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28 G - 3

A RANDOM ACCELERATION MODEL FOR FILTERING POLYNOMIAL-LIKE SIGNALS

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25 October 1963

LJ/FS:sm

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-500.

LEXINGTON MASSACHUSETTS

ERRATA SHEET

for 28G-3

The authors have detected the following errors in 28G-3 (L. Jones, F. Schweppe "A Random Acceleration Model for Filtering Polynomial-Like Signals," 25 October 1963). Kindly insert this page into your copy of that report.

Page 1 Equation (1. 1) should read

$$x(nT+T) = x(nT) + \dot{x}(nT)\,T + \ddot{x}(nT)\frac{T^2}{2}$$

Page 4 Equation (2. 6) should read

$$\underline{F} = \sum_{i=1}^{\tau} \underline{W}_i \underline{H}\,\underline{\phi}^{-i+\alpha+1} - \underline{I} = 0$$

Page 5 The first line of Eq. (2. 13) should read

$$\sum_{k=\beta}^{\alpha-1} \underline{H}\underline{\phi}^{-i+k}\,\underline{R}(\underline{H}\,\underline{\phi}^{-j+k})' , \quad i < \alpha+1,\ j < \alpha+1$$

14 November 1963

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ABSTRACT

Random type models for filtering discrete time, polynomial-like signals from noise are discussed. The particular model analyzed is a second-degree polynomial with the acceleration (second derivative) considered to be a random process. The optimum filter corresponding to this model is derived. Results include curves of filter frequency response, estimate variance, and filter impulse response for various choices of the system parameters. Some of the factors that affect the choice of a model for polynomial-like signals are discussed.

1. INTRODUCTION

There are many applications in which it is necessary to filter polynomial-like signals from the presence of additive random noise. The term, polynomial-like, refers to the class of signals that have the general appearance of a low degree polynomial in time although the exact form of the signals is not known. For example, in radar tracking of space vehicles, the true slant range (or angle or cartesian position) is often a smooth time function with the appearance of a low degree polynomial in time over a sufficiently short period of time. A standard filter design procedure for such signals models the signal as a fixed-degree polynomial observed in the presence of additive noise. The filter is then designed to provide minimum variance, unbiased estimates of the state of the signal at some desired time. This report introduces an alternate type of model for the same class of problems wherein the signal is assumed to be a particular stochastic process.

The explicit signal model analyzed assumes a discrete time signal, x, of the form of a second-degree polynomial in time

$$x(nT+T) = x(nT) + \dot{x}(nT)\, nT + \ddot{x}(nT)\, \frac{(nT)^2}{2} \tag{1.1}$$

with the acceleration being changed by a zero mean, random process u

$$\ddot{x}(nT) = \ddot{x}(nT-T) + bu\,(nT). \tag{1.2}$$

Time is represented by nT where $n = 0, 1, 2, \ldots,$ and T is the sampling period. Henceforth, the model of Eqs. (1.1) and (1.2) is referred to as the random acceleration model. The case, b = 0, represents the standard model for a fixed, second-degree polynomials.

The filter for the random acceleration model is derived in Sec. 2 to provide minimum variance, unbiased estimates of the state of the signal at some specified time. The performance characteristics of the optimum filter for the random acceleration model are presented in Sec. 3 along with those of the fixed polynomial model for the case of a second-degree polynomial. There are many references which consider different degrees of fixed polynomials. See for example, Refs. 1 through 6. (The last two of these references are to the statistical theory of linear regression of which the standard polynomial model is a very special case.) Section 4 discusses some of the factors that determine the applicability of the random type model.

2. DERIVATION OF THE OPTIMAL FILTERS

The random acceleration model and the criteria of filter optimality are explicitly defined and the filter derived. State variable techniques are used to represent the model.

First let us define discrete white noise. The random process y(n) is discrete white noise if

$$E[y(n)y(m)] = \delta_{nm}$$

$$E[y(n)] = 0$$

where E denotes the expected value and δ_{nm} is the Kronecker delta.

