TG 511

July 1963

DERIVATION OF EQUATIONS FOR CONVERTING FROM GEODETIC COORDINATES TO GEOCENTRIC COORDINATES

by F. T. Heuring

THE JOHNS HOPKINS UNIVERSITY
APPLIED PHYSICS LABORATORY
8621 GEORGIA AVENUE SILVER SPRING, MARYLAND

TG 511

Copy No. 19

July 1963

Derivation Of Equations For Converting From Geodetic Coordinates To Geocentric Coordinates

by F. T. Heuring

THE JOHNS HOPKINS UNIVERSITY
APPLIED PHYSICS LABORATORY
8621 GEORGIA AVENUE SILVER SPRING, MARYLAND

Operating under Contract NOw 62-0604-c with the Bureau of Naval Weapons, Department of the Navy

TABLE OF CONTENTS

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

DERIVATION OF EQUATIONS FOR CONVERTING FROM GEODETIC COORDINATES TO GEOCENTRIC COORDINATES

F. T. Heuring

In the A.P.L. orbit computation programs, the TRANET Tracking Stations are specified in a geocentric coordinate system, whereas, particular positions (such as a TRANET Tracking Site) over the Earth are expressed initially in a geodetic coordinate system. In order to acquire geocentric coordinates from a given set of geodetic coordinates a set of transformation equations were derived.

Section I will define the notation, and Section II will embody the derivation of the transformation equations.

I. Notation*

Let:

φ_{G_i} = geodetic latitude of i-th tracking site in its local datum,

λ_{G_i} = geodetic longitude of i-th tracking site in its local datum,

h_i = elevation of i-th tracking site above (below) geoid,

H_i = geoidal height of i-th tracking site in its local datum,

ξ_i = deflection in meridian at i-th tracking site,

η_i = deflection in prime vertical at i-th tracking site,

a_i = equatorial radius of the local datum spheroid of the i-th tracking site, scaled by R_o,

b_i = polar radius of the local datum spheroid on the i-th tracking site, scaled by R_o,

*See References 1 and 2 for definition of geodetic, datum, etc.

$x_{G_i}, y_{G_i}, z_{G_i}$ = cartesian coordinates, scaled by R_o, on i-th datum spheroid as specified by tracking site φ_G and λ_G, (cartesian origin identical to i-th datum origin),

$x_{H_i}, y_{H_i}, z_{H_i}$ = cartesian coordinates, scaled by R_o, on geoid as specified by tracking site H, (cartesian origin identical to i-th datum origin),

$x_{E_i}, y_{E_i}, z_{E_i}$ = cartesian coordinates, scaled by R_o, of tracking site on earth's surface, (cartesian origin identical to i-th datum origin),

$\Delta x_i, \Delta y_i, \Delta z_i$ = center of spheroid of the i-th tracking site datum in the A.P.L. Datum, scaled by R_o,

$\zeta_{G_i} = (x_{G_i}^2 + y_{G_i}^2)^{\frac{1}{2}}$

R_o = equatorial radius of A.P.L. Datum spheroid,

$x_{c_i}, y_{c_i}, z_{c_i}$ = cartesian coordinates of tracking site in A.P.L. geocentric coordinates, scaled by R_o,

r_{c_i} = radius of i-th tracking site in A.P.L. geocentric coordinates, scaled by R_o,

φ_{c_i} = latitude of i-th tracking site in A.P.L. geocentric coordinates,

λ_{c_i} = longitude of i-th tracking site in A.P.L. geocentric coordinates,

$\zeta_{c_i} = (x_{c_i}^2 + y_{c_i}^2)^{\frac{1}{2}}$.

II. Derivation

A. Given φ_{G_i}, λ_{G_i}, a_i and b_i, conversion to x_{G_i}, y_{G_i}, z_{G_i} and ζ_{G_i} is as follows. Using the equation for an ellipse

$$\frac{\zeta_{G_i}^2}{a_i^2} + \frac{z_{G_i}^2}{b_i^2} = 1,$$

in particular the ellipse is a meridional plane of the i-th datum; differentiate z_{G_i} with respect to ζ_{G_i}

$$\frac{\partial z_{G_i}}{\partial \zeta_{G_i}} = - \frac{b_i^2}{a_i^2} \frac{\zeta_{G_i}}{z_{G_i}} .$$

