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STUDIES IN FUNCTIONAL EQUATIONS OCCURRING IN DECISION PROCESSES,

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PREFACE

As an outgrowth of the study of certain decision making models, one is led to consider various functional equations. These functional equations have an interest of their own, and a study of their solutions is important in connection with the aforementioned decision processes. In this collection of three papers, a fairly complete discussion is given for certain problems arising in connection with the Bales decision making model as treated by Householder ("The Binomial Decision Maker" (unpub.)). Since this work is being continued by a group under M. M. Flood, with respect to various different fusion models, it is hoped that the methods developed herein will be of some assistance in this direction.

In the first paper, T. E. Harris analyzes the relevant "Markov" process associated with the Bales model, proves the existence and continuity of the "limit distribution", and derives a functional equation for this distribution.

The second paper by R. Bellman considers general functional equations of the type occurring in the Harris paper and studies questions of existence and uniqueness of continuous solutions. Various other properties of the solutions are also considered.

The last paper by H. N. Shapiro focuses its attention on the "one-dimensional case", that is, on the functional equation

$$f(x) = xf(\alpha + (1-\alpha)x) + (1-x)f(\sigma x).$$

Among other things, the question of calculating the solution is considered, and the behavior of the solution as a function of the parameters (α, σ) is discussed.

That these papers have been allowed to remain overlapping in content, at some points, is due to a sincere effort to achieve some measure of coherency in each individual paper, while preserving the overall connection between them.

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Summary: In these three papers certain problems are considered which arise from a study of the Bales decision making model. The Markov process associated with this model is considered and existence and continuity, as well as a functional equation, are obtained for the limit distribution. Generalizations of this functional equation are discussed and a detailed treatment is given for the one-dimensional case.

I. A PROBABILITY MODEL FOR A DECISION MAKING PROCESS

T. E. Harris

The following problem in probability arises in the decision making and learning models suggested by Bales, Householder, Bush, Mosteller, Flood, and others (see Flood's RM-853 for some background material).

Let $\xi^{(0)} = (z_1^{(0)}, z_2^{(0)}, z_3^{(0)})$ be a vector with $z_i^{(0)} \geq 0$, $\sum z_i = 1$. Let A_i, $i = 1,2,3$, be three "Markov" matrices; i.e. each A_i has non-negative elements and the sum of the elements in each row is 1. One of the three matrices A_i is selected by chance, $z_i^{(0)}$ being the probability that A_i is selected and a new vector $\xi^{(1)}$ is defined by

$$\xi^{(1)} = \xi^{(0)}B$$

where B is that one of the A_i which is selected. Next one of the A_i is selected again, this time with probabilities given by the components of $\xi^{(1)}$, and the matrix is multiplied by $\xi^{(1)}$. In general, at the k^{th} stage the matrix is selected in accordance with the components of $\xi^{(k)}$, and is then pre-multiplied by $\xi^{(k)}$ to give $\xi^{(k+1)}$.

In the model which we are considering the A_i have the form

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1-\alpha & 0 \\ \beta & 0 & 1-\beta \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1-\rho_2 & \rho_2 & 0 \\ 0 & 1 & 0 \\ 0 & \sigma & 1-\sigma \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $0 < \alpha, \beta, \rho < 1$.

In order to have a generic name for the A_i we shall let B_1 be that one of the A_i which is chosen first, B_2 the one chosen next, and so on. We notice that A_1 has the eigenvector $(1,0,0)$ corresponding to the eigenvalue 1 and that A_2 has the eigenvector $(0,1,0)$ corresponding to the eigenvalue 1. In both these cases these eigenvalues are simple and the other eigenvalues are less than 1 in magnitude.

We shall say that a sequence $B_1, B_2, \ldots$ <u>concludes</u> (A_1) if all the B_i, for i greater than some integer, are equal to A_1; a similar definition holds for <u>conclusion</u> (A_2). A sequence <u>concludes</u> if it concludes (A_1) or concludes (A_2).

<u>Theorem 1</u>. <u>For any</u> $\xi^{(0)} \neq (0,0,1)$ <u>the sequence</u> $B_1, B_2, \ldots$ <u>concludes with probability</u> 1. <u>Let</u> $\pi(\xi)$ <u>be the probability that the sequence concludes</u> (A_1), <u>if</u> $\xi^{(0)} = \xi$. <u>Then, for</u>

$\xi \neq (0,0,1)$, $\pi(\xi)$ is a continuous function of ξ which is 0 when $\xi = (0,1,0)$ and 1 when $\xi = (1,0,0)$. Also $0<\pi(\xi)<1$ for ξ not equal $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$; and $\pi(\xi)$ satisfies the functional equation

$$(1) \quad \pi(\xi) = z_1\pi(\xi A_1) + z_2\pi(\xi A_2) + z_3\pi(\xi)$$

where $\xi = (z_1, z_2, z_3)$.

The proof of Theorem 1 is quite similar to that of analogous classical theorems about random walks, requiring in addition only elementary matrix theory. The general type of argument required is set forth in Chapters 15 and 16 of Feller's book, "Introduction to Probability Theory".

It should be noticed that certain other forms of the matrices A_i give rise to an essentially different kind of behavior, where the vector $\xi^{(n)}$ has a limiting distribution, as $n \to \infty$, which is independent of ξ^0. This case will not be considered here.

Proof of Theorem 1. If $\xi^{(0)} = (1,0,0)$ it is clear that the sequence $\{B_i\}$ concludes (A_1) and that $\xi^{(i)} = \xi^{(0)}$ for all i. Similarly if $\xi^{(0)} = (0,1,0)$ or $(0,0,1)$ then $\xi^{(i)} = \xi^{(0)}$ for all i and all the $B_i = A_2$ or A_3 respectively. In what follows we exclude these trivial cases.

For convenience we shall refer to a vector $\xi = (z_1, z_2, z_3)$ as a probability vector if $z_1, z_2, z_3 \geqslant 0$ and $z_1 + z_2 + z_3 = 1$.

It follows from elementary matrix theory (in the present case it is also easily seen directly) that there exist constants c and d, $c > 0$, $0 < d < 1$, such that for any probability vector

(2) $\quad |(1,0,0) - \xi A_1^n| < cd^n, \qquad n = 0,1,2,\ldots,$

(3) $\quad |(0,1,0) - \xi A_2^n| < cd^n, \qquad n = 0,1,2,\ldots,$

where c and d are independent of ξ . (By the "absolute value" of a vector we mean the sum of the absolute values of its components.)

Suppose then that $\xi(0) = (z_1^{(0)}, z_2^{(0)}, z_3^{(0)})$ is given with $z_3^{(0)} \neq 1$. Notice that an application of A_1 increases z_1, A_2 increases z_2, and both decrease z_3. Let N be an integer such that $cd^N < 1$. Now either $z_1^{(0)}$ or $z_2^{(0)}$ must be $\geq \frac{1}{2}(1-z_3^{(0)})$; suppose that $z_1^{(0)}$ is. Then the probability that $B_1 = B_2 = \ldots = B_N = A_1$ is greater than $\left[\frac{1}{2}(1-z_3^{(0)})\right]^N$. Since from (2), $z_1^{(i)}$ is greater than $1 - cd^i$, the probability that $B_i = A_1$ for all $i > N$ is then greater than $\prod_{r=1}^{\infty}(1-cd^{N+r}) > 0$. A similar argument applies if $z_2^{(0)} \geq \frac{1}{2}(1-z_3^{(0)})$. Thus for any initial $\xi^{(0)} \neq (0,0,1)$, the probability is greater than $\left[\frac{1}{2}(1-z_3^{(0)})\right]^N \prod_{r=1}^{\infty} (1-cd^{N+r})$ that all the B_i from the very beginning will be identical and equal to A_1, or all equal to A_2. From this we see that conclusion must occur. For suppose an unbroken sequence is not obtained at the beginning. Then at the time the first break occurs, we will have the same situation as initially except that the vector $\xi^{(0)}$ will have changed to some vector $\xi^{(k)}$. However, since $z_3^{(k)} \leq z_3^{(0)}$ we will have the same lower bound as before for the probability that an unbroken sequence will be obtained immediately, and so on. Thus there are an unlimited number of opportunities for an unbroken sequence to begin, all having a probability greater than a fixed positive number, and hence conclusion occurs with probability 1.

It is clear from the type of argument used above that $\pi(\xi)$ is positive if $\xi \neq (0,1,0)$ or $(0,0,1)$.

To show continuity of $\pi(\xi)$ it is convenient to define the function $\pi_0(\xi)$ as the probability that $B_i = A_1$ for all i, if ξ is the initial vector. Clearly,

$$(4) \qquad \pi_0(\xi) = \prod_{r=0}^{\infty} z_1^{(r)}$$

where $\xi^{(r)} = (z_1^{(r)}, z_2^{(r)}, z_3^{(r)}) = \xi A_1^r$.

Because of the simple form of A_1 it is convenient to write $\pi_0(\xi)$ explicitly as

$$(5) \qquad \pi_0(\xi) = z_1 \prod_{r=1}^{\infty} \left[1-z_2(1-\alpha)^r - z_3(1-\beta)^r\right] .$$

It is clear from (5) that $\pi_0(\xi)$ is a continuous function of ξ, as ξ ranges over all probability vectors, since the infinite product converges uniformly.

To show that $\pi(\xi)$ is continuous for $\xi \neq (0,0,1)$, one first shows that if $f(n, \xi)$ is the probability that conclusion has not begun on or before the n^{th} step, the initial vector being ξ, then $f(n, \xi) \to 0$ uniformly in ξ, as $n \to \infty$, provided $|\xi - (0,0,1)|$ is greater than some fixed positive number. The argument for this step is simple and details will be omitted here. Thus if we write

$$\pi(\xi) = \pi_n'(\xi) + \pi_n''(\xi)$$

where $\pi_n''(\xi)$ is the probability that the sequence (B_i) concludes A_1 but that conclusion begins at the n^{th} step or later, we know that $\pi_n''(\xi) \to 0$ uniformly as $n \to \infty$. The quantity $\pi_n'(\xi)$, the probability that conclusion (A_1) begins before n, can be written explicitly as a finite sum of continuous functions. For example,

$$\pi_2(\xi) = \pi_o(\xi) + z_2\pi_o(\xi A_2) + z_3\pi_o(\xi) .$$

Continuity of $\pi(\xi)$ now follows.

The functional equation (1) is obtained by merely examining the possibilities for the first step of the process.

II. ON A CERTAIN CLASS OF FUNCTIONAL EQUATIONS

Richard Bellman

§1. Introduction.

In this paper, the main objective is to discuss the existence and uniqueness of continuous solutions of the functional equation

$$f(\underline{X}) = \sum_{i=1}^{N} x_i f(T_i \underline{X}) \tag{1.1}$$

where $\underline{X} = (x_1, \ldots, x_{N-1})$, $x_i \geq 0$, $\sum_{i=1}^{N} x_i = 1$, and T_i is an inhomogeneous linear transformation of the form

$$T_i \underline{X} = \left(\sum_{j=1}^{N-1} a_{ij}^{(i)} x_j + c_{i1}, \ldots, \sum_{j=1}^{N-1} a_{N-1,j}^{(i)} x_j + c_{i,N-1} \right), \tag{1.2}$$

subject to some additional conditions which will be given below.

The one dimensional case $N = 2$, and the two dimensional case $N = 3$ will be discussed in detail. The general case presents no features not already present in the case $N = 3$, and may be treated in an entirely analagous fashion.

§2. The One Dimensional Case.

In the one dimensional case the functional equation (1.1) assumes the form

(2.1) $$f(x) = xf(c_1+c_2x) + (1-x)f(c_3+c_4x).$$

We impose the boundary conditions $f(0) = 0$, $f(1) = 1$, and focus our attention on the interval $0 \leqslant x \leqslant 1$. As for the transformations $T_1(x) = c_1 + c_2x$, $T_2(x) = c_3 + c_4x$ we assume

(2.2) $$0 \leqslant T_1(x) \leqslant 1 \text{ and } 0 \leqslant T_2(x) \leqslant 1 \qquad \text{for } 0 \leqslant x \leqslant 1,$$

(2.3) $$\lim_{n \to \infty} T_1^{(n)}(x) = 1 \qquad \text{for } 0 \leqslant x \leqslant 1,$$

(2.4) $$\lim_{n \to \infty} T_2^{(n)}(x) = 0 \qquad \text{for } 0 \leqslant x \leqslant 1,$$

where $T_1^{(n)}$, $T_2^{(n)}$ denote the n^{th} iterates of the transformations T_1, T_2 respectively. The assumption (2.3) implies that $T_1^{(1)} = 1$, i.e. $c_1 + c_2 = 1$. Since $T_1(0) . c_1$, $0 \leqslant c_1 \leqslant 1$, we have

$$T_1(x) = \alpha + (1-\alpha)x \quad \text{where } 0 \leqslant \alpha \leqslant 1.$$

Similarly, (2.4) implies that

$$T_2(x) = \sigma x \quad \text{where } 0 \leqslant \sigma \leqslant 1.$$

Finally, the functional equation takes the form

(2.5) $$f(x) = xf[\alpha + (1-\alpha)x] + (1-x)f(\sigma x).$$

(a) Existence and Continuity.