The model of the signal is a second-degree polynomial, which in state variable representation requires three states (here chosen to be position, velocity, and acceleration). The model is defined

$$\underline{X}(nT) = \underline{\phi}\underline{X}(nT - T) + \underline{B}u(nT) \tag{2.1}$$

where nT denotes the particular sample time with T the sampling period. Henceforth, the period is assumed unity, and the model is

$$\begin{aligned}
\underline{X}(n) &= \underline{\phi}\underline{X}(n-1) + \underline{B}u(n) \\
\underline{X}(n) &: \text{3-vector (the state)} \\
\underline{\phi} &: \text{3 x 3 constant state transition matrix} = \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
\underline{B} &: \text{3-vector coefficient of } u(n) = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \\
u(n) &: \text{discrete white noise} \\
\underline{R} &= \text{3 x 3 covariance matrix of } \underline{B}u(n) \\
&= E[\underline{B}u(n)u'(n)\underline{B}'] = \underline{B}\underline{B}'
\end{aligned} \tag{2.1'}$$

where prime denotes transposition. Thus, the model of the signal has the form of a linear system driven by discrete white noise. Define

$$
\begin{aligned}
z(n) &= \underline{H}\underline{X}(n) + cv(n) \\
z(n) &: \text{ measured quantity} \\
\underline{H} &: 1 \times 3 \text{ vector} = [1\ 0\ 0] \\
c &: \text{ coefficient of measurement noise} \\
v(n) &: \text{ discrete white noise} \\
Q &= \text{variance of measurement noise, } cv(n) \\
&= c^2
\end{aligned}
\tag{2.2}
$$

The random processes, u(n) and v(n) are uncorrelated; that is,

$$E[u(n)\,v(m)] = 0 \qquad \text{all } n \text{ and } m.$$

This completes the definition of the model and the form of the measurements. The parameter of great interest is b. As mentioned in the Introduction, the case $b = 0$ is the standard second-degree polynomial model and comparison of the two models is carried out for various b values.

The filter to be designed is to be a linear, weighted sum of the last τ measurements, designed to yield minimum variance, unbiased estimates of the state of the system. The estimate $\hat{\underline{X}}(n-\alpha)$ is defined as

$$\hat{\underline{X}}(n-\alpha) = \sum_{i=1}^{\tau} \underline{W}_i\, z(n+1-i) \tag{2.3}$$

where $\underline{W}_i$ is the i^{th} column of a $3 \times \tau$ matrix of weights. The variable α specifies the time at which the estimate is desired; that is, α is -1 for prediction one time step ahead, 0 for smoothing, and $(\tau-1)/2$ for midpoint estimation.

Substituting Eqs. (2.1) and (2.2) into (2.3) yields

$$\hat{\underline{X}}(n-\alpha) = \sum_{i=1}^{\tau} \underline{W}_i \underline{H}\underline{\phi}^{-i+\alpha+1} \underline{X}(n-\alpha)$$

$$+ \sum_{i=1}^{\alpha} \underline{W}_i \sum_{k=i}^{\alpha} \underline{H}\underline{\phi}^{-i+k} \underline{B}\, u(n+1-k)$$

$$- \sum_{i=\alpha+2}^{\tau} \underline{W}_i \sum_{k=\alpha+1}^{i-1} \underline{H}\underline{\phi}^{-i+k} \underline{B}\, u(n+1-k) \qquad (2.4)$$

$$+ \sum_{i=1}^{\tau} \underline{W}_i\, cv(n+1-i) \quad .$$

Considering $\underline{X}(n-\alpha)$ to be an unknown vector, an unbiased estimate requires that

$$E[\hat{\underline{X}}(n-\alpha)] = \underline{X}(n-\alpha) = \sum_{i=1}^{\tau} \underline{W}_i \underline{H}\underline{\phi}^{-i+\alpha+1} \underline{X}(n-\alpha) \quad . \qquad (2.5)$$

For this to apply for all n, the constraint equation F has the form

$$\underline{F} = \sum_{i=1}^{\tau} \underline{W}_i \underline{H}\underline{\phi}^{-i+w+1} - \underline{I} = 0 \qquad (2.6)$$

where $\underline{F}$ is a 3 x 3 matrix. Define the matrix

$$\underline{M} = (\underline{M}_i) \quad , \quad i = 1, \ldots, \tau \qquad (2.7)$$

where $\underline{M}_i$ is the i^{th} row of a τ x 3 matrix, and

$$\underline{M}_i = \underline{H}\underline{\phi}^{-i+\alpha+1} \qquad (2.8)$$

Then the constraint equation becomes

$$\underline{F} = \underline{W}\underline{M} - \underline{I} = 0 \tag{2.9}$$

The covariance matrix of the errors is

$$\underline{\Sigma}(n) = \underline{\Sigma} = E\{[\underline{\hat{X}}(n-\alpha) - \underline{X}(n-\alpha)][\underline{\hat{X}}(n-\alpha) - \underline{X}(n-\alpha)]'\} \tag{2.10}$$