But (see Figure 1A),

$$\frac{\partial z_{G_i}}{\partial \zeta_{G_i}} = - \frac{1}{\tan \varphi_{G_i}}$$

from which by algebraic manipulation (Figure 1B),

$$\zeta_{G_i} = \frac{a_i}{\left(1 + \left(\frac{b_i}{a_i}\right)^2 \tan^2 \varphi_{G_i}\right)^{\frac{1}{2}}} ,$$

after which,

$$\left.\begin{aligned} x_{G_i} &= \zeta_{G_i} \cos \lambda_{G_i} \\ y_{G_i} &= \zeta_{G_i} \sin \lambda_{G_i} \\ z_{G_i} &= \zeta_{G_i} \frac{b_i^2}{a_i^2} \tan \varphi_{G_i} . \end{aligned}\right\} \quad (1)$$

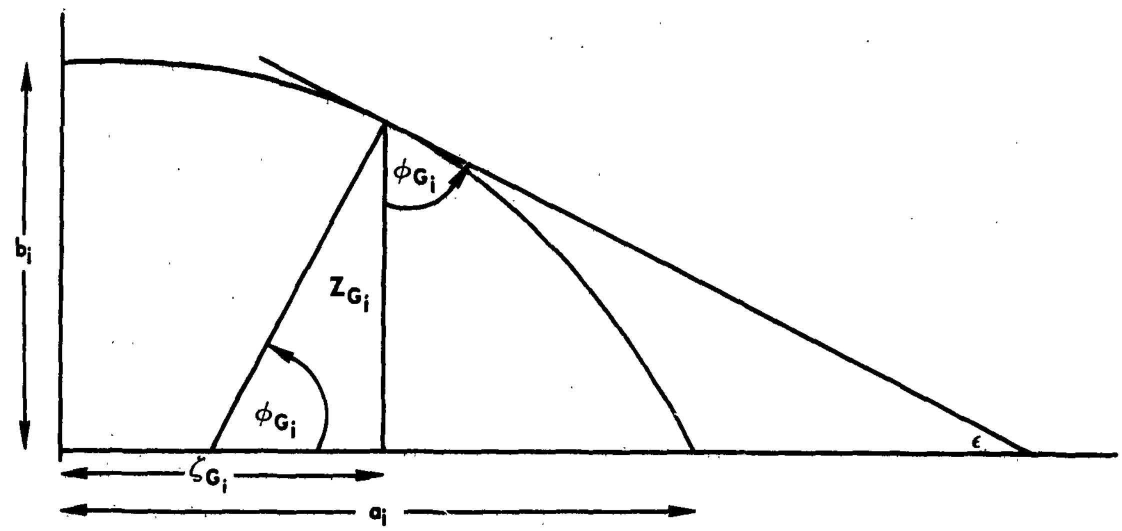

Figure 1A **Meridian Plane in i-th Datum**

$$\frac{1}{\tan \phi_{G_i}} = \cot \phi_{G_i} = \tan \epsilon - \frac{\partial z_{G_i}}{\partial \zeta_{G_i}}$$