To obtain a solution by iteration we define

$$f_0(x) = x$$

$$f_{n+1}(x) = x\, f_n(\alpha + (1-\alpha)x) + (1-x) f_n(\sigma x), \quad n = 0,1,2,\ldots, .$$

It is clear that for each n, $f_n(0) = 0$, $f_n(1) = 1$. Furthermore,

$$f_1(x) = x\left[\alpha + (1-\alpha)x\right] + \sigma x(1-x)$$

$$= x\left[x(1-\alpha-\sigma) + \alpha + \sigma\right] .$$

Then, if $\alpha + \sigma \geqslant 1$, $f_1(x) \geqslant f_0(x) = x$; whereas if $\alpha + \sigma \leqslant 1$, $f_1(x) \leqslant f_0(x) = x$. By induction it follows that

$$1 \geqslant f_{n+1}(x) \geqslant f_n(x) \geqslant 0 \quad \text{if } \alpha + \sigma \geqslant 1 ,$$

$$0 \leqslant f_{n+1}(x) \leqslant f_n(x) \leqslant 1 \quad \text{if } \alpha + \sigma \leqslant 1.$$

Thus in any case the sequence $f_n(x)$ converges as $n \to \infty$ for all x in $< 0,1 >$. The limit $f(x)$ of this sequence is clearly a solution of the functional equation (2.5).

That the above solution $f(x)$ is continuous will now be demonstrated by showing that $|f_n'(x)|$ is uniformly bounded. We have

(2.6) $f_{n+1}'(x) = (1-\alpha)x\, f_n'(\alpha + (1-\alpha)x) + \sigma(1-x) f_n'(\sigma x) + f_n(\alpha + 1 - \alpha)x) - f_n(\sigma x).$

Letting $u_n = \max_{0 \leqslant x \leqslant 1} |f_n'(x)|$, (2.6) implies

$$u_{n+1} \leqslant c\, u_n + 1 ,$$

where $c = \max(\sigma, 1-\alpha)$. This in turn yields

$$u_n \leqslant \frac{1}{1-c} \quad ,$$

since $u_o = 1$.

Hence the sequence $\{f_n(x)\}$ is equicontinuous, and consequently possesses a uniformly convergent subsequence which converges to a continuous function. Since this function must be $f(x)$ we have that $f(x)$ is continuous.

(b) Monotonicity.

Let us now demonstrate that $f(x)$ is monotone in x, which is to be expected from the original probabilistic derivation of the previous paper. We have from (2.6), since $\alpha + (1-\alpha)x \geqslant \sigma x$ for $0 \leqslant x \leqslant 1$, that if $f_n'(x) \geqslant 0$, then $f_{n+1}'(x) \geqslant 0$, i.e. if $f_n(x)$ is monotone increasing so is $f_{n+1}(x)$. Since $f_o(x) = x$ is monotone increasing, we conclude that all $f_n(x)$ are monotone increasing, and hence their limit, $f(x)$, is also monotone increasing.

(c) Convexity.

Next we consider the convexity (or concavity) of $f(x)$ and show that it is concave for $\alpha + \sigma \leqslant 1$. That it is convex for $\alpha + \sigma \geqslant 1$ then follows from (2.4) of paper III, on page 28 . We have

$$f_{n+1}''(x) = 2(1-\alpha)f_n'(\alpha+(1-\alpha)x) + (1-\alpha)^2 x\, f_n''(\alpha+(1-\alpha)x)$$
$$+ \sigma(1-x)f_n''(\sigma x) - 2\sigma f_n'(\sigma x)$$

and for $\alpha + \sigma \leqslant 1$, $f''_n(x) \geqslant 0$ implies $f''_{n+1}(x) \geqslant 0$. Since $f''_o(x) = 0$, it follows that $f''_n(x) \geqslant 0$ for all n. Thus all of the $f_n(x)$ are concave, and consequently the same is true of their limit f(x).

(d) Analyticity.

The next question we shall consider briefly is that of the analytic continuation of f(x). First it will be shown that for $\alpha + \sigma \leqslant 1$, f(x) is absolutely monotonic in the interval < 0,1 >, i.e. that

$$f^{(k)}(x) \geqslant 0, \quad k = 0,1,2,\dots . \tag{2.7}$$

Via the result of paper III referred to above it suffices to consider the case $\alpha + \sigma \leqslant 1$.

We first prove the analogue of (2.7) for each element of the sequence $\{f_n(x)\}$. We have

$$f_{n+1}^{(k)}(x) = x(1-k)^{k+1} f_n^{(k+1)}(\alpha+(1-\alpha)x) + (1-x)\sigma^{k+1} f_n^{(k+1)}(\sigma x) + (k+1)(1-\alpha)^k f_n^{(k)}(\alpha+(1-\alpha)x - (k+1)\sigma^k f_n^{(k)}(\sigma x) \tag{2.8}$$

It is clear that $f_o^{(k)}(x) \geqslant 0$. Assume that the result has been established for n = 0,1,...,N; and all k, It then follows first that

$$f_N^{(k)}(\alpha+(1-\alpha)x) \geqslant f_N^{(k)}(\sigma x),$$

and then from (2.8) that $f_{N+1}^{(k)}(x) \geqslant 0$ which completes the induction.

Now since $f_n^{(k)}(x) \geqslant 0$ for all x in $< 0,1 >$, it follows that $(-1)^k f_n^{(k)}(e^{-x}) \geqslant 0$ for $0 \leqslant x \leqslant \infty$. Therefore $f_n(e^{-x})$ is completely monotone in $(0,\infty)$ and by the Bernstein-Widder Theorem

$$f_n(e^{-x}) = \int_0^\infty e^{-xt}\, d\alpha_n(t). \tag{2.9}$$

where $d\alpha_n \geqslant 0$ and $\int_0^\infty d\alpha_n(t) = 1$, since $f_n(1) = 1$ for all n.

Using Helly's Theorem we may choose a subsequence of the $\alpha_n(t)$ which converge to a function $\alpha(t)$ and thus

$$f(e^{-x}) = \lim_{n \to \infty} f_n(e^{-x}) = \int_0^\infty e^{-xt} d\alpha(t) , \tag{2.10}$$

or

$$f(x) = \int_0^\infty x^t d\alpha(t), \quad 0 \leqslant x \leqslant 1, \tag{2.11}$$

where $d\alpha(t) \geqslant 0$, $\int_0^\infty d\alpha(t) = 1$.

This then establishes the absolute monotonicity of f(x) and using (2.10) or (2.11) it is easy to see that $f(e^{-z})$ as defined by $\int_0^\infty e^{-zt} d\alpha(t)$ is analytic for $Re(z) \geqslant 0$.

Finally, we note, it is not difficult to see how the functional equation may be used ab initio to show that f(z) may be continued to an entire function of z.

(e) <u>Uniqueness</u>.

Finally, we turn to the demonstration of the uniqueness of the continuous solution to (2.5). Let $f(x)$, $\varphi(x)$ be two continuous solutions of (2.5) which are 0 at $x = 0$ and 1 at $x = 1$. Then

$$(2.12) \quad f(x) - \varphi(x) = x\{f(\alpha + (1-\alpha)x) - \varphi(\alpha + 1 - \alpha)x)\} + (1-x)\{f(\sigma x) - \varphi(\sigma x)\}$$

Let x_0 be a point at which $f - \varphi$ assume its maximum absolute value. Then from (2.12), $\alpha + (1-\alpha)x_0$ and σx_0 must also be such points. Thus for any integer $k > 0$, $|f(\sigma^k x_0) - \varphi(\sigma^k x_0)|$ is this maximum absolute value. Letting $k \to \infty$, and using the continuity at 0 it follows that this maximum must be zero. Consequently $f(x) = g(x)$ in $[0,1]$, and the desired uniqueness is proved.

§3. The Two Dimensional Case.

We consider the linear homogeneous functional equation

$$(3.1) \quad f(x_1, x_2) = f(\underline{X}) = x_1 f(T_1 \underline{X}) + x_2 f(T_2 \underline{X}) + x_3 f(T_3 \underline{X}),$$

where $x_1 + x_2 + x_3 = 1$, $x_i \geq 0$, and T_i is a linear inhomogeneous transformation which takes the point $\underline{X}$ into a transformed point $\underline{X}_i$ given by

$$(3.2) \quad \begin{cases} x_1^{(i)} = c_1^{(i)} + a_{11}^{(i)} x_1 + a_{12}^{(i)} x_2, \\ x_2^{(i)} = c_2^{(i)} + a_{21}^{(i)} x_1 + a_{22}^{(i)} x_{23} \end{cases}$$

and has the further properties:

(3.3) (a) If $\underline{X} \in D$, then $T_i \underline{X} \in D$.

(b) $\lim_{n \to \infty} T_i^{(n)} \underline{X} = p_i$, where $T^{(n)}$ denotes the n^{th} iterate of T_i, p_i is the point $(\delta_{1i}, \delta_{2i}, \delta_{3i})$, ($\delta_{ij}$ is the Kronecker delta symbol), and P is any point in the domain, D: $x_1 + x_2 + x_3 = 1$, $x_i \geq 0$.

These are analogous to the conditions which we imposed in the one dimensional case. We shall not give the cannonical forms of T_i since we use only the properties (a) and (b). It follows from (b) that p_i is a fixed-point of T_i.

We are interested in solutions of (3.1) satisfying the boundary conditions

(3.4) $$f_i(p_j) = \delta_{ij}$$

It will be shown that there exists a unique, continuous solution of (3.1) satisfying (3.4). The proof will be given for f_1, since the proofs for the other components of $f(\underline{X})$ are exactly similar.

We employ the method of successive approximations. It is essential, however, that we choose a useful first approximation. Such an approximation is given by a solution of

(3.5) $$g(x_1,x_2) = x_1\, g(T_1 P),\ g(p_1) = 1,\ g(p_2) = 0,\ g(p_3) = 0\ .$$

We obtain the solution by iteration,

$$g(x_1,x_2) = \prod_{n=0}^{\infty} (T_1^{(n)}P)_{x_1} , \tag{3.6}$$

where $(T_1^{(n)}P)_{x_1}$ denotes the x_1 components of $T_1^{(n)}P$ and $(T_1^{(0)}P)_{x_1} = x_1$. It is necessary to show that this infinite product converges, and we shall also, in the process, show that $g(x_1,x_2)$ is continuous.

Let us now employ a small amount of vector-matrix notation.

$$P = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} c_1' \\ c_2' \end{pmatrix}, \quad A = \begin{pmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{pmatrix}. \tag{3.7}$$

Then, since p_1 is a fixed point of T_1 we have

$$\begin{aligned} T_1P-p_1 &= c + AP - p_1 = c + AP - (c+Ap_1) \\ &= A(P-p_1) . \end{aligned} \tag{3.8}$$

From this we obtain

$$T_1^2 P-p_1 = A(T_1P-p_1) = A^2(P-p_1), \tag{3.9}$$

and, generally,

$$T_1^n P = p_1 + A^n(P-p_1) . \tag{3.10}$$

Since, according to 3(b), $\lim_{n\to\infty} T_1^n P = p_1$, we must have $\lim_{n\to\infty} A^n = 0$, which means that the characteristic roots of A are less than one in absolute value. If A has distinct characteristic roots, we have

$$A = T^{-1}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T, \tag{3.11}$$

while, if not, we have, in general,

$$A = T^{-1}\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix} T. \tag{3.12}$$

In the first case,

$$A^n = T^{-1}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T, \tag{3.13}$$

while in the second case,

$$A^n = T^{-1}\begin{pmatrix} \lambda_1^n & 0 \\ n\lambda_1^{n-1} & \lambda_1^n \end{pmatrix} T. \tag{3.14}$$

In both cases we see that

$$T_1^n P = p_1 + O(\lambda_2^n), \quad 0 < \lambda_2 < 1, \tag{3.15}$$

for some λ_2,

where $O(\lambda_2^n)$ represents a vector where components are $O(\lambda_2^n)$ as $n \to \infty$. Hence, uniformly for $P \in D$,

$$(T_1^n P)_{x_1} = 1 + O(\lambda_2^n). \tag{3.16}$$

Consequently, $\prod_{n=0}^{\infty} (T_1{}^n P)_{x_1}$ converges uniformly and represents a continuous function, $g(P)$. Since $T_1 p_1 = p_1$, $g(p_1) = 1$, and clearly $g(p_2) = g(p_3) = 0$.

