then

(1) $$\sum_{i=1}^{n} \vec{F}_i = \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{v}_i$$

and

(2) $$\sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i = \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{r}_i \times \vec{v}_i$$

where the derivatives are taken relative to an inertial frame.

Suppose we wish to establish the equations of motion for a rocket at a given instant t_o. Let S be the system of particles comprising the fixed parts of the rocket plus the gases in the combustion chamber at t_o. Then at instants later than t_o, S contains exhausted gases as well as the fixed parts and the contents of the combustion chamber. For $t \geq t_o$, let

$$\vec{P}(t) = \sum_{i=1}^{n} m_i \vec{v}_i ,$$

where the sum on the right is evaluated at the instant t, and where the vector $\vec{P}_{(t)}$ is expressed at all instants of time relative to a single inertial frame. Principle 1 applies over any interval $t_o \leq t \leq t_o + \Delta t$, and in particular, we have

$$\vec{F}_{(t_o)} = \lim_{\Delta t \to o} \frac{1}{\Delta t} \left[\vec{P}(t_o + \Delta t) - \vec{P}(t_o) \right],$$

where $\vec{F}_{(t_o)}$ is the total external force on S at t_o. The above result is an equation holding at the single instant t_o. Since t_o is arbitrary, this equation must hold over the entire period of thrust.

These calculations can be carried through to completion by regarding the rocket system as a finite collection of particles; of course, this is precisely what the missile is. However, since the interpretation of various quantities which appear is somewhat awkward, a parallel approach will be taken.

In this report, the system is regarded as a continuously distributed mass with density $\mu(\vec{r}, t)$ and velocity distribution $\vec{V}(\vec{r}, t)$. The finite sums can be replaced with vector volume integrals by writing $\mu(\vec{r}, t)\, d\tau$ for m_i, $\vec{V}(\vec{r}, t)$ for $\vec{v}_i$, and so on.

For example, the expressions for the linear and angular momenta carry over to

$$\sum_{i=1}^{n} m_i \vec{v}_i \longleftrightarrow \int_V \mu(\vec{r}, t)\, \vec{V}(\vec{r}, t)\, d\tau$$

and

$$\sum_{i=1}^{n} m_i \vec{r}_i \times \vec{v}_i \longleftrightarrow \int_V \mu(\vec{r}, t)\, \vec{r} \times \vec{V}(\vec{r}, t)\, d\tau$$

where V is the volume occupied by the system.

DEFINITIONS AND NOTATION CONVENTIONS

Let S be the system consisting of all fixed rocket parts plus burned and unburned propellant forward of the exit plane at a given instant t_o. The exit plane is a plane, perpendicular to the missile axis of symmetry, in contact with the after surface of the endplate. At a later instant $t \geq t_o$, S is the sum of three subsystems:

S_1 fixed rocket parts and unburned propellant

S_g those particles of S not in S_1 that are forward of the exit plane (S_g consists of the exiting gases in the combustion chamber)

S_e those particles of S that have crossed the exit plane during $t - t_o$ (S_e consists of the gases exhausted during Δt)

In this report, time derivatives of vectors are taken relative to four right-handed rectangular Cartesian coordinate systems:

$x_I\ y_I\ z_I$ a coordinate system (assumed to be inertial) having its origin at the center of the earth, and irrotational with respect to the stars (assumed to be fixed)

$x_o\ y_o\ z_o$ a Cartesian system fixed to the surface of the earth; the origin is taken somewhere in the vicinity of the launching area with the y_o axis vertical, y_o increasing with altitude

$y_1\ y_2\ y_3$ a system with its origin at the center of mass of S_1, y_1 taken along the axis of symmetry (y_1 taken positive toward the nose of the missile) and irrotational with respect to S_1

$x_1\ x_2\ x_3$ a system obtained from the $y_1\ y_2\ y_3$ frame by rotating that coordinate system about its own y_1 axis in such a way that after the rotation, y_3 is in the horizontal plane and y_2 is (at launch) positive away from the earth's surface. The new y_1, y_2, y_3 axes are called x_1, x_2, x_3, respectively

If $\vec{A}$ is a time-varying vector in space, than we shall write

$$\left.\frac{d\vec{A}}{dt}\right|_I, \quad \frac{d\vec{A}}{dt}, \quad \dot{\vec{A}}, \text{ and } \frac{\delta\vec{A}}{\delta t}$$

for the time derivative of $\vec{A}$ as seen by an observer fixed to the $x_I\ y_I\ z_I$, $x_o\ y_o\ z_o$, $y_1\ y_2\ y_3$, and $x_1\ x_2\ x_3$ frames, respectively. More precisely,

if $\vec{I}\,\vec{J}\,\vec{K}$, $\vec{i}\,\vec{j}\,\vec{k}$, $\vec{u}_1\,\vec{u}_2\,\vec{u}_3$, $\vec{v}_1\,\vec{v}_2\,\vec{v}_3$ are unit vectors along the $x_I\ y_I\ z_I$, $x_o\ y_o\ z_o$, $y_1\ y_2\ y_3$, and $x_1\ x_2\ x_3$ axes, respectively, then $\vec{A}$ may be resolved into components relative to any of these four frames.
The components are given by

$$\vec{A} = A_1\vec{I} + A_2\vec{J} + A_3\vec{K}$$

$$\vec{A} = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$$

$$\vec{A} = C_1\vec{u}_1 + C_2\vec{u}_2 + C_3\vec{u}_3$$

$$\vec{A} = D_1\vec{v}_1 + D_2\vec{v}_2 + D_3\vec{v}_3 \quad .$$

The definition of the various derivatives may be restated as

$$\left.\frac{d\vec{A}}{dt}\right|_I = \dot{A}_1\vec{I} + \dot{A}_2\vec{J} + \dot{A}_3\vec{K}$$

$$\frac{d\vec{A}}{dt} = \dot{B}_1\vec{i} + \dot{B}_2\vec{j} + \dot{B}_3\vec{k}$$

$$\dot{\vec{A}} = \dot{C}_1\vec{u}_1 + \dot{C}_2\vec{u}_2 + \dot{C}_3\vec{u}_3$$

$$\frac{\delta\vec{A}}{\delta t} = \dot{D}_1\vec{v}_1 + \dot{D}_2\vec{v}_2 + \dot{D}_3\vec{v}_3 \quad .$$

where $\dot{A}$ for a scalar A denotes the time derivative of A in the usual sense.

The symbols employed for the unit vectors above were chosen solely for this definition, and are not used in this sense elsewhere in this report.

A discussion of the rate of change of vectors is included in Appendix B.

Let μ be the mass density of S, as a function of position and time. This functional dependence is sometimes indicated explicitly by writing $\mu(x_1, x_2, x_3, t)$ or, more simply, $\mu(\vec{x}, t)$, where (x_1, x_2, x_3) are the coordinates of the terminus of the vector $\vec{x}$ from the origin of one of the coordinate system origins to the point in question.

Define the center of mass $\vec{x}_o$ of an arbitrary collection of particles as

$$\vec{x}_o = \frac{\int \vec{x}\mu(\vec{x}, t)d\tau}{\int \mu(\vec{x}, t)d\tau}$$

where both integrals are extended over the volume containing the particles.

In the following, let $\vec{r}$ be the vector from the c.m. of S_1 to a mass element $\mu d\tau$; i.e., μ is the mass density at the terminus of the vector $\vec{r}$, and $d\tau$ is a differential of volume.

We make the following definitions:

$$M = \int_S \mu d\tau, \quad M_1 = \int_{S_1} \mu d\tau, \quad M_g = \int_{S_g} \mu d\tau, \quad M_e = \int_{S_e} \mu d\tau$$

$$\vec{R}_g = \frac{1}{M_g} \int_{S_g} \mu \vec{r} d\tau, \quad \vec{R}_e = \frac{1}{M_e} \int_{S_e} \mu \vec{r} d\tau,$$

$$\vec{R}_p = \frac{1}{\int p_e dA} \int_{A_e} \vec{r} p_e dA, \quad \vec{V}_g = \frac{1}{M_g} \int_{S_g} \mu \dot{\vec{r}} d\tau,$$

$$\vec{V}_e = \frac{1}{M_e} \int_{S_e} \mu \dot{\vec{r}} d\tau, \quad \vec{V}_{cg} = \frac{-1}{M_1} \int_{S_1} \mu \dot{\vec{r}} d\tau \ .$$

$\vec{R}_g$ and $\vec{R}_e$ may be interpreted as the coordinates of the c.m. of S_g and S_e, respectively, with respect to the c.m. of S_1. $\vec{V}_g$ and $\vec{V}_e$ are the "effective velocities" of the masses in S_g and S_e, relative to the c.m. of S_1. M_1, M_g, and M_e are the respective masses of the fixed parts, the burned gases in transit, and the gases already exhausted during the interval in question. Clearly, M is the sum $M_1 + M_g + M_e$, and is therefore the mass of the entire system. Of course, M is constant in any interval $t_o \le t \le t_o + \Delta t$. $\vec{V}_{cg}$ is the velocity of the c.m. of S_1 relative to the fixed parts. $\vec{R}_p$ is, in the limit, the effective center of pressure on the exit plane.

More precisely, the center of pressure (c.p.) at t_o is given by

$$\lim_{\Delta t \to o} \vec{R}_p(t_o + \Delta t) \ .$$

It is appropriate to observe now a subtle yet important distinction between these quantities as they have been defined here, in order to establish the equations, and as they appear in the finished equations.

Let us consider $\vec{R}_e$ as an example. If $t > 0$ is a positive increment of time, then at $t_o + t$ the exhausted gases S_e extend, in a rather diffuse and irregular column, for a considerable distance aft of the missile. The terminus of $R_e(t_o + t)$, located at the center of mass (c.m.) of this volume of gases, may not be near the exit plane of the missile. However, if we consider the limit of $\vec{R}_e(t_o + t)$ as $t \to 0$, it is clear that the resulting vector terminates at the c.m. of the gases at the exit plane at t_o. This point is sometimes referred to as the "effective center of thrust."

We can define the center of thrust at other instants of time, as we have at t_o, by redefining the system of particles and going through the same limiting process. For example, to get the center of thrust at $t_o + t$, let S' be the system consisting of the fixed parts and the unburned gases forward of the exit plane at $t_o + t$. Then at later instants $t_o + t + \Delta t$, define S_1, S_g, and S_e as they were defined at t_o. Take the vector from the c.m. of S_1 to the c.m. of S_e at $t_o + t + \Delta t$ and let $\Delta t \to 0$; the limiting vector is defined as the center of thrust.

The center of thrust at $t_o + t$ defined in this way is of course not the same as $\vec{R}_e(t_o + t)$ mentioned above. It is a vector whose terminus remains in the vicinity of the exit plane throughout burning.

The same interpretation applies to both $\vec{R}_p$ and $\vec{V}_e$. For example, the "effective exit velocity" at t_o is given by the $\lim_{t \to o} \vec{V}_e(t_o + t)$ for the system as defined at t_o. $\lim \vec{R}_p$ is the "effective center of pressure." The same symbols $\vec{R}_e$, $\vec{V}_e$, and $\vec{R}_p$ are used to denote the corresponding limiting functions. The distinction should be clear in context.

Let us now consider the quantity M_e, representing the exhausted mass. In some discussions of rocket ballistics, this mass is regarded as a "small" differential quantity, and is denoted by ΔM or dM. Although some care must be exercised in order to obtain the complete equations using differentials, there is of course nothing incorrect in this point of view. On the other hand, it is not necessary to treat M_e as a differential; in fact, it is much more straightforward mathematically and intuitively to handle M_e in a manner more parallel to the treatment given the other quantities. In this vein, we list the properties of M_e.

M_e is a function of time, defined at all instants later than t_o. It is equal to zero at t_o, and increases monotonically from this value at later instants of time. It has a derivative $\dot{M}_e$ (the rate at which mass crosses the exit plane) and if there is no accumulation of mass in the combustion chamber S_g, this derivative is just equal to $-\dot{M}_l$ (the rate at which mass leaves the burning surface). Hereafter, we shall write merely $\dot{M}$ for either of the quantities $\dot{M}_e$ or $-\dot{M}_l$. $\dot{M}$ is therefore positive.

In the following, it becomes necessary to calculate derivatives of the form $d(M_e\vec{A})/dt$, where $\vec{A}$ is a vector function of time. This form may be evaluated simply and directly, using the above properties of M_e, with no appeal at all to the concept of a differential.

We have

$$\frac{d}{dt}(M_e\vec{A})\bigg|_{t_o} = M_e(t_o)\frac{d\vec{A}}{dt}\bigg|_{t_o} + \dot{M}(t_o)\,\vec{A}(t_o) \ .$$

The first term on the right is zero since $M_e(t_o) = 0$.

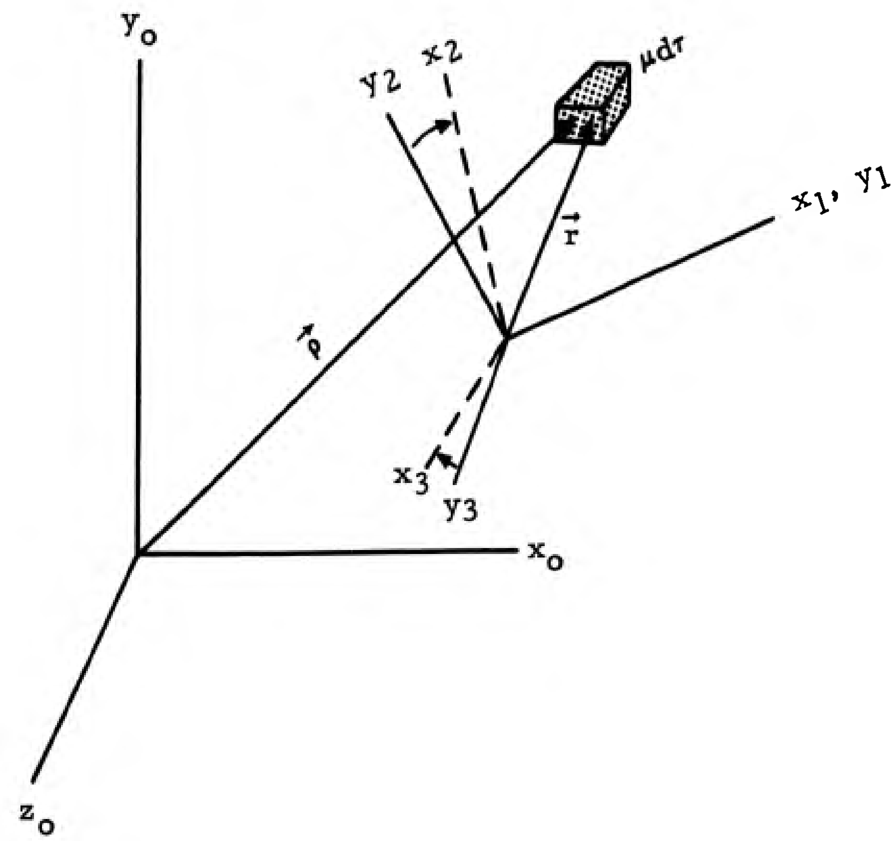
Hence,

$$\frac{d}{dt}(M_e\vec{A})\bigg|_{t_o} = \dot{M}(t_o)\,\vec{A}(t_o) \ .$$

One further point should be mentioned concerning the symbols denoting mass velocity. It is seen in the definition of $\vec{V}_g$, $\vec{V}_e$, and $\vec{V}_{cg}$ that $\dot{\vec{r}}$ is used to denote the velocity of the mass element $\mu d\tau$, located at the terminus of the vector $\vec{r}$, relative to the c.m. of S_l. The customary symbol denoting such a vector field is $\vec{v}(\vec{r}, t)$. In such a form, the definition of $\vec{V}_e$, say, would appear as

$$\vec{V}_e = \frac{1}{M_e}\int_{S_e} \mu(\vec{r}, t)\,\vec{v}(\vec{r}, t)\,d\tau \ .$$

However, in this report, it is necessary to refer mass velocities to various coordinate systems in motion relative to one another. Of course, each coordinate system requires its own distinct velocity function, and the convention introduced above becomes extremely cumbersome. The following symbols appear to be more descriptive: $\delta\vec{r}/\delta t$, $\dot{\vec{r}}$, $d\vec{r}/dt$, $\delta\vec{\rho}/\delta t$, $\dot{\vec{\rho}}$, $d\vec{\rho}/dt$. Here the vector argument ($\vec{r}$ or $\vec{\rho}$) denotes the coordinate system in which the particle is referenced, and the differential operator designates the coordinate system in which the rate of change is observed.



For example, the rate of change of the vector $\vec{\rho}$, as seen by an observer fixed to, say, the $y_1\ y_2\ y_3$ frame, is given by $\delta\vec{\rho}/\delta t$. The velocities of $\mu d\tau$ relative to observers fixed at the origins of the $x_o\ y_o\ z_o$, $x_1\ x_2\ x_3$, and $y_1\ y_2\ y_3$ are $d\vec{\rho}/dt$, $\delta\vec{r}/\delta t$, and $\dot{\vec{r}}$, respectively. A more complete discussion is included in Appendix B.

Most of the remaining symbols and terms used in this report are listed below.

A	Exterior surface of the volume occupied by the system S at the instant t. At $t = t_o$, this is the exterior surface of the missile, including the surface at the exit plane.
A_e	That part of A in exit plane. This symbol is also used to denote the numerical area of this plane surface.
$A - A_e$	That part of the surface of S not in the exit plane; i.e., the exterior surface of the missile, excluding the region in the vicinity of the exhaust nozzles.
$d\tau$	Volume differential.
$\vec{e}_x\ \vec{e}_y\ \vec{e}_z$	Unit vectors along the $x_o\ y_o\ z_o$ axes, respectively.
$\vec{e}_1\ \vec{e}_2\ \vec{e}_3$	Unit vectors along the $x_1\ x_2\ x_3$ axes, respectively.
$\vec{f}$	Total force per unit area acting on the surface of the missile during a flight. $\vec{f}$ consists of atmospheric pressure, exhaust pressure, skin friction (drag), etc.
$\vec{f}_a$	The "pure" aerodynamic force per unit area acting on the surface of the missile during a flight. More precisely, $\vec{f}_a = \vec{f} - \vec{f}_s$.
$\vec{f}_o$	Force per unit area exerted on the surface of the missile as a function of position and time for a statically fired missile at sea level. This amounts to the exhaust pressure at the exit plane, acting along the missile axis, and to atmospheric pressure acting along the surface normal at other points of the surface.
$\vec{f}_s$	Force per unit area that would be exerted on the missile if it were held <u>fixed</u> at the instantaneous position and orientation it occupies at t, during a flight. This amounts to exhaust pressure acting normally to the exit plane, and to atmospheric pressure (at the missile's altitude) acting normally to the surface at other points. $\vec{f}_s$ is sometimes called the "static contribution" to the total force per unit area acting on the surface of the missile.
$\vec{p}$	Vector from the c.m. of S to the mass element $\mu\, d\tau$.

p_a Atmospheric pressure at the missile.

$\vec{p}_a$ A vector acting along the inward normal to the surface A, with its magnitude given by $|\vec{p}_a| = p_a$.

p_e Exhaust pressure at the exit plane during thrust. This pressure is assumed to be a function of position on the exit plane as well as time, and it is assumed to be independent of the state of motion of the missile and the surrounding environment.

$\vec{p}_e$ Force per unit area on the exit plane during thrust. Hence, $\vec{p}_e = p_e\vec{e}_1$.

P_e Defined by the integral $P_e = \frac{1}{A_e} \int_{A_e} p_e dA$ and is, therefore, a function of time only. P_e is the effective exhaust pressure at the exit plane.

p_o Atmospheric pressure at sea level.

$\vec{p}_o$ A vector acting along the inward normal to the surface A, with its magnitude given by $|\vec{p}_o| = p_o$.

Q Matrix of the transformation from $x_o\ y_o\ z_o$ to $x_1\ x_2\ x_3$ coordinates.

Q^t Transpose of Q.

$\vec{q}$ Vector from the c.m. of S to the c.m. of S_1.

$\vec{r}$ Vector from the c.m. of S_1 to $\mu\, d\tau$.