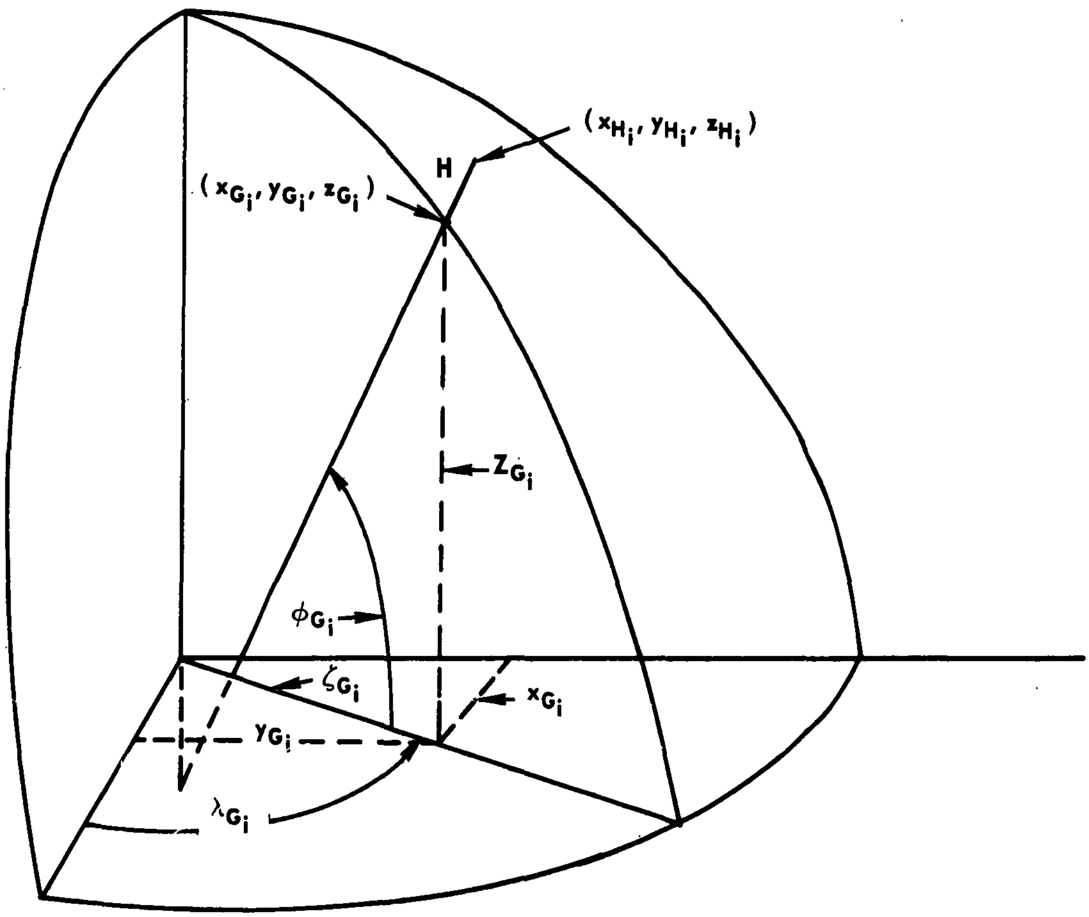

Figure 1B

Pictorial view of geodetic ($\phi_{G_i}, \lambda_{G_i}$), cartesian "geodetic" ($x_{G_i}, y_{G_i}, z_{G_i}$) and cartesian "geoidal" ($x_{H_i}, y_{H_i}, z_{H_i}$) coordinates.

B. Compute x_{H_i}, y_{H_i}, z_{H_i} (Figure 1B). H_i is an extension of the normal to the spheroid, consequently,

$$\begin{aligned} x_{H_i} &= x_{G_i} + H_i \cos \varphi_{G_i} \cos \lambda_{G_i} \\ y_{H_i} &= y_{G_i} + H_i \cos \varphi_{G_i} \sin \lambda_{G_i} \\ z_{H_i} &= z_{G_i} + H_i \sin \varphi_{G_i} . \end{aligned} \tag{2}$$

C. Compute x_{E_i}, y_{E_i}, z_{E_i} by considering h_i, ξ_i and η_i (see Figure 2).

$$\begin{aligned} x_{E_i} &= x_{H_i} + h_i \cos (\varphi_{G_i} + \xi_i) \cos (\lambda_{G_i} + \Delta\lambda_i) \\ y_{E_i} &= y_{H_i} + h_i \cos (\varphi_{G_i} + \xi_i) \sin (\lambda_{G_i} + \Delta\lambda_i) \\ z_{E_i} &= z_{H_i} + h_i \sin (\varphi_{G_i} + \xi_i). \end{aligned} \tag{3}$$

From law of cosines for spherical triangles (Figure 2), $\Delta\lambda_i$ can be approximated.

$$\cos \Delta\lambda_i = \frac{\cos \eta_i - \sin^2 (\varphi_{G_i} + \xi_i)}{\cos^2 (\varphi_{G_i} + \xi_i)}. \tag{4}$$

(Restrict $\Delta\lambda_i$ to have the same sign as η_i).