We now consider the successive approximations given by

$$(3.17)\qquad \begin{aligned} f_o &= g \\ f_{n+1} &= x_1\, f_n(T_1P) + x_2\, f_n(T_2P) + x_3\, f_n(T_3P) \end{aligned}$$

It follows immediately that $1 \geqslant f_1(P) \geqslant f_o(P) \geqslant 0$, for all $P \in D$, and thence, inductively, that $1 \geqslant f_{n+1}(P) \geqslant f_n(P) \geqslant 0$. Hence $f_n(P)$ converges, to $f(P)$, say, for all $P \in D$. To show that $f(P)$ is continuous we proceed as in the previous section. That each $f_n(P)$ satisfies the prescribed boundary conditions is clear.

We have

$$(3.18)\qquad \begin{aligned} \frac{\partial f_{n+1}}{\partial x_1} = {} & x_1 \frac{\partial f_n(T_1P)}{\partial x_1'} \frac{\partial x_1'}{\partial x_1} + x_1 \frac{\partial f_n(T_1P)}{\partial x_2'} \frac{\partial x_2'}{\partial x_1} \\ & + x_2 \frac{\partial f_n(T_2P)}{x_2'} \frac{\partial x_1'}{x_1} + x_2 \frac{\partial f_n(T_2P)}{x_2'} \frac{\partial x_2'}{x_1} \\ & + x_3 \frac{\partial f_n(T_3P)}{\partial x_1'} \frac{\partial x_1'}{\partial x_1} + x_3 \frac{\partial f_n(T_3P)}{\partial x_2'} \frac{\partial x_2'}{\partial x_1} \\ & + f_n(T_1P) - f_n(T_3P) \end{aligned}$$

Let

$$(3.19)\qquad \begin{aligned} u_n &= \max_P \left| \frac{\partial f_n}{\partial x_1} \right| \\ v_n &= \max_P \left| \frac{\partial f_n}{\partial x_2} \right| \end{aligned}$$

Then from (3.18) we obtain

(3.20) $$u_{n+1} \leqslant a_{11}' u_n + a_{21}' v_n + 1,$$

and similarly

(3.21) $$v_{n+1} \leqslant a_{12}' u_n + a_{22}' v_n + 1.$$

The vector (u_n, v_n) is majorized by the vector solution of

(3.22) $$a_{n+1} = A' a_n + c,$$

where $c = (1,0)$. The solution of (3.22) is given by

(3.23) $$a_n = c + (A' + A'^2 + \dots A'^n) a_o .$$

Since the characteristic roots of A' are the same as A which we know to be less than 1 in absolute value, it follows that a_n is uniformly bounded by $c + (I-A)^{-1} a_o$. Hence u_n and v_n are bounded uniformly. Thus the sequence f_n is equicontinuous in D and permits the extraction of a uniformly convergent subsequence converging to a continuous function, which must be f(P). The proof of the uniqueness of this solution is as before.

III. ON THE FUNCTIONAL EQUATION:

$f(x) = x\, f(\alpha+(1-\alpha)x) + (1-x)f(\sigma x)$

Harold N. Shapiro

§1. Introduction.

In this paper a detailed investigation will be carried out for the functional equation of the title. Since it would take up a distressingly large amount of space to describe, in this introduction, all of the material contained in this note, we will content ourselves with a list of a few of the highlights.

(1) A necessary and sufficient condition that for an initial approximation $f_o(x)$, $f_o(0) = 0$, $f_o(1) = 1$, we have that the method of successive approximations converges uniformly to the continuous solution of the functional equation, is that $f_o(x)$ be bounded in $\langle 0,1 \rangle$ and continuous at $x = 0$ and $x = 1$.

(2) If $f_o(x)$ has finite right hand derivative numbers at $x = 0$, and finite left hand derivative numbers at $x = 1$, then the successive approximants $f_n(x)$ converge uniformly to the continuous solution at a rate which is $O(c^n)$, $0 < c < 1$; (provided $\sigma \neq 1$, $\alpha \neq 0$).

(3) If $f_{(\alpha,\sigma)}(x)$ denotes the solution to the functional

equation for the parameter values (α, σ), then as $(\alpha, \sigma) \longrightarrow (0,1)$ a necessary and sufficient condition that $f_{(\alpha,\sigma)}(x) \longrightarrow x$ is that

$$\sigma = 1 - \alpha + o(\alpha^2) .$$

At the conclusion of the paper, the generalization of many of the results to functional equations of the form

$$f(x) = p(x)f(G(x)) + (1-p(x)f(H(x)),$$

will be considered.

§ 2. Uniqueness Theorems.

In this section we shall undertake an analysis and extension of the uniqueness theorem for the solution of the functional equation in question. As for notation, we will write

$$L_\alpha(x) = \alpha + (1-\alpha)x,$$

so that the functional equation is

(2.1) $$f(x) = x\, f(L_\alpha(x)) + (1-x)f(\sigma x).$$

Theorem: <u>If $f(x)$ is any function continuous in $\langle 0,1 \rangle$, (no boundary conditions) such that (2.1) is satisfied, then</u>

$$\max_{0 \leq x \leq 1} |f(x)| = \max\left\{|f(0)|, |f(1)|\right\} .$$

The proof is precisely as in the proof of the uniqueness theorem in paper II. The uniqueness theorem is an immediate corollary of the above, since if $f(0) = f(1) = 0$ then it yields $f(x) \equiv 0$. This also yields immediately in the case $f(0) = 0$, $f(1) = 1$ that $|f(x)| \leqslant 1$, (which is also obvious directly).

The above theorem represents a simple extraction of the content of the uniqueness theorem of paper II. However, we now will extend this theorem as follows:

<u>Theorem</u>: <u>Let</u> $f(x)$ <u>be a function bounded in</u> $< 0,1 >$, <u>continuous at</u> $x = 0$ <u>and</u> $x = 1$ <u>such that</u> (2.1) <u>holds</u>. <u>Then</u>

(2.2) $$||f|| = \sup_{0 \leqslant x \leqslant 1} |f(x)| = \max(|f(0)|, |f(1)|) .$$

<u>Proof</u>: Choose a sequence of $\epsilon_i > 0$ such that $\epsilon_i \longrightarrow 0$. For each ϵ_i choose $x_i = x_i(\epsilon_i)$ such that $|f(x_i)| > ||f|| - \epsilon_i$. We then have two alternatives.

<u>Case (1)</u>. There is a subsequence of the x_i which converges to 0 or 1, (i.e. 0 or 1 are limit points). Suppose then $x_{i_l} \longrightarrow 0$. From $|f(x_{i_l})| > ||f|| - \epsilon_{i_l}$ we get upon letting $l \longrightarrow \infty$, $|f(0)| \geqslant ||f||$, whence $|f(0)| = ||f||$. Similarly, if $x_{i_l} \longrightarrow 1$, $|f(1)| = ||f||$.

<u>Case (2)</u>. The sequence x_i stays away from 0 and 1, i.e. there is a $\delta > 0$ such that $\delta \leqslant x_i \leqslant 1 - \delta$ for all sufficiently large i (we may assume this for all i). Then from

$$f(x_i) = x_i f(L(x_i)) + (1-x_i) f(\sigma x_i)$$

we conclude that

$$(2.3)\qquad \left.\begin{array}{l} |f(L(x_i))| \geqslant ||f|| - \dfrac{\epsilon_i}{x_i} \\ \text{and} \\ |f(\sigma x_i)| \geqslant ||f|| - \dfrac{\epsilon_i}{1-x_i} \end{array}\right\}.$$

The inequalities (2.3) in turn imply

$$|f(L^{(2)}(x_i))| \geqslant ||f|| - \frac{\epsilon_i}{x_i L(x_i)} \quad \text{and} \quad |f(\sigma^2 x_i)| \geqslant ||f|| - \text{[illegible]}$$

Iterating this procedure yields for each $k \geqslant 1$

$$|f(L^{(k)}(x_i))| \geqslant ||f|| - \frac{\epsilon_i}{x_i L(x_i) L^{(2)}(x_i) \cdots L^{(k-1)}(x_i)}$$

and

$$|f(\sigma^k x_i)| \geqslant ||f|| - \frac{\epsilon_i}{(1-x_i)(1-\sigma x_i)\cdots(1-\sigma^{k-1} x_i)} .$$

Letting $k \to \infty$ we obtain

$$|f(1) \geqslant ||f|| - \frac{\epsilon_i}{\prod_{j=0}^{\infty} L^{(j)}(x_i)} \geqslant ||f|| - \frac{\epsilon_i}{\prod_{j=0}^{\infty} \{1-(1-\alpha)^j (1-\delta)\}}$$

$$|f(0)| \geqslant ||f|| - \frac{\epsilon_i}{\prod_{j=0}^{\infty} (1-\sigma^j x_i)} \geqslant ||f|| - \frac{\epsilon_i}{\prod_{j=0}^{\infty} \{1-\sigma^j (1-\delta)\}} .$$

Finally, letting $i \to \infty$ we get $|f(1)| \geq ||f||$ and $|f(0)| \geq ||f||$ whence

$$|f(1)| = |f(0)| = ||f||,$$

Corollary: If $f(0) = f(1) = 0$, $f(x)$ bounded in $< 0,1 >$ and satisfying (2.1), then $f(x) \equiv 0$.

Corollary: If $f(0) = 0$, $f(1) = 1$, $f(x)$ bounded in $< 0,1 >$ and satisfying (2.1), then if it is continuous at 0 and at 1, it is unique. Consequently, of course, it has to be the unique continuous solution.

Thus we see that we've extended the uniqueness theorem in that there is at most one solution among the bounded functions continuous only at $x = 0$ and $x = 1$. Another very brief proof of the above theorem will be given later, which depends on other ideas.

At this point we might note an amusing consequence of the uniqueness theorem. Taking $\alpha = 1$, and σ fixed < 1, the functional equation assumes the form

$$f(x) = x + (1-x)f(\sigma x).$$

Since

$$x + \sigma x(1-x) + \sigma^2 x(1-x)(1-\sigma x) + \cdots$$

and

$$1 - \prod_{i=0}^{\infty}(1-\sigma^i x)$$

are clearly both continuous solutions of this functional equation, they must be identical in $0 \leqslant x \leqslant 1$; i.e.

$$\prod_{i=0}^{\infty} (1-\sigma^i x) = 1 - x - \sigma x(1-x) - \sigma^2 x(1-x)(1-\sigma x) - \cdots$$

Since it is easy to show that both sides of this equality represent entire functions, this equality holds for all x.

In conclusion, we shall derive a very simple but illuminating consequence of the uniqueness theorem. Writing $f(x) = f_{(\alpha,\sigma)}(x)$ for the unique continuous solution of (2.1) which satisfies $f(0) = 0$, $f(1) = 1$, $(\alpha \neq 0,\ \sigma \neq 1)$, it is easily verified that $1 - f_{(1-\sigma,1-\alpha)}(1-x)$ satisfies (2.1) and the requisite boundary conditions. Thus via the uniqueness theorem

(2.4) $$f_{(\alpha,\sigma)}(x) = 1 - f_{(1-\sigma,1-\alpha)}(1-x).$$

For $\alpha = 0$, $\sigma \neq 1$, it is easily seen that(requiring continuity at $x = 0$)

$$f_*(x) = \begin{cases} 0 & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$$

is the unique solution of (2.1). Also for $\sigma = 1$, $\alpha \neq 0$,

$$f^*(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x \neq 0 \end{cases}$$

is the unique solution of (2.1) continuous at $x = 1$. Thus (2.4) is verified in these cases also, and in fact for all cases with the exception of $(\alpha,\sigma) = (0,1)$.

The effect of the transformation $(\alpha,\sigma) \rightarrow (1-\sigma,\ 1-\alpha)$

is to interchange the triangles ($\alpha + \sigma \leqslant 1$, $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \sigma \leqslant 1$) and ($\alpha + \sigma \geqslant 1$, $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \sigma \leqslant 1$). Thus (2.4) gives us a tool for carrying over results proved for $\alpha + \sigma \leqslant 1$ or $\alpha + \sigma \geqslant 1$ to the other half of the parameter square.

