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ECONOMICS OF INCREASING KNOWLEDGE ABOUT MISSILE RELIABILITY

TECHNICAL DOCUMENTARY REPORT NO. ESD-TDR- 63-290

OCTOBER 1963

S. A. Sobel

Prepared for

DIRECTORATE OF PROGRAMS  
ELECTRONIC SYSTEMS DIVISION  
AIR FORCE SYSTEMS COMMAND  
UNITED STATES AIR FORCE

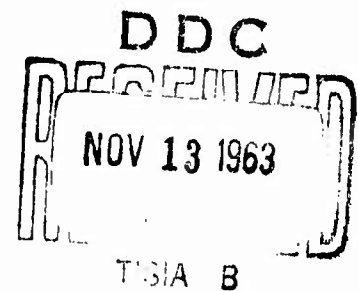
L. G. Hanscom Field, Bedford, Massachusetts



Project 850

Prepared by

THE MITRE CORPORATION  
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Contract AF19(628)-2390



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## ABSTRACT

A small number of rocket boosters are to be deployed in a military system on a standby basis. The system's performance specifications require that in time of need at least  $Q$  missiles fire successfully with  $\gamma$  probability. The reliability of these missiles is not known for certain, but is described, by the analyst, in terms of a subjective unimodal probability distribution.

A technique is described in this report to provide the answer to two questions that can be raised about the situation described above:

- 1) With existing information about the booster, what is the required deployment quantity to meet performance specifications ?
- 2) Is it expected that missile test firings to increase knowledge of missile reliability will reduce deployment costs sufficiently to more than offset test firing costs ?

## ECONOMICS OF INCREASING KNOWLEDGE ABOUT MISSILE RELIABILITY

### SECTION I

#### INTRODUCTION

A small number of rocket boosters are to be deployed in a military system on a standby basis. The system's performance specifications require that, in time of need, at least  $Q$  missiles fire successfully with  $\gamma$  probability. The reliability of these missiles is not known for certain. Nevertheless, the analyst, on the basis of expert opinion, is able to describe the reliability in terms of a subjective unimodal probability distribution.

A technique is developed which incorporates the system performance specification and the assumed prior distribution to answer the following two questions:

- 1) What is the number of missiles which must be deployed to meet the performance specification on the basis of the above distribution with no test program to acquire additional information?
- 2) If test firings are conducted, will the increased knowledge about the actual missile reliability reduce the expected deployment cost by an amount more than the additional costs of testing?

The sequential testing procedure proposed allows the decision to cease further tests to be made at any time that the expected value of the information realizable from the next test is less than the incremental testing cost.

Section II presents, for illustrative purposes, a statement of the problem with a hypothetical set of parameters. Section III offers the general solution, and Section IV solves the hypothetical problem. An author's comment appears in Section V.

## SECTION II

### THE PROBLEM

#### 2.1 Deployment

The performance specifications of a military system employing rocket boosters on a standby basis require that, in time of need, the probability that at least three of the boosters will fire successfully is 95 percent. Missile experts have not had sufficient experience with the booster to know its exact reliability figure, but believe 0.6 is the most likely value, and are 80 percent confident that the reliability lies between 0.40 and 0.75. With this information, it is necessary to determine the number of missiles that should be deployed to meet performance requirements.

#### 2.2 Initial Test Firing

Because of all the costs accompanying the deployment of each missile over the life of the system, a deployed missile is estimated to cost twice as much as one which is test-fired. Since greater knowledge of the missile's reliability might reduce the deployment required to meet performance specifications, the question is raised as to whether a single test shot would more than pay for itself in terms of reduced deployment costs.

#### 2.3 Subsequent Test Firing

After  $n_t$  test firings have been conducted, yielding  $r_t$  successes, the question again can be posed as to whether one more test shot would be economical.

## SECTION III

## GENERAL SOLUTION

3.1 Notation

As a convenience in solving the problems discussed in the preceding Section, the following list of variables will be employed:

|          |                                                                                                                         |
|----------|-------------------------------------------------------------------------------------------------------------------------|
| $R_1$    | expert opinion of most likely value of reliability                                                                      |
| $R_c$    | the difference between the upper and lower bounds to the 80 percent confidence ranges for reliability                   |
| $P_o(r)$ | the prior probability distribution for reliability                                                                      |
| $P_s(r)$ | the posterior probability distribution for reliability, given one successful test firing                                |
| $P_F(r)$ | the posterior probability distribution for reliability, given one unsuccessful test firing                              |
| $D_o$    | the required number of deployed missiles to meet system specifications, given no test firings                           |
| $D_s$    | the required number of deployed missiles, given one successful test firing                                              |
| $D_F$    | the required number of deployed missiles, given one unsuccessful test firing                                            |
| $C_T$    | total cost incurred when a missile is test-fired                                                                        |
| $C_D$    | total system cost associated with procuring and deploying a missile                                                     |
| $C_R$    | the ratio of $C_T/C_D$                                                                                                  |
| $Q$      | the required minimum number of missiles to be successfully fired with some probability of meeting system specifications |
| $\gamma$ | the probability demanded by system specifications that at least $Q$ missiles will be successfully fired                 |