$\vec{R}$ Vector from the origin of $x_o\ y_o\ z_o$ to the c.m. of S_1.

$\vec{T}$ $-\dot{M}\vec{V}_e + \int_A \vec{f}_o dA$ or $-\dot{M}\vec{V}_e + \int_{A_e} (\vec{p}_e - \vec{p}_o) dA$. $\vec{T}$ is the static thrust at sea level. (See the section on Thrust for a discussion of this vector.)

α, β — Angles defined by $\cos\alpha = \frac{1}{R_e}\left|\vec{R}_e \cdot \vec{e}_1\right|$, $\cos\beta = \frac{1}{T}\left|\vec{T} \cdot \vec{e}_1\right|$. They are, respectively, the angle between $\vec{R}_e$ and the x_1 axis, and the angle between the thrust vector and the x_1 axis.

δ_R, δ_T — Thrust offset distance and the normal component of thrust. In terms of α and β, $\delta_R = R_e \sin\alpha$, $\delta_T = T \sin\beta$.

μ — Mass density of the system S, as a function of position and time.

ρ — Radial distance from the axis of symmetry to the center line of any one of the exhaust ports. (This parameter arises solely in connection with the ASROC missile. See section on Rotational Equations.)

ϕ_L, ψ_L — Defined by the vector equation $\vec{\Omega} = \Omega(\cos\phi_L \cos\psi_L \vec{e}_x + \sin\phi_L \vec{e}_y + \cos\phi_L \sin\psi_L \vec{e}_z)$ where ϕ_L is the latitude of the origin of the $x_o\ y_o\ z_o$ system, and ψ_L is its azimuth, taken positive in the sense of a rotation from north to east.

ϕ_R, ϕ_T — Angles that relate to thrust misalignment; a complete definition can be found in the section on Thrust Misalignment.

$\vec{\omega}$ — Angular velocity vector of the missile (i.e., S_1) with respect to the $x_o\ y_o\ z_o$ system.

$\vec{\omega}_c$ — Angular velocity of the $x_1\ x_2\ x_3$ coordinate system relative to the "ground" system $x_o\ y_o\ z_o$.

$\omega_1\ \omega_2\ \omega_3$ — Components of $\vec{\omega}$ along the $x_1\ x_2\ x_3$ axes. Or, $\vec{\omega} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3$.

$\omega_{c1}\ \omega_{c2}\ \omega_{c3}$ — Components of $\vec{\omega}_c$ along the $x_1\ x_2\ x_3$ axes; i.e., $\vec{\omega}_c = \omega_{c1} \vec{e}_1 + \omega_{c2} \vec{e}_2 + \omega_{c3} \vec{e}_3$.

Ω — Angular rate $|\vec{\Omega}|$. Or $\Omega = 2\pi/24$ hours $\doteq 0.0000727\ \text{sec}^{-1}$.

$\vec{\Omega}$ Angular velocity vector of the earth about its own axis of revolution, relative to the inertial system $x_I \; y_I \; z_I$.

ASSUMPTIONS ON THE SYSTEM "S"

1. $\dot{M}_g \doteq 0$ This assumption states that the mass of the burned gases in the combustion chamber remains reasonably constant throughout burning. For a propellant that is burning erratically, one might expect considerable fluctuations in M_g. However, in light of the order of magnitude of M_g relative to M, say, this effect is negligible.

2. $\dot{\vec{V}}_g \doteq 0$ This assumption is that the effective velocity of the gas in the propellant chamber is constant. This assumption is justified by an argument similar to that in 1.

3. $\dot{\vec{V}}_{cg} \doteq 0$ The c.m. of the missile moves with a uniform velocity with respect to the fixed parts during burning.

4. S_1 has an axis of symmetry. This assumption is that axial asymmetries in the missile caused by fins, canards, the shape of the propellant charge, etc., are insignificant in their effect on the moments of inertia and related quantities. It is implicit in this assumption that the c.m. of S_1 lies on this symmetry axis.

5. S_1 is rigid. More precisely, if two particles are in S_1 during an interval, then the distance between them remains constant during that interval. It should be observed that S_1 is not the rigid body of classical mechanics, since it loses particles at the burning surface.

In the next two assumptions, $\vec{R}_e$ and $\vec{R}_p$ are the "center of thrust" and the "center of pressure" functions. (See the section on Definitions and Notation, page 4.)

6. $\vec{R}_p \doteq -R_p \vec{e}_1$ The center of pressure lies very close to the missile axis.

7. $\vec{R}_e$ and $\vec{T}$ are fixed relative to S_1. It follows that $\dot{\phi}_R = \dot{\phi}_T = \dot{\phi}$. This fact is used in the section on thrust misalignment.

In the next three assumptions, let $\vec{e}_1$ be a unit vector along the axis of symmetry of S_1, directed from the exit plane to the c.m. of S_1.

8. $\vec{R}_g \doteq -R_g \vec{e}_1$ $\vec{R}_g$ is parallel to the missile axis. In other words, the c.m. of the gases in the combustion chamber lies very close to the axis of symmetry.

9. $\vec{V}_g \doteq -V_g \vec{e}_1$ The gases in the propellant chamber are moving with an effective velocity nearly parallel to the missile axis.

10. $\vec{V}_{cg} \doteq V_{cg} \vec{e}_1$ The missile c.m. moves parallel to its axis of symmetry, relative to the fixed parts.

In the next two assumptions, let $\vec{p}$ and $\vec{q}$ be the vectors from the c.m. of S_g and S_e, respectively, to the mass element $\mu \, d\tau$.

11. $\int_{S_g} \mu \vec{p} x \dot{\vec{r}} d\tau \doteq 0$* Since $\dot{\vec{R}}_g \doteq 0$, $\dot{\vec{p}} = \dot{\vec{r}} - \dot{\vec{R}}_g = \dot{\vec{r}}$, this assumption is equivalent to $\int_{S_g} \mu \vec{p} x \dot{\vec{p}} d\tau \doteq 0$. The angular momentum of the gases in the propellant chamber about their own c.m. is negligible.

12. $\lim_{M_e \to 0} \frac{1}{M_e} \int_{S_e} \mu \vec{q} x \dot{\vec{r}} d\tau \doteq 0$* Roughly, this assumption states that the exiting gases carry off angular momentum at a negligible rate.

Let I and I_g be the moments of inertia of S_1 (fixed missile parts) and S_g (gases in the combustion chamber), respectively.

*For a more complete discussion of Assumptions 11 and 12, see the section on Rotational Equations, page 34.

13. $I + I_g \doteq I$ In components, this assumption is

$$\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} + \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \doteq \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} .$$

Even though I_g is not diagonal, its terms are so small that they cause an error in $(I + I_g)\,\vec{\omega}$ which is less than that introduced by uncertainty in the accepted values of I_1 and I_2.

The following assumptions are obviously satisfied.

14. $|\vec{V}_{cg}| << |\vec{V}_g|$, $|\vec{V}_{cg}| << |\vec{V}_e|$ The velocity of the missile c. g. relative to the fixed parts is negligible compared to either the mean gas velocity in the combustion chamber or the gas exit velocity. Although chamber velocities are, on the average, lower than exit velocities, this assumption is clearly valid.

15. $|\vec{\omega}| << \dfrac{|\vec{V}_g|}{|\vec{R}_g|}$, $|\vec{\omega}| << \dfrac{|\vec{V}_e|}{|\vec{R}_g|}$ This assumption becomes apparent if one considers the orders of magnitude of $\vec{\omega}$, $\vec{R}_g$, and the gas velocity in the combustion chamber. It follows from this assumption that $|\vec{\omega} \times \vec{R}_g| << |\vec{V}_g|$ and $|\vec{\omega} \times \vec{R}_g| << |\vec{V}_e|$.

16. $|\vec{\Omega}| << |\vec{\omega}|$ The magnitude of $\vec{\Omega}$ is given by $|\vec{\Omega}| = 360°/24$ hours. This is clearly negligible compared to the mean angular rate of the missile.

17. The centripetal force, due to the earth's rotation, that would be felt by the system S if it were held fixed relative to the earth's surface is negligible.

18. $M_g << M$ or $M_1 \doteq M$ The gases in the combustion chamber have a mass that is negligible compared to the mass of the entire system.

19. $\cos\alpha \doteq 1$, $\cos\beta \doteq 1$ The angles α and β are small.

20. $\dot{\vec{R}}_g \doteq - \vec{V}_{cg}$ The c.m. of the mass in the combustion chamber is fixed with respect to the fixed parts.

THRUST

In attempting to define thrust, one encounters a considerable number of ambiguities. In the first place, of course, thrust is not an external force at all; it is the result of an internal exchange of momentum between the fixed rocket parts and the remaining particles of the system. The net effect of this exchange is acceleration of the rocket, so it is tempting to regard this acceleration as being produced by an external force, which we call thrust. This point of view of thrust as an external force on the system is formally a poor one, since in order to derive the correct equations of motion from Newtonian principles, one has to perform some rather devious and unsatisfying mental gymnastics. Needless to say, this approach leaves much room for conceptual errors.

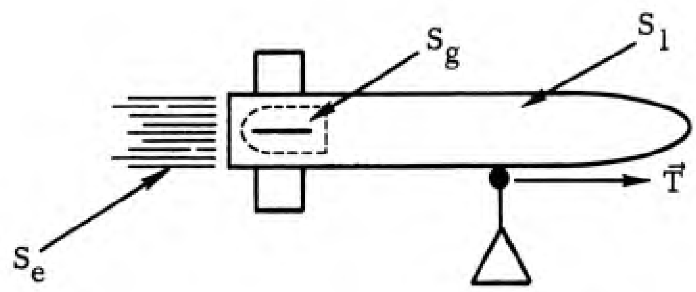
It is less confusing and, in the long run, it seems less artificial to regard the symbol T appearing in the equations merely as a quantity, representing several momentum and pressure terms, which is known as a function of time.

There still remains a larger question. Which terms in the dynamic equations should be absorbed into this "thrust function?" Let us recall the method commonly used to simulate thrust in the ballistic equations for a rocket. A number of static firings are made, presumably at sea level, with rocket engines of the same type as that used in the vehicle. The static thrust (i.e., the force exerted on the test stand) is recorded as a function of time for each motor, and an "average" function is derived from these data. This function, the so-called thrust curve, is the forcing function $T_{(t)}$ to be used in the ballistic equations.

It is relatively easy to write down the momentum and pressure terms that contribute to the static thrust, and to convince oneself that these are the only significant terms entering into this reaction. However, for the dynamic case of a missile in flight during burning, the situation is considerably different. Not only are additional terms involved (relating to the exhausted gases), but it is likely that, because of the complicated motion of the missile, the terms involved in the tabulated thrust curve are altered from their values for a statically fired missile at the same stage of burning. The problem now, restated, is how to apply this function, determined from static firings, to the dynamic case.

In this report, it is assumed that the sum of the terms in the complete equations that produce static thrust is unaffected by the motion of the missile; this sum is replaced by the statically determined function $T_{(t)}$. The additional terms, relating to the exhaust gases, may therefore be regarded as other forces and moments affecting the motion of the system. An example of one of the latter terms that is not entirely insignificant is the so-called "jet damping" torque.

We first derive an expression for the static thrust of the missile. Assume that the missile is mounted on a stationary test stand with the motor exhausting horizontally, and that it is possible to measure the horizontal component of the force exerted by the missile on the test stand. The effect of gravity is superfluous in this connection, and is disregarded in what follows.



The inertial coordinate system $x_o\ y_o\ z_o$ to be used is fixed relative to the missile parts, and hence moves with velocity $-\vec{V}_{cg}$ relative to the c.m. of the fixed parts. x_o is taken parallel to the missile axis. Let $\vec{T}$ be the force exerted by the missile on the test stand. We will calculate $\vec{T}$ at an arbitrary instant t_o.

Given t_o, let S be the system described in the section on Definitions and Notation. We shall write an expression for the momentum $\vec{P}$ of S at any instant t later than t_o. Then we will calculate $\frac{d\vec{P}_{(t_o)}}{dt}$ and apply Newton's law $\frac{d\vec{P}_{(t_o)}}{dt} = \sum$ forces on S at t_o; equivalently

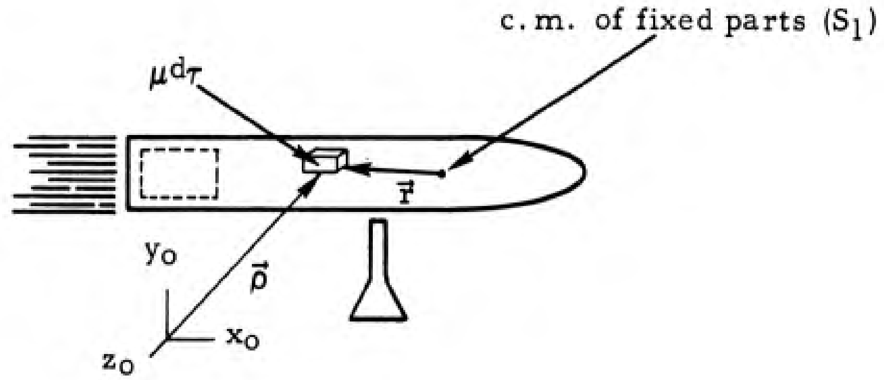
$$-\vec{T} + \int_A \vec{f}_o dA = \frac{d\vec{P}_{(t_o)}}{dt}$$ where A is the surface enclosing the mass

system S at t_o, and $\vec{f}_o$ is the force per unit area acting along the surface normal at all points of A.

It should be kept in mind that $\vec{f}_o$ is purely a pressure term, being equal in magnitude to the exhaust pressure at the exit plane, and to atmospheric pressure at sea level at other points of the surface.

Merely rewriting the equation above, we see that $\vec{T}$ (at t_o) is given by

(3) $$\vec{T} = \int_A \vec{f}_o dA - \frac{d\vec{P}(t_o)}{dt} \ .$$



At any instant t later than t_o, the momentum $\vec{P}$ of S relative to the given inertial coordinate system is

(4) $$\vec{P}_{(t)} = \int_S \mu \frac{d\vec{\rho}}{dt} d\tau = \int_{S_1} \mu \frac{d\vec{\rho}}{dt} d\tau + \int_{S_g} \mu \frac{d\vec{\rho}}{dt} d\tau + \int_{S_e} \mu \frac{d\vec{\rho}}{dt} d\tau$$

where $\vec{\rho}$ is the vector from the origin of the fixed system to the mass element $\mu\, d\tau$.*

Since S_1 is motionless relative to $x_o\ y_o\ z_o$, $\frac{d\vec{\rho}}{dt} = 0$ everywhere on S_1. Hence

$$\int_{S_1} \mu \frac{d\vec{\rho}}{dt} d\tau = 0 \ .$$

*For a discussion of velocities and time derivative notation, see the definitions and Appendix B.

Also, at points of S_g, $d\vec{\rho}/dt$ = velocity of $\mu\, d\tau$ relative to fixed parts = velocity of $\mu\, d\tau$ relative to c.m. of S_1 + velocity of c.m. of S_1 relative to fixed parts. Or, $d\vec{\rho}/dt = \dot{\vec{r}} + \vec{V}_{cg}$. Hence

$$\int_{S_g} \mu \frac{d\vec{\rho}}{dt} d\tau = \int_{S_g} \mu \dot{\vec{r}} d\tau + V_{cg} \int_{S_g} \mu d\tau = M_g(\vec{V}_g + \vec{V}_{cg}) \; .$$

Similarly,

$$\int_{S_e} \mu \frac{d\vec{\rho}}{dt} d\tau = M_e(\vec{V}_e + \vec{V}_{cg}), \text{ so that Eq. 4 becomes}$$

(5) $$\vec{P}_{(t)} = M_g(\vec{V}_g + \vec{V}_{cg}) + M_e(\vec{V}_e + \vec{V}_{cg}) \; .$$

Differentiating Eq. 5,

$$\frac{d\vec{P}_{(t_o)}}{dt} = M_g(\dot{\vec{V}}_g + \dot{\vec{V}}_{cg}) + \dot{M}_g(\vec{V}_g + \vec{V}_{cg}) + \dot{M}(\vec{V}_e + \vec{V}_{cg}) \; .$$

By Assumptions 1, 2, and 3, $\dot{M}_g \doteq 0$, $\dot{\vec{V}}_g \doteq 0$, $\dot{\vec{V}}_{cg} \doteq 0$. And by Assumption 14, $\vec{V}_g + \vec{V}_{cg} \doteq \vec{V}_y$ and $\vec{V}_e + \vec{V}_{cg} \doteq \vec{V}_e$. So that we have, very nearly,

(6) $$\frac{d\vec{P}_{(t_o)}}{dt} \doteq \dot{M}\vec{V}_e \; .$$

This means that Eq. 3 has the form

(7) $$\underline{\vec{T} = - \dot{M}\vec{V}_e + \int_A \vec{f}_o dA} \; .$$

This formula is adopted as a definition of thrust.

We can obtain a slightly different (though equivalent) form for Eq. 7 by carrying out the following steps.

Define a new function $\vec{g}_o$, equal to the force per unit area on the surface of the missile with the rocket motor shut down. In this case, disregarding gravity, the test stand exerts no force at all on the missile in maintaining equilibrium. So we must have $\int_A \vec{g}_o dA = 0$.

That is, the x_o, y_o, and z_o components of the vector integral are separately zero. Now assume that the motor is ignited. The pressure function becomes $\vec{f}_o$. The only change that has occurred, ideally, is that the pressure in the vicinity of the exhaust nozzles has jumped from atmospheric pressure to exhaust pressure; and in this region, the pressure acts normally to the exit plane, in the direction of the $+x_o$ axis.

In other words, $\vec{f}_o$ and $\vec{g}_o$ are equal at all points of the surface except at the exit plane. On the exit plane, $\vec{f}_o = \vec{p}_e$, $\vec{g}_o = \vec{p}_o$, where $\vec{p}_e$ and $\vec{p}_o$ are, respectively, exhaust pressure and atmospheric pressure at sea level, acting along the missile axis.

Then since $\int_A \vec{g}_o dA = 0$,

$$\int_A \vec{f}_o dA = \int_A \vec{f}_o dA - \int_A \vec{g}_o dA = \int_A (\vec{f}_o - \vec{g}_o) dA$$

$$= \int_{A_e} (\vec{f}_o - \vec{g}_o) dA = \int_A (\vec{p}_e - \vec{p}_o) dA \, .$$

So we can write Eq. 7 in the equivalent form

$$\vec{T} = -\dot{M}\vec{V}_e + \int_{A_e} (\vec{p}_e - \vec{p}_o) dA \, . \qquad (8)$$

Either Eq. 7 or 8 may be used as the definition of thrust, whichever is more appropriate. The important thing is that Eq. 7 or 8 allows us to replace $\dot{M}V_e$ in the motion equations by an expression involving only an empirically determined function $\vec{T}$ and a "known" quantity.