§3. Initial Approximation for which the Iteration Scheme Converges Uniformly.

In this and ensuing sections we consider the linear operator $\Lambda = \Lambda_{(\alpha,\sigma)}$ defined via

$$\Lambda\varphi = x\,\varphi(L_\alpha(x)) + (1-x)\,\varphi(\sigma x)$$

where $\varphi(x)$ is any function defined in $< 0,1>$. Thus (2.1) may be written as $\Lambda f = f$.

For a given $f_o(x)$ defined and bounded in $< 0,1>$, with $f(0) = 0$, $f_o(1) = 1$, as initial approximation, the method of successive approximations dictates that we form

$$\Lambda^{(n)} f_o = f_n(x)$$

(where $\Lambda^{(n)}$ is the n^{th} iterate of Λ) and examine whether or not $f_n(x)$ converges to a solution of $\Lambda f = f$. The problem we propose to solve here is that of determining those $f_o(x)$ for which $\Lambda^{(n)} f_o$ converges to $f_{(\alpha,\sigma)}(x)$ uniformly in x, $0 \leqslant x \leqslant 1$.

We shall first prove a lemma, which is called here

"The Principal Lemma", simply to highlight the fact that most of the results of the paper, from this point on, depend on this lemma.

<u>The Principal Lemma.</u>

$$(3.1)\quad \sum_{i=0}^{\infty} \Lambda^{(i)}\{x(1-x)\} = \begin{cases} \dfrac{f_{(\alpha,\sigma)}(x)-x}{\alpha+\sigma-1} & \text{for } \alpha+\sigma-1 \neq 0,\ \alpha \neq 0, \sigma \neq 1. \\ x(1-x)/\alpha^2 & \text{for } \alpha+\sigma-1 = 0,\ \alpha \neq 0. \end{cases}$$

<u>Proof</u>: Taking $f_0(x) = x$ in the iteration scheme we have

$$(3.2)\qquad \Lambda x - x = (\alpha+\sigma-1)\, x(1-x)\ .$$

This in turn implies for $k \geqslant 1$

$$(3.3)\qquad \Lambda^{(k)}x - \Lambda^{(k-1)}x = (\alpha+\sigma-1)\ \Lambda^{(k-1)}\{x(1-x)\}\ .$$

Summing this for $k = 1,\ldots,n$ yields

$$(3.4)\qquad \Lambda^{(n)}x - x = (\alpha+\sigma-1) \sum_{i=0}^{n-1} \Lambda^{(i)}\{x(1-\alpha)\}\ .$$

Now $\Lambda^{(n)}x$ converges to $f_{(\alpha,\sigma)}(x)$ as $n \longrightarrow \infty$ (see paper II) so that for $\alpha+\sigma-1 \neq 0$,

$$\frac{f_{(\alpha,\sigma)}-x}{\alpha+\sigma-1} = \sum_{i=1}^{\infty} \Lambda^{(i)}\{x(1-x)\}\ ,$$

which is the desired result.

For $\alpha+\sigma-1 = 0$, we argue as follows:

$$\begin{aligned} \Lambda\{x(1-x)\} &= x\,L(x)(1-L(x)) + (1-x)\sigma\, x(1-\sigma x) \\ &= x(1-x)\,[(1-\alpha)L(x) + \sigma - \sigma^2 x] \\ &= (1-\alpha^2)\ x(1-x)\ . \end{aligned}$$

Thus

$$(3.5)\qquad \Lambda^{(k)}\left\{x(1-x)\right\} = (1-\alpha^2)^{k-1}x(1-x), \quad k = 1,\ldots$$

so that

$$\sum_{k=0}^{\infty} \Lambda^{(k)}\left\{x(1-x)\right\} = x(1-x) \sum_{k=0}^{\infty}(1-\alpha^2)^k = \frac{x(1-x)}{\alpha^2}$$

which completes the proof of the lemma.

Remarks:

(1) In the case $\alpha + \sigma = 1, \alpha \neq 0$, it is clear from (3.5) that for $0 \leqslant x \leqslant 1$

$$\Lambda^{(k)}\left\{x(1-x)\right\} \geqslant 0, \quad k = 1,\ldots$$

Then since $\sum_{k=0}^{\infty} \Lambda^{(k)}\left\{x(1-x)\right\} \leqslant \frac{1}{\alpha^2}$ is a uniformly bounded (in $< 0,1 >$ series of positive terms it is uniformly convergent in $< 0,1 >$ (via Dini's Theorem).

(2) Similarly, for $\alpha + \sigma - 1 \neq 0$, $\alpha \neq 0$, $\sigma \neq 1$ we see from (3.2) and (3.3) that

$$\Lambda^{(k)}\left\{x(1-x)\right\} \geqslant 0,\ 0 \leqslant x \leqslant 1,\ k = 0,1,2,\ldots$$

Thus since

$$\sum_{i=0}^{\infty} \Lambda^{(i)}\left\{x(1-x)\right\} \leqslant \frac{1}{|\alpha+\sigma-1|},$$

we have as in (1) that

$$\sum_{i=0}^{\infty} \Lambda^{(i)}\left\{x(1-x)\right\}$$

converges uniformly in x, $0 \leqslant x \leqslant 1$.

(3) From the above remarks and (3.4) and (3.2) it is clear that $\Lambda^{(n)}x$ converges <u>uniformly</u> to $f_{(\alpha,\sigma)}(x)$ for $\alpha \neq 0$, $\sigma \neq 1$.

(4) As a corollary of remarks (2), (3) it follows that in all cases with the exception of ($\alpha = 0$ or $\sigma = 1$), $\Lambda^{(k)}x(1-x)$ goes uniformly to zero, $0 \leqslant x \leqslant 1$, as $k \rightarrow \infty$. This simple fact is the key to the proofs of several of the theorems which follow.

We next prove a lemma which is more particularly related to the proof of the theorem of this section.

<u>Lemma.</u> If $f_o(x)$ has finite right hand derivative numbers at $x = 0$, and finite left hand derivative numbers at $x = 1$, $f_o(0) = 0$, $f_o(1) = 1$, and is bounded in $< 0,1 >$, then for $\alpha \neq 0$, $\sigma \neq 1$, $\Lambda^{(n)}f_o$ converges uniformly to $f_{(\alpha,\sigma)}(x)$.

<u>Proof</u>: The hypothesis that $f_o(x)$ is bounded ($f_o(0) = 0$), and has finite right hand derivative numbers at $x = 0$ implies that $h(x) = \dfrac{f_o(x)}{x}$ is bounded in $< 0,1 >$. Next since

$$\frac{f_o(1)-f_o(x)}{1-x} = \frac{h(1)-h(x)x}{1-x} = \frac{h(1)-h(x)}{1-x} + h(x)$$

$$\sim \frac{h(1)-h(x)}{1-x} + 1 \text{ for } x \sim 1,$$

the finiteness of the derivative numbers of $f_o(x)$ at $x = 1$

imply the same thing for $h(x)$, and since $h(x)$ is bounded in $< 0,1 >$ we obtain that

$$g(x) = \frac{1-h(x)}{1-x}$$

is bounded in $< 0,1 >$. Combining the above substitutions we have

$$f_o(x) = x - x(1-x)g(x) \tag{3.6}$$

or

$$x - f_o(x) = x(1-x)g(x) \ . \tag{3.7}$$

Since $g(x)$ is bounded in $< 0,1 >$ (3.7) yields for an appropriate constant c,

$$|x - f_o(x)| \leq c\ x(1-x) \ . \tag{3.8}$$

Applying $\Lambda^{(k)}$ to both sides of (3.8) it follows that

$$|\Lambda^{(k)}x - \Lambda^{(k)}f_o(x)| \leq c\ \Lambda^{(k)}x(1-x) \ .$$

Then since $\Lambda^{(k)}x(1-x)$ goes uniformly to zero as $k \longrightarrow \infty$, and $\Lambda^{(k)}x$ goes uniformly to $f_{(\alpha,\sigma)}(x)$ it follows that $\Lambda^{(k)}f_o$ goes uniformly to $f_{(\alpha,\sigma)}(x)$.

<u>Theorem</u>: <u>For a function</u> $f_o(x)$, $f_o(0) = 0$, $f_o(1) = 1$, <u>bounded in</u> $< 0,1 >$, $\alpha \neq 0$, $\sigma \neq 1$, <u>a necessary and sufficient</u>

condition that $\Lambda^{(k)}f_o$ *converge uniformly to* $f_{(\alpha,\sigma)}(x)$ *is that* $f_o(x)$ *be continuous at* $x = 0$ *and at* $x = 1$.

Proof: We first consider the necessity. Suppose that $f_o(x)$ is not continuous at $x = 0$, and has at this point a saltus $s_o > 0$. We then have a sequence of points $x_i \to 0$ such that

$$\overline{\lim_i} \, |f_o(x_i)| \geq s_o .$$

Now $\Lambda f_o = x \, f_o(L(x) + (1-x)f_o(\sigma x)$, so that

$$\overline{\lim_i} \, | \Lambda f_o|x = \frac{x_i}{\sigma} = \overline{\lim_i} \, |f_o(x_i)| \geq s_o$$

Hence the saltus s_1 of Λf_o at $x = 0$ is $\geq s_o$. Continuing this argument we have that the saltus s_k of $\Lambda^{(k)}f_o$ at $x = 0$ is also $\geq s_o$. Now since $\Lambda^{(k)}f_o$ converges uniformly to $f_{(\alpha,\sigma)}$, for sufficient large k,

$$| \Lambda^{(k)}f_o - f_{(\alpha,\sigma)}(x)| < \frac{s_o}{4}$$

for all x, $0 \leq x \leq 1$; whence

$$(3.9) \qquad | \Lambda^{(k)}f_o| < \frac{s_o}{4} + |f_{(\alpha,\sigma)}(x)| .$$

But we can find a point $x = x_o$ as close to the origin as we please such that

$$|\Lambda^{(k)}f_o|_{x=x_o} \geq \frac{s_o}{2}$$

and since

$$|f_{(\alpha,\sigma)}(x_o)| < \frac{s_o}{4}$$

for x_o sufficiently close to 0, (3.9) yields a contradiction an analogous agreement works for $x = 1$. This completes the proof of the necessity.

We next proceed to the sufficiency. Letting $f_o(x)$ be any function satisfying the hypothesis of the theorem, and given any $\epsilon > 0$, we first show that we can find an $f_o^*(x)$, $f_o^*(0) = 0$, $f_o^*(1) = 1$, bounded in $< 0,1 >$, finite derivative numbers at $x = 0$ and $x = 1$, such that

$$(3.10) \qquad |f_o(x) - f_o^*(x)| < \epsilon/2, \quad 0 \leqslant x \leqslant 1 .$$

Since $f_o(x)$ is continuous at $x = 0$, we can find a $\delta > 0$ ($\delta < \frac{1}{3}$) such that for $0 \leqslant x \leqslant \delta$, $|f_o(x)| < \epsilon/2$. Define

$$f^*(x) = 0 \quad \text{for } 0 \leqslant x \leqslant \delta$$

Define $f_o^*(x)$ analogously in a suitable neighborhood $1 - \delta' \leqslant x \leqslant 1$ of $x = 1$ and in $\delta < x < 1 - \delta'$ define $f_o^*(x) = f_o(x)$. It then follows that $f_o^*(x)$ has finite derivative numbers at $x = 0$ and $x = 1$, is bounded, $f_o^*(0) = 0$, $f_o^*(1) = 1$, and (3.10) is satisfied.

Next operate on (3.10) with $\Lambda^{(k)}$ so that we get

$$|\Lambda^{(k)} f_o - \Lambda^{(k)} f_o^*| < \epsilon/2, \quad 0 \leqslant x \leqslant 1.$$

This in turn gives

$$(3.11)\quad |\Lambda^{(k)}f_0 - f_{(\alpha,\sigma)}(x)| < \frac{\epsilon}{2} + |\Lambda^{(k)}f_0^* - f_{(\alpha,\sigma)}(x)|, \quad 0 \leq x \leq 1 .$$

However, by the lemma above,

$$|\Lambda^{(k)}f_0^* - f_{(\alpha,\sigma)}(x)| < \frac{\epsilon}{2}, \quad 0 \leq x \leq 1$$

for sufficiently large k. Consequently,

$$|\Lambda^{(k)}f_0 - f_{(\alpha,\sigma)}(x)| < \epsilon, \quad 0 \leq x \leq 1,$$

and the proof of the sufficiency is completed.