for all n. $\underline{\Sigma}$ is constant with respect to time because $\underline{\phi}, b^2, \underline{H}$, and Q are constants. It is a function of τ, b^2, and Q, decreasing with τ. Theoretical results, see Ref. 7 indicate that for the case $b \neq 0$, $\underline{\Sigma}$ approaches a constant as $\tau \to \infty$. Substituting Eq. (2.4) into (2.10) and taking expectations, gives

$$\underline{\Sigma} = \underline{W}\underline{N}\underline{W}' . \tag{2.11}$$

The $\underline{N}$ matrix has the form

$$\underline{N} = (n_{ij}) \qquad \begin{matrix} i = 1, \ldots, \tau \\ j = 1, \ldots, \tau \end{matrix} \tag{2.12}$$

$$n_{ij} = \begin{cases} \sum_{k=\beta}^{\alpha} \underline{H}\underline{\phi}^{-i+k}\underline{R}(\underline{H}\underline{\phi}^{-j+k})' , & i < \alpha+1,\ j < \alpha+1 \\ 0 , & i < \alpha+1,\ j > \alpha+1 \\ 0 , & i > \alpha+1,\ j < \alpha+1 \\ 0 , & i = j = \alpha+1 \\ \sum_{k=\alpha+1}^{\gamma-1} \underline{H}\underline{\phi}^{-i+k}\underline{R}(\underline{H}\underline{\phi}^{-j+k})' , & i > \alpha+1,\ j > \alpha+1 \end{cases} + \begin{cases} Q & i = j \\ 0 & i \neq j \end{cases} \tag{2.13}$$

where β is the larger of i and j, γ is the smaller of i and j, and $\underline{N}$ is symmetric and nonsingular.

The estimates are to have minimum variance in each of the components of the state, position, velocity, and acceleration. This means that the diagonal terms of $\underline{\Sigma}$ are to be minimum. The minimization, subject to the constraint of unbiased estimation (see the Appendix) yields

$$\underline{\Sigma} = (\underline{M}'\underline{N}^{-1}\underline{M})^{-1} \tag{2.14}$$

$$\underline{W} = \underline{\Sigma}\underline{M}'\underline{N}^{-1} \tag{2.15}$$

where the $\underline{\Sigma}$ of Eq. (2.14) is a minimum and the weights of Eq. (2.15) form the desired optimal filter.

The filter transfer function $G(\bar{f})$ is (see Ref. 8)

$$G(\bar{f}) = \sum_{i=1}^{\tau} \underline{W}_i e^{-j2\pi\bar{f}[i-(\alpha+1)]} \tag{2.16}$$

where $\bar{f} = fT$ is normalized frequency and $\underline{W}_i$ is the i^{th} column of the $\underline{W}$ matrix. As in all discrete time systems, the magnitude of the frequency response is a periodic, even function of frequency. Therefore, $|G(\bar{f})|$ plotted over the range $0 \leq \bar{f} \leq 0.5$ furnishes knowledge of the entire spectrum.

3. RESULTS

Curves of frequency response, variance, and filter weights are presented. These are solutions to Eqs. (2.14), (2.15), and (2.16) for various values for b^2, Q, τ, and α. Three values of α are considered: -1 for Prediction filters, 0 for Smoothing filters, and $(\tau-1)/2$ for Midpoint filters. Filter length is τ. Variances are normalized with respect to the variance Q of measurement noise. Families of curves are parameterized by the ratio $b^2/Q = \rho$.

Figures 1, 2, and 3 illustrate the manner in which the variances in position, velocity, and acceleration vary with filter length. The Smoothing filter was chosen for representation here, but the Midpoint and Prediction filters have similarly shaped families of curves. With $b^2 = 0$ ($\rho = 0$), the variances approach zero as filter length increases to infinity. With $b^2 \neq 0$, the variances approach a constant value (a function of $\rho = b^2/Q$ and α) as τ increases. This illustrates the theoretical results for $b^2 \neq 0$ that $\underline{\Sigma}$ approaches a constant as τ increases. From these curves the conclusion can be drawn that for $\rho = 0.001$, the filter changes but slightly as τ increases beyond 15. For larger ρ, the corresponding τ is smaller

Frequency responses (magnitude of transfer function), of Smoothing and Midpoint filters are presented in Figures 4 and 5. Filter length is 25; however, except for the case $\rho = 0$, these curves are valid for all $\tau \geq 15$ to a very good approximation. The responses for $b^2 = 0$ ($\rho = 0$) have ripples at high frequencies. As b^2 (ρ) increases, the responses smooth out but increase in magnitude.