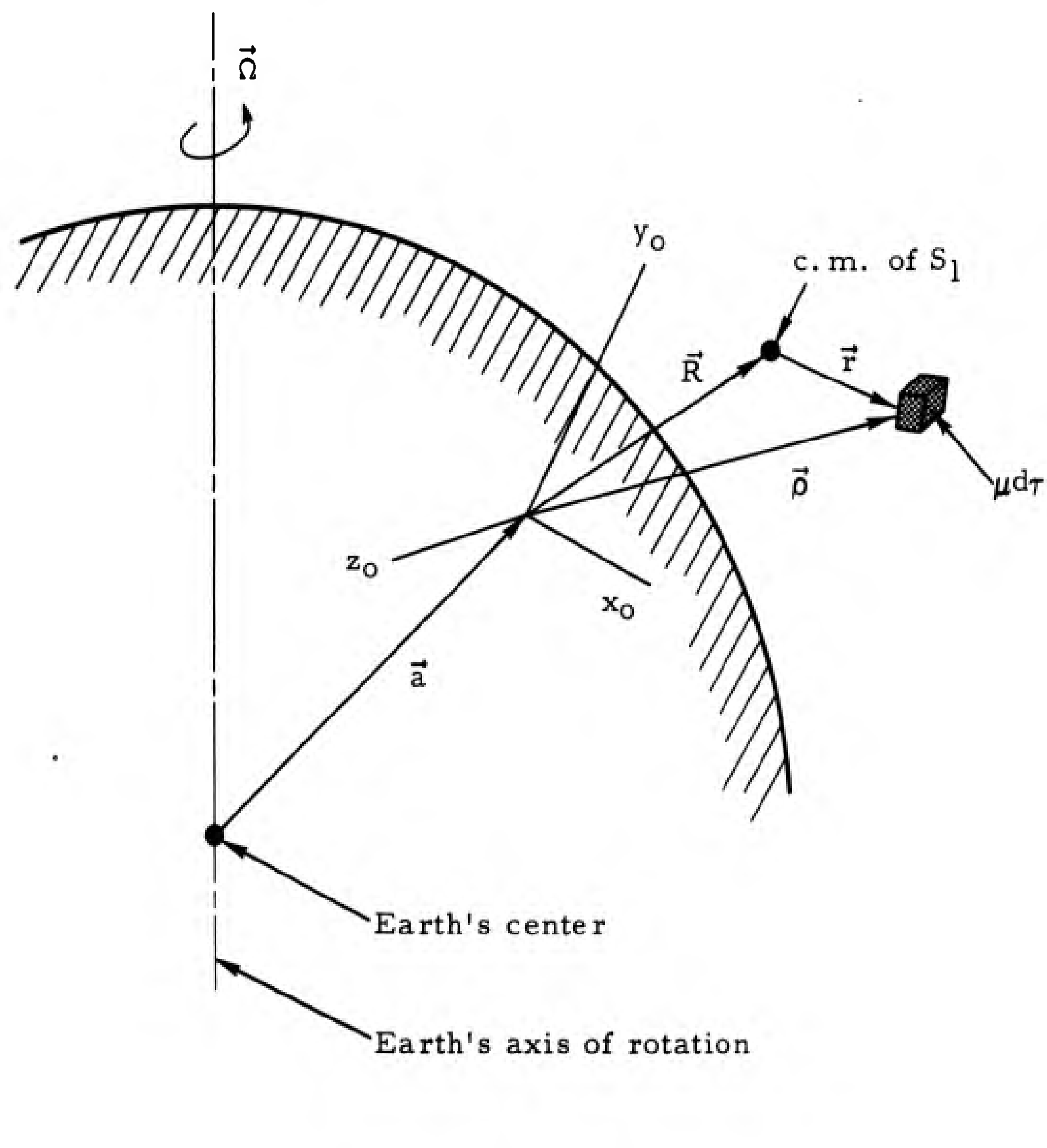
TRANSLATORY EQUATIONS

Consider a coordinate system with its origin at the center of the earth, irrotational with respect to the stars, to be an inertial frame.

Let $\vec{a}$ be the vector from the origin of the inertial system to the origin of the $x_o\ y_o\ z_o$ system. Let $\vec{R}$ be the vector from the origin of the $x_o\ y_o\ z_o$ system to the c.m. of S_1, let $\vec{\rho}$ be the vector from the $x_o\ y_o\ z_o$ origin to the mass element $\mu\, d\tau$, and let $\vec{r} = \vec{\rho} - \vec{R}$. Let $\vec{\Omega}$ be the angular velocity vector of the earth relative to the inertial system (i.e., relative to the stars). Time derivatives of vectors will be denoted by the symbols defined earlier.

The velocity of $\mu\, d\tau$ relative to the inertial frame is $\left.\frac{d}{dt}(\vec{a} + \vec{\rho})\right|_I$ so if S is the system defined at the instant t_o, then the momentum $\vec{P}$ of S at $t \geq t_o$ is given by

$$\vec{P}_{(t)} = \int_V \mu \left.\frac{d}{dt}(\vec{a} + \vec{\rho})\right|_I d\tau$$



and the equations of motion at t_o are given by

$$\frac{dP_{(t_o)}}{dt}\Bigg|_I = \sum \text{forces on S at } t_o \ .$$

We will first evaluate $\vec{P}_{(t)}$. By the formula in Appendix C,

$$\frac{d}{dt}(\vec{a}+\vec{\rho})\Bigg|_I = \frac{d\vec{a}}{dt} + \frac{d\vec{\rho}}{dt} + \vec{\Omega}x(\vec{a}+\vec{\rho})$$

$$= \frac{d\vec{\rho}}{dt} + \vec{\Omega}x(\vec{a}+\vec{\rho})$$

since $\vec{a}$ is fixed relative to $x_o\ y_o\ z_o$.

Writing $\vec{\rho} = \vec{R} + \vec{r}$, there results

$$\vec{P}_{(t)} = \int_S \mu \frac{d\vec{R}}{dt} d\tau + \int_S \mu \frac{d\vec{r}}{dt} d\tau + \vec{\Omega} x \left[\int_S \mu(\vec{a}+\vec{R})d\tau + \int_S \mu\vec{r}d\tau\right].$$

Since $\int_S \mu d\tau = M$, this takes the form

$$(9) \qquad \vec{P}_{(t)} = M\left[\frac{d\vec{R}}{dt} + \vec{\Omega}x(\vec{a}+\vec{R})\right] + \int_S \mu \frac{d\vec{r}}{dt} d\tau + \vec{\Omega} x \int_S \mu\vec{r}d\tau \ .$$

Now $d\vec{r}/dt = \dot{\vec{r}} + \vec{\omega}x\vec{r}$, where $\vec{\omega}$ is the angular velocity of $y_1\ y_2\ y_3$ (the missile) with respect to $x_o\ y_o\ z_o$, so Eq. 9 becomes

$$(10) \qquad \vec{P}_{(t)} = M\left[\frac{d\vec{R}}{dt} + \vec{\Omega}x(\vec{a}+\vec{R})\right] + \int_S \mu\dot{\vec{r}}d\tau + (\vec{\Omega}+\vec{\omega})x \int_S \mu\vec{r}d\tau \ .$$

Write

$$\int_S \mu\dot{\vec{r}}d\tau = \int_{S_1} \mu\dot{\vec{r}}d\tau + \int_{S_g} \mu\dot{\vec{r}}d\tau + \int_{S_e} \mu\dot{\vec{r}}d\tau \ .$$

We observe that $\int_{S_1} \mu\dot{\vec{r}}d\tau = -M_1\vec{V}_{cg}$ since $\dot{\vec{r}} = -\vec{V}_{cg}$ everywhere in S_1; and by definition,

$$\int_{S_g} \mu\dot{\vec{r}}d\tau = M_g\vec{V}_g \text{ and } \int_{S_e} \mu\dot{\vec{r}}d\tau = M_e\vec{V}_e \ .$$

Hence,

$$\int_S \mu \vec{r} d\tau = -M_1 \vec{V}_{cg} + M_g \vec{V}_g + M_e \vec{V}_e \ .$$

Similarly,

$$\int_S \mu \vec{r} d\tau = M_g \vec{R}_g + M_e \vec{R}_e \ .$$

Substitution of these in Eq. 10 gives

$$\text{(11)} \qquad \vec{P}_{(t)} = M \left[\frac{d\vec{R}}{dt} + \vec{\Omega} x (\vec{a} + \vec{R}) \right] - M_1 \vec{V}_{cg} + M_g \left[\vec{V}_g + (\vec{\Omega} + \vec{\omega}) x \vec{R}_g \right] + M_e \left[\vec{V}_e + (\vec{\Omega} + \vec{\omega}) x \vec{R}_e \right] .$$

By Assumptions 15 and 16,

$$\vec{V}_g + (\vec{\Omega} + \vec{\omega}) x \vec{R}_g \doteq \vec{V}_g \quad \text{and} \quad \vec{V}_e + (\vec{\Omega} + \vec{\omega}) x \vec{R}_e \doteq \vec{V}_e \ .$$

This reduces Eq. 11 to

$$\text{(12)} \qquad \vec{P}_{(t)} = M \left[\frac{d\vec{R}}{dt} + \vec{\Omega} x (\vec{a} + \vec{R}) \right] - M_1 \vec{V}_{cg} + M_g \vec{V}_g + M_e \vec{V}_e$$

where all terms on the right are, of course, evaluated at the instant t_o.

Now since $\left. \frac{d}{dt} \right|_I = \frac{d}{dt} + \vec{\Omega} x$, we have

$$\text{(13)} \qquad \sum \text{forces on S at } t_o = \left. \frac{d\vec{P}_{(t_o)}}{dt} \right|_I = \frac{d\vec{P}_{(t_o)}}{dt} + \vec{\Omega} x \vec{P}_{(t_o)} \ .$$

And since $M_e(t_o) = 0$ and $\dot{M}_e = \dot{M}$,

$$\left. \frac{d}{dt} (M_e \vec{V}_e) \right|_{t_o} = \dot{M} \vec{V}_e \ .$$

This means that Eq. 13 can be written in the form

$$\text{(14)} \qquad \sum \text{forces} = \frac{d\vec{Q}}{dt} + \dot{M} \vec{V}_e + \vec{\Omega} x \vec{Q}$$

where

$$\vec{Q} = M\left[\frac{d\vec{R}}{dt} + \vec{\Omega}x(\vec{a} + \vec{R})\right] - M_1\vec{V}_{cg} + M_g\vec{V}_g \ .$$

The calculations involved in carrying out the operations in Eq. 14 are simple but tedious.

Recalling that M is constant and that $d\vec{a}/dt = d\vec{\Omega}/dt = 0$ (i. e., $\vec{a}$ and $\vec{\Omega}$ are fixed relative to $x_o\ y_o\ z_o$ coordinates), we can write

$$\frac{d\vec{Q}}{dt} = M\left(\frac{d^2\vec{R}}{dt^2} + \vec{\Omega}x\frac{d\vec{R}}{dt}\right) - \dot{M}_1\vec{V}_{cg} - M_1(\dot{\vec{V}}_{cg} + \vec{\omega}x\vec{V}_{cg}) + \dot{M}_g\vec{V}_g$$
$$+ M_g(\dot{\vec{V}}_g + \vec{\omega}x\vec{V}_g)$$

where $\dot{\vec{V}}_{cg} + \vec{\omega}x\vec{V}_{cg}$ and $\dot{\vec{V}}_g + \vec{\omega}x\vec{V}_g$ have been written for $d\vec{V}_{cg}/dt$ and $d\vec{V}_g/dt$, respectively.

In accordance with Assumptions 1, 2, and 3, $\dot{M}_g \doteq 0$, $\dot{\vec{V}}_g \doteq 0$, and $\dot{\vec{V}}_{cg} \doteq 0$. Also, since $-\dot{M}_1 = \dot{M}$, $d\vec{Q}/dt$ may be written

(15) $$\frac{d\vec{Q}}{dt} = M\left(\frac{d^2\vec{R}}{dt^2} + \vec{\Omega}x\frac{d\vec{R}}{dt}\right) + \dot{M}\vec{V}_{cg} + \vec{\omega}x(M_g\vec{V}_g - M_1\vec{V}_{cg}) \ .$$

For $\vec{\Omega}x\vec{Q}$, we have

$$\vec{\Omega}x\vec{Q} = M\left\{\vec{\Omega}x\frac{d\vec{R}}{dt} + \vec{\Omega}x[\vec{\Omega}x(\vec{a} + \vec{R})]\right\} + \vec{\Omega}x(M_g\vec{V}_g - M_1\vec{V}_{cg}) \ .$$

The term $M\vec{\Omega}x[\vec{\Omega}x(\vec{a} + \vec{R})]$ is just the centripetal force on the missile due to the earth's rotation. This negligible quantity is mentioned in Assumption 17.

Hence,

(16) $$\vec{\Omega}x\vec{Q} = M\vec{\Omega}x\frac{d\vec{R}}{dt} + \vec{\Omega}x(M_g\vec{V}_g - M_1\vec{V}_{cg}) \ .$$

Substitution of Eq. 15 and Eq. 16 into Eq. 14 gives

(17) $$\sum \text{forces} = M\left(\frac{d^2\vec{R}}{dt^2} + 2\vec{\Omega} \times \frac{d\vec{R}}{dt}\right) + \dot{M}(\vec{V}_e + \vec{V}_{cg}) + (\vec{\Omega} + \vec{\omega}) \times (M_g\vec{V}_g - M_1\vec{V}_{cg}) .$$

By Assumptions 14 and 16, $\vec{V}_e + \vec{V}_{cg} \doteq \vec{V}_e$ and $\vec{\Omega} + \vec{\omega} \doteq \vec{\omega}$. So Eq. 17 may be written

$$\sum \text{forces} = M\left(\frac{d^2\vec{R}}{dt^2} + 2\vec{\Omega} \times \frac{d\vec{R}}{dt}\right) + \dot{M}\vec{V}_e + \vec{\omega} \times (M_g\vec{V}_g - M_1\vec{V}_{cg}) .$$

By the definition of thrust, $\dot{M}\vec{V}_e$ is just $-\vec{T} + \int_A \vec{f}_o dA$, where $\vec{T}$ is the thrust function and $\vec{f}_o$ is the force per unit area over the surface of a statically fired missile at sea level. Furthermore, the total force acting on S at time t_o is just $M\vec{g} + \int_A \vec{f} dA$, where $\vec{g}$ is the gravity vector and $\vec{f}$ is the force per unit area acting on the surface A. Using these facts, the equation above becomes

(18) $$\vec{T} + M\vec{g} + \vec{F} = M\left(\frac{d^2\vec{R}}{dt^2} + 2\vec{\Omega} \times \frac{d\vec{R}}{dt}\right) + \vec{\omega} \times (M_g\vec{V}_g - M_1\vec{V}_{cg})$$

where $\vec{F} = \int_A (\vec{f} - \vec{f}_o) dA$.

We observe that a part of the integral for $\vec{F}$ can be discarded. If we let A_e represent the surface area of the exit plane and let $A - A_e$ represent the rest of the surface area of the missile, then

$$\vec{F} = \int_{A_e} (\vec{f} - \vec{f}_o) dA + \int_{A-A_e} (\vec{f} - \vec{f}_o) dA .$$

Now on A_e, $\vec{f}_o$ is the exhaust pressure for a statically fired missile at sea level, and $\vec{f}$ is the exhaust pressure for a missile in flight at the same stage of burning. It is commonly accepted that these pressures are equal; i.e., that the pressure in the vicinity of the exhaust nozzles is relatively independent of the state of motion of the rocket and of the surrounding environment, depending only on time (stage of burning). This means that

$$\int_{A_e} \vec{f} dA = \int_{A_e} \vec{f}_o dA,$$

or equivalently, that

$$\vec{F} = \int_{A-A_e} (\vec{f} - \vec{f}_o)dA .$$

We now separate this integral into two parts: the "pure" aerodynamic force and a static pressure force. If $\vec{f}_s$ represents the force per unit area that would exist on the surface of the missile if it were held fixed in its configuration at time t_o, and if $\vec{f}_a$ is the pure aerodynamic force per unit area, defined by $\vec{f}_a = \vec{f} - \vec{f}_s$, then the integral for $\vec{F}$ may may be written

$$\vec{F} = \vec{F}_a + \int_{A-A_e} (\vec{f}_s - \vec{f}_o)dA$$

where

$$\vec{F}_a = \int_{A-A_e} \vec{f}_a dA$$

is the pure aerodynamic force on the system.

The second term can be simplified further. It follows from the definitions of $\vec{p}_a$ and $\vec{p}_o$ that $\int_A \vec{p}_a dA = \int_A \vec{p}_o dA = 0$, $\vec{p}_a = \vec{f}_s$ on $A - A_e$, and $\vec{p}_o = \vec{f}_o$ on $A - A_e$. Hence,

$$\int_{A-A_e} \vec{f}_s dA = \int_{A-A_e} \vec{f}_s dA - \int_A \vec{p}_a dA = \int_{A-A_e} (\vec{f}_s - \vec{p}_a)dA$$

$$- \int_{A_e} \vec{p}_a dA = - \int_{A_e} \vec{p}_a dA .$$

Similarly,

$$\int_{A-A_e} \vec{f}_o dA = - \int_{A_e} \vec{p}_o dA .$$

So the integral $\int_{A-A_e} (\vec{f}_s - \vec{f}_o)dA$ is given by

$$\int_{A-A_e} (\vec{f}_s - \vec{f}_o)dA = \int_{A_e} (\vec{p}_o - \vec{p}_a)dA = (p_o - p_a) A_e \vec{e}_1$$

and

$$\underline{\vec{F} = \vec{F}_a + (p_o - p_a) A_e \vec{e}_1} .$$

To reiterate, p_o and p_a are, respectively, atmospheric pressure at sea level and at the missile. For a low-altitude rocket such as that used in ASROC, $p_o \doteq p_a$, and the equation above reduces to

$$\vec{F} \doteq \vec{F}_a \, .$$

For high-altitude rockets during burning, this term does not drop out. In addition, one must take into account variations in other terms involved in the thrust function. The upshot of this is that one must use a thrust function that is corrected from its measured value at sea level to allow for these altitude variations. This point is not taken up in this report.

Using the above result, Eq. 18 can be rewritten as

$$\text{(19)} \qquad \vec{T} + M\vec{g} + \vec{F}_a = M\left(\frac{d^2\vec{R}}{dt^2} + 2\vec{\Omega} x \frac{d\vec{R}}{dt}\right) + \vec{\omega} x (M_g \vec{V}_g - M_1 \vec{V}_{cg})$$

where

$$\vec{F}_a = \int_{A-A_e} \vec{f}_a dA$$

is the pure aerodynamic force on the missile at time t_o.

Each of the terms in Eq. 19 will now be expressed in components relative to the $x_1\ x_2\ x_3$ system.

If $d\vec{R}/dt$ has components $V_1\ V_2\ V_3$ relative to $x_1\ x_2\ x_3$, i.e., if

$$\frac{d\vec{R}}{dt} = V_1\vec{e}_1 + V_2\vec{e}_2 + V_3\vec{e}_3 = \vec{V}$$

then

$$\frac{d^2\vec{R}}{dt^2} = \frac{d\vec{V}}{dt} = \frac{\delta\vec{V}}{\delta t} + \vec{\omega}_c x \vec{V} = \dot{V}_1\vec{e}_1 + \dot{V}_2\vec{e}_2 + \dot{V}_3\vec{e}_3 + \vec{\omega}_c x \vec{V}$$

where $\vec{\omega}_c$ is the angular velocity of the $x_1\ x_2\ x_3$ frame relative to $x_o\ y_o\ z_o$.