The above theorem yields a very simple proof of the second corollary on page 27. For if $f(x)$, $f(0) = 0$, $f(1) = 1$ is bounded in $< 0,1 >$, continuous at 0, and 1, then by the above theorem $\Lambda^{(k)}f$ converges uniformly to $f_{(\alpha,\sigma)}$. Thus if $\Lambda f = f$, $\Lambda^{(k)}f = f(x)$ and consequently $f(x) = f_{(\alpha,\sigma)}(x)$.

§4. Calculating the Solution.

At this point, we pause in the general development, and give a small application of the result of the preceding section to the problem of the numerical calculation of the solution.

It has already been observed that (for $\alpha \neq 0$, $\sigma \neq 1$) for the sequence $\Lambda^{(k)}x$

$$x \leq \Lambda x \leq \Lambda^{(2)}x \leq \ldots \leq \Lambda^{(k)}x \leq \Lambda^{(k+1)}x \leq \ldots \leq f_{(\alpha,\sigma)}(x) \text{ for } \alpha+\sigma \geq 1,$$

and

$$x \geq \Lambda x \geq \Lambda^{(2)}x \geq \ldots \geq \Lambda^{(k)}x \geq \Lambda^{(k+1)}x \geq \ldots \geq f_{(\alpha,\sigma)}(x) \text{ for } \alpha+\sigma \leq 1.$$

Thus if in the case $\alpha + \sigma \geq 1$ we provide an $f_o(x)$ such that $\Lambda^{(k)} f_o$ converges in a monotone decreasing fashion to $f_{(\alpha,\sigma)}(x)$; and for $\alpha + \sigma \leq 1$ an $f_o(x)$ which converges to $f_{(\alpha,\sigma)}(x)$ in a monotone increasing fashion, then the "computer" will have an effective procedure for approximating to the solution $f_{(\alpha,\sigma)}(x)$ with a known degree of error. In the following theorem we provide such initial approximants $f_o(x)$.

Theorem: For $\alpha \neq 0$, $\sigma \neq 1$, let

$$(4.1) \qquad f_o(x) = \begin{cases} \prod_{i=1}^{\infty} L_{\alpha}^{(i)}(x) & \text{for } \alpha + \sigma \leq 1, \\ 1 - \prod_{i=0}^{\infty} L_{1-\sigma}(1-x) & \text{for } \alpha + \sigma \geq 1 . \end{cases}$$

Then:

$$(4.2) \qquad f_o(x) \leq \Lambda f_o \leq \cdots \leq \Lambda^{(k)} f_o \leq \Lambda^{(k+1)} f_o \leq \cdots \leq f_{(\alpha,\sigma)}(x)$$

for $\alpha + \sigma \leq 1$

and

$$(4.3) \qquad f_o(x) \geq \Lambda f_o \geq \cdots \geq \Lambda^{(k)} f_o \geq \Lambda^{(k+1)} f_o \geq \cdots \geq f_{(\alpha,\sigma)}(x)$$

for $\alpha + \sigma \geq 1$.

Furthermore, $\Lambda^{(k)} f_o$ converges uniformly to $f_{(\alpha,\sigma)}(x)$.

Proof: We need only consider the case $\alpha + \sigma \leq 1$, the case $\alpha + \sigma \geq 1$ then following as a consequence of (2.4) Since in this case

$$f_o(x) = \prod_{i=0}^{\infty} L_{\alpha}^{(i)}(x) = \prod_{i=1}^{\infty} \left[1-(1-\alpha)^i(1-x)\right]$$

is clearly continuous at $x = 0,1$, and bounded in $< 0,1 >$; By the theorem of the preceding section $\Lambda^{(k)} f_o$ converges uniformly to $f_{(\alpha,\sigma)}(x)$. In order to establish (4.2) we need only verify that $f_o(x) \leqslant \Lambda f_o$. But this is clear since

$$\Lambda f_o = x f_o(L(x) + (1-x) f_o(\sigma x) = f_o(x) + (1-x) f_o(\sigma x) \ .$$

§ 5. The Geometric Convergence Theorem.

Apart from the computational method suggested above it is desirable to have an estimate for the rate of convergence of the successive approximants in terms of the parameters α, σ. For reasons which will become clear shortly attention is focused on $\Lambda^{(k)} x$, and in order to shed some light on the problem we shall first obtain a partial result. We restrict ourselves to $\alpha + \sigma \geqslant 1$ and observe that

$$0 \leqslant \Lambda x - x = x(1-x)(\alpha+\sigma-1)$$

$$\begin{aligned} 0 \leqslant \Lambda^{(2)} x - \Lambda x &= (\alpha+\sigma-1) x (1-x) \{(1-\alpha) L(x) + \sigma(1-\sigma x) \\ &= (\alpha+\sigma-1) x (1-x) \{[(1-\alpha)\alpha + \sigma] + [(1-\alpha)^2 - \sigma^2] x\} \\ &\leqslant x(1-x)(\alpha+\sigma-1)[(1-\alpha)\alpha + \sigma]. \end{aligned}$$

Writing $1 - \tau = (1-\alpha)\alpha + \sigma$, this yields

(5.1) $$0 \leqslant \Lambda^{(k)} x - \Lambda^{(k-1)} x \leqslant (\alpha+\sigma-1)(1-\tau)^{k-1} \ .$$

Then since

$$f_{(\alpha,\sigma)}(x) - \Lambda^{(n)}x = \sum_{k=n}^{\infty}\left\{\Lambda^{(k+1)}x - \Lambda^{(n)}x\right\},$$

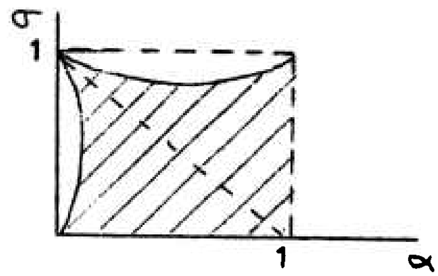
if $\tau < 1$ we obtain from (5.1)

$$(5.2) \qquad |f_{(\alpha,\sigma)}(x) - \Lambda^{(n)}x| \leq \frac{(\alpha+\sigma-1)}{\tau}(1-\tau)^n$$

By using (2.4) we have in the case $\alpha + \sigma \leq 1$ the result (5.2) with

$$\tau = (1-\sigma)^2 - (1-\alpha-\sigma).$$

This result provides what we might call "geometric convergence" for the parameter values (α, σ) lying in the interior of the shaded region of the parameter space as pictured below:



The above discussion leads one to prove that we have this kind of "geometric convergence" for any point (α, σ) of the parameter square, $\alpha \neq 0$, $\sigma \neq 1$. This is in fact true and is established in the following theorem. We first, however, establish a lemma which will be needed in the proof.

Lemma: For $0 \leq \sigma < 1$,

(5.3) $$\lim_{k \to \infty} \left\{ \frac{d}{dx} \Lambda^{(k)} x \right\}_{x=0} = \frac{f(\alpha)}{1-\sigma} .$$

Proof: For any $\varphi(x)$, such that $\varphi(0) = 0$, $\varphi(1) = 1$, φ differentiable,

$$\Lambda \varphi = x \; \varphi(L(x)) + (1-x) \; \varphi(\sigma x)$$

$$\left\{ \frac{d}{dx} \Lambda \varphi \right\}_{x=0} = \varphi(\alpha) + \sigma \varphi'(0)$$

Setting $\varphi(x) = \Lambda^{(k-1)} x$ we get

(5.4) $$\left\{ \frac{d}{dx} \Lambda^{(k)} x \right\}_{x=0} = (\Lambda^{(k-1)} x)(\alpha) + \sigma \left\{ \frac{d}{dx} \Lambda^{(k-1)} x \right\}_{x=0} .$$

Writing
$$u_k = \left\{ \frac{d}{dx} \Lambda^{(k)} x \right\}_{x=0}$$
$$v_k = \left\{ (\Lambda^{(k-1)} x(\alpha) \right\}$$

(5.4) may be written as

(5.5) $$u_k = v_{k-1} + \sigma \; u_{k-1} .$$

This in turn implies

(5.6) $$u_k = v_{k-1} + \sigma v_{k-2} + \sigma^2 v_{k-3} + \dots + \sigma^{k-2} v_1 .$$

Now clearly $v_k \to f(\alpha)$ as $k \to \infty$, whence it easily follows from (5.6), since $0 \leq \sigma < 1$, that

$$u_k \to \frac{f(\alpha)}{1-\sigma} \quad \text{as } k \to \infty.$$

Theorem: For (α, σ) any parameter values, $\alpha \neq 0$, $\sigma \neq 1$, there exists constants $K = K(\alpha, \sigma)$, $\lambda = \lambda(\alpha, \sigma)$, $0 \leqslant \lambda < 1$ such that

$$(5.7) \qquad |\Lambda^{(n)} x - f_{(\alpha,\sigma)}(x)| \leqslant K \lambda^n (x(1-x), \ n = 0,1,2,\ldots, \ 0 \leqslant x \leqslant 1.$$

Proof: To begin with we restrict ourselves to the case $\alpha + \sigma > 1$. We introduce a new positive linear operator Γ defined by

$$(5.8) \qquad \Gamma\varphi(x) = (1-\alpha)L(x)\varphi(x) + \sigma(1-\sigma x)\varphi(\sigma x)$$

and let $\varphi_o(x) \equiv 1$ in $0 \leqslant x \leqslant 1$. Next it will be shown that

$$(5.9) \qquad \Lambda^{(n)} x(1-x) = x(1-x)\, \Gamma^{(n)} \varphi_o .$$

Since the result is clearly true for $n = 0$, we proceed by induction, assuming (5.9) for n. Then

$$\begin{aligned} \Lambda^{(n+1)} x(1-x) &= xL(x)(1-L(x))(\Gamma^{(n)}\varphi_o)(L(x)) + \sigma x(1-x)(1-\sigma x)(\Gamma^{(n)}\varphi_o)(\sigma x) \\ &= x(1-x)\left\{(1-\alpha)L(x)(\Gamma^{(n)}\varphi_o)(L(x)) + \sigma(1-\sigma x)(\Gamma^{(n)}\varphi_o)(\sigma x)\right\} \\ &= x(1-x)\,\Gamma^{(n+1)}\varphi_o \end{aligned}$$

which completes the induction, and the proof of (5.9).

Next we demonstrate that for each n, $\Gamma^{(n)}\varphi_o$ is a monotone decreasing function of x for $0 \leqslant x \leqslant 1$. This is trivial for $n = 0$, and we again proceed by induction. It clearly suffices to show that if $\varphi(x)$ is monotone decreasing and differentiable in $< 0,1 >$ then $\Gamma\varphi$ inherits these properties. With regard to the differentiability this is clear. With regard to the monotonicity,

$$\frac{d}{dx}\Gamma\varphi = (1-\alpha)^2 L(x)\,\varphi'(L(x)) + \sigma^2(1-\sigma x)\,\varphi'(\sigma x) + \left[(1-\alpha)^2\,\varphi(L(x)) - \sigma^2\,\varphi(\sigma x)\right]$$

and since $\varphi'(x) \leqslant 0$, and $\sigma > 1-\alpha$ it follows that this is also $\leqslant 0$ in $< 0,1 >$.