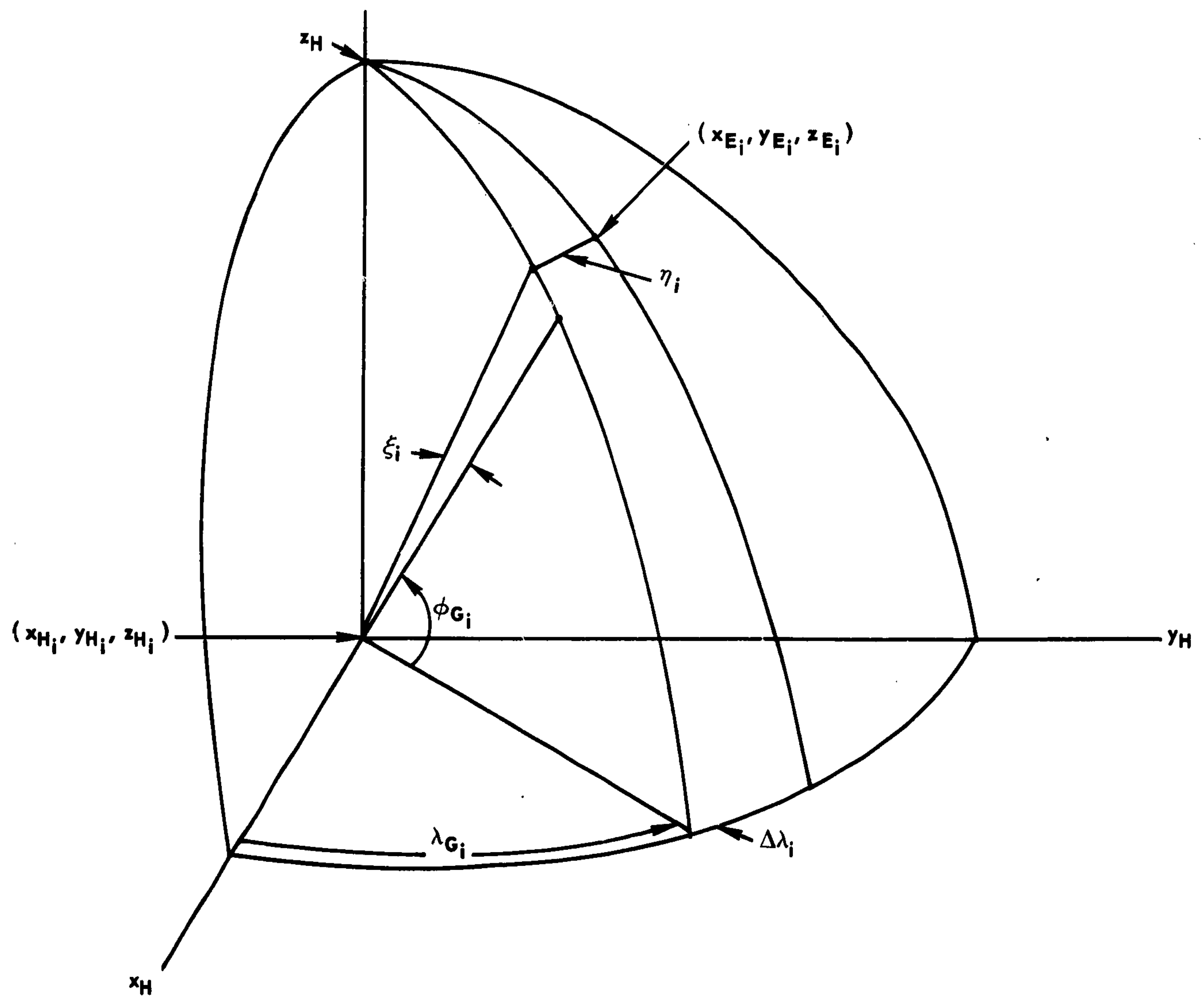

Figure 2
Diagram of the deflections of the vertical (ξ_i and η_i) and the associated quantities necessary to acquire the cartesian coordinates on the geoid from earth surface cartesian coordinates.

D. Let us simplify by expanding small quantities. Assume:

$$\xi_i, \ \eta_i \leqq 30''^{*} \text{ (of arc);}$$

and

$$1^\circ < |\varphi_{G_i}| < 89^\circ;$$

and only take quantities of magnitude ξ_i, η_i and $\Delta\lambda_i$ to second order.

$$\cos \xi_i \doteq 1 - \frac{\xi_i^2}{2}, \ \sin \xi_i \doteq \xi_i$$

$$\cos \eta_i \doteq 1 - \frac{\eta_i^2}{2}, \ \sin \eta_i \doteq \eta_i$$

$$\cos \Delta\lambda_i \doteq 1 - \frac{\Delta\lambda_i^2}{2}$$

thus,

$$\cos^2 (\varphi_{G_i} + \xi_i) = [\cos \varphi_{G_i} (1 - \frac{\xi_i^2}{2}) - \xi_i \sin \varphi_{G_i}]^2$$

$$= (1 - \frac{\xi_i^2}{2})^2 \cos^2 \varphi_{G_i} + \xi_i^2 \sin^2 \varphi_{G_i}$$

$$- 2 \xi_i (1 - \frac{\xi_i^2}{2}) \sin \varphi_{G_i} \cos \varphi_{G_i}$$

$$= \cos^2 \varphi_{G_i} - \xi_i \sin 2\varphi_{G_i} - \xi_i^2 \cos 2\varphi_{G_i} + \text{3rd order} \qquad (5)$$

---

*From a personal communication with Mr. L. Simmons, U.S.C. and G.S., deflection of 30" exist but are in general uncommon.