Writing this out in components, we obtain

$$\frac{d^2\vec{R}}{dt^2} = (\dot{V}_1 + \omega_{c2}V_3 - \omega_{c3}V_2)\vec{e}_1 + (\dot{V}_2 + \omega_{c3}V_1 - \omega_{c1}V_3)\vec{e}_2 + (\dot{V}_3 + \omega_{c1}V_2 - \omega_{c2}V_1)\vec{e}_3 \, .$$

In Appendix C it is shown that $\omega_1\ \omega_2\ \omega_3$ and $\omega_{c1}\ \omega_{c2}\ \omega_{c3}$ are related by

$$\begin{pmatrix} \omega_{c1} \\ \omega_{c2} \\ \omega_{c3} \end{pmatrix} = \begin{pmatrix} \omega_2 \tan\theta \\ \omega_2 \\ \omega_3 \end{pmatrix} .$$

So the expression for $d^2\vec{R}/dt^2$ becomes

$$\text{(20)} \quad \frac{d^2\vec{R}}{dt^2} = (\dot{V}_1 + \omega_2 V_3 - \omega_3 V_2)\vec{e}_1 + (\dot{V}_2 + \omega_3 V_1 - \omega_2 V_3 \tan\theta)\vec{e}_2 + (\dot{V}_3 + \omega_2 V_2 \tan\theta - \omega_2 V_1)\vec{e}_3 \, .$$

The term $2M\vec{\Omega} x d\vec{R}/dt$ may be recognized as the Coriolis force on the missile. It is evaluated as follows. In terms of the $x_o\ y_o\ z_o$ system,

$$\frac{d\vec{R}}{dt} = \dot{x}_o\,\vec{e}_{x_o} + \dot{y}_o\,\vec{e}_{y_o} + \dot{z}_o\,\vec{e}_{z_o} \, .$$

If ϕ_L is the latitude of the missile ($\phi_L = 0$ at the equator) and ψ_L its azimuth (ψ_L taken positive in the sense of a rotation from north to east), then

$$\vec{\Omega} = \Omega(\cos\phi_L \cos\psi_L\,\vec{e}_{x_o} + \sin\phi_L\,\vec{e}_{y_o} + \cos\phi_L \sin\psi_L\,\vec{e}_{z_o})$$

where Ω is the angular rate of the earth about its own axis, relative to the sun. So if we put

$$a_x = 2\cos\phi_L \cos\psi_L, \quad a_y = 2\sin\phi_L, \quad a_z = 2\cos\phi_L \sin\psi_L$$

the cross product $2\vec{\Omega} x \frac{d\vec{R}}{dt}$ can be calculated directly, giving

$$2\vec{\Omega}x\frac{d\vec{R}}{dt} = \Omega(a_y\dot{z}_o - a_z\dot{y}_o)\vec{e}_{x_o} + \Omega(a_z\dot{x}_o - a_x\dot{z}_o)\vec{e}_{y_o} + \Omega(a_x\dot{y}_o - a_y\dot{x}_o)\vec{e}_{z_o} .$$

If Q is the matrix of the transformation from $x_o\ y_o\ z_o$ to $x_1\ x_2\ x_3$ (derived in Appendix A), the components of $2\vec{\Omega}x\frac{d\vec{\Omega}}{dt}$ relative to $x_1\ x_2\ x_3$ are given by

(21) $$\Omega Q \begin{pmatrix} a_y\ \dot{z}_o - a_z\ \dot{y}_o \\ a_z\ \dot{x}_o - a_x\ \dot{z}_o \\ a_x\ \dot{y}_o - a_y\ \dot{x}_o \end{pmatrix} .$$

The quantity $\vec{\omega}x(M_g\vec{V}_g - M_1\vec{V}_{cg})$ can be calculated by observing that $\vec{V}_g \doteq -V_g\vec{e}_1$ and $\vec{V}_{cg} = V_{cg}\vec{e}_1$, by Assumptions 9 and 10. The result is

(22) $$\vec{\omega}x\ (M_g\vec{V}_g - M_1\vec{V}_{cg}) = -\ (M_gV_g + M_1V_{cg})(\omega_3\vec{e}_2 - \omega_2\vec{e}_3) .$$

It is shown in the section on thrust misalignment that

(23) $$\vec{T} \doteq T\vec{e}_1 - \delta_T \sin\phi_T\vec{e}_2 + \delta_T \cos\phi_T\vec{e}_3$$

where δ_T is the normal component of thrust and ϕ_T is the angle between the thrust misalignment plane and the $x_1\ x_3$ plane. And, in accordance with a later section on "g" corrections,

$$\vec{g} = -\ g_o\left[\frac{x_o}{r_e}\vec{e}_{x_o} + \left(1 - 2\frac{y_o}{r_e}\right)\vec{e}_{y_o}\right] .$$

In components relative to $x_1\ x_2\ x_3$, this is

$$-\ g_oQ\begin{pmatrix} x_o/r_e \\ 1 - 2y_o/r_e \\ 0 \end{pmatrix} .$$

Finally, using Eq. 20, 21, 22, and 23, Eq. 19 may be written out in components in the form

$$(24)\begin{cases}\dot{V}_1 = \dfrac{T}{M} + C_1 + \dfrac{F_1}{M} - \omega_2 V_3 + \omega_3 V_2 \\[2ex] \dot{V}_2 = -\dfrac{\delta_T}{M}\sin\phi_T + C_2 + \dfrac{F_2}{M} - \omega_3\left(V_1 - V_{cg} - \dfrac{M_g}{M}V_g\right) + \omega_1 V_3 \tan\theta \\[2ex] \dot{V}_3 = \dfrac{\delta_T}{M}\cos\phi_T + C_3 + \dfrac{F_3}{M} + \omega_2\left(V_1 - V_{cg} - \dfrac{M_g}{M}V_g\right) - \omega_1 V_2 \tan\theta\end{cases}$$

where the fact that $M_1/M \doteq 1$ has been tacitly used. The auxiliary equations are

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = Q \begin{pmatrix} -g_o \dfrac{x_o}{r_e} - \Omega(a_y\dot{z}_o - a_z\dot{y}_o) \\ -g_o\left(1 - \dfrac{2y_o}{r_e}\right) - \Omega(a_z\dot{x}_o - a_x\dot{z}_o) \\ -\Omega(a_x\dot{y}_o - a_y\dot{x}_o) \end{pmatrix},$$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = 2\begin{pmatrix} \cos\phi_L \cos\psi_L \\ \sin\phi_L \\ \cos\phi_L \sin\psi_L \end{pmatrix}, \quad F_i = \vec{F}_a \cdot \vec{e}_i ,$$

$$\vec{F}_a = \int_{A-A_e} \vec{f}_a dA ,$$

and where $V_1\ V_2\ V_3$ are related to $\dot{x}_o\ \dot{y}_o\ \dot{z}_o$ by

$$\begin{pmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{z}_o \end{pmatrix} = Q^t \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}.$$

As mentioned before, $\vec{f}_a$ is to be regarded as the pure aerodynamic force per unit area, produced solely by the dynamic agents acting on the surface of the missile.

Two of the terms in Eq. 24 are so insignificant that they may be discarded immediately. These are V_{cg} and $(M_g/M)V_g$. They were calculated at nominal values for the ASROC missile and, as one might expect, they were insignificant compared to V_1 (which is very nearly the tangential speed of the missile along its trajectory).

Observing, therefore, that

$$V_1 - V_{cg} - \frac{M_g}{M} V_g \doteq V_1$$

in Eq. 24, we obtain

$$(25) \qquad \begin{cases} \dot{V}_1 = \dfrac{T}{M} + C_1 + \dfrac{F_1}{M} - \omega_2 V_3 + \omega_3 V_2 \\ \dot{V}_2 = -\dfrac{\delta_T}{M} \sin \phi_T + C_2 + \dfrac{F_2}{M} - \omega_3 V_1 + \omega_1 V_3 \tan \theta \\ \dot{V}_3 = \dfrac{\delta_T}{M} \cos \phi_T + C_3 + \dfrac{F_3}{M} + \omega_2 V_1 - \omega_1 V_2 \tan \theta \end{cases}$$

where

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = Q \begin{pmatrix} -g_o \dfrac{x_o}{r_e} - \Omega(a_y \dot{z}_o - a_z \dot{y}_o) \\ -g_o \; 1 - 2 \dfrac{y_o}{r_o} - \Omega(a_z \dot{x}_o - a_x \dot{z}_o) \\ -\Omega(a_x \dot{y}_o - a_y \dot{x}_o) \end{pmatrix}$$

and

$$F_i = \vec{F}_a \cdot \vec{e}_i \, .$$

The system (Eq. 25) is still of greater generality than one ordinarily needs in short-range ballistic work.

If we neglect the curvature of the earth, Coriolis effects, and the variation of gravity with altitude, the expression for C_1, C_2, and C_3 becomes

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = Q \begin{pmatrix} 0 \\ -g_o \\ 0 \end{pmatrix};$$

or equivalently, $C_1 = -g_o \sin\theta$, $C_2 = -g_o \cos\theta$, $C_3 = 0$.

So Eq. 25 may be written

$$(26) \quad \begin{cases} \dot{V}_1 = \frac{T}{M} - g_o \sin\theta + \frac{F_1}{M} - \omega_2 V_3 + \omega_3 V_2 \\ \dot{V}_2 = -\frac{\delta_T}{M} \sin\phi_T - g_o \cos\theta + \frac{F_2}{M} - \omega_3 V_1 + \omega_1 V_3 \tan\theta \\ \dot{V}_3 = \frac{\delta_T}{M} \cos\phi_T + \frac{F_3}{M} + \omega_2 V_1 - \omega_1 V_2 \tan\theta \end{cases}$$

where $F_i = \vec{F}_a \cdot \vec{e}_i$.

If there is no thrust misalignment, $\delta_T = 0$ and

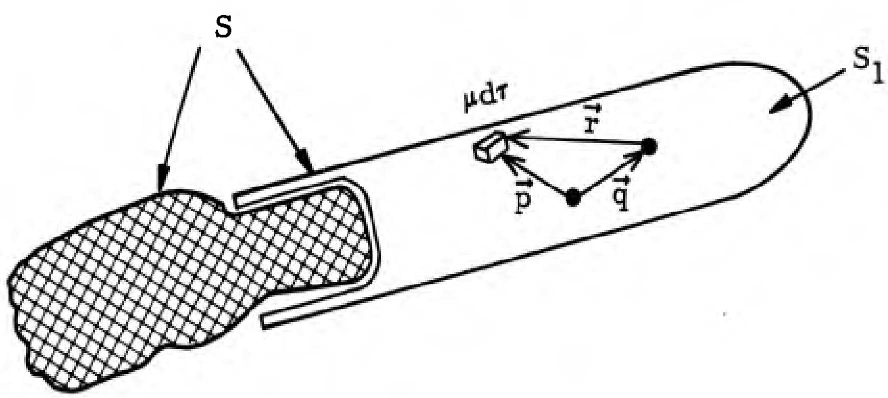
$$(27) \quad \begin{cases} \dot{V}_1 = \frac{T}{M} - g_o \sin\theta + \frac{F_1}{M} - \omega_2 V_3 + \omega_3 V_2 \\ \dot{V}_2 = - g_o \cos\theta + \frac{F_2}{M} - \omega_3 V_1 + \omega_1 V_3 \tan\theta \\ \dot{V}_3 = \frac{F_3}{M} + \omega_2 V_1 - \omega_1 V_2 \tan\theta \end{cases}$$

where $F_i = \vec{F}_a \cdot \vec{e}_i$.

ROTATIONAL EQUATIONS

In this section, the $x_o y_o z_o$ system will be considered an inertial frame. The error introduced into the angular terms by this assumption is exceedingly small. From a purely logical standpoint, it would be better to take an approach more consistent with the previous section, regarding $x_I y_I z_I$ as the inertial frame in deriving the rotational equations, discarding the insignificant terms at the end of the analysis. However, the final result is exactly the same as that obtained in this section; if the rotation of the earth is taken into account in the rotational terms, the derivation takes on a formidable complexity.

Let $\vec{p}$ be the vector from the c.m. of S to the mass element $\mu d\tau$, $\vec{q}$ the vector from the c.m. of S to the c.m. of S_1, and $\vec{r}$ the vector from the c.m. of S_1 to $\mu d\tau$.



Then the angular momentum $\vec{L}$ of S about its c.m. is

$$\vec{L} = \int_S \mu \vec{p} \times \frac{d\vec{p}}{dt} \, d\tau \tag{28}$$

relative to $x_o y_o z_o$.

Observing that $\vec{p} = \vec{q} + \vec{r}$, we put

$$\frac{d\vec{p}}{dt} = \frac{d\vec{q}}{dt} + \frac{d\vec{r}}{dt}$$

in Eq. 28, obtaining

$$\vec{L} = -\frac{d\vec{q}}{dt} \times \int_S \mu \vec{p} \, d\tau + \int_S \mu \vec{p} \times \frac{d\vec{r}}{dt} \, d\tau .$$

But since $\vec{p}$ originates from the c.m. of S, $\int_S \mu\vec{p}d\tau = 0$. Hence

$$\vec{L} = \int_S \mu\vec{p} \times \frac{d\vec{r}}{dt} \, d\tau .$$

Using $\vec{p} = \vec{q} + \vec{r}$ once again, Eq. 28 becomes

$$\vec{L} = \vec{q} \times \int_S \mu \frac{d\vec{r}}{dt} \, d\tau + \int_S \mu\vec{r} \times \frac{d\vec{r}}{dt} \, d\tau . \tag{29}$$

Equation 29 expresses the fact that the angular momentum of S about its own c.m. is equal to a correction term plus the angular momentum of S about the c.m. of S_1. It will be shown first that the correction term may be dropped; that its contribution to the final equations is negligible.

More precisely, since the final equations are equivalent to

$$\frac{d\vec{L}(t_o)}{dt} = \sum \text{exterior moments at } t_o,$$

what we need to show is that

$$\frac{d\vec{L}}{dt} \doteq \frac{d}{dt} \int_S \mu\vec{r} \times \frac{d\vec{r}}{dt} \, d\tau$$

at t_o, or equivalently,

$$\frac{d}{dt}\left(\vec{q} \times \int_S \mu \frac{d\vec{r}}{dt} \, d\tau \right) \doteq 0 \text{ at } t_o .$$

Since $d\vec{r}/dt = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$, it follows that

$$\vec{q} \times \int_S \mu \frac{d\vec{r}}{dt} \, d\tau = \vec{q} \times \int_S \mu\dot{\vec{r}}d\tau + \vec{q} \times (\vec{\omega} \times \int_S \mu\vec{r}d\tau) . \tag{30}$$

Furthermore, $\vec{r} = \vec{p} - \vec{q}$ and $\int_S \mu\vec{p}d\tau = 0$, so that Eq. 30 becomes

$$\begin{aligned} \vec{q} \times \int_S \mu \frac{d\vec{r}}{dt} \, d\tau &= \vec{q} \times \int_S \mu\dot{\vec{r}}d\tau - \vec{q} \times (\vec{\omega} \times \vec{q} \int_S \mu d\tau) \\ &= \vec{q} \times \int_S \mu\dot{\vec{r}}d\tau - M\vec{q} \times (\vec{\omega} \times \vec{q}) . \end{aligned} \tag{31}$$

Obviously,

$$\int_S \mu\dot{\vec{r}}d\tau = \int_{S_1} \mu\dot{\vec{r}}d\tau + \int_{S_g} \mu\dot{\vec{r}}d\tau + \int_{S_e} \mu\dot{\vec{r}}d\tau .$$

But $\dot{\vec{r}} = -\vec{V}_{cg}$ everywhere on S_1; and by definition of $\vec{V}_g$ and $\vec{V}_e$.

$$\int_S \mu\dot{\vec{r}}d\tau = -M_1\vec{V}_{cg} + M_g\vec{V}_g + M_e\vec{V}_e .$$

So Eq. 31 may be written

$$(32) \quad \vec{q} \times \int_S \mu \frac{d\vec{r}}{dt} d\tau = \vec{q} \times (-M_1\vec{V}_{cg} + M_g\vec{V}_g + M_e\vec{V}_e) - M\vec{q} \times (\vec{\omega} \times \vec{q}) .$$

It is easily seen by evaluating

$$\int_S \mu\vec{p}d\tau = \int_S \mu(\vec{q} + \vec{r})d\tau = 0$$

that $\vec{q}$ is given by

$$\vec{q} = \frac{M_g}{M}\vec{R}_g + \frac{M_e}{M}\vec{R}_e .$$

So $\vec{q}$ is a vector of very small amplitude, lying nearly parallel to the x_1 axis; that is,

$$\vec{q} \doteq -\left(\frac{M_g}{M}R_g + \frac{M_e}{M}R_e\right)\vec{e}_1 .$$

By Assumptions 9 and 10, $\vec{V}_g \doteq -V_g\vec{e}_1$ and $\vec{V}_{cg} \doteq V_{cg}\vec{e}_1$, so that $\vec{q} \times (-M_1\vec{V}_{cg} + M_g\vec{V}_g + M_e\vec{V}_e) \doteq 0$, and Eq. 32 takes the form

$$(33) \quad \vec{q} \times \int_S \mu \frac{d\vec{r}}{dt} d\tau \doteq -Mq^2\vec{\omega}_N$$

where

$$q = \frac{M_g}{M}R_g + \frac{M_e}{M}R_e$$

and where $\vec{\omega}_N$ is the component of $\vec{\omega}$ normal to the x_1 axis.

This term is of second order in q; when it is differentiated, the result, given by $-2Mq\dot{q}\vec{\omega}_N - M_q^2(\dot{\vec{\omega}}_N + \vec{\omega} \times \vec{\omega}_N)$, is negligible.

In view of this, we may just as well write Eq. 29 as

$$\vec{L} = \int_S \mu\vec{r} \times \frac{d\vec{r}}{dt} d\tau . \tag{34}$$

The right side of Eq. 34 will now be evaluated at an arbitrary $t \geq t_o$. Writing $d\vec{r}/dt = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$,

$$\vec{L} = \int_S \mu\vec{r} \times \dot{\vec{r}} d\tau + \int_S \mu\vec{r} \times (\vec{\omega} \times \vec{r}) d\tau = \int_S \mu\vec{r} \times \dot{\vec{r}} d\tau \tag{35}$$

$$+ \int_{S_1} \mu\vec{r} \times (\vec{\omega} \times \vec{r}) d\tau + \int_{S_g} \mu\vec{r} \times (\vec{\omega} \times \vec{r}) d\tau + \int_{S_e} \mu\vec{r} \times (\vec{\omega} \times \vec{r}) d\tau .$$

Utilizing the double cross product expansion formula $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$, the second term on the right in Eq. 35 can be written

$$\int_{S_1} \mu\vec{r} \times (\vec{\omega} \times \vec{r}) d\tau = \int_{S_1} \mu[r^2\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}] d\tau$$

$$= [\omega_1 \int_{S_1} \mu(y_2^2 + y_3^2) d\tau - \omega_2 \int_{S_1} \mu y_1 y_2 d\tau - \omega_3 \int_{S_1} \mu y_1 y_3 d\tau]\vec{e}_{y_1}$$

$$+ [-\omega_1 \int_{S_1} \mu y_1 y_2 d\tau - \omega_2 \int_{S_1} \mu(y_1^2 + y_3^2) d\tau - \omega_3 \int_{S_1} \mu y_2 y_3 d\tau]\vec{e}_{y_2}$$

$$+ [-\omega_1 \int_{S_1} \mu y_1 y_3 d\tau - \omega_2 \int_{S_1} \mu y_2 y_3 d\tau + \omega_3 \int_{S_1} \mu(y_1^2 + y_2^2) d\tau]\vec{e}_{y_3} .$$

Or in matrix notation, we have

$$(36) \qquad \int_{S_1} \mu\vec{r}\times(\vec{\omega}\times\vec{r})d\tau = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = I\vec{\omega}$$

where $I_{11} = \int_{S_1} \mu\left(y_2^2 + y_3^2\right)d\tau$, $I_{12} = -\int_{S_1} \mu y_1 y_2 d\tau$, etc., and it is understood that the vectors $\vec{\omega}$ and $I\vec{\omega}$ are expressed in the coordinate system fixed to the missile. By axial symmetry, all entries off the diagonal of I are zero, and $I_{22} = I_{33}$. So I can be written in the form

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}.$$

The integral $\int_{S_g} \mu\vec{r}\times(\vec{\omega}\times\vec{r})d\tau$ can be evaluated in a somewhat similar fashion. If we let $\vec{r}'$ be the vector from the c.m. of S_g to the mass increment $\mu d\tau$, then $\vec{r} = \vec{R}_g + \vec{r}'$, and the integral can be expanded to

$$\int_{S_g} \mu\vec{r}\times(\vec{\omega}\times\vec{r})d\tau = \int_{S_g} \mu(\vec{R}_g + \vec{r}')\times[\vec{\omega}\times(\vec{R}_g + \vec{r}')]\,d\tau$$

$$= \vec{R}_g \times(\vec{\omega}\times\vec{R}_g)\int_{S_g}\mu d\tau + \vec{R}_g\times\left(\vec{\omega}\times\int_{S_g}\mu\vec{r}'d\tau\right)$$

$$- (\vec{\omega}\times\vec{R}_g)\times\int_{S_g}\mu\vec{r}'d\tau + \int_{S_g}\mu\vec{r}'\times(\vec{\omega}\times\vec{r}')d\tau.$$

Clearly $\int_{S_g} \mu d\tau = M_g$ and $\int_{S_g} \mu\vec{r}'d\tau = 0$; and by the same steps as those taken above,

$$\int_{S_g} \mu\vec{r}'\times(\vec{\omega}\times\vec{r}')d\tau = I_g\vec{\omega}$$

where I_g is the moment of inertia matrix (not necessarily diagonal) of the subsystem S_g about its own c.m. relative to the system fixed to the missile. Hence,

(37) $$\int_{S_g} \mu\vec{r} \times (\vec{\omega} \times \vec{r})d\tau = M_g\vec{R}_g \times (\vec{\omega} \times \vec{R}_g) + I_g\vec{\omega} .$$

Similarly, the integral $\int_{S_e} \mu\vec{r} \times (\vec{\omega} \times \vec{r})d\tau$ evaluates as

$$\int_{S_e} \mu\vec{r} \times (\vec{\omega} \times \vec{r})d\tau = M_e\vec{R}_e \times (\vec{\omega} \times \vec{R}_e) + I_e\vec{\omega}$$

where I_e is the moment of inertia of S_e about its own center of mass, relative to the system fixed to the missile. Combining these results, we see that

(38) $$\int_S \mu\vec{r} \times (\vec{\omega} \times \vec{r})d\tau = (I + I_g + I_e)\vec{\omega} + M_g\vec{R}_g \times (\vec{\omega} \times \vec{R}_g) + M_e\vec{R}_e \times (\vec{\omega} \times \vec{R}_e).$$

It remains to evaluate $\int_S \mu\vec{r} \times \dot{\vec{r}}d\tau$. Again we write $\int_S = \int_{S_1} + \int_{S_g} + \int_{S_e}$, and evaluate the right side term by term.