From the fact that $\Gamma^{(k)}\varphi_o$ is monotone decreasing we have clearly

$$(5.10) \qquad \|\Gamma^{(k)}\varphi_o\| = \max_{0\leqslant x\leqslant 1} \Gamma^{(k)}\varphi_o = \Gamma^{(k)}\varphi_o|_{x=0} \;.$$

Next combining (3.3) and (5.9) we obtain for $0 \leqslant x \leqslant 1$

$$(5.11) \qquad \frac{\Lambda^{(k+1)}x - \Lambda^{(k)}x}{x} = (\alpha + \sigma - 1)(1-x)\,\Gamma^{(k)}\varphi_o.$$

Letting $x \longrightarrow 0$, (5.11) in turn yields

$$\left\{\frac{d}{dx}\Lambda^{(k+1)}x\right\}_{x=0} - \left\{\frac{d}{dx}\Lambda^{(k)}x\right\}_{x=0} = (\alpha + \sigma - 1)\,\Gamma^{(k)}\varphi_o|_{x=0}$$

Finally, letting $k \longrightarrow \infty$, applying the lemma, and recalling our assumption $\alpha + \sigma > 1$, we get

$$\lim_{k\longrightarrow\infty}\left\{\Gamma^{(k)}\varphi_o|_{x=0}\right\} = 0.$$

From (5.10) this implies

$$\lim_{k\to\infty} ||\Gamma^{(k)}\varphi_o|| = 0 . \tag{5.12}$$

Thus there exists a smallest positive integer k_o such that

$$\rho = ||\Gamma^{(k_o)}\varphi_o|| < 1. \tag{5.13}$$

Combining (5.13) with (5.9) we obtain

$$\Lambda^{(k_o)}x(1-x) \leqslant \rho x(1-x)$$

operating repeatedly with $\Lambda^{(k_o)}$ on this yields

$$\Lambda^{(k_o m)}x(1-x) \leqslant \rho^m x(1-x); \quad m = 0,1,2,3,\ldots \tag{5.14}$$

On the other hand, letting $\beta = \max_{0\leqslant k\leqslant k_o} \left\{||\Gamma^{(k)}\varphi_o||, 1\right\}$ we have from (5.9)

$$\Lambda^{(k)}x(1-x) \leqslant \beta x(1-x); \quad k = 1,\ldots,k_o - 1.$$

Operating on both sides of this repeatedly with Λ^{k_o} we get

$$\Lambda^{(k_o m+k)}x(1-x) \leqslant \beta\rho^m x(1-x), \quad k = 1,\ldots,k_o - 1. \tag{5.15}$$

Since the progressions $k_o m + k$, $k = 0,1,\ldots,k_o-1$ include all integers (5.14) and (5.15) imply that for all $n \geqslant 0$

$$\Lambda^{(n)}x(1-x) \leqslant \beta' \rho^{\frac{n}{k_o}} x(1-x) . \quad (\beta' = \tfrac{\beta}{\rho}) \tag{5.16}$$

From (3.1) it follows that

$$f_{(\alpha,\sigma)}(x) - \Lambda^{(n)}x = (\alpha + \sigma - 1) \sum_{k=n}^{\infty} \Lambda^{(k)}x(1-x),$$

and using (5.16) this gives

$$|f_{(\alpha,\sigma)}(x) - \Lambda^{(n)}x| \leqslant \frac{(\alpha+\sigma-1)\beta}{1-\rho^{1/k_o}} \left(\rho^{1/k_o}\right)^n .$$

Thus the theorem is proved in this case with $K = \dfrac{\alpha+\sigma-1}{1-\rho^{1/k_o}}$ and $\lambda = \rho^{1/k_o}$. Via (2.4), for $\alpha + \sigma < 1$, (5.7) then follows with

$$K = \frac{1-\alpha-\sigma}{1-\bar{\rho}^{1/\bar{k}_o}} \;, \text{ and } \lambda = \bar{\rho}^{1/\bar{k}_o}$$

where $\bar{\rho}$ and $\bar{k}_o$ are determined analogously from the operator

$$\Gamma\varphi = \sigma[(1-\sigma)+\sigma x]\varphi(x) + (1-\alpha)[1-(1-\alpha)x]\varphi((1-\alpha)x) .$$

Finally, for $\alpha + \sigma = 1$, $\alpha \neq 0$, $\sigma \neq 1$, the result is true with any K and λ; and the proof of (5.7) is complete in all cases.

The above theorem can now be extended easily to

Theorem: Let $f_o(x)$ be any bounded function such that $f_o(0) = 0$, $f_o(1) = 1$, where $f_o(x)$ has finite derivative numbers at $x = 0,1$. Then for $\alpha \neq 0$, $\sigma \neq 1$, there exist constants $\tilde{K} = \tilde{K}(\alpha, \sigma, f_o)$ $\tilde{\lambda} = \tilde{\lambda}(\alpha, \sigma)$, $0 \leqslant \tilde{\lambda} < 1$, such that

(5.17) $$|\Lambda^{(n)}f_o - f_{(\alpha,\sigma)}(x)| \leqslant \tilde{K}\,\tilde{\lambda}^n x(1-x),\quad n = 0,1,\ldots,\ 0\leqslant x\leqslant 1.$$

Proof: For such $f_o(x)$ we have from (3.1)

$$|f_o(x) - x| \leqslant c\, x(1-x)$$

so that operating with $\Lambda^{(n)}$ we obtain using (5.16)

$$|\Lambda^{(n)} f_o - \Lambda^{(n)} x| \leq c\,\Lambda^{(n)} x(1-x) \leqslant c\beta(\rho^{1/k_o})^n\, x(1-x),\ 0 \leqslant x \leqslant 1\ .$$

Combining this with the previous theorem yields

$$|\Lambda^{(n)} f_o - f_{(\alpha,\sigma)}| \leqslant (c\beta + K)\,\lambda^n x(1-x)$$

and (5.17) follows with $K = c\beta + K$ and $\tilde{\lambda} = \lambda$.

It is interesting to note that $\tilde{\lambda} = \lambda$ is independent of $f_o(x)$ and depends only on (α, σ). Also, we might remark in passing that the finiteness of the derivative numbers at $x = 0,1$ is necessary in the sense that this is a consequence of (5.17) with $n = 0$.

§ 6. <u>Behavior of the Solution as a Function of the Parameters</u> α, σ.

In this section an investigation will be carried out concerning the variation of $f_{(\alpha,\sigma)}(x)$ as a function of (α, σ).

(a). We begin with a consideration of the question of the continuity of $f_{(\alpha,\sigma)}(x)$ as a function of α, σ at those points of the parameter square on the lines $\alpha = 0$, and $\sigma = 1$, excluding the point (0,1). We shall refer to these lines as the "critical boundary". If we could assert on the basis of some obvious remark that for a fixed $\alpha > 0$, (6.1) $$\lim_{\sigma \to 1-} \left\{ f_{(\alpha,\sigma)}(x) - f_{(\alpha,\sigma)}(\sigma x) \right\} = 0$$

Then the proofs of the following theorems could be contracted considerably. However, since the proofs are simple, their slightly devious route is not too distressing.

First we recall that for

$$\alpha = 0, \ \sigma \neq 1, \ f_*(x) = \begin{cases} 0 \text{ for } x \neq 1 \\ 1 \text{ for } x = 1 \end{cases}$$

$$\alpha \neq 0, \ \sigma = 1, \ f^*(x) = \begin{cases} 1 \text{ for } x = 1 \\ 0 \text{ for } x \neq 1 \end{cases}$$

is the unique solution of $\Lambda_{(\alpha,\sigma)} f = f$, (subject to suitable continuity assumption at one end point).

Lemma: For $\alpha \neq 0$, $\sigma \neq 0$, $f_{(\alpha,\sigma)}(x)$ has a uniformly bounded difference quotient in $0 \leqslant x \leqslant 1$; and in fact a bound is

$$\frac{1}{\min(\alpha, 1-\sigma)} .$$

Proof: This result is inherent in the proof of the existence theorem given in paper II since $\frac{1}{\min(\alpha, 1-\sigma)}$ is obtained as a uniform bound for all the difference quotients of all the $\Lambda^{(n)}x$. From this the result is immediate.

Lemma: For fixed α, and x, $\alpha > 0$; $f_{(\alpha,\sigma)}(x)$ is a non-decreasing function of σ. Also for fixed σ and x, $\sigma < 1$, $f_{(\alpha,\sigma)}(x)$ is a non-decreasing function of α.

Proof: The second statement follows from the first via (2.4). Thus consider $\alpha > 0$, fixed. Then for $\sigma' \geq \sigma$, we have

$$(6.2) \quad \Lambda_{(\alpha,\sigma)} f_{(\alpha,\sigma')} = f_{(\alpha,\sigma')}(x) - (1-x)\left\{f_{(\alpha,\sigma')}(\sigma' x) - f_{(\alpha,\sigma')}(\sigma x)\right\} .$$

Since $f_{(\alpha,\sigma')}(x)$ is a monotone increasing function this yields

$$\Lambda_{(\alpha,\sigma)} f_{(\alpha,\sigma')} \leq f_{(\alpha,\sigma')}(x) .$$

Operating repeatedly with $\Lambda_{(\alpha,\sigma)}$ this yields

$$\Lambda^{(k)}_{(\alpha,\sigma)} f_{(\alpha,\sigma')} \leq f_{(\alpha,\sigma')}(x)$$

for every integer $K \geq 1$. Since $f_{(\alpha,\sigma')}(x)$ is continuous, letting $K \to \infty$,

$$\Lambda^{(k)}_{(\alpha,\sigma)} f_{(\alpha,\sigma')} \longrightarrow f_{(\alpha,\sigma)}(x)$$

and the lemma follows.

Theorem: If $\sigma \longrightarrow 1-$ and $\alpha \geq \varkappa > 0$, ($\varkappa$ fixed), then

$$(6.3) \qquad f_{(\alpha,\sigma)}(x) \longrightarrow f^*(x)$$

Also, if $\alpha \longrightarrow 0+$ and $\sigma \leq 1-\varkappa$, $\varkappa > 0$ fixed, then

$$(6.4) \qquad f_{(\alpha,\sigma)}(x) \longrightarrow f_*(x) .$$

Proof: We consider first the case where $\alpha > 0$ is fixed, and we let $\sigma \longrightarrow 1-$. From the lemma, $\lim_{\sigma \to 1-} f_{(\alpha,\sigma)}(x)$ exists for each x, $0 \leq x \leq 1$, and we define

$$G_\alpha(x) = \lim_{\sigma \to 1-} f_{(\alpha,\sigma)}(x) . \tag{6.5}$$

Clearly $G_\alpha(0) = 0$, and $G_\alpha(1) = 1$.

We now break up the proof into several steps.

1) $$\lim_{\delta \to 1-} G_\alpha(\delta x) = G_\alpha(x) .$$

Since $G_\alpha(x)$ is clearly a non-decreasing function of x, we have $0 \leqslant G_\alpha(x) - G_\alpha(\delta x)$. Thus assuming 1) to be false for some $x \neq 0$, we have a sequence of $\delta_i \to 1-$ and an $\epsilon > 0$ such that

$$G_\alpha(x) > G_\alpha(\delta_i x) + \epsilon .$$

Then
$$G_\alpha(x) \geqslant \sup_i G_\alpha(\delta_i x) + \epsilon$$
$$> \sup_i G_\alpha(\delta_i x) + \frac{\epsilon}{2} .$$

Thus for all σ sufficiently close to 1 (we fix one such σ for the following argument), we have

$$f_{(\alpha,\sigma)}(x) > \sup_i G_\alpha(\delta_i x) + \frac{\epsilon}{2}$$

so that

$$f_{(\alpha,\sigma)}(x) > G_\alpha(\delta_i x) + \frac{\epsilon}{2} \quad \text{for all i.}$$

From this last inequality we get that for <u>all</u> i, and all $\sigma' < 1$,

$$f_{(\alpha,\sigma)}(x) > f_{(\alpha,\sigma')}(\delta_i x) + \frac{\epsilon}{2} . \tag{6.6}$$

Now applying our estimate for the difference quotients of

$f_{(\alpha,\sigma')}(x)$ we get

$$f_{(\alpha,\sigma')}(\delta_i x) = f_{(\alpha,\sigma')}(x) - \left\{f_{(\alpha,\sigma')}(x) - f_{(\alpha,\sigma')}(\delta_i x)\right\}$$

$$\geq f_{(\alpha,\sigma')}(x) - \frac{1-\delta_i}{\min(\alpha, 1-\sigma')} \quad .$$

Inserting this in (6.6) yields

$$(6.7) \qquad f_{(\alpha,\sigma)}(x) > f_{(\alpha,\sigma')}(x) + \frac{\epsilon}{2} - \frac{(1-\delta_i)}{\min(\alpha, 1-\sigma')}$$

for all i, and all σ'.

Take $\sigma' = 1 - (1-\delta_i)^{\frac{1}{2}}$ in (6.7). Then for i sufficiently large, $1 - \sigma' < \alpha$, and $\sigma' > \sigma$ so that (6.7) together with the lemma, preceding the theorem, implies

$$f_{(\alpha,\sigma)}(x) > f_{(\alpha,\sigma')}(x) + \frac{\epsilon}{2} - \frac{1-\delta_i}{1-\sigma'}$$

$$> f_{(\alpha,\sigma)}(x) + \frac{\epsilon}{2} - (1-\delta_i)^{\frac{1}{2}}$$

or

$$(1-\delta_i)^{\frac{1}{2}} > \frac{\epsilon}{2}$$

for all sufficiently large i. But this is clearly false for i large enough, since $\delta_i \to 1-$. This completes the proof of 1).