$a, b$  parameters of the beta probability density distribution of the form

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} r^{a-1} (1-r)^{b-1} = P_{\text{BETA}}(r; a, b)$$

$a_o, b_o; a_s, b_s; a_F, b_F$  parameters of the beta function given no test firing, one successful test firing, and one unsuccessful test firing, respectively

$N, n, k, x$  parameters of the Hypergeometric probability distribution of the form

$$\frac{C_x^n C_{k-x}^{N-n}}{C_k^N} = P_{\text{HYPER}}(x; N, n, k)$$

where

$$C_j^i = \frac{i!}{j! (i-j)!} \quad \text{for } i \geq j$$

$N_o, n_o, k_o, x_o$

parameters of the hypergeometric function given no test firing, one successful test firing, and one unsuccessful test firing, respectively

$N_s, n_s, k_s, x_s$

$N_F, n_F, k_F, x_F$

$P_o(S)$  the probability distribution of the number of deployed missiles which will be fired successfully, given  $D_o$  are deployed, and  $a_o$  and  $b_o$  as the parameters of the prior Beta distribution for the missile reliability

$P_s(S)$  equivalent to  $P_o(S)$ , but for  $a_s, b_s$ , and  $D_s$

$P_F(S)$  equivalent to  $P_o(S)$ , but for  $a_F, b_F$ , and  $D_F$

### 3.2 The Prior Distribution of the Reliability

It will be assumed that the prior distribution of the reliability can be satisfactorily represented by a beta distribution. \* In this case, Appendix A2 of Reference 3 demonstrates how  $R_1$  and  $R_c$  can be used to calculate the two parameters of the Beta function. The calculation requires the simultaneous solution of the following equations:

$$(0.007 + 0.38R_c)^2 \frac{a_o b_o}{(a_o + b_o)^2 (a_o + b_o + 1)} = \text{variance} \quad (1)$$

$$R_1 = \frac{a_o - 1}{a_o + b_o - 2} \quad (2)$$

Subsequent computations are simplified by approximating  $a_o$  and  $b_o$  by use of the nearest integral values.

### 3.3 The Posterior Distribution of the Reliability

Reference 1\*\* shows that a prior beta distribution supplemented by sample information from a Bernoulli process produces a Beta posterior distribution. Thus

$$P_S(r) = P_{\text{BETA}}(r; a_S, b_S) \quad (3)$$

where  $a_S = a_o + 1$ ,  $b_S = b_o$ ;

---

\* See Reference 1 for a discussion of the adequacy of this assumption, pp. 43-76.

\*\* Ibid., p. 263.

$$P_F(r) = P_{\text{BETA}}(r; a_F, b_F) \quad (4)$$

where  $a_F = a_O$ ,  $b_F = b_O + 1$ .

### 3.4 The Distribution for Successfully Fired Deployed Missiles

Reference 1\* shows that the distribution for successfully fired deployed missiles, given their reliability is represented by a beta distribution, will be a "beta-binomial mass function." The cumulative form of the latter can be expressed as a cumulative hypergeometric. Reference 1\* demonstrates:

$$P_O(S < Q) = P_{\text{HYPER}}(x \leq Q - 1; N_O, n_O, k_O) \quad (5)$$

where

$$\begin{aligned} N_O &= a_O + b_O - 1 + D_O = A_O + D_O \\ n_O &= D_O \\ k_O &= Q + a_O - 1 \end{aligned} \quad (5a)$$

$$P_S(S < Q) = P_{\text{HYPER}}(x \leq Q - 1; N_S, n_S, k_S) \quad (6)$$

where

$$\begin{aligned} N_S &= A_O + 1 + D_S \\ n_S &= D_S \\ k_S &= k_O + 1 \end{aligned} \quad (6a)$$

---

\* Ibid., pp. 265 and 238.

$$P_F(S < Q) = P_{\text{HYPER}}(x \leq Q - 1; N_F, n_F, k_F) \quad (7)$$

where

$$\begin{aligned} N_F &= A_o + 1 + D_F \\ n_F &= D_F \\ k_F &= k_o \end{aligned} \quad (7a)$$

To find  $D_o$ ,  $D_S$ , and  $D_F$ , it is necessary to consult cumulative Hypergeometric tables\* to locate a compatible set of parameters that will yield a cumulative probability of  $1 - \gamma$ .