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

$$\sin^2(\varphi_{G_i} + \xi_i) \doteq [\sin\varphi_{G_i}(1 - \frac{\xi_i^2}{2}) + \xi_i \cos\varphi_{G_i}]^2$$

$$\doteq (1 - \frac{\xi_i^2}{2}) \sin^2\varphi_{G_i} + \xi_i^2 \cos^2\varphi_{G_i} + 2\,\xi_i \sin\varphi_{G_i} \cos\varphi_{G_i}$$

$$= \sin^2\varphi_{G_i} + \xi_i \sin 2\varphi_{G_i} + \xi_i^2 \cos 2\varphi_{G_i} + \text{3rd order} \qquad (6)$$

$$\cos(\varphi_{G_i} + \xi_i) = \cos\varphi_{G_i}(1 - \frac{\xi_i^2}{2}) - \xi_i \sin\varphi_{G_i} = \cos\varphi_{G_i} - \xi_i \sin\varphi_{G_i} - \frac{\xi_i^2}{2}\cos\varphi_{G_i} \qquad (7)$$

$$\sin(\varphi_{G_i} + \xi_i) = \sin\varphi_{G_i}(1 - \frac{\xi_i^2}{2}) + \xi_i \cos\varphi_{G_i} = \sin\varphi_{G_i} + \xi_i \cos\varphi_{G_i} - \frac{\xi_i^2}{2}\sin\varphi_{G_i} \qquad (8)$$

from equation (4):

$$1 - \frac{\eta_i^2}{2} = \sin^2(\varphi_{G_i} + \xi_i) + \cos^2(\varphi_{G_i} + \xi_i)[1 - \frac{\Delta\lambda_i^2}{2}],$$

$$= 1 - \frac{\Delta\lambda_i^2}{2}\cos^2(\varphi_{G_i} + \xi_i),$$

$$\Delta\lambda_i = \frac{\eta_i}{\cos(\varphi_{G_i} + \xi_i)},$$

and using equation (7),

$$\Delta\lambda_i = \frac{\eta_i}{\cos\varphi_{G_i}}\left[\frac{1}{1 - \xi_i \tan\varphi_{G_i} - \frac{\xi_1^2}{2}}\right] = \frac{\eta_i}{\cos\varphi_{G_i}}\left[1 + \xi_i \tan\varphi_{G_i} + \frac{\xi_i^2}{2} + \frac{\xi_i^2}{2}\tan^2\varphi_{G_i}\right]$$

$$= \eta_i \sec\varphi_{G_i}\left[1 + \xi_i \tan\varphi_{G_i}\right] + \text{3rd order.} \tag{9}$$

Further,

$$\sin(\lambda_{G_i} + \Delta\lambda_i) = \sin\lambda_{G_i}\left(1 - \frac{\eta_i^2 \sec^2\varphi_{G_i}}{2}\right) + \cos\varphi_{G_i}\,\eta_i \sec\varphi_{G_i}\,(1 + \xi_i \tan\varphi_{G_i})$$

$$= \sin\lambda_{G_i} + \eta_i \frac{\cos\lambda_{G_i}}{\cos\varphi_{G_i}} + \frac{\eta_i}{\cos^2\varphi_{G_i}}\left[\xi_i \cos\lambda_{G_i} \sin\varphi_{G_i} - \frac{\eta_i}{2}\sin\lambda_{G_i}\right] + \text{3rd order} \tag{10}$$

$$\cos(\lambda_{G_i} + \Delta\lambda_i) = \cos\lambda_{G_i} - \sin\lambda_{G_i} \sec\varphi_{G_i}\,(1 + \xi_i \tan\varphi_{G_i})\,\eta_i - \frac{\eta_i^2}{2}\sec^2\varphi_{G_i}\cos\lambda_{G_i}$$

$$= \cos\lambda_{G_i} - \eta_i \frac{\sin\lambda_{G_i}}{\cos\varphi_{G_i}} - \frac{\eta_i}{\cos^2\varphi_{G_i}}\left[\xi_i \sin\lambda_{G_i} \sin\varphi_{G_i} + \frac{\eta_i}{2}\cos\lambda_{G_i}\right] + \text{3rd order.} \tag{11}$$

E. Using equations (1), (2), (3), (7), (8), (9), (10), and (11), x_{E_i}, y_{E_i} and z_{E_i} can be expressed as functions of the geodetic inputs (φ_{G_i}, λ_{G_i}, h_i, H_i, η_i, ξ_i, a_i, and b_i).