By definition, $\dot{\vec{r}}$ is the velocity of the mass element $\mu d\tau$ relative to the fixed missile parts. So on S_1, we obviously have $\dot{\vec{r}} = -\vec{V}_{cg}$. Hence,

$$\int_{S_1} \mu\vec{r} \times \dot{\vec{r}}d\tau = \vec{V}_{cg} \times \int_{S_1} \mu\vec{r}d\tau = \vec{V}_{cg} \times \vec{0} = 0 .$$

Before integrating over S_g, we again transform to center of mass (of S_g) coordinates by way of the relation $\vec{r} = \vec{R}_g + \vec{r}'$.

$$\int_{S_g} \mu\vec{r} \times \dot{\vec{r}}d\tau = \int_{S_g} \mu(\vec{R}_g + \vec{r}') \times \dot{\vec{r}}d\tau = \vec{R}_g \times \int_{S_g} \dot{\vec{r}}d\tau + \int_{S_g} \mu\vec{r}' \times \dot{\vec{r}}d\tau$$

$$= M_g\vec{R}_g \times \vec{V}_g + \int_{S_g} \mu\vec{r}' \times \dot{\vec{r}}d\tau .$$

The last term is the angular momentum of the propellant particles in transit relative to their own center of mass, with the velocities taken relative to the c.m. of S_1 (the moving c.g.).

Similarly,

$$\int_{S_e} \mu\vec{r} \times \dot{\vec{r}}d\tau = M_e\vec{R}_e \times \vec{V}_e + \int_{S_e} \mu\vec{r}'' \times \dot{\vec{r}}d\tau$$

where $\vec{r}''$ is the vector from the c.m. of S_e to $\mu d\tau$, given by $\vec{r}'' = \vec{r} - \vec{R}_e$. Again, the term $\int_{S_e} \mu \vec{r}'' \times \dot{\vec{r}} d\tau$ is the angular momentum of S_e about its own c.m., with velocities taken relative to the c.m. of S_1.

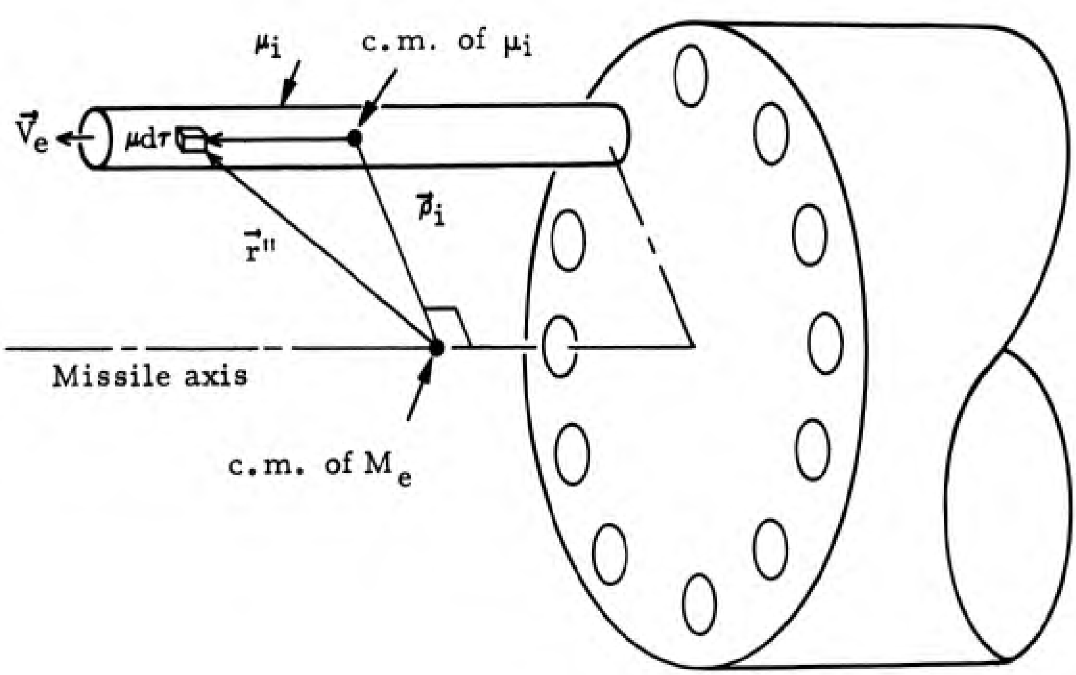
Summarizing all these results, Eq. 35 can be written

$$(39)\quad \vec{L} = M_g\vec{R}_g \times (\vec{V}_g + \vec{\omega} \times \vec{R}_g) + M_e\vec{R}_e \times (\vec{V}_e + \vec{\omega} \times \vec{R}_e) + (I + I_g + I_e)\vec{\omega} + \int_{S_g} \mu \vec{r}' \times \dot{\vec{r}} d\tau + \int_{S_e} \mu \vec{r}'' \times \dot{\vec{r}} d\tau .$$

In the first place, it is not clear how the last two terms in Eq. 39 can be estimated. Ideally, one would of course enjoy finding a uniform velocity and mass profile in Sg and Se; for in this case both terms vanish. However, it is likely that local variations in these quantities will occur in a missile experiencing yaw oscillations, for example. Even so, the cumulative effect of these terms is no doubt small, and they will be neglected in what follows. Thus Eq. 39 reduces to

$$(40)\quad \vec{L} = M_g\vec{R}_g \times (\vec{V}_g + \vec{\omega} \times \vec{R}_g) + M_e\vec{R}_e \times (\vec{V}_e + \vec{\omega} \times \vec{R}_e) + (I + I_g + I_e)\vec{\omega} .$$

The quantity $I_e\vec{\omega}$ will be estimated first.



The following assumptions are made only to obtain an estimate of the order of magnitude of this term, and do not apply elsewhere in this report. Assume that the missile has twelve exhaust nozzles, equally spaced on a circle centered at the missile axis and in a plane perpendicular to that axis. Let μ_i be that part of M_e that has been exhausted through the ith nozzle during Δt. Further, assume that the mass flow is uniformly distributed among the μ_i (i. e., $\mu_i = \frac{1}{12} M_e$, $1 \leq i \leq 12$), and that the moment of inertia of each μ_i about its own c.m. is negligible compared to the moment of inertia of M_e about its c.m. Let v_i be the volume occupied by μ_i, let $\vec{\rho}_i$ be the vector from the c.m. of M_e to the c.m. of μ_i, $\vec{r}''$ the vector from the c.m. of M_e to the mass element $\mu d\tau$; and if $\mu d\tau$ is in μ_i, let $\vec{q} = \vec{r}'' - \rho_i$. Assume that all the $\vec{\rho}_i$ are perpendicular to the missile axis.

$I_e\vec{\omega}$ can now be written in the form

$$I_e\vec{\omega} = \int_{S_e} \mu\vec{r}'' \times (\vec{\omega} \times \vec{r}'')d\tau = \sum_{i=1}^{12} \int_{v_i} \mu\vec{r}'' \times (\vec{\omega} \times \vec{r}'')d\tau .$$

Substituting $\vec{r}'' = \vec{\rho}_i + \vec{q}$ and observing that $\int_{v_i} \mu(\vec{r}'' - \vec{\rho}_i)d\tau$ $= \int_{v_i} \mu\vec{q}d\tau = 0$, there results

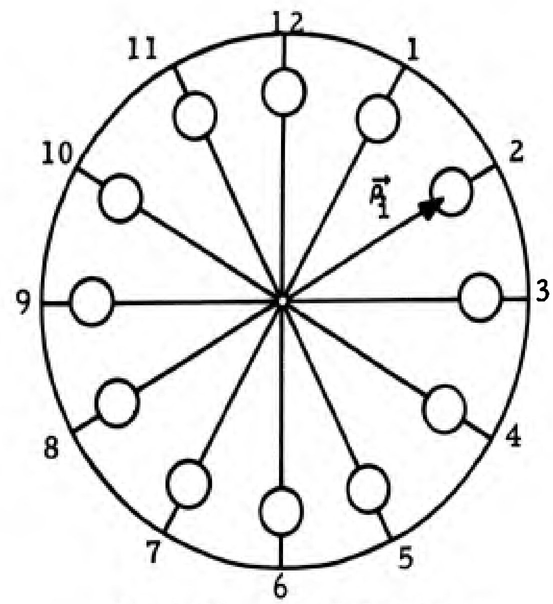
$$I_e\vec{\omega} = \sum_{i=1}^{12} \mu_i\vec{\rho}_i \times (\vec{\omega} \times \vec{\rho}_i) + \sum_{i=1}^{12} \int_{v_i} \mu\vec{q} \times (\vec{\omega} \times \vec{q})d\tau .$$

Each term in the second sum is of the form $I_i\vec{\omega}$, where I_i is the moment of inertia of μ_i about its own c.m. These terms have been assumed negligible. Observing that $\mu_i = \frac{1}{12} M_e$, $I_e\vec{\omega}$ becomes

$$I_e\vec{\omega} = \frac{1}{12} M_e \sum_{i=1}^{12} \vec{\rho}_i \times (\vec{\omega} \times \vec{\rho}_i) . \tag{41}$$

This sum may be evaluated explicitly by considering the geometry of the nozzle arrangement.

Let $\vec{e}_i$ be the unit vector in the direction of $\vec{\rho}_i$. By the assumptions on the symmetry of the μ_i, all the $\vec{\rho}_i$ have the same length ρ. Writing $\vec{\rho}_i = \rho\vec{e}_i$, Eq. 41 becomes



End View of Exit Plane

(42) $$I_e\vec{\omega} = \frac{1}{12} M_e \rho^2 \sum_{i=1}^{12} \vec{e}_i \times (\vec{\omega} \times \vec{e}_i) .$$

Since $\vec{e}_i \times (\vec{\omega} \times \vec{e}_i) = \vec{\omega} - (\vec{\omega} \cdot \vec{e}_i)\vec{e}_i$, the sum in Eq. 42 becomes

$$12\vec{\omega} - \sum_{i=1}^{12} (\vec{\omega} \cdot \vec{e}_i)\vec{e}_i .$$

Now $\vec{e}_1$ and $\vec{e}_4$ are perpendicular, as are $\vec{e}_2$ and $\vec{e}_5$, $\vec{e}_3$ and $\vec{e}_6$, $\vec{e}_7$ and $\vec{e}_{10}$, $\vec{e}_8$ and $\vec{e}_{11}$, $\vec{e}_9$ and $\vec{e}_{12}$. So $(\vec{\omega} \cdot \vec{e}_1)\vec{e}_1 + (\vec{\omega} \cdot \vec{e}_4)\vec{e}_4$ is merely the projection of $\vec{\omega}$ on the exit plane, which is of course normal to the missile axis. So we have $(\vec{\omega} \cdot \vec{e}_1)\vec{e}_1 + (\vec{\omega} \cdot \vec{e}_4)\vec{e}_4 = \vec{\omega}_N$, where ω_N is the component of $\vec{\omega}$ normal to the missile axis. The same argument applies to each of the six pairs mentioned above, so that

$$\sum_{i=1}^{12} (\vec{\omega} \cdot \vec{e}_i)\vec{e}_i = 6\vec{\omega}_N .$$

Equation 42 can now be written as

(43) $$\underline{I_e\vec{\omega} = M_e \rho^2 \left(\vec{\omega} - \frac{1}{2} \vec{\omega}_N\right)} .$$

It should be kept in mind that this expression for $I_e\vec{\omega}$ applies only to a missile with the exhaust configuration described above.

We now rewrite Eq. 40 as

$$(44) \qquad \vec{L} = M_g\vec{R}_g \times (\vec{V}_g + \vec{\omega} \times \vec{R}_g) + (I + I_g)\vec{\omega} + M_e\left[\vec{R}_e \times (\vec{V}_e + \vec{\omega} \times \vec{R}_e) + \rho^2\left(\vec{\omega} - \frac{1}{2}\vec{\omega}_N\right)\right].$$

This formula holds for all instants of time later than t_o.

The only step remaining is to apply Newton's law Σ moments $= \frac{d\vec{L}}{dt}$. But first, some remarks should be made concerning the interpretation of the terms in Eq. 44 when they are expressed in a new coordinate system.

There is a considerable advantage to be had in writing the final equations in a moving system different from the one rotating with the missile. The advantage lies in the fact that the transformation from inertial ($x_o y_o z_o$) coordinates to this frame takes on a much simpler form. This new frame is the $x_1x_2x_3$ system defined in the section on Definitions and Notation.

The vectors $\vec{R}_g$, $\vec{R}_e$, $\vec{V}_g$, $\vec{V}_e$, and $\vec{\omega}$ have components relative to $x_1x_2x_3$ which are, of course, different from those they had in the original $y_1y_2y_3$ system. Both moments of inertia I and I_g might also be expected to have different components relative to $x_1x_2x_3$; that is, we may conceivably have to use a matrix different from

$$\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}$$

for I in Eq. 44, if we regard Eq. 44 as a vector equation in the new coordinates. It turns out, as it might be expected, that the new matrix is identical with the old. This fact is now brought out in detail.

If $(\omega_1, \omega_2, \omega_3)$ and $(\omega_1', \omega_2', \omega_3')$ represent the components of $\vec{\omega}$ relative to $y_1y_2y_3$ and $x_1x_2x_3$, respectively, then $\int_{S_1} \mu\vec{r} \times (\vec{\omega} \times \vec{r})d\tau$ (i.e., $I\vec{\omega}$) may be written in either of the two ways:

$$(45) \qquad I\vec{\omega} = I_1\omega_1\vec{e}_{y_1} + I_2\omega_2\vec{e}_{y_2} + I_2\omega_3\vec{e}_{y_3}$$

$$= (I'_{11}\omega'_1 + I'_{12}\omega'_2 + I'_{13}\omega'_3)\vec{e}_{x_1}$$

$$+ (I'_{21}\omega'_1 + I'_{22}\omega'_2 + I'_{23}\omega'_3)\vec{e}_{x_2}$$

$$+ (I'_{31}\omega'_1 + I'_{32}\omega'_2 + I'_{33}\omega'_3)\vec{e}_{x_3}$$

where $I'_{11} = \int_{S_1} \mu(x_2^2 + x_3^2)d\tau$, $I'_{12} = -\int_{S_1} \mu x_1 x_2 d\tau$, ... , $I'_{33} = \int_{S_1} \mu(x_1^2 + x_2^2)d\tau$ are the components of the inertia tensor in the new system. From Eq. 45, it is clear that we can write $I\vec{\omega}$ as

$$\begin{pmatrix} I'_{11} & I'_{12} & I'_{13} \\ I'_{21} & I'_{22} & I'_{23} \\ I'_{31} & I'_{32} & I'_{33} \end{pmatrix} \begin{pmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{pmatrix} = I'\vec{\omega}$$

if, on carrying out the matrix multiplication, we regard the result as giving the components of a vector relative to the new $x_1x_2x_3$ system.

Now since the x_1 axis coincides with y_1, the same symmetry arguments apply to the terms I'_{ij} as were used in evaluating the terms of

$$\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} .$$

Hence the two matrices are equal; $I'_{11} = I_1$, $I'_{22} = I'_{33} = I_2$, $I'_{ij} = 0$ for $i \neq j$. Another way to see the equivalence of I and I' is to express I' explicitly in terms of I. This can be done by using Eq. 45. The result is found to be

$$I' = P^{-1}\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} P$$

where P is the matrix of the transformation from $y_1y_2y_3$ coordinates to $x_1x_2x_3$ coordinates. In other words, I' is obtained from I by a similarity transformation about the axis of symmetry. It is a well-known fact that the moment of inertia is invariant under such a transformation.

Unfortunately, the gases in the combustion chamber do not possess this symmetry, and their moment of inertia changes under the transformation. This means that if we regard Eq. 44 as a vector equation having components in $x_1x_2x_3$ rather than $y_1y_2y_3$, we must replace $(I + I_g)\vec{\omega}$ by $(I + I_g')\vec{\omega}$, where I_g' is the moment of inertia of S_g relative to the $x_1x_2x_3$ system.

However in <u>either</u> system, the individual entries of I_g (or I_g') are so small <u>compared to</u> the diagonal entries of I that their cumulative effect on the equations is negligible, if not altogether undetectable. So in accordance with Assumption 13, I_g' will be dropped from the expression for $\vec{L}$.

We now differentiate Eq. 44, obtaining

$$(46) \quad \frac{d\vec{L}_{(t_o)}}{dt} = \frac{d}{dt}\left[M_g\vec{R}_g \times (\vec{V}_g + \vec{\omega} \times \vec{R}_g)\right] + \frac{d}{dt}(I\vec{\omega})$$

$$+ \frac{d}{dt}\left\{M_e\left[\vec{R}_e \times (\vec{V}_e + \vec{\omega} \times \vec{R}_e) + \rho^2\left(\vec{\omega} - \frac{1}{2}\vec{\omega}_N\right)\right]\right\}$$

where all derivatives are evaluated at t_o.

If we observe that $\dot{M}g \doteq 0$, $\dot{\vec{R}}g \doteq -Vcg\,\vec{e}_1$, $\dot{\vec{V}}g \doteq 0$, $\vec{R}g \doteq -Rg\vec{e}_1$, and $\vec{V}g \doteq -Vg\,\vec{e}_1$, then it follows after some rather lengthy (but simple) calculations that

$$\frac{d}{dt}\left[M_g\vec{R}_g \times (\vec{V}_g + \vec{\omega} \times \vec{R}_g)\right] \doteq \frac{d}{dt}\left[M_g\vec{R}_g \times (\vec{\omega} \times \vec{R}_g)\right].$$

Furthermore, since $\dot{M}_e(t_o) = 0$, we have for every vector $\vec{A}$,

$$\frac{d}{dt}(M_eA)\Big|_{t_o} = \dot{M}_{(t_o)}A_{(t_o)} \quad .$$

The right side of Eq. 46 may be written more simply as

$$\frac{d}{dt}[M_g\vec{R}_g \times (\vec{\omega} \times \vec{R}_g) + I\vec{\omega}] + \dot{M}\left[R_e \times (\vec{V}_e + \vec{\omega} \times \vec{R}_e) + \rho^2\left(\vec{\omega} - \frac{1}{2}\vec{\omega}_N\right)\right].$$

So if $\vec{N}$ represents the total external moment applied to S about its c.m.,

$$\text{(47)} \qquad \vec{N} = \frac{d}{dt}[M_g\vec{R}_g \times (\vec{\omega} \times \vec{R}_g) + I\vec{\omega}] + \dot{M}\left[\vec{R}_e \times (\vec{V}_e + \vec{\omega} \times \vec{R}_e) + \rho^2\left(\vec{\omega} - \frac{1}{2}\vec{\omega}_N\right)\right].$$

The terms on the right side of Eq. 47 are now evaluated.