Figures 6, 7, and 8 show how the normalized variances vary with ρ for the three types of filters Filter length was chosen as 25, but again it is safe to consider these curves valid for $\tau \geq 15$. Because the logarithmic scale prevents showing the point $\rho = 0$, the variance value is written on each curve. The obvious conclusion here is that the Midpoint filter yields the smallest errors.

Referring now to Eq. (2.3) it is seen that the filters are groups of weights $\underline{W}_i$ which are parameterized by $i = 1, \ldots, \tau$. The larger i values correspond to the older measurements, with $i = 1$ corresponding to the most current measurement. Figures 9 and 10 are curves of Smoothing weights to estimate position and velocity versus i. Although shown as smooth curves, they are valid only at discrete values of i. Filter length is 25. These curves are actually the impulse response of the filter. As ρ increases, the Smoothing filter de-emphasizes the older data, that is, the weights approach zero as i approaches τ. For the Midpoint filters (not shown) the weights approach zero as i decreases to 1 and increases to τ, with greatest emphasis on the data about the Midpoint of the filter. Therefore, since the filters are forming estimations based on data derived from an assumed random acceleration model, the values nearest to the point being estimated are considered the most valuable. For $\rho = 0$ ($b^2 = 0$) this isn't true, because the model is entirely deterministic.

4. DISCUSSION

The random acceleration model has been presented as an alternate to the approach of modeling the signal as a fixed degree polynomial. Thus, the two models must be compared.

It should be first emphasized that the explicit random acceleration model analyzed here is only an example of a broad class of models for polynomial-like signals. For example, instead of random acceleration, a random third derivative could be assumed. Alternately, the white noise of the random acceleration could be replaced by a first order Markov process. Thus, much of the following discussion applies to this broad class of random type models rather than to just the results explicitly given in this report.

The model of a fixed degree polynomial is easy to analyze and easy to mechanize. The random type model is computationally more difficult to analyze but this is not a severe limitation. However, the actual mechanization of the two different types of filters are equivalent with the random type model having a possible advantage if a recursive formulation is employed.* The filters for both types of models are derived under the same mathematical criteria, minimum variance and unbiased estimates. Both designs can be extended to include correlated measurement noise if desired.

Thus, the important question is: "Which model best represents the signal in the sense of providing more accurate estimates?" If an exact model for the signal for all time were available, a simple answer would be possible. However, by definition, we are dealing with a polynomial-like signal; that is, one whose explicit form is not known but which has the general appearance of a polynomial in time.† In some cases, the random type model is a representation of the physical process generating the signal. For example, a re-entry space vehicle experiences random accelerations due to atmospheric drag variations. However, such criteria for the choice of model are often not available and other factors must be considered.

* References 7 and 9 discuss recursive formulations. The advantage occurs for the case when τ is large enough to make $\underline{\Sigma}$ independent of τ. Then the filter can be mechanized as a growing memory filter.

† The fact that the random acceleration model satisfies this criteria was verified by Monte Carlo trials.

Figures 4 and 5 illustrate the fact that the frequency response corresponding to a fixed polynomial filter ($\rho = 0$) has high frequency ripples. The spectrum of the signal is, of course, not exactly known, but it is often very difficult to associate such high frequency behavior with any known or expected physical characteristic of the signals. Thus, these ripples often indicate a possible basic fault in the fixed polynomial model. The random acceleration model smooths out and can remove these ripples.

If the state of a signal is to be estimated at a certain time and if the exact form of the signal is uncertain, it is intuitively reasonable to weigh most heavily the observations taken at times close to the time of estimation. Figure 9 illustrates the relative behavior of the weights for the two types of model.

Figures 6, 7 and 8 show that the fixed polynomial model provides much smaller variances in the estimates. However, these variances are calculated under the assumption the signal is of the actual form used in the model and thus are not necessarily indicative of the actual performance of the filter.

The random acceleration model requires the specification of the value of $\rho = b^2/Q$. In most problems b^2 is not known and often neither is Q. The fact that the fixed polynomial filter is independent of both b^2 and Q is a definite advantage from an operational point of view. However, the polynomial filter is strongly dependent on the filter memory, τ, while for many cases, the random acceleration model is not. In addition, the degree of the polynomial to be used is often a difficult decision. Thus, the filters for both models have parameters that are often difficult to set.