2) $\lim_{\sigma \to 1-} f_{(\alpha,\sigma)}(\sigma x)$ exists and is $\geq G_\alpha(x)$.

The existence of the limit follows from the fact that for $\sigma' \geq \sigma$

$$f_{(\alpha,\sigma')}(\sigma' x) \geq f_{(\alpha,\sigma')}(\sigma x) \geq f_{(\alpha,\sigma)}(\sigma x) .$$

Alternately, the existence of this limit also follows from

$$(6.8) \qquad f_{(\alpha,\sigma)}(x) = x\, f_{(\alpha,\sigma)}(L_\alpha(x)) + (1-x) f_{(\alpha,\sigma)}(\sigma x)$$

since the limits of the other terms exist as $\sigma \longrightarrow 1-$.

Given any δ, $0 < \delta < 1$, for $\sigma \geq \delta$ we have

$$f_{(\alpha,\sigma)}(\sigma x) \geq f_{(\alpha,\sigma)}(\delta x) .$$

Letting $\sigma \longrightarrow 1-$ we get

$$\lim_{\sigma \to 1-} f_{(\alpha,\sigma)}(\sigma x) \geq G_\alpha(\delta x) .$$

Next letting $\delta \longrightarrow 1-$ and applying 1), it follows that

$$\lim_{\sigma \to 1-} f_{(\alpha,\sigma)}(\sigma x) \geq G_\alpha(x).$$

3) $\qquad G_\alpha(x) = f^*(x)$ for $\alpha > 0$.

Starting with (6.8), letting $\sigma \longrightarrow 1-$ and using 2) we get

$$G_\alpha(x) \geq x\, G_\alpha(L_\alpha(x)) + (1-x) G_\alpha(x)$$

or

$$x\, G_\alpha(x) \geq x\, G_\alpha(L_\alpha(x)).$$

Then for $x \neq 0$ we have

$$1 \geq G_\alpha(x) \geq G_\alpha(L_\alpha(x))$$

which in turn gives

$$1 \geq G_\alpha(x) \geq G_\alpha(L_\alpha^{(k)}(x)) \geq f_{(\alpha,\sigma)}(L_\alpha^{(k)}(x))$$

for any fixed σ, $0 < \sigma < 1$, and all integers $k \geq 1$. Finally,

$$1 \geq G_\alpha(x) \geq \lim_{k \to \infty} f_{(\alpha,\sigma)}(L_\alpha^{(k)}(x)) = f_{(\alpha,\sigma)}(1) = 1.$$

Thus for all $x \neq 0$, $G_\alpha(x) = 1$. Since we've already noted that $G_\alpha(0) = 0$, we see that $G_\varkappa(x) = f^*(x)$.

4) The above section 3) proves the theorem if $\sigma \to 1-$ with α fixed. Now suppose $\sigma \to 1-$ and $\alpha \geq \varkappa > 0$, α not necessarily fixed. We note that for σ fixed, $\alpha \geq \varkappa$,

$$f_{(\alpha,\sigma)}(x) \geq f_{(\varkappa,\sigma)}(x)$$

so that for $x \neq 0$

$$1 \geq \overline{\lim_{\substack{\sigma \to 1- \\ \alpha \geq \varkappa}}} f_{(\alpha,\sigma)}(x) \geq \underline{\lim_{\substack{\sigma \to 1- \\ \alpha \geq \varkappa}}} f_{(\alpha,\sigma)}(x) \geq \lim_{\sigma \to 1-} f_{(\varkappa,\sigma)}(x) = G_\varkappa(x) = f^*(x) = 1 .$$

Thus it follows that for $x \neq 0$, as $\sigma \to 1-$,

$$f_{(\alpha,\sigma)}(x) \to 1.$$

Since for $x = 0$, $f_{(\alpha,\sigma)}(0) = 0 \to 0$, the proof of this part of the theorem is completed.

The second assertion concerning $\alpha \to 0+$ as $\sigma \leq 1-\varkappa$ follows from the first part by means of (2.4).

(b) <u>A Lipschitz condition in (α,σ); uniform continuity off the critical boundary</u>.

In this section we shall derive a "Lipschitz condition" for $f_{(\alpha,\sigma)}(x)$ as a function of (α,σ), uniformly in x. From this will follow information concerning the continuity of $f_{(\alpha,\sigma)}(x)$, as a function of (α,σ), uniformly in x.

We will write $\|\varphi(x)\| = \sup_{0\leq x\leq 1} |\varphi(x)|$; and denote by $\mathcal{S}$ the parameter square, excluding the lines $\alpha = 0$, $\sigma = 1$.

<u>Theorem</u>: <u>For</u> (α,σ), (α',σ') <u>in</u> $\mathcal{S}$;

$$\|f_{(\alpha,\sigma)}(x) - f_{(\alpha',\sigma')}(x)\| \leq A|\alpha-\alpha'| + B|\sigma-\sigma'| \tag{6.9}$$

<u>where</u>

$$A = A(\alpha,\sigma,\alpha',\sigma') = \left\{\min[1-\sigma', \max(\alpha,\alpha')]\,\left[\min[1-\sigma', \min(\alpha,\alpha')]\right]^3\right\}^{-1}$$

$$B = B(\alpha,\sigma,\alpha',\sigma') = \left\{\min[\alpha, 1-\min(\sigma,\sigma')]\,\left[\min[\alpha, 1-\max(\sigma,\sigma')]\right]^3\right\}^{-1}$$

<u>Proof</u>: Let $\Lambda = \Lambda_{(\alpha,\sigma)}$ and <u>assume</u> $\sigma \geq \sigma'$. Then

$$\Lambda f_{(\alpha,\sigma')} - f_{(\alpha,\sigma')} = (1-x)\left\{f_{(\alpha,\sigma')}(\sigma x) - f_{(\alpha,\sigma')}(\sigma' x)\right\},$$

and using our estimate for the difference quotient, we get

$$0 \leq \Lambda f_{(\alpha,\sigma')} - f_{(\alpha,\sigma')} \leq \frac{\sigma-\sigma'}{c_1}\, x(1-x); \quad c_1 = \min(\alpha, 1-\sigma'). \tag{6.10}$$

Operating repeatedly with Λ on (6.10) gives for $k \geq 1$

$$0 \leq \Lambda^{(k)} f_{(\alpha,\sigma)} - \Lambda^{(k-1)} f_{(\alpha,\sigma')} \leq \frac{\sigma-\sigma'}{c}\, \Lambda^{(k-1)} x(1-x). \tag{6.11}$$

Summing (6.11) over k = 1,...,n, and letting $n \to \infty$, we have, since $\Lambda^{(n)} f_{(\alpha,\sigma')} \to f_{(\alpha,\sigma)}$,

$$(6.12) \qquad 0 \leqslant f_{(\alpha,\sigma)}(x) - f_{(\alpha,\sigma')}(x) \leqslant \frac{\sigma - \sigma'}{c_1} \sum_{i=0}^{\infty} \Lambda^{(i)} x(1-x) .$$

Combining (3.1) with (6.12)

$$(6.13) \qquad 0 \leqslant f_{(\alpha,\sigma)} - f_{(\alpha,\sigma')} \leqslant \frac{\sigma - \sigma'}{c_1} \begin{cases} \dfrac{f_{(\alpha,\sigma)}(x) - x}{\alpha + \sigma - 1}, & \text{if } \alpha + \sigma - 1 \neq 0 \\ \dfrac{1}{\alpha^2}, & \text{if } \alpha + \sigma - 1 = 0 \end{cases}$$

Finally, removing the assumption $\sigma \geqslant \sigma'$ from (6.13) we get in all cases:

$$(6.14) \qquad |f_{(\alpha,\sigma)}(x) - f_{(\alpha,\sigma')}| \leqslant \frac{|\sigma - \sigma'|}{d_1} \begin{cases} \dfrac{f_{\alpha,\sigma^*}(x) - x}{\alpha + \sigma^* - 1}, & \text{if } \alpha + \sigma^* - 1 \neq 0 \\ \dfrac{1}{2}, & \text{if } \alpha + \sigma^* - 1 = 0 \end{cases}$$

where

$$d_1 = \min[\alpha, 1 - \min(\sigma, \sigma')]$$

and

$$\sigma^* = \max(\sigma, \sigma').$$

Suppose then that $\sigma \geqslant \sigma'$, and $\alpha + \sigma - 1 = 0$. Then from (6.13)

$$(6.15) \qquad ||x - f_{(\alpha,\sigma')}|| \leqslant \frac{|\sigma - \sigma'|}{c_1} \cdot \frac{1}{\alpha^2} = \frac{|1 - \alpha - \sigma'|}{\min(\alpha, 1 - \sigma')} \cdot \frac{1}{\alpha^2}$$

From this we conclude that for $1 - \alpha \geqslant \sigma$

$$(6.16) \qquad ||x - f_{(\alpha,\sigma)}|| \leqslant \frac{|1 - \alpha - \sigma|}{\alpha^3} .$$

On the other hand for $1 - \alpha \leqslant \sigma$, we have via (2.4) and (6.16), since $1-(1-\sigma) \geq 1-\alpha$,

$$(6.17) \qquad ||x - f_{(\alpha,\sigma)}(x)|| = ||x - f_{1-\sigma,1-\alpha}(x)|| \leq \frac{|1-\alpha-\sigma|}{(1-\sigma)^3} .$$

From (6.16) and (6.17) we see that we have in all cases

$$(6.18) \qquad ||x - f_{(\alpha,\sigma)}(x)|| \leq \frac{|1-\alpha-\sigma|}{\{\min(\alpha,1-\sigma)\}^3}$$

Inserting this back in (6.14) we have for $\alpha + \sigma^* - 1 \neq 0$

$$(6.19) \qquad ||f_{(\alpha,\sigma)}(x) - f_{(\alpha,\sigma')}(x)|| \leq \frac{|\sigma-\sigma'|}{d_1} \cdot \frac{1}{\{\min(\alpha,1-\sigma^*)\}^3}$$

From the second part of (6.14) it is clear that (6.19) holds as is for $\alpha + \sigma^* - 1 = 0$ also.

Next, using (6.19) and (2.4) we have

$$(6.20) \qquad ||f_{(\alpha,\sigma)}(x) - f_{(\alpha',\sigma)}(x)|| = ||f_{(1-\sigma,1-\alpha)}(x) - f_{(1-\sigma,1-\alpha')}(x)||$$

$$\leq \frac{|\alpha-\alpha'|}{d_2} \cdot \frac{1}{\{\min(1-\sigma,\alpha^*)\}^3}$$

where

$$d_2 = \min[1-\sigma, \max(\alpha,\alpha')]$$

$$\alpha^* = \min(\alpha,\alpha') .$$

Finally, from (6.19) and (6.20) we have

$$||f_{(\alpha,\sigma)}(x) - f_{(\alpha',\sigma')}(x)|| \leq f_{(\alpha,\sigma)}(x) - f_{(\alpha,\sigma')}(x)|| + ||f_{(\alpha,\sigma')} - f_{(\alpha',\sigma')}(x)||$$

$$\leq |\sigma-\sigma'| \left\{\frac{1}{d_1\{\min(\alpha,1-\sigma^*)\}^3}\right\} + |\alpha-\alpha'| \left\{\frac{1}{d_2\{\min(1-\sigma',\alpha^*)\}^3}\right\}$$

which is the result stated in the theorem.

From the above theorem we can now deduce the following:

<u>Corollary</u>: For any δ, $0 < \delta < 1$, however small; if $(\alpha, \sigma), (\alpha', \sigma')$ belong to the closed region $\mathcal{S}_\delta$: $\delta \leq \alpha \leq 1$, $0 \leq \sigma \leq 1-$; then (6.9) may be replaced by

$$(6.21) \qquad \| f_{(\alpha,\sigma)} - f_{(\alpha',\sigma')} \| \leq K \left\{ |\alpha - \alpha'| + |\sigma - \sigma'| \right\}$$

where $K = K(\delta)$ is independent of (α, σ), (α', σ').

<u>Proof</u>: In order to obtain the above from the theorem simply requires the observation that for (α, σ) and (α', σ') in $\mathcal{S}_\delta$ we have

$$A \leq \delta^{-4} \text{ and } B \leq \delta^{-4} .$$

Thus (6.21) follows, with $K(\delta) = \delta^{-4}$.