### 3.5 Cost Savings from Test Firing

The expected number of successes,  $E(r)$ , from a Bernoulli trial of size  $n$  where the probability of success is represented by a beta distribution (with parameters  $a_o$ ,  $b_o$ ) equals:\*\*

$$E(r) = \frac{na_o}{a_o + b_o}$$

For  $n = 1$ ,  $E(r)$  equals the probability of a success. Hence, the expected cost with one test firing is:\*\*\*

$$C_D \left( \frac{a_o}{a_o + b_o} D_S + \frac{b_o}{a_o + b_o} D_F \right) + C_T$$

---

\* Reference 2.

\*\* Reference 1, p. 237.

\*\*\* This expression should be modified if the cost function is nonlinear.

which must be less than the cost associated with deployment without a test firing,  $C_D D_o$ . Thus, the relation,

$$\frac{a_o D_S + b_o D_F}{a_o + b_o} + C_R < D_o \quad (8)$$

must hold if savings are expected by a test firing.

### 3.6 Economics of the $n + 1$ th Test After Conducting $n_t$ Test Shots Which Yield $r_t$ Successes

The fact that a prior beta distribution, augmented by sample information from a Bernoulli process, yields a beta posterior was pointed out in Paragraph 3.3. If  $n$  trials produce  $r$  successes, the parameters of the new beta become

$$\begin{aligned} a'_o &= a_o + r \\ b'_o &= b_o + n - r \end{aligned} \quad (9)$$

The analysis described in Paragraphs 3.1 through 3.5 can be repeated with  $a'_o$  substituted for  $a_o$  and  $b'_o$  substituted for  $b_o$  to determine whether the  $n + 1$ th shot will yield an expected savings.

## SECTION IV

## SOLUTION WITH SPECIFIC VALUES CITED IN SECTION II

To solve the particular problem described in Section II, the variables listed below take on the indicated values:

$$R_1 = 0.6 \quad R_c = 0.35 \quad C_R = 0.5 \quad Q = 3 \quad \gamma = 0.95$$

4.1 Values for the Parameters of the Prior Distribution

If Equations (1) and (2) are solved (or if a set of beta tables\* is used) it is found that  $a_o = 7$  and  $b_o = 5$ .

4.2 Determination of the Required Deployment

Solution of Equations (5a), (6a), and (7a), yields:

$$\begin{array}{lll} N_o = 11 + D_o & N_S = 12 + D_S & N_F = 12 + D_F \\ n_o = D_o & n_S = D_S & n_F = D_F \\ k_o = 9 & k_S = 10 & k_F = 9 \end{array}$$

If the cumulative hypergeometric tables\*\* are consulted to determine the smallest deployment quantities that provide at least a 95 percent probability of at least three successful firings, it is found that:

$$D_o = 11 \quad D_S = 10 \quad D_F = 12$$

---

\* Reference 4.

\*\* Reference 2.

4.3 Determination as to Whether a Test Shot Realizes a Cost Saving

The left-hand side of Relation (8) becomes

$$\frac{7.10 + 5.12}{12} + 0.5 = 11.33$$

The right-hand side of Relation (8) is 11. Therefore, a first test is uneconomic.

## SECTION V

## AUTHOR'S COMMENT

5.1 Validity of the Criterion

The reader is reminded that the approach presented in this paper uses as its optimization criterion the minimization of expected costs. This approach, therefore, should only be employed if this criterion is acceptable to the decision-maker. Reference 1\* discusses the use of this criterion.

5.2 Sensitivity of the Results to the Prior Distribution

The approach used allows the analyst to insert his a priori feelings into the analysis in the form of a prior subjective probability distribution. If the analyst experiences some uneasiness in the distribution that he specifies, he will be well advised to examine the effect of making plausible changes to his prior distribution results.

5.3 The Decision Not to Deploy

Budgetary constraints may limit the total costs that can be spent on deployment and testing. Hence, the decision might be made at the outset, or at some point during the testing program, not to deploy the booster under consideration, since the costs of a sufficient deployment to meet performance specifications has too high a probability of exceeding the budgetary limitation. If the budgetary constraint and the threshold probability are known, the decision not to deploy may be determined as follows:

- (1) With the original prior and the results of the tests which have already been conducted, form a new prior (see (9)).

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\* Chapter 1.

- (2) Consider all possible outcomes from future tests and follow through each chain of events until it does not pay to conduct an additional test (see Paragraph 3.6).
- (3) Determine the required deployment for each chain (see Paragraphs 3.1 to 3.4).
- (4) Compute the total testing and deployment cost implied by each chain.
- (5) Using the prior generated in (1), find the probability for each chain (use expression for  $E(r)$  in Paragraph 3.5 and Equation (9) respectively).
- (6) Identify those chains which have costs in excess of the budgetary constraint, and total their probabilities to see if the total exceeds the threshold probability.

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