$$\left.\begin{aligned}
x_{E_i} &= \zeta_{G_i} \cos \lambda_{G_i} + H_i \cos \varphi_{G_i} \cos \lambda_{G_i} + h_i \left(\cos \varphi_{G_i} - \xi_i \sin \varphi_{G_i}\right) \left(\cos \lambda_{G_i} - \eta_i \frac{\sin \lambda_{G_i}}{\cos \varphi_{G_i}}\right) \\
&= \zeta_{G_i} \cos \lambda_{G_i} + (H_i + h_i) \cos \varphi_{G_i} \cos \lambda_{G_i} - h_i(\xi_i \sin \varphi_{G_i} \cos \lambda_{G_i} + \eta_i \sin \lambda_{G_i}) + \text{3rd order.} \\
y_{E_i} &= \zeta_{G_i} \sin \lambda_{G_i} + H_i \cos \varphi_{G_i} \sin \lambda_{G_i} + h_i(\cos \varphi_{G_i} - \xi_i \sin \varphi_{G_i}) \left(\sin \lambda_{G_i} + \eta_i \frac{\cos \lambda_{G_i}}{\cos \varphi_{G_i}}\right) \\
&= \zeta_{G_i} \sin \lambda_{G_i} + (H_i + h_i) \cos \varphi_{G_i} \sin \lambda_{G_i} - h_i(\xi_i \sin \varphi_{G_i} \sin \lambda_{G_i} - \eta_i \cos \lambda_{G_i}) + \text{3rd order.} \\
z_{E_i} &= \zeta_{G_i} \frac{b_i^2}{a_i^2} \tan \varphi_{G_i} + H_i \sin \varphi_{G_i} + h_i(\sin \varphi_{G_i} + \xi_i \cos \varphi_{G_i}) \\
&= \zeta_{G_i} \frac{b_i^2}{a_i^2} \tan \varphi_{G_i} + (H_i + h_i) \sin \varphi_{G_i} + h_i \xi_i \cos \varphi_{G_i} + \text{3rd order.}
\end{aligned}\right\} \quad (12)$$

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

F. The cartesian coordinates in the A.P.L. geocentric system are:

$$x_{c_i} = x_{E_i} + \Delta x_i$$

$$y_{c_i} = y_{E_i} + \Delta y_i$$

$$z_{c_i} = z_{E_i} + \Delta z_i$$

where Δx_i, Δy_i, and Δz_i are of second order, at best.

H. The cylindrical coordinates $(z_{c_i}, \zeta_{c_i}, \lambda_{c_i})$ in the A.P.L. geocentric system are:

$$z_{c_i} = \zeta_{G_i} \frac{b_i^2}{2} \tan \varphi_{G_i} + (H_i + h_i) \sin \varphi_{G_i} + h_i \xi_i \cos \varphi_{G_i} + \Delta z_i + \text{3rd order} \quad (13)$$

$$\zeta_{c_i}^2 = x_{c_i}^2 + y_{c_i}^2$$

After some algebraic manipulation and using the binominal expansion

$$\zeta_{c_i} = \zeta_{G_i} + (H_i + h_i) \cos \varphi_{G_i} + \Delta x_i \cos \lambda_{G_i} + \Delta y_i \sin \lambda_{G_i} - h_i \xi_i \sin \varphi_{G_i}$$

$$+ (H_i + h_i) \cos \varphi_{G_i} \cdot \frac{1}{\zeta_{G_i}} (\Delta x_i \cos \lambda_{G_i} + \Delta y_i \sin \lambda_{G_i}) \quad (14)$$

$$+ \text{3rd order.}$$

In the derivation of λ_{G_i}, no previously derived quantities were used as was the case with ζ_{c_i}. From Figure 3A, h_i is considered to be zero, thus the angle σ can be approximated as follows:

$$\epsilon_1 + \epsilon_2 = \Delta x_i \sin \lambda_{G_i}$$

$$\epsilon_2 \doteq \Delta y_i \cos \lambda_{G_i}$$ where ϵ_1 and ϵ_2 are normal to ζ_{G_i}, and

$$\epsilon_1 \doteq \Delta x_i \sin \lambda_{G_i} - \Delta y_i \cos \lambda_{G_i}.$$

Since ϵ_1 considered, at best, second order,

$$\sigma \doteq \frac{\epsilon_1}{\zeta_{G_i}}$$

and from the geometry,

$$\lambda_{c_i} = \lambda_{G_i} - \sigma = \lambda_{G_i} - \frac{1}{\zeta_{G_i}} (\Delta x_i \sin \lambda_{G_i} - \Delta y_i \cos \lambda_{G_i}) \qquad (15)$$