Since $\vec{R}_g = -R_g\vec{e}_1$ and $\vec{V}_g = -V_g\vec{e}_1$,

$$M_g\vec{R}_g \times (\vec{\omega} \times \vec{R}_g) = M_gR_g^2\vec{e}_1 \times (\vec{\omega} \times \vec{e}_1) = M_gR_g^2\vec{\omega}_N .$$

Also, if $\delta/\delta t$ is the time derivative operator relative to $x_1x_2x_3$ and if $\vec{\omega}_c$ is the angular velocity of the $x_1x_2x_3$ system relative to $x_oy_oz_o$, then

$$\frac{d}{dt} = \frac{\delta}{\delta t} + \vec{\omega}_c \times .$$

Hence,

$$\frac{d}{dt}[M_g\vec{R}_g \times (\vec{\omega} \times \vec{R}_g)] = \frac{\delta}{\delta t}(M_gR_g^2\vec{\omega}_N) + M_gR_g^2\vec{\omega}_c \times \vec{\omega}_N .$$

Now by Assumptions 1 and 20, $\dot{M}_g \doteq 0$ and $\dot{R}_g \doteq V_{cg}$. So we have

$$\text{(48)} \quad \frac{d}{dt}[M_g\vec{R}_g \times (\vec{\omega} \times \vec{R}_g)] \doteq 2M_gR_gV_{cg}\vec{\omega}_N + M_gR_g^2\left(\frac{\delta\vec{\omega}_N}{\delta t} + \vec{\omega}_c \times \vec{\omega}_N\right).$$

Similarly,

$$\frac{d}{dt}(I\vec{\omega}) = \frac{\delta}{\delta t}(I\vec{\omega}) + \omega_c \times (I\vec{\omega}) .$$

It is easily seen by writing out the components of $I\vec{\omega}$ in $x_1x_2x_3$ coordinates that

$$\frac{\delta}{\delta t}(I\vec{\omega}) = \dot{I}\vec{\omega} + I\frac{\delta\vec{\omega}}{\delta t}$$

where

$$\dot{I} = \begin{pmatrix} \dot{I}_1 & 0 & 0 \\ 0 & \dot{I}_2 & 0 \\ 0 & 0 & \dot{I}_2 \end{pmatrix} \qquad \vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad \text{and} \quad \frac{\delta\vec{\omega}}{\delta t} = \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix}$$

and $\omega_1\omega_2\omega_3$ are the components of $\vec{\omega}$ relative to $x_1x_2x_3$ coordinates. There results

$$\frac{d}{dt}(I\vec{\omega}) = \dot{I}\vec{\omega} + I\frac{\delta\vec{\omega}}{\delta t} + \vec{\omega}_c \times (I\vec{\omega}) \tag{49}$$

where the right side is regarded as a vector expressed in its $x_1x_2x_3$ components.

Using Eq. 48 and 49, Eq. 47 becomes

$$\vec{N} = 2M_gR_gV_{cg}\vec{\omega}_N + M_gR_g^2\left(\frac{\delta\vec{\omega}_N}{\delta t} + \vec{\omega}_c \times \vec{\omega}_N\right) \tag{50}$$

$$+ \dot{I}\vec{\omega} + I\frac{\delta\vec{\omega}}{\delta t} + \vec{\omega}_c \times (I\vec{\omega}) + \dot{M}\left[\vec{R}_e \times (\vec{V}_e + \vec{\omega}\times\vec{R}_e) + \rho^2\left(\vec{\omega} - \frac{1}{2}\vec{\omega}_N\right)\right].$$

Thrust has been defined as

$$\vec{T} = -\dot{M}\vec{V}_e + \int_{A_e}(\vec{p}_e - \vec{p}_o)dA.$$

So $\dot{M}\vec{R}_e \times \vec{V}_e$ can be written as

$$\vec{R}_e \times (\dot{M}\vec{V}_e) = -\vec{R}_e \times \vec{T} + \int_{A_e}\vec{R}_e \times (\vec{p}_e - \vec{p}_o)dA.$$

Using this in Eq. 50, there results

(51) $\vec{N} - \int_{A_e} \vec{R}_e \times (\vec{p}_e - \vec{p}_o) dA + \vec{R}_e \times \vec{T} = 2M_g R_g V_{cg} \vec{\omega}_N$

$$+ M_g R_g^2 \left(\frac{\delta \vec{\omega}_N}{\delta t} + \vec{\omega}_c \times \vec{\omega}_N \right) + \dot{I}\vec{\omega} + I \frac{\delta \vec{\omega}}{\delta t} + \vec{\omega}_c \times (I\vec{\omega})$$

$$+ \dot{M} \left[\vec{R}_e \times (\vec{\omega} \times \vec{R}_e) + \rho^2 \left(\vec{\omega} - \frac{1}{2} \vec{\omega}_N \right) \right].$$

Now $\vec{N}$ is the total moment applied to S about its c.m. This includes the moment produced by the exhaust pressure at the exit plane as well as the "pure" aerodynamic moment. These two components of $\vec{N}$ will now be separated and examined.

$\vec{N}$ is defined as the total moment applied to S about its center of mass by all the external forces acting on S, at the instant t. Now at time t_o, S consists solely of the fixed missile parts S_1 and the gases in the combustion chamber. These gases obviously have negligible mass, hence the c.m. of S and the c.m. of S_1 are very nearly coincident. So at t_o, we may as well take $\vec{N}$ as the total moment about the c.m. of S_1 (Assumption 18).

Since gravity makes no contribution to the moment, we can regard $\vec{N}$ as the moment produced exclusively by the pressure and drag forces acting on the exterior surface of S. Let A represent this entire surface. Then an element dA of A contributes an increment $d\vec{N} = \vec{r} \times \vec{f}\, dA$ to the total moment $\vec{N}$, where $\vec{r}$ is the vector from the c.m. of S_1 to dA, and where $\vec{f}$ is the total external force per unit area at this point. So $\vec{N}$ is given by

$$\vec{N} = \int_A \vec{r} \times \vec{f}\, dA .$$

We split this integral into two parts. If A_e represents the surface area of the exit plane and $A - A_e$ represents the rest of the surface of the missile, then

(52) $$\vec{N} = \int_{A-A_e} \vec{r} \times \vec{f}\, dA + \int_{A_e} \vec{r} \times \vec{f}\, dA .$$

Now at all points of $A - A_e$, $\vec{f}$ can be thought of as the sum of a "static" pressure and an "aerodynamic" pressure. The static pressure is the force per unit area that would be exerted at points of $A - A_e$ if the missile were held motionless in its current configuration. This static pressure is denoted by $\vec{f}_s$. The aerodynamic pressure $\vec{f}_a$ on $A - A_e$ is defined by $\vec{f}_a = \vec{f} - \vec{f}_s$. So $\vec{f}_a$ may be thought of as that part of f that is due solely to motion.

Writing $\vec{f} = \vec{f}_a + \vec{f}_s$ in Eq. 52, we obtain

(53) $$\vec{N} = \vec{N}_a + \int_{A-A_e} \vec{r} \times \vec{f}_s \, dA + \int_{A_e} \vec{r} \times \vec{f} \, dA$$

where

$$\vec{N}_a = \int_{A-A_e} \vec{r} \times \vec{f}_a \, dA \, .$$

$\vec{N}_a$ is ordinarily called the total aerodynamic moment.

We can go a step further in Eq. 53. If a missile is held fixed in the configuration of S at time t, it is clear that the atmospheric pressure exerts no net moment about the c.m. This means that

$$\int_A \vec{r} \times \vec{f}_s dA = \int_{A_e} \vec{r} \times \vec{f}_s dA + \int_{A-A_e} \vec{r} \times \vec{f}_s dA = 0 \, .$$

Hence,

$$\int_{A-A_e} \vec{r} \times \vec{f}_s dA = - \int_{A_e} \vec{r} \times \vec{f}_s dA$$

and Eq. 53 becomes

$$\vec{N} = \vec{N}_a + \int_{A_e} \vec{r} \times (\vec{f} - \vec{f}_s) dA \, .$$

If we observe that $\vec{f} = \vec{p}_e$ and $\vec{f}_s = \vec{p}_a$ on A_e, then

(54) $$\vec{N} = \vec{N}_a + \int_{A_e} \vec{r} \times (\vec{p}_e - \vec{p}_a) dA \, .$$

Using Eq. 54, we can write the quantity

$$\vec{N} - \int_{A_e} \vec{R}_e \times (\vec{p}_e - \vec{p}_o) dA$$

in Eq. 51 as

$$\vec{N} - \int_{A_e} \vec{R}_e \times (\vec{p}_e - \vec{p}_o) dA = \vec{N}_a + \int_{A_e} (\vec{r} - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a) dA + R_e \times \int_{A_e} (\vec{p}_o - \vec{p}_a) dA \, .$$

Since $\vec{p}_o = p_o\vec{e}_1$ and $\vec{p}_a = p_a\vec{e}_1$,

$$\vec{N} - \int_{A_e} \vec{R}_e \times (\vec{p}_e - \vec{p}_o)dA = \vec{N}_a + \int_{A_e} (\vec{r} - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a)dA$$

$$+ (p_o - p_a)A_e\vec{R}_e \times \vec{e}_1 .$$

Now for a low-altitude missile, $p_o - p_a$ is small. And since $\vec{R}_e$ and $\vec{e}_1$ are practically colinear, $\vec{R}_e \times \vec{e}_1$ is very small. Hence

$$\vec{N} - \int_{A_e} \vec{R}_e \times (\vec{p}_e - \vec{p}_o)dA \doteq \vec{N}_a + \int_{A_e} (\vec{r} - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a)dA.$$

So Eq. 51 may be written

$$(55) \quad \vec{N}_a + \int_{A_e} (\vec{r} - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a)dA + \vec{R}_e \times \vec{T} = 2M_gR_gV_{cg}\vec{\omega}_N$$

$$+ M_gR_g^2\left(\frac{\delta\vec{\omega}_N}{\delta t} + \vec{\omega}_c \times \vec{\omega}_N\right) + \dot{I}\vec{\omega} + I\,\frac{\delta\vec{\omega}}{\delta t} + \vec{\omega}_c \times (I\vec{\omega})$$

$$+ \dot{M}\left[\vec{R}_e \times (\vec{\omega} \times \vec{R}_e) + \rho^2\left(\vec{\omega} - \frac{1}{2}\,\vec{\omega}_N\right)\right].$$

Let us now examine the term $\int_{A_e}(\vec{r} - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a)dA$ in Eq. 55. The integrand may be written as

$$(\vec{r} - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a) = (\vec{r} - \vec{R}_p) \times (\vec{p}_e - \vec{p}_a) + \vec{R}_p - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a)$$

$$= [(\vec{r} - \vec{R}_p)(p_e - p_a) + (\vec{R}_p - \vec{R}_e)(p_e - p_a)] \times \vec{e}_1 ,$$

since $\vec{p}_e = p_e\vec{e}_1$ and $\vec{p}_a = p_a\vec{e}_1$, and where

$$\vec{R}_p = \frac{1}{\int_{A_e} p_e dA} \int_{A_e} \vec{r}p_e dA$$

is the "center of pressure."

We carry out the integration, observing that $\int_{A_e}(\vec{r} - \vec{R}_p)p_e dA = 0$, and obtain

$$\int_{A_e}(\vec{r} - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a)dA = [-p_a \int_{A_e}(\vec{r} - \vec{R}_p)dA + (\vec{R}_p - \vec{R}_e)(p_e - p_a)] \times \vec{e}_1 .$$

By symmetry and Assumption 6, $\int_{A_e}(\vec{r} - R_p)dA \doteq 0$. Furthermore, $\vec{R}_p - \vec{R}_e$ is small, and for a well-designed rocket nozzle, $p_o \doteq p_a$. Hence $(\vec{R}_p - \vec{R}_e)(p_e - p_a)$ is a second-order term and it may be discarded.

This means that the integral $\int_{A_e}(\vec{r} - \vec{R}_e) \times (\vec{p}_e - \vec{p}_a)dA$ can be dropped out of Eq. 55, leaving

$$(56)\quad \vec{N}_a + \vec{R}_e \times \vec{T} = 2M_g R_g V_{cg}\vec{\omega}_N + M_g R_g^2 \left(\frac{\delta\vec{\omega}_N}{\delta t} + \vec{\omega}_c \times \vec{\omega}_N\right) + \dot{I}\vec{\omega} + I\frac{\delta\vec{\omega}}{\delta t}$$

$$+ \vec{\omega}_c \times (I\vec{\omega}) + \dot{M}\left[\vec{R}_e \times (\vec{\omega} \times \vec{R}_e) + \rho^2\left(\vec{\omega} - \frac{1}{2}\vec{\omega}_N\right)\right] .$$

The individual terms in Eq. 56 will now be expressed in terms of their components relative to $x_1x_2x_3$.

It is shown in the section on Thrust Misalignment that

$$(57)\quad \vec{R}_e \times \vec{T} = \delta_R\delta_T \sin(\varphi_{T_o} - \varphi_{R_o})\vec{e}_1 + (T\delta_R \cos\varphi_R + R_e\delta_T \sin\varphi_T)\vec{e}_2$$

$$+ (T\delta_R \sin\varphi_R + R_e\delta_T \sin\varphi_T)\vec{e}_3$$

where δ_R, δ_T, φ_R, and φ_T are as defined in that section.

Since $\vec{\omega} = \omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3$,

$$\frac{\delta\vec{\omega}}{\delta t} = \dot{\omega}_1\vec{e}_1 + \dot{\omega}_2\vec{e}_2 + \dot{\omega}_3\vec{e}_3$$

$$\vec{\omega}_N = \omega_2\vec{e}_2 + \omega_3\vec{e}_3$$

and

$$\frac{\delta \vec{\omega}_N}{\delta t} = \dot{\omega}_2 \vec{e}_2 + \dot{\omega}_3 \vec{e}_3 \ ,$$

the following identities are obvious:

$$(58) \quad \begin{cases} 2M_g R_g V_{cg} \vec{\omega}_N = 2M_g R_g V_{cg} (\omega_2 \vec{e}_2 + \omega_3 \vec{e}_3) \\ M_g R_g^2 \dfrac{\delta \vec{\omega}_N}{\delta t} = M_g R_g^2 (\dot{\omega}_2 \vec{e}_2 + \dot{\omega}_3 \vec{e}_3) \\ \dot{I}\vec{\omega} = \dot{I}_1 \omega_1 \vec{e}_1 + \dot{I}_2 \omega_2 \vec{e}_2 + \dot{I}_2 \omega_3 \vec{e}_3 \\ I \dfrac{\delta \vec{\omega}}{\delta t} = I_1 \dot{\omega}_1 \vec{e}_1 + I_2 \dot{\omega}_2 \vec{e}_2 + I_3 \dot{\omega}_3 \vec{e}_3 \\ \dot{M}\rho^2 \left(\vec{\omega} - \dfrac{1}{2} \vec{\omega}_N \right) = \dot{M}\rho^2 \left(\omega_1 \vec{e}_1 + \dfrac{1}{2} \omega_2 \vec{e}_2 + \dfrac{1}{2} \omega_3 \vec{e}_3 \right) . \end{cases}$$

To evaluate $\dot{M}\vec{R}_e \times (\vec{\omega} \times \vec{R}_e)$, let $\vec{e}_N = -\sin \phi_R \vec{e}_2 + \cos \phi_R \vec{e}_3$. Then $\vec{R}_e$ can be written $\vec{R}_e \doteq -R_e \vec{e}_1 + \delta_R \vec{e}_N$ (see section on Thrust Misalignment). The triple cross product can be expanded to

$$\vec{R}_e \times (\vec{\omega} \times \vec{R}_e) = R_e^2 \vec{e}_1 \times (\vec{\omega} \times \vec{e}_1) - \delta_R R_e [\vec{e}_N \times (\vec{\omega} \times \vec{e}_1) + \vec{e}_1 \times (\vec{\omega} \times \vec{e}_N)] + \delta_R^2 \vec{e}_N \times (\vec{\omega} \times \vec{e}_N).$$

The third term is of second order in δ_R and is negligible. Using the expansion formula $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$, we obtain

$$\vec{e}_N \times (\vec{\omega} \times \vec{e}_1) + \vec{e}_1 \times (\vec{\omega} \times \vec{e}_N) = (\omega_2 \sin \phi_R - \omega_3 \cos \phi_R) \vec{e}_1 + \omega_1 \sin \phi_R \vec{e}_2 - \omega_1 \cos \phi_R \vec{e}_3$$

and

$$\vec{e}_1 \times (\vec{\omega} \times \vec{e}_1) = \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3 \ .$$

Hence,

$$\dot{M}\vec{R}_e \times (\vec{\omega} \times \vec{R}_e) = \dot{M}\delta_R R_e(\omega_3 \cos\varphi_R - \omega_2 \sin\varphi_R)\vec{e}_1 + \dot{M}(R_e^2\omega_2 - \delta_R R_e \omega_1 \sin\varphi_R)\vec{e}_2 + \dot{M}(R_e^2\omega_3 + \delta_R R_e\omega_1 \cos\varphi_R)\vec{e}_3 \; . \tag{59}$$

The right side of Eq. 59 is the so-called "jet damping" moment. This can be seen by putting $\delta_R = 0$ (i.e., no thrust offset), obtaining

$$\dot{M}\vec{R}_e \times (\vec{\omega} \times \vec{R}_e) = \dot{M}R_e^2(\omega_2\vec{e}_2 + \omega_3\vec{e}_3) \; .$$

This is the expression ordinarily taken as the jet damping term.

It is shown in Appendix C that $\vec{\omega}_c = \omega_2 \tan\theta\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3$. The following cross products can easily be calculated:

$$M_gR_g^2\vec{\omega}_c \times \vec{\omega}_N = M_gR_g^2(-\omega_2\omega_3 \tan\theta\vec{e}_2 + \omega_2^2 \tan\theta\vec{e}_3)\vec{\omega}_c \times I\vec{\omega} = (I_1\omega_1\omega_3 - I_2\omega_2\omega_3 \tan\theta)\vec{e}_2 + (I_2\omega_2^2 \tan\theta - I_1\omega_1\omega_2)\vec{e}_3 \; . \tag{60}$$

Using Eq. 57 to 60, we may write Eq. 56 out in components in the form

$$(61)\quad \begin{cases} N_1 + \delta_R\delta_T \sin(\phi_{T_o} - \phi_{R_o}) - \dot{M}\delta_R R_e(\omega_3 \cos\phi_R - \omega_2 \sin\phi_R) = (\dot{I}_1 + \dot{M}\rho^2)\omega_1 + I_1\dot{\omega}_1 \\ N_2 + \delta_R(T\cos\phi_R + \dot{M}R_e\omega_1 \sin\phi_R) + \delta_T R_e \cos\phi_T \\ \quad = \left[\dot{I}_2 + \dot{M}\left(R_e^2 + \frac{1}{2}\rho^2\right) + 2M_gR_gV_{cg}\right]\omega_2 + (I_2 + M_gR_g^2)(\dot{\omega}_2 - \omega_2\omega_3 \tan\theta) + I_1\omega_1\omega_3 \\ N_3 + \delta_R(T \sin\phi_R - \dot{M}R_e\omega_1 \cos\phi_R) + \delta_T R_e \sin\phi_T \\ \quad = \left[\dot{I}_2 + \dot{M}\left(R_e^2 + \frac{1}{2}\rho^2\right) + 2M_gR_gV_{cg}\right]\omega_3 + (I_2 + M_gR_g^2)(\dot{\omega}_3 + \omega_2^2 \tan\theta) - I_1\omega_1\omega_3 . \end{cases}$$

Here $N_1N_2N_3$ are the components of the "pure" aerodynamic moment $\vec{N}_a$ about the center of mass of the fixed parts of the missile, defined by

$$\vec{N}_i = \vec{N}_a \cdot \vec{e}_i$$

where

$$\vec{N}_a = \int_{A-A_e} \vec{r} \times \vec{f}_a dA,$$

and

$$\vec{f}_a = \vec{f} - \vec{f}_s$$

is the "pure" aerodynamic force per unit area acting on the missile surface. This moment must be written out as an explicit function of the system variables $V_1V_2V_3$, ω_1, ω_2, ω_3, etc., when it is desired to actually obtain a solution to the motion equations.