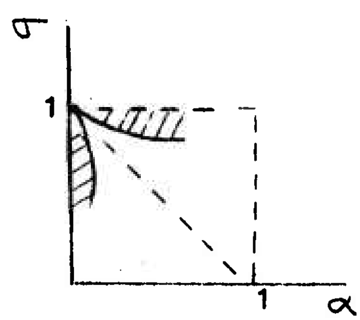
From (6.21) we see that in any $\mathcal{S}_\delta$, $f_{(\alpha,\sigma)}(x)$ is a continuous function of (α, σ) uniformly in x. From this in turn it follows that at any point (α_0, σ_0) $\alpha_0 \neq 0$, $\sigma_0 \neq 1$, $f_{(\alpha,\sigma)}(x)$ is a continuous function of (α, σ) uniformly in x.

6(c). <u>Behavior of the solution as</u> $(\alpha, \sigma) \rightarrow (0,1)$.

The point $(\alpha, \sigma) = (0,1)$ of the parameter square, henceforth referred to as the point P, has the property that at this point the functional equation degenerates to the identity $f(x) = f(x)$. It is thus of some interest to study what happens to $f_{(\alpha,\sigma)}(x)$ as (α, σ) approaches P. Since this point is the intersection of three lines along which we have, respectively, the discontinuous solutions f^*, and f_* and the continuous solution x, one might suspect that the behavior of $f_{(\alpha,\sigma)}(x)$ in the neighborhood of P is rather chaotic and depends upon the mode of approach to P. That this is so will be revealed in the following discussion.

From the theorem of section 6(a) we first obtain the following:

<u>Theorem</u>: <u>There exists a "stolz-like" neighborhood of</u> P



<u>bordering</u> $\sigma = 1$, <u>such that as</u> $(\alpha, \sigma) \to P$, (α, σ) <u>in this neighborhood</u>, $f_{(\alpha,\sigma)}(x) \to f^*(x)$. <u>Also, there exists such a neighborhood of</u> P <u>bordering</u> $\alpha = 0$ <u>such that as</u> $(\alpha, \sigma) \to P$ <u>in this neighborhood</u>

$$f_{(\alpha,\sigma)}(x) \to f_*(x) .$$

<u>Proof</u>: As usual we need only prove the first assertion, the second following from it via an application of (2.4).

For each $\alpha > 0$, choose $\sigma = \sigma(\alpha)$ such that

$$f_{(\alpha,\sigma(\alpha))}(\alpha) \geq 1 - \alpha \quad . \tag{6.22}$$

This can be done since $\lim_{\sigma \to 1-} f_{(\alpha,\sigma)}(\alpha) = 1$. Now if

$(\alpha,\sigma) \to P$ inside the region bounded by $\sigma = \sigma(\alpha)$ and $\sigma = 1$, we clearly have $f_{(\alpha,\sigma)}(0) \to 0$, and for any given $x \neq 0$, $x \geq \alpha$ for α sufficiently small, so that

$$1 \geq f_{(\alpha,\sigma)}(x) \geq f_{(\alpha,\sigma)}(\alpha) \geq f_{(\alpha,\sigma(\alpha))}(\alpha) \geq 1 - \alpha .$$

Thus in the limit, since $\alpha \to 0$, $f_{(\alpha,\sigma)}(x) \to 1$.

With regard to the possibility of $f_{(\alpha,\sigma)}(x) \to x$ as $(\alpha,\sigma) \to P$, it is an easy consequence of (6.9) that this is so if the mode of approach is such that $\sigma = 1 - \alpha + O(\alpha^3)$. However, by utilizing the method which led to (6.9) in a sharper form, we obtain the following more complete result:

<u>Theorem</u>: <u>A necessary and sufficient condition that as</u> $(\alpha,\sigma) \to P$ <u>we have</u>

(6.23) $$f_{(\alpha,\sigma)}(x) \to x,$$

<u>is that the mode of approach is such that</u>

(6.24) $$\sigma = 1 - \alpha + o(\alpha^2).$$

<u>Proof</u>: Since substituting $(1-\sigma, 1-\alpha)$ for (α,σ) in (6.24) gives

$$1-\alpha = \sigma + o(1-\sigma)^2$$

which is precisely the same as (6.24) for $(\alpha,\sigma) \to P$, we see that it suffices to prove the whole theorem under the assumption that

(6.25) $$g(\alpha) = \alpha + \sigma - 1$$

is greater than zero.

Sufficiency; i.e. assume $g(\alpha) = o(\alpha^2)$. We recall from (3.4) that

(6.26) $$f_{(\alpha,\sigma)}(x) - x = (\alpha+\sigma-1)\sum_{i=0}^{\infty}\Lambda^{(i)}[x(1-x)]$$

also via the calculation preceding (5.1), since $\alpha+\sigma > 1$, $\sigma > 1-\alpha$, and we have

(6.27) $$\Lambda[x(1-x)] \leqslant x(1-x)\;[(1-\alpha)\alpha+\sigma].$$

Now note that $(1-\alpha)\alpha+\sigma = 1-(\alpha^2-g(\alpha))$ so that

$$\Lambda[x(1-x)] \leqslant x(1-x)\;[1-(\alpha^2-g(\alpha))].$$

Then

$$\Lambda^{(i)}[x(1-x)] \leqslant x(1-x)\;[1-(\alpha^2-g(\alpha))]^i,$$

and since $g(\alpha) = o(\alpha^2)$, $\alpha^2-g(\alpha) > 0$ for α sufficiently small. Thus summing over i, yields

(6.28) $$\sum_{i=0}^{\infty}\Lambda^{(i)}[x(1-x)] \leqslant \frac{x(1-x)}{\alpha^2-g(\alpha)}.$$

Inserting (6.28) in (6.26) we obtain

(6.29) $$0 \leqslant f_{(\alpha,\sigma)}(x)-x \leqslant \frac{g(\alpha)}{\alpha^2-g(\alpha)}\,x(1-x) \leqslant \frac{g(\alpha)}{\alpha^2-g(\alpha)}$$

But as $\alpha \to 0$, $\dfrac{g(\alpha)}{\alpha^2-g(\alpha)} \to 0$ since $g(\alpha) = o(\alpha^2)$. Thus (6.29) yields that $f_{(\alpha,\sigma)}(x) \to x$ uniformly in x.

Necessity; i.e. assume $f_{(\alpha,\sigma)}(x) \to x$ for any one fixed $x \neq 0,1$. We have since $\alpha + \sigma > 1$

$$\Lambda x(1-x) = x(1-x)\left\{[(1-\alpha)\alpha+\sigma] + [(1-\alpha)^2 - \sigma^2]x\right\}$$
$$\geq x(1-x)\ [(1-\alpha)\alpha+\sigma + (1-\alpha)^2 - \sigma^2]$$

so that

$$(6.30)\qquad \Lambda^{(i)}[x(1-x)] \geq x(1-x)\ [(1-\alpha)\alpha+\sigma+(1-\alpha)^2-\sigma^2]^i \ .$$

Since $(1-\alpha)\alpha + \sigma + (1-\alpha)^2 - \sigma^2 = 1 - g(\alpha) - (\alpha - g(\alpha))^2$

it is clear that for sufficiently small α, and σ close to 1, since $g(\alpha) \to 0$, this quantity lies between 0 and 1. Thus summing (6.30) over i yields

$$(6.31)\qquad \sum_{i=0}^{\infty} \Lambda^{(i)}[x(1-x)] \geq \frac{x(1-x)}{g(\alpha)+(\alpha-g(\alpha))^2} \ .$$

Furthermore, since $g(\alpha) = \sigma + \alpha - 1 < 2\alpha$, $-\alpha < \alpha - g(\alpha) < \alpha$, so that

$$(\alpha - g(\alpha))^2 < \alpha^2$$

and (6.31) implies

$$(6.32)\qquad \sum_{i=0}^{\infty} \Lambda^{(i)}x(1-x) \geq \frac{x(1-x)}{\alpha^2+g(\alpha)} \ .$$

Combining (6.32) with (6.26) results in

$$(6.33)\qquad f_{(\alpha,\sigma)}(x) - x \geq \frac{g(\alpha)}{\alpha^2+g(\alpha)}\ x(1-x) \geq 0.$$

Thus if for fixed $x \neq 0,1$, $f_{(\alpha,\sigma)}(x) - x \to 0$ as $(\alpha, \sigma) \to P$, then

$$\frac{g(\alpha)}{\alpha^2+g(\alpha)} = \frac{1}{1+\frac{\alpha^2}{g(\alpha)}} \to 0,$$

whence $g(\alpha) = o(\alpha^2)$. This completes the proof of the theorem.

The following corollary may be extracted from the proof of the theorem.

Corollary: If $\alpha^2 > g(\alpha) > 0$ for $\alpha > 0$, then for $\sigma = 1 - \alpha + g(\alpha)$

$$\frac{g(\alpha)}{\alpha^2-g(\alpha)} x(1-x) \geq f_{(\alpha,\sigma)}(x) - x \geq \frac{g(\alpha)}{\alpha^2+g(\alpha)} x(1-x).$$

Corollary: If $f_{(\alpha,\sigma)}(x) \to x$, as $(\alpha, \sigma) \to P$, for any one particular $x \neq 0,1$, then $f_{(\alpha,\sigma)}(x) \to x$ uniformly in x for all x in $< 0,1 >$.

From the first corollary we see that if we take as our mode of approach to P the curve Γ_λ given by:

$$\Gamma_\lambda : \sigma = 1 - \alpha + \lambda \alpha^2, \quad 0 < \lambda < 1$$

where λ is fixed. In this case $g(\alpha) = \lambda \alpha^2$ and the corollary yields

$$(6.34) \qquad x + \frac{\lambda}{1-\lambda} x(1-x) \geq f_{(\alpha,\sigma)_\lambda}(x) \geq x + \frac{\lambda}{1+\lambda} x(1-x)$$

where $(\alpha, \sigma)_\lambda$ denotes a point on Γ_λ). Now we can choose $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots \to 0$ such that the intervals $\left(\frac{\lambda_i}{1-\lambda_i}, \frac{\lambda_i}{1+\lambda_i}\right)$ are mutually disjoint. Then for all $x \neq 0,1$, and $i \neq j$

$$f_{(\alpha,\sigma)_{\lambda_i}}(x) \neq f_{(\alpha,\sigma)_{\lambda_j}}(x) .$$

In fact

$$|f_{(\alpha,\sigma)_{\lambda_i}}(x) - f_{(\alpha,\sigma)_{\lambda_j}}(x)| > \delta_{ij} > 0$$

uniformly for all $(\alpha, \sigma)_{\lambda_i}$ on Γ_{λ_i} and $(\alpha, \sigma)_{\lambda_j}$ on Γ_{λ_j}, and all x bounded away from 0 and 1. Thus along the different Γ_{λ_i} we cannot get the same limit function (if the limit exists) as $\alpha \to 0$. In any event the above indicates the variety of possible outcomes as one approaches P from within the parameter square and observes the behavior of $f_{(\alpha,\sigma)}(x)$. In fact one might venture the conjecture that given any monotone function $h(x)$ such that $h(0) = 0$, $h(1) = 1$, there exists a sequence $(\alpha_i, \sigma_i) \to P$ such that

$$f_{(\alpha_i,\sigma_i)}(x) \to h(x).$$

§7. Generalizations.

In conclusion a few remarks will be added concerning some slightly more general function equations for which an analysis analogous to the above may be carried out.

First of all, if we let $\alpha(x)$ be any continuous function such that

$$0 < x \leqslant \alpha(x) \leqslant 1$$

for $0 \leqslant x \leqslant 1$, and $\sigma(x)$ a continuous function such that $0 \leqslant \sigma(x) \leqslant 1 - x$ for $0 \leqslant x \leqslant 1$, then the existence and uniqueness theorem goes through for the continuous solution of

$$f(x) = xf[\alpha(x) + (1-\alpha(x))x] + (1-x)\, f[\sigma(x)x],$$

such that $f(0) = 0$, $f(1) = 1$. More generally, if $p(x)$ is any continuous function such that $0 \leqslant p(x) \leqslant 1$, $p(0) = 0$, $p(1) = 1$ analogous theorems hold for

(7.1) $f(x) = p(x)\, f[\alpha(x) + (1-\alpha(x))x] + (1-p(x))\, f(\sigma(x)x)$.

Finally, if $H(x)$ and $G(x)$ are continuous, $H(0) = 0$, $G(1) = 1$, and certain conditions are placed on their behavior so that $H(x)$ may be mimicked by a function of the form $x\sigma(x)$, and $G(x)$ by $\alpha(x) + (1-\alpha(x))x$, the results for (7.1) may be carried over to

(7.2) $f(x) = p(x)f(G(x)) + (1-p(x))f(H(x))$.