Upon including the station elevation (h_i) and deflection in the prime vertical (η_i) (see Figure 3B)

$$\tau = h_i \sin \eta_i = h_i \eta_i \quad (\eta_i \text{ is of magnitude } \pm 30'' \text{ of arc})$$

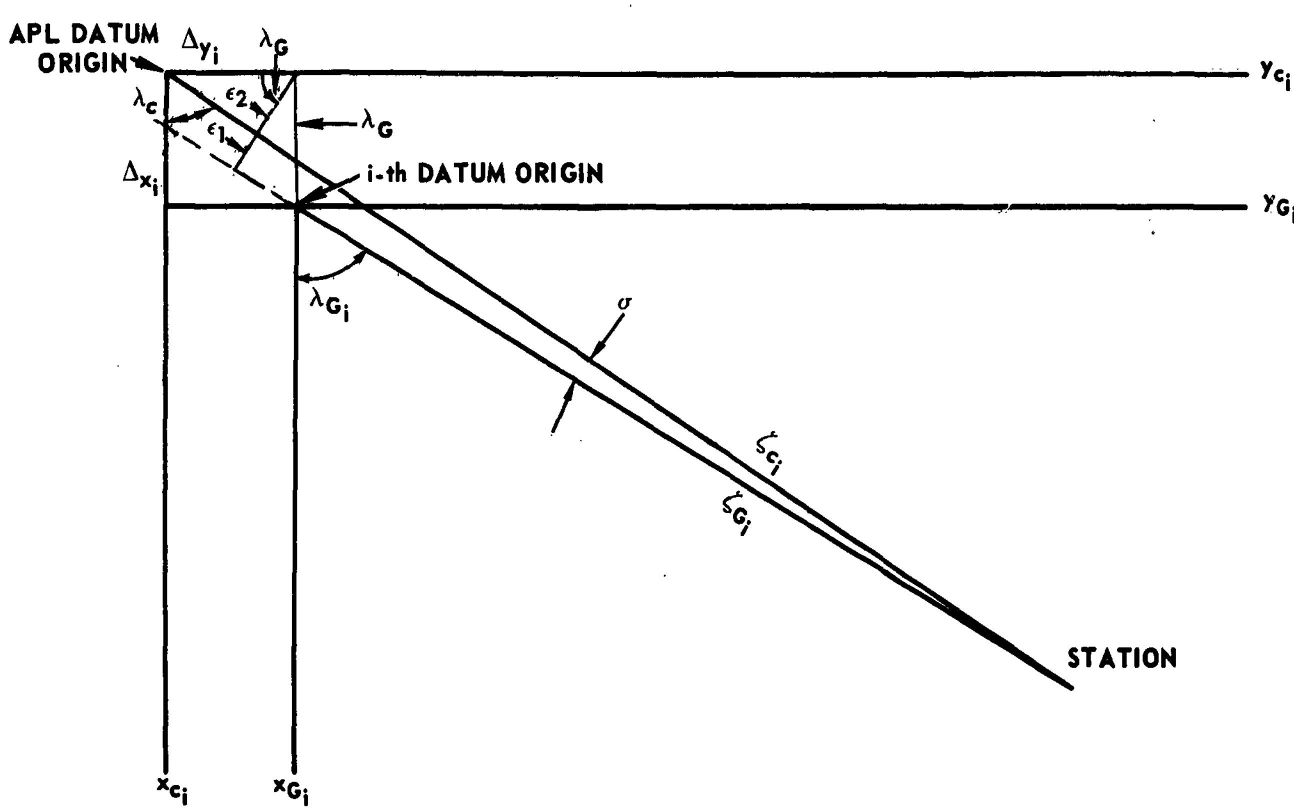

Figure 3A Diagram showing means of determining λ_c when $h_i = 0$.