To reiterate the auxiliary conditions, we have

$$\delta_T = T \sin \beta$$

$$\omega_1 = \dot{\psi} \sin \theta - \dot{\varphi}$$

$$\omega_2 = \dot{\psi} \cos \theta$$

$$\omega_3 = \dot{\theta}$$

$$\varphi_T(t) = [\varphi_{T_0} - \varphi_{(0)}] + \varphi_{(t)}$$

$$\varphi_R(t) = [\varphi_{R_0} - \varphi_{(0)}] + \varphi_{(t)} \ .$$

T, I_1, I_2, and M are to be given as functions of time in general.

δ_R, R_e, ρ, M_g, and R_g are given as fixed quantities for a given flight. All appropriate initial conditions are, of course, to be specified as well.

The system (Eq. 61) is much more general than is ordinarily necessary to solve ballistic problems. Furthermore, the precision one needs in his solution varies from problem to problem. For this reason, we shall carry the system of equations through a series of successive simplifications.

Many of the terms on the right side of Eq. 61, being exceedingly small in relation to their companions, may be discarded immediately. The terms I_2, R_e, R_g, ρ, M_g, $\dot{M}$, and V_{cg} were evaluated at nominal values for the ASROC missile. It was found that $M_g R_g^2 << I_2$, $\frac{1}{2}\rho^2 << R_e^2$, and $2M_g R_g V_{cg} << \dot{M} R_e^2$. So we conclude that

$$\dot{I}_2 + \dot{M}\left(R_e^2 + \frac{1}{2}\rho^2\right) + 2M_g R_g V_{cg} \doteq \dot{I}_2 + \dot{M} R_e^2$$

and

$$I_2 + M_g R_g^2 \doteq I_2 \; .$$

In the light of these approximations, Eq. 61 becomes

$$(62) \quad \begin{cases} N_1 + \delta_R \delta_T \sin(\phi_{T_o} - \phi_{R_o}) - \dot{M}\delta_R R_e(\omega_3 \cos\phi_R - \omega_2 \sin\phi_R) \\ \qquad = (\dot{I}_1 + \dot{M}\rho^2)\omega_1 + I_1\dot{\omega}_1 \\ N_2 + \delta_R(T\cos\phi_R + \dot{M}R_e\omega_1 \sin\phi_R) + \delta_T R_e \cos\phi_T = (\dot{I}_2 + \dot{M}R_e^2)\omega_2 \\ \qquad + I_2(\dot{\omega}_2 - \omega_2\omega_3 \tan\theta) + I_1\omega_1\omega_3 \\ N_3 + \delta_R(T\sin\phi_R - \dot{M}R_e\omega_1 \cos\phi_R) + \delta_T R_e \sin\phi_T = (\dot{I}_2 + \dot{M}R_e^2)\omega_3 \\ \qquad + I_2(\dot{\omega}_3 + \omega_2^2 \tan\theta) - I_1\omega_1\omega_2 . \end{cases}$$

In many studies of the effect of thrust misalignment, it is assumed that $\delta_R \doteq 0$: that there is no thrust offset, the thrust vector intersecting the missile axis at some small angle. Under this assumption, the equations simplify further to

$$(63) \quad \begin{aligned} N_1 &= (\dot{I}_1 + \dot{M}\rho^2)\omega_1 + I_1\dot{\omega}_1 \\ N_2 + \delta_T R_e \cos\phi_T &= (\dot{I}_2 + \dot{M}R_e^2)\omega_2 + I_2(\dot{\omega}_2 - \omega_2\omega_3\tan\theta) + I_1\omega_1\omega_3 \\ N_3 + \delta_T R_e \sin\phi_T &= (\dot{I}_2 + \dot{M}R_e^2)\omega_3 + I_2(\dot{\omega}_3 + \omega_2^2\tan\theta) - I_1\omega_1\omega_2 . \end{aligned}$$

The simplest set of equations is obtained by setting $\delta_T \doteq 0$ (no thrust misalignment at all), $\dot{M}R_e^2 \doteq 0$ (no jet damping), and $\dot{I}_1 + \dot{M}\rho^2 \doteq 0$. There results:

$$N_1 = I_1\dot{\omega}_1$$

(64) $$N_2 = \dot{I}_2\omega_2 + I_2(\dot{\omega}_2 - \omega_2\omega_3 \tan\theta) + I_1\omega_1\omega_3$$

$$N_3 = \dot{I}_2\omega_3 + I_2(\dot{\omega}_3 + \omega_2^2 \tan\theta) - I_1\omega_1\omega_2 .$$

It may be noted, in the last case, that if $N_1 = 0$ (no bent fins, etc.), ω_1 turns out to be constant and that one need solve only the two remaining equations.

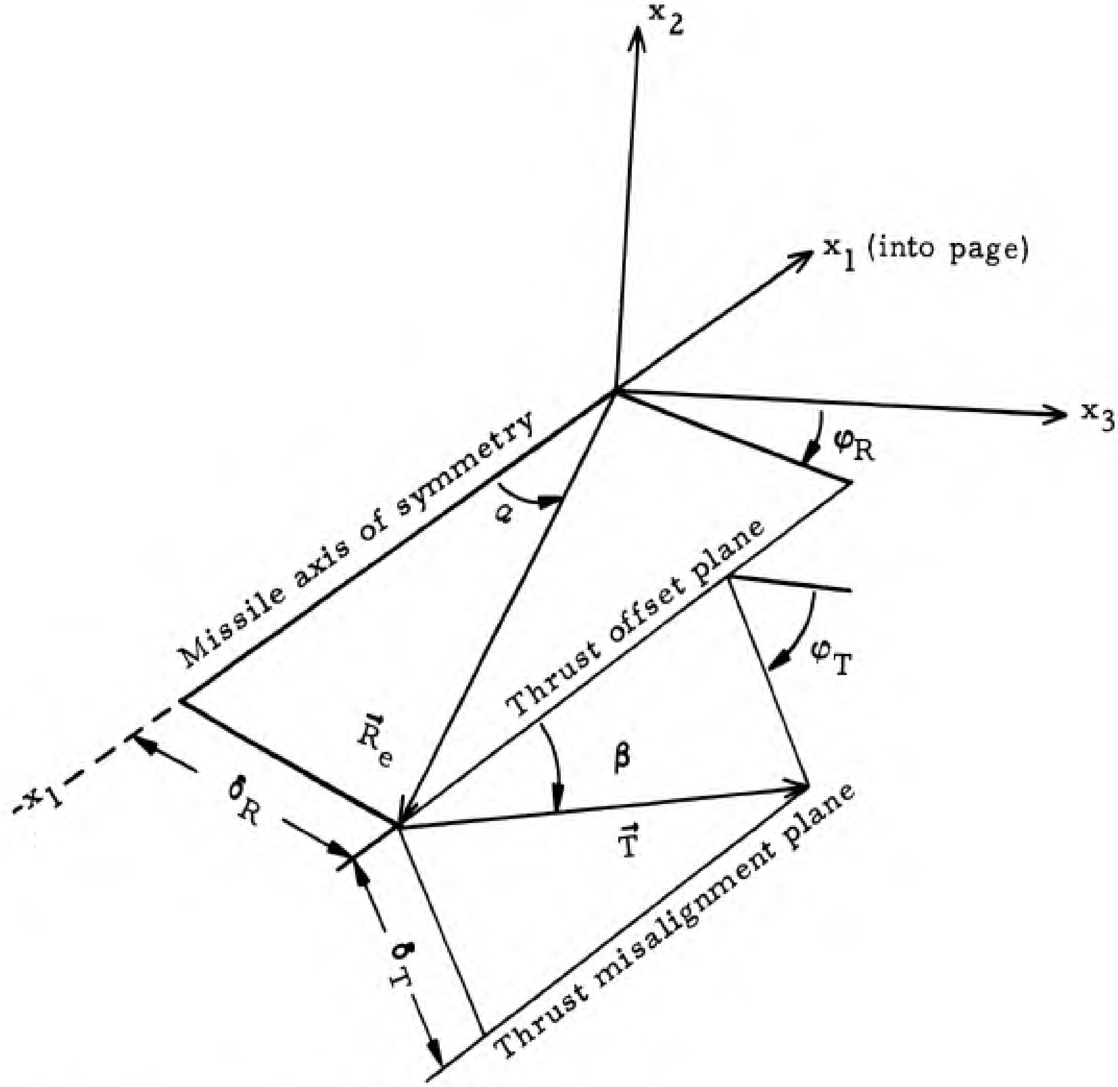
THRUST MISALIGNMENT

We first make a definition. The thrust misalignment of a rocket is the vector $\vec{R}_e \times \vec{T}$. Roughly speaking, thrust misalignment is the moment of the thrust "force" about the c.g. of the missile. This moment may be resolved into components perpendicular to and parallel to the missile axis. The first component introduces additional pitching and yawing into the motion of the missile; the second introduces spin.

In many expositions on rocket ballistics, the spin component is either tacitly disregarded or overlooked entirely by making a somewhat loaded assumption. This assumption is that the vectors associated with thrust, i.e., $\vec{T}$ and $\vec{R}_e$, are coplanar with the missile axis. Another consequence of this assumption is that the thrust offset (the distance, measured on the exit plane, between the missile axis and the terminus of the vector $\vec{R}_e$) and the direction of the thrust vector are closely related to one another: that the effects of both of these anomalies can be represented by a single lumped parameter, called the "effective thrust offset." It should be emphasized, however, that thrust offset and thrust direction are entirely independent in their effect on the equations of motion. These ideas are clarified in the following discussion.

We wish to calculate $\vec{R}_e \times \vec{T}$. Assuming that neither $\vec{R}_e$ nor $\vec{T}$ is parallel to x_1, let the thrust offset plane be the plane determined by $\vec{R}_e$ and the x_1 axis, and the thrust misalignment plane be the plane containing $\vec{T}$ and a line parallel to the x_1 axis (see figure below). Let ϕ_R and ϕ_T be the angles between these planes, respectively, and the x_3 axis, where both angles are taken positive in the sense of a positive rotation about x_1.

Let α and β be the angles between $\vec{R}_e$ and $-\vec{e}_1$, and between $\vec{T}$ and $+\vec{e}_1$, respectively. Let δ_R and δ_T be the magnitudes of the components of $\vec{R}_e$ and $\vec{T}$ normal to the x_1 axis, respectively. δ_R and δ_T are called the thrust offset distance and the normal component of thrust. It follows that $\delta_R = R_e \sin \alpha$ and $\delta_T = T \sin \beta$.



Resolving $\vec{R}_e$ and $\vec{T}$ into components relative to $x_1x_2x_3$, we obtain

$$\vec{R}_e = R_e(-\cos\alpha\,\vec{e}_1 - \sin\alpha\sin\varphi_R\vec{e}_2 + \sin\alpha\cos\varphi_R\,\vec{e}_3)$$

$$\vec{T} = T(\cos\beta\,\vec{e}_1 - \sin\beta\,\sin\varphi_T\vec{e}_2 + \sin\beta\cos\varphi_T\vec{e}_3)\ .$$

In terms of δ_R and δ_T, and observing that $\cos\alpha \doteq 1$, $\cos\beta \doteq 1$, this becomes

(65) $$\vec{R}_e = (-R_e\vec{e}_1 - \delta_R\sin\varphi_R\vec{e}_2 + \delta_R\cos\varphi_R\vec{e}_3)$$

$$\vec{T} = (T\,\vec{e}_1 - \delta_T\sin\varphi_T\vec{e}_2 + \delta_T\cos\varphi_T\vec{e}_3)\ .$$

The cross product $\vec{R}_e \times \vec{T}$ may be calculated directly, giving

$$(66) \quad \vec{R}_e \times \vec{T} = \delta_R \delta_T \sin(\varphi_T - \varphi_R)\vec{e}_1 + (T\delta_R \cos\varphi_R + R_e\delta_T \cos\varphi_T)\vec{e}_2 + (T\delta_R \sin\varphi_R + R_e\delta_T \sin\varphi_T)\vec{e}_3 \, .$$

Since it is assumed that $\vec{R}_e$ and $\vec{T}$ are fixed relative to the missile, it follows that $\dot{\varphi}_T = \dot{\varphi}$ and $\dot{\varphi}_R = \dot{\varphi}$. Hence

$$\varphi_T = \varphi_{T_o} + \int_o^t \dot{\varphi}\, dt, \quad \varphi_R = \varphi_{R_o} + \int_o^t \dot{\varphi} dt$$

and

$$\varphi_T - \varphi_R = \varphi_{T_o} - \varphi_{R_o} \, .$$

It is easily seen that Eq. 66 holds true in the case where one or both of $\vec{T}$ and $\vec{R}_e$ are parallel to the x_1 axis, so long as the correct values are used for δ_R and δ_T.

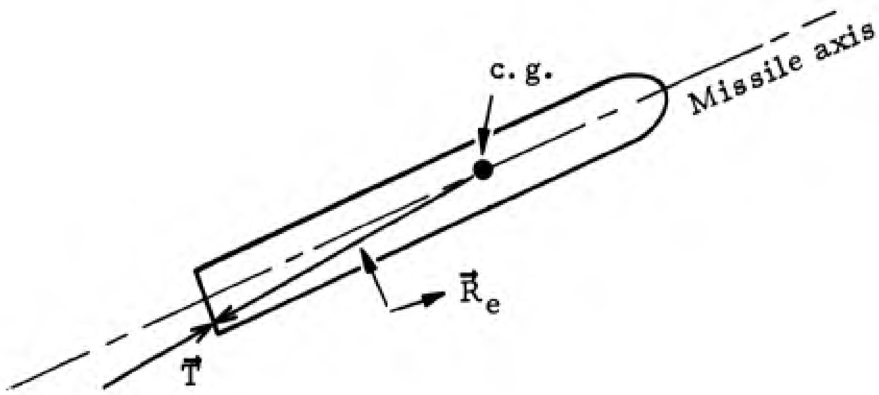
Now if we assume that $\vec{R}_e$, $\vec{T}$, and x_1 are all coplanar, then it follows that $\varphi_T = \varphi_R$, and Eq. 66 takes the form

$$(67) \quad \vec{R}_e \times \vec{T} = (T\delta_R + R_e\delta_T) \cos\varphi_R \, \vec{e}_2 + (T\delta_R + R_e\delta_T) \sin\varphi_R \, \vec{e}_3 \, .$$

Hence we see that δ_R and δ_T effectively coalesce into a single parameter $T\delta_R + R_e\delta_T$, producing a moment perpendicular to the x_1 axis. However, in the general case, δ_R and δ_T act independently. In fact, it can be shown directly from Eq. 66 that the effect of δ_R and δ_T cannot be reproduced by a single lumped parameter.

One more point deserves mention. It is entirely possible for $\vec{R}_e \times \vec{T}$ to be zero without having $\vec{T}$ act along the axis of the missile. For such a case, we must indeed have $\vec{R}_e$ and $\vec{T}$ colinear, and this line must intersect the c.m. of the missile. But this line need not be parallel to the axis of the missile.

Although there is no thrust misalignment term in the rotational equations in this case, there will of course be a normal component of thrust in the translatory equations.



GRAVITATIONAL FORCE (g) CORRECTIONS

In many applications of ballistic equations, it is sufficient to assume that the gravity vector remains constant in magnitude and remains oriented parallel to the y_o axis as the missile proceeds down range. However, for long-range firings, it may be desirable to have available the correction terms needed to reduce the error in this assumption.

During an actual flight, the magnitude of the gravitational force acting on a missile varies with altitude, according to the inverse square law; it varies in direction, acting always along the radius vector to the center of the earth rather than along the y_o axis. In this section, these variations will be considered and the first-order approximation will be derived for their magnitudes.

It is assumed that the earth is a sphere, generating a central inverse square gravitational field at its geometric center (see sketch on next page).

Let $\vec{g}_{(r)}$ be the gravitational vector at the missile, let $g_{(r)}$ be its magnitude, and let g be the acceleration of gravity at sea level.

Then by the inverse square law,

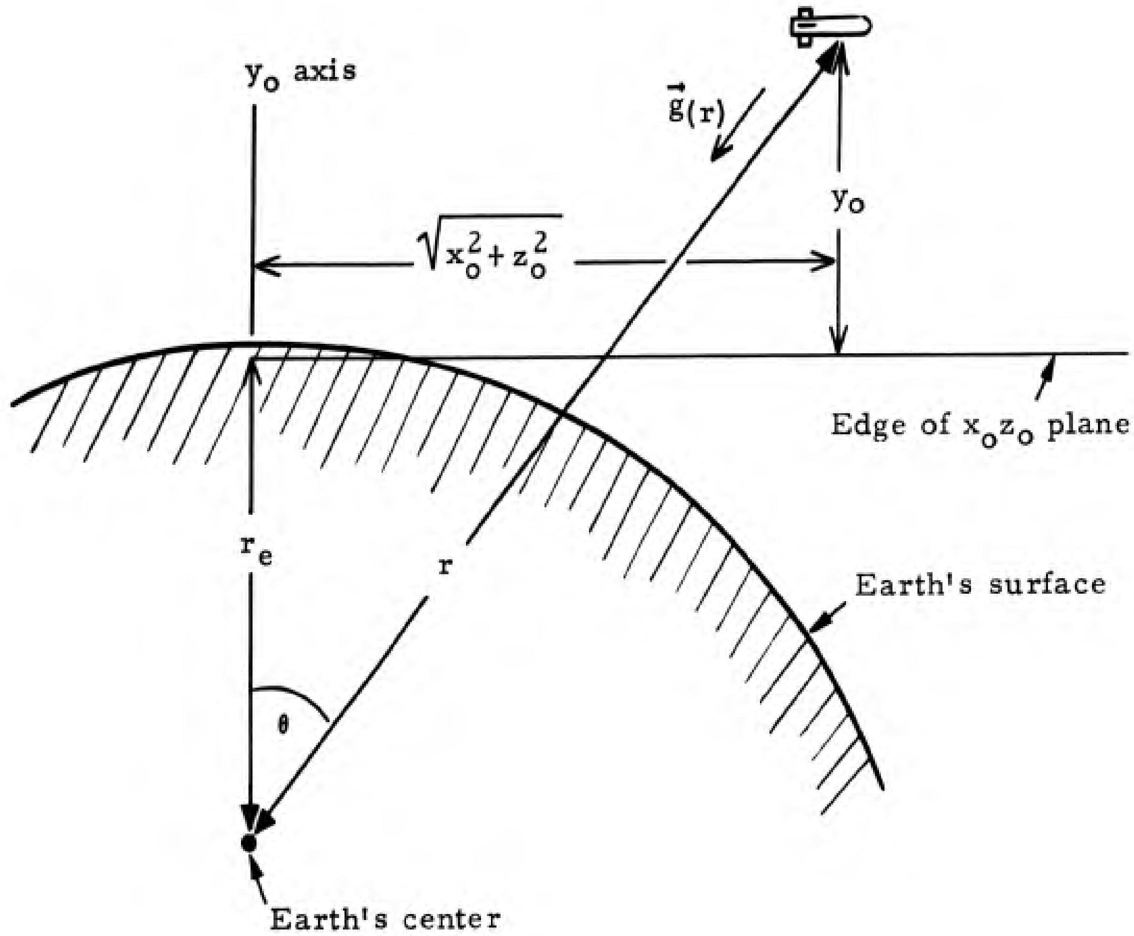
$$g_{(r)} = \left(\frac{r_e}{r}\right)^2 g$$

From the sketch it is evident that $r \cos\theta = y_o + r_e$, so that

$$\left(\frac{r_e}{r}\right)^2 = \left(\frac{r_e}{y_o + r_e}\right)^2 \cos^2\theta = \left(1 + \frac{y_o}{r_e}\right)^{-2} (1 - \sin^2\theta)$$

or

$$\left(\frac{r_e}{r}\right)^2 = \left(1 + \frac{y_o}{r_e}\right)^{-2} \left(1 - \frac{x_o^2 + z_o^2}{r^2}\right)$$



The right side of the equation above may be expanded as a power series in $\frac{1}{r_e}$. Retaining only the first-order terms, there results

$$\left(\frac{r_e}{r}\right)^2 \doteq 1 - 2\left(\frac{y_o}{r_e}\right)$$

so that

(68)
$$g(r) \doteq g\left(1 - 2\,\frac{y_o}{r_e}\right)$$

This expresses the magnitude of the acceleration of gravity as a function of altitude. It remains to compute the direction of $\vec{g}$.