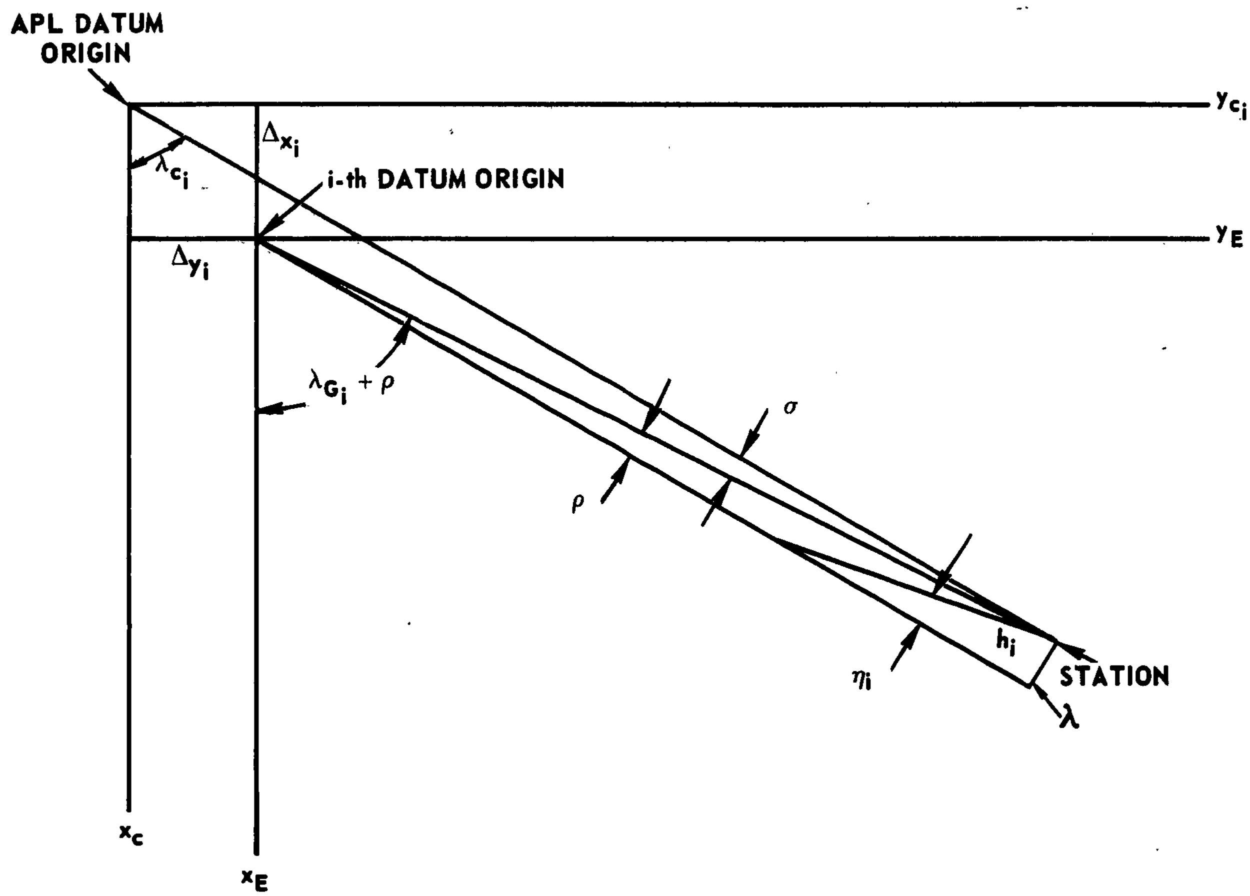

Figure 3B Diagram showing determination of λ_c when $h_i \neq 0$.

and it follows similarly

$$\rho \doteq \frac{\tau}{\zeta_{G_i}} = \frac{h_i \eta_i}{\zeta_{G_i}}$$

From equation (15) and Figure 3B,

$$\lambda_{c_i} = \lambda_{G_i} + \rho - \sigma$$

$$= \lambda_{G_i} + \frac{h_i \eta_i}{\zeta_{G_i}} - \frac{1}{\zeta_{G_i}} [\Delta x_i \sin (\lambda_{G_i} + \rho) - \Delta y_i \cos (\lambda_{G_i} + \rho)]$$

Assuming $\cos \rho = 1 - \frac{\rho^2}{2}$, $\sin \rho = \rho$,

$$\lambda_{c_i} = \lambda_{G_i} + \frac{1}{\zeta_{G_i}} [h_i \eta_i - (\Delta x_i \sin \lambda_{G_i} - \Delta y_i \cos \lambda_{G_i})] + \text{3rd order.} \qquad (16)$$

Equations (13), (14), and (16) are the cylindrical coordinates z_{c_i}, ζ_{c_i}, λ_{c_i} in the A.P.L. Earth fixed coordinate system expressed as a function of the geodetic coordinates of a tracking station.

References

1. Bomford, Brigadier G., "Geodesy", Clarendon Press, 1952.

2. Hasner, George L., "Geodesy", Wiley, Second Edit., 1930, (Chap. V - Properties of the Spheroid and Chap. VIII - Figure of the Earth).

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

Initial distribution of this document has been made in accordance with a list on file in the Technical Reports Group of The Johns Hopkins University, Applied Physics Laboratory.