Let $\vec{e}_g$ be a unit vector in the direction of $\vec{g}(r)$, defined by $\vec{e}_g = \frac{1}{g(r)}\,\vec{g}(r)$. The components of $\vec{e}_g$ relative to the $x_o y_o z_o$ axes are easily calculated, giving

$$\vec{e}_g \cdot \vec{e}_{x_o} = -\sin\theta \frac{x_o}{\sqrt{x_o^2 + z_o^2}} = -\frac{x_o}{r}$$

$$\vec{e}_g \cdot \vec{e}_{y_o} = -\cos\theta = -\left(1 - \frac{x_o^2 + z_o^2}{r^2}\right)^{1/2}$$

$$\vec{e}_g \cdot \vec{e}_{z_o} = -\sin\theta \frac{z_o}{\sqrt{x_o^2 + z_o^2}} = -\frac{z_o}{r} \; .$$

But

$$\frac{x_o}{r} = \frac{x_o}{r_e}\left(1 + \frac{y_o}{r_e}\right)^{-1} = \frac{x_o}{r_e}\left(1 - \frac{y_o}{r_e} + \frac{y_o^2}{r_e^2} - \ldots\right) .$$

So to first order in $1/r_e$,

$$\frac{x_o}{r} \doteq \frac{x_o}{r_e} \; .$$

Similarly,

$$\frac{z_o}{r} \doteq \frac{z_o}{r_e} \quad \text{and} \quad \left(1 - \frac{x_o^2 + y_o^2}{r^2}\right)^{1/2} \doteq 1 \; .$$

Hence

$$\vec{e}_g \doteq -\frac{x}{r_e}\vec{e}_{x_o} - \vec{e}_{y_o} - \frac{z_o}{r_e}\vec{e}_{z_o} \; .$$

And again neglecting second-order terms, we can write

$$(69) \quad \vec{g}_{(r)} = g_{(r)}\vec{e}_g \doteq -g\vec{e}_{y_o} - g\left(\frac{x_o}{r_e}\vec{e}_{x_o} - 2\frac{y_o}{r_e}\vec{e}_{y_o} + \frac{z_o}{r_e}\vec{e}_{z_o}\right) .$$

The first term on the right side of Eq. 69 is what is usually taken for $\vec{g}_{(r)}$. The second term is the first-order correction.

It should be noticed that, since in most applications the x_o axis is taken in the direction of the launcher line, z_o displacements are usually very much smaller than x_o displacements. For such a case, the coefficient of $\vec{e}_{z_o}$ in Eq. 69 is insignificant compared to x_o/r_e or y_o/r_e. Disregarding this term reduces Eq. 69 to

$$\vec{g}_{(r)} = -g\vec{e}_{y_o} - g\left(\frac{x_o}{r_e}\vec{e}_{x_o} - 2\frac{y_o}{r_e}\vec{e}_{y_o}\right) . \tag{70}$$

ACCELEROMETERS

Many ballistic missiles carry timing devices to activate certain components at preset velocities or distances from launch. One of the simplest is the integrating accelerometer.

Although there are many types of integrating accelerometers, they all share a common feature: they perform some operation on the acceleration which they sense. Of course, the sensed acceleration is not identical with the inertial acceleration of the missile; but the conversion from one to the other is a simple matter.

The accelerometer is assumed to be a force meter, attached to the fixed parts of the missile on the symmetry axis, which measures the force it exerts on a mass m_a in carrying m_a along with the missile. In addition, this device is assumed to sense only the component of force (acceleration) parallel to the missile axis.

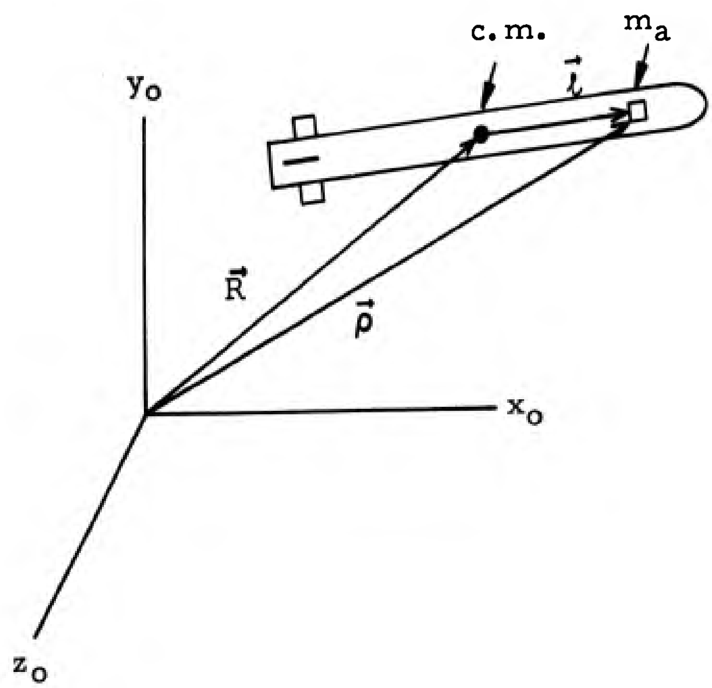
During a ballistic flight, the only mechanical agents acting on m_a are the accelerometer and gravity. So if $\vec{F}_a$ represents the force exerted on m_a by the accelerometer, the total force acting on m_a is $\vec{F}_a + m_a\vec{g}$, and the sensed acceleration is

$$\frac{1}{m_a}\vec{F}_a \cdot \vec{e}_1 .$$

If $\vec{\rho}$ is the position vector of m_a relative to the ground coordinate system $x_o y_o z_o$ (assumed to be inertial), Newton tells us that

$$\vec{F}_a + m_a\vec{g} = m_a\frac{d^2\vec{\rho}}{dt^2} . \tag{71}$$

$d^2\vec{\rho}/dt^2$ may be evaluated in terms of the missile acceleration as follows. Let $\vec{\ell}$ be the vector from the c.m. of S_1 to the accelerometer, and let $\vec{R}$ be the position vector of the c.m. of S_1 in the $x_o y_o z_o$ system.



Then $\vec{R} = \vec{\rho} - \vec{\ell}$, and

$$\frac{d^2\vec{\rho}}{dt^2} = \frac{d^2\vec{R}}{dt^2} + \frac{d^2\vec{\ell}}{dt^2} . \tag{72}$$

d^2R/dt^2 is the translatory acceleration of the missile, and a simplified expression for this is obtained from the section on Translatory Equations as

$$\frac{d^2\vec{R}}{dt^2} = \frac{\vec{T}}{M} + \frac{\vec{F}}{M} + \vec{g} \tag{73}$$

where $\vec{F}$ is the aerodynamic force on the missile.

Applying the rule

$$\frac{d\vec{A}}{dt} = \dot{\vec{A}} + \vec{\omega} \times \vec{A}$$

we obtain

$$\frac{d\vec{\ell}}{dt} = \dot{\vec{\ell}} + \vec{\omega} \times \vec{\ell} = -\vec{V}_{cg} + \vec{\omega} \times \vec{\ell}$$

and

$$\frac{d^2\vec{\ell}}{dt^2} = -\dot{V}_{cg} + \dot{\vec{\omega}} \times \vec{\ell} + \omega \times \dot{\vec{\ell}} - \vec{\omega} \times \vec{V}_{cg} + \vec{\omega} \times (\vec{\omega} \times \vec{\ell}) .$$

Since $\dot{\vec{V}}_{cg} \doteq 0$,

$$\frac{d^2\vec{\ell}}{dt^2} = \vec{\omega} \times (\vec{\omega} \times \vec{\ell}) - 2\vec{\omega} \times \vec{V}_{cg} + \dot{\vec{\omega}} \times \vec{\ell} . \tag{74}$$

Equation 71 may now be written

$$\vec{F}_a = m_a \left(\frac{d^2\vec{\rho}}{dt^2} - \vec{g} \right) \tag{75}$$

$$= m_a \left[\frac{\vec{T}}{M} + \frac{\vec{F}}{M} + \vec{\omega} \times (\vec{\omega} \times \ell) - 2\vec{\omega} \times \vec{V}_{cg} + \dot{\vec{\omega}} \times \vec{\ell} \right] .$$

The acceleration sensed by the accelerometer is the x_1 component of

$$\frac{1}{m_a} \vec{F}_a$$

or

$$\frac{1}{m_a} \vec{F}_a \cdot \vec{e}_1 = \left[\frac{\vec{T}}{M} + \frac{\vec{F}}{M} + \vec{\omega} \times (\vec{\omega} \times \vec{\ell}) - 2\vec{\omega} \times \vec{V}_{cg} + \dot{\vec{\omega}} \times \vec{\ell} \right] \cdot \vec{e}_1 . \tag{76}$$

Using the facts that $\vec{\ell} = \ell \vec{e}_1$ and $\vec{V}_{cg} \doteq V_{cg}\vec{e}_1$, $\vec{T} \cdot \vec{e}_1 \doteq T$, there results

$$\vec{T} \cdot \vec{e}_1 \doteq T$$

$$\vec{F} \cdot \vec{e}_1 = -D$$

where D is the magnitude of the axial component of drag,

$$[\vec{\omega} \times (\vec{\omega} \times \vec{\ell})] \cdot \vec{e}_1 = (\omega_1 \ell \vec{\omega} - \omega^2 \vec{\ell}) \cdot \vec{e}_1 = \ell(\omega_1^2 - \omega^2) = -\ell(\omega_2^2 + \omega_3^2),$$

$$(\vec{\omega} \times \vec{V}_{cg}) \cdot \vec{e}_1 = \begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ V_{cg} & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 ,$$

and

$$(\dot{\vec{\omega}} \times \vec{\ell}) \cdot \vec{e}_1 = \begin{vmatrix} \dot{\omega}_1 & \dot{\omega}_2 & \dot{\omega}_3 \\ \ell & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 .$$

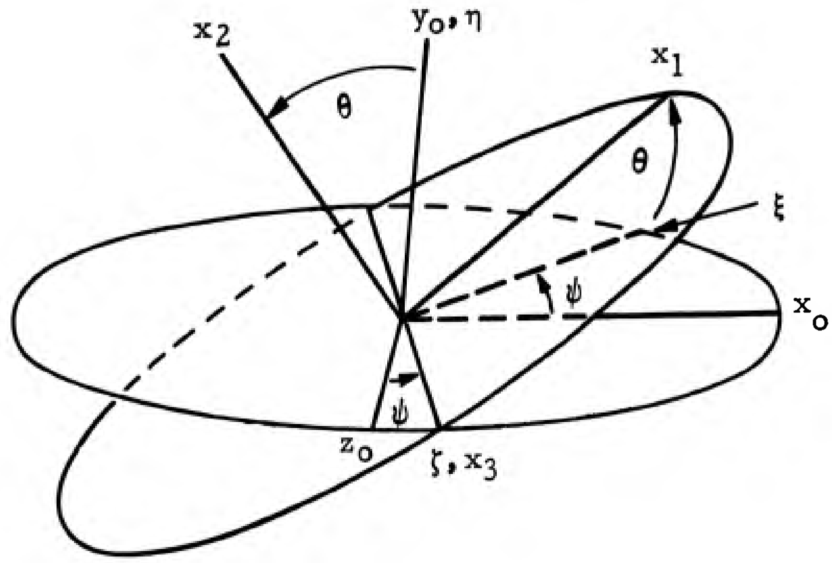
Finally, we can write the acceleration sensed by the accelerometer as

$$\frac{1}{m_a} \vec{F}_a \cdot \vec{e}_1 = \frac{T - D}{M} - \ell(\omega_2^2 + \omega_3^2) \tag{77}$$

Appendix A

THE TRANSFORMATION MATRIX

A vector may be resolved into components relative to either the "ground" system $x_o y_o z_o$ or the moving system $x_1 x_2 x_3$. These two sets of components are related by a linear transformation; the matrix of this transformation is derived below.



Let $\vec{V}$ be an arbitrary vector, and let $V_{x_o} = \vec{V} \cdot \vec{e}_{x_o}$, $V_{y_o} = \vec{V} \cdot \vec{e}_{y_o}$, $V_{z_o} = \vec{V} \cdot \vec{e}_{z_o}$, $V_\xi = \vec{V} \cdot \vec{e}_\xi$, etc. What is desired, then, is the matrix Q of the expression

$$\begin{pmatrix} V_{x_1} \\ V_{x_2} \\ V_{x_3} \end{pmatrix} = Q \begin{pmatrix} V_{x_o} \\ V_{y_o} \\ V_{z_o} \end{pmatrix}$$

It is clear that if T_1 and T_2 are the matrices

$$\begin{pmatrix} V_{x_1} \\ V_{x_2} \\ V_{x_3} \end{pmatrix} = T_1 \begin{pmatrix} V_\xi \\ V_\eta \\ V_\zeta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} V_\xi \\ V_\eta \\ V_\zeta \end{pmatrix} = T_2 \begin{pmatrix} V_{x_o} \\ V_{y_o} \\ V_{z_o} \end{pmatrix}$$

then $Q = T_1 T_2$.

T_1 and T_2 are easily calculated, giving

$$T_1 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Hence

$$Q = \begin{pmatrix} \cos\theta\ \cos\psi & \sin\theta & -\cos\theta\ \sin\psi \\ -\sin\theta\ \cos\psi & \cos\theta & \sin\theta\ \sin\psi \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Since T_1 and T_2 are both orthogonal, so is Q; and its inverse is simply the transposed matrix

$$Q^{-1} = Q^t = \begin{pmatrix} \cos\theta\ \cos\psi & -\sin\theta\ \cos\psi & \sin\psi \\ \sin\theta & \cos\theta & 0 \\ -\cos\theta\ \sin\psi & \sin\theta\ \sin\psi & \cos\psi \end{pmatrix}.$$

Appendix B

ROTATING COORDINATE SYSTEMS AND TIME DERIVATIVES OF VECTORS

It was stated in the first section that Newton's law applies only to mechanical systems referenced to one of a very special family of coordinate systems: the inertial frames. More precisely, to apply the principle $\vec{F} = m \dfrac{d^2\vec{r}}{dt^2}$ to a particle, one must first define the position vector $\vec{r}$ in an inertial frame, then he must take the time derivatives of $\vec{r}$ relative to a coordinate system which is irrotational with respect to the inertial frame.

It is convenient, in practice, to employ rotating coordinate systems in deriving the equations of a mechanical system (e.g., the body axes of a missile). Of course, the rate of change of a vector as seen by an observer in a rotating coordinate system is not the same as that observed from a fixed frame. However, there is a simple relation between these two vector quantities.

Let $x_1x_2x_3$ and $x_1'x_2'x_3'$ be two Cartesian frames, in motion relative to one another, and let $\vec{\omega}$ be the angular velocity of $x_1'x_2'x_3'$ relative to $x_1x_2x_3$. If $\vec{e}_1\vec{e}_2\vec{e}_3$ and $\vec{e}_1'\vec{e}_2'\vec{e}_3'$ are unit vectors along the respective axes, and if d/dt represents the time derivative as seen by an observer fixed to $x_1x_2x_3$, then it is easy to see that for the unit vectors,

$$\frac{d\vec{e}_i'}{dt} = \vec{\omega} \times \vec{e}_i' \text{ for } i = 1,\ 2,\ 3$$

by resolving $\vec{e}_i'$ into components in the $x_1x_2x_3$ frame.

Now let a vector $\vec{V}$ be resolved into components relative to both frames:

$$\vec{V} = V_1\vec{e}_1 + V_2\vec{e}_2 + V_3\vec{e}_3 = V_1'\vec{e}_1' + V_2'\vec{e}_2' + V_3'\vec{e}_3' .$$

Then in terms of the $x_1x_2x_3$ components,

$$\frac{d\vec{V}}{dt} = \dot{V}_1\vec{e}_1 + \dot{V}_2\vec{e}_2 + \dot{V}_3\vec{e}_3;$$

and in terms of the moving system,

$$\frac{d\vec{V}}{dt} = \dot{V}'_1\vec{e}'_1 + \dot{V}'_2\vec{e}'_2 + \dot{V}'_3\vec{e}'_3 + V'_1\frac{d\vec{e}'_1}{dt} + V'_2\frac{d\vec{e}'_2}{dt} + V'_3\frac{d\vec{e}'_3}{dt}$$

$$= \dot{V}'_1\vec{e}'_1 + \dot{V}'_2\vec{e}'_2 + \dot{V}'_3\vec{e}'_3 + V'_1\vec{\omega} \times \vec{e}'_1 + V'_2\vec{\omega} \times \vec{e}'_2 + V'_3\vec{\omega} \times \vec{e}'_3$$

$$= \dot{V}'_1\vec{e}'_1 + \dot{V}'_2\vec{e}'_2 + \dot{V}'_3\vec{e}'_3 + \vec{\omega} \times \vec{V} .$$

But $\dot{V}'_1\vec{e}'_1 + \dot{V}'_2\vec{e}'_2 + \dot{V}'_3\vec{e}'_3$ is just the rate of change of $\vec{V}$ as seen by an observer fixed to the rotating system $x'_1x'_2x'_3$. If we denote this by $\dot{\vec{V}}$, then the equation above is

$$\underline{\frac{d\vec{V}}{dt} = \dot{\vec{V}} + \omega \times \vec{V}}$$

Appendix C

THE RELATION BETWEEN THE ANGULAR VELOCITY OF THE MISSILE AND ITS ORIENTATION ANGLES

We first notice that, relative to the $x_1x_2x_3$ frame, the missile rotates about the x_1 axis at the rate $\dot{\varphi}$. So it is clear that the angular velocity $\vec{\omega}$ of the missile is related to the angular velocity $\vec{\omega}_c$ of the $x_1x_2x_3$ frame by the formula

$$\vec{\omega} = \vec{\omega}_c + \dot{\varphi}\vec{e}_1 \quad . \tag{78}$$

So to evaluate the vector $\vec{\omega}$, it suffices merely to evaluate $\vec{\omega}_c$. The figure in Appendix A shows that $\vec{\omega}_c$ is given by

$$\vec{\omega}_c = \dot{\psi}\vec{e}_\eta + \dot{\theta}\,\vec{e}_3,$$

where $\vec{e}_\eta$ and $\vec{e}_3$ are unit vectors along the η and x_3 axes, respectively.

Hence

$$\omega_{c1} = \vec{\omega}_c \cdot \vec{e}_1 = \dot{\psi}\vec{e}_3 \cdot \vec{e}_1 + \dot{\theta}\vec{e}_3 \cdot \vec{e}_1 = \dot{\psi}\sin\theta$$

$$\omega_{c2} = \vec{\omega}_c \cdot \vec{e}_2 = \dot{\psi}\cos\theta$$

$$\omega_{c3} = \vec{\omega}_c \cdot \vec{e}_3 = \dot{\theta} \quad .$$

Using these relations and Eq. 78, it follows that

$$\begin{pmatrix} \omega_{c1} \\ \omega_{c2} \\ \omega_{c3} \end{pmatrix} = \begin{pmatrix} \dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta \\ \dot{\theta} \end{pmatrix} ; \quad \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\psi}\sin\theta + \dot{\varphi} \\ \dot{\psi}\cos\theta \\ \dot{\theta} \end{pmatrix} . \tag{79}$$

These relations may be expressed in the equivalent form

$$\begin{pmatrix} \omega_{c1} \\ \omega_{c2} \\ \omega_{c3} \end{pmatrix} = \begin{pmatrix} \omega_2\tan\theta \\ \omega_2 \\ \omega_3 \end{pmatrix} ; \quad \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\psi}\sin\theta + \dot{\varphi} \\ \dot{\psi}\cos\theta \\ \dot{\theta} \end{pmatrix} . \tag{80}$$

INDEX

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