In problems where Q is unknown and is to be estimated, the polynomial model has computational advantages. This is also true when the state of the signal is to be estimated over some range of time rather than at just one time point. Such problems are an extension of the filter design problem explicitly considered in this report but they are often of importance.

The preceding paragraphs have listed certain of the relevant features of a random type and fixed polynomial model. However, there are other alternates to the fixed polynomial and all of these have relative advantages and disadvantages. Reference 10 discusses filters designed to fit a predetermined frequency response, with no attempt at minimizing the variance. Reference 11 compares such filters to minimum variance filters, assuming a fixed degree polynomial signal and a random signal. Reference 12 provides a statistical design for what engineers would call an adaptive filter. An n^{th} degree polynomial is the basic assumption, but the exact degree used by the filter is computed and adjusted using the data itself. Reference 13 uses an exponential type of weighting.

The following conclusions on the choice of model are thus drawn. Since we are concerned with signals of uncertain character, there is no "best" model and the choice ultimately depends on the designer's "feel" for particular nature of the actual problem being considered. Thus, the random type of model, as illustrated by the random acceleration results of this report, is not the correct model for all polynomial-like signals. However, it's many potential advantages definitely require its consideration.

APPENDIX

In order to arrive at the results presented in Sec. 2 a slightly more basic approach is used. The optimal filter is derived for each of the three states of $\underline{X}$ separately, then combined in one concise matrix form. The estimate is defined as

$$\hat{x}_\nu(n-\alpha) = \sum_{i=1}^{\tau} w_{\nu i}\, z(n+1-i)\,, \qquad \nu = 1,2,3 \tag{A.1}$$

where $\hat{x}_\nu$ is the estimate of the ν^{th} state of $\underline{X}$ and the $w_{\nu i}$'s are individual weights in the 3 x τ $\underline{W}$ matrix. The $\underline{F}$ equation (constraint of unbiased estimation) has the form

$$\underline{F}_\nu = \sum_{i=1}^{\tau} w_{\nu i}\, \underline{H}\underline{\phi}^{-i+\alpha+1} - \underline{H}_\nu \tag{A.2}$$

$$= \underline{W}_\nu \underline{M} - \underline{H}_\nu = 0 \tag{A.2)'$$

where $\underline{F}_\nu$ is a 1 x 3 row vector, the $\underline{H}_\nu$'s are

$$\underline{H}_1 = [\,1 \;\; 0 \;\; 0\,]\;,\quad \underline{H}_2 = [\,0 \;\; 1 \;\; 0\,],\; \underline{H}_3 = [\,0 \;\; 0 \;\; 1\,],$$

and $\underline{W}_\nu$ is a $1 \times \tau$ row vector of weights. In Sec. 2, $\underline{W}_i$ was the i^{th} column of the $\underline{W}$ matrix; here, $\underline{W}_\nu$ is the ν^{th} row of the same matrix.

The criteria of optimality for the filter are minimum variance in each state and unbiased estimates. The minimization occurs only on the diagonal of $\underline{\Sigma}$. Each of the three diagonal terms has the form

$$\sigma^2_{\nu\nu} = E\left\{[\hat{x}_\nu(n-\alpha) - \underline{H}_\nu \underline{X}(n-\alpha)]\ [\hat{x}_\nu(n-\alpha) - \underline{H}_\nu \underline{X}(n-\alpha)]'\right\} \qquad (A.3)$$

$$= E[\underline{D}_\nu \underline{D}'_\nu] + E[\underline{S}_\nu \underline{S}'_\nu] + E[T_\nu T'_\nu] \qquad (A.3)'$$

$$\underline{D}_\nu = \sum_{i=1}^{\tau} w_{\nu i}\ cv(n+1-i)$$

$$\underline{S}_\nu = \sum_{i=1}^{\alpha} w_{\nu i} \sum_{k=i}^{\alpha} \underline{H}\,\underline{\phi}^{-i+k} \underline{B}\ u(n+1-k)$$

$$\underline{T}_\nu = \sum_{i=\alpha+2}^{\alpha} w_{\nu i} \sum_{k=\alpha+1}^{i-1} \underline{H}\,\underline{\phi}^{-i+k} \underline{B}\ u(n+1-k)$$

The first term of $\sigma^2_{\nu\nu}$ is obviously

$$E\left\{\underline{D}_\nu \underline{D}'_\nu\right\} = E\left\{\sum_{i=1}^{\tau} w_{\nu i}\ cv(n+1-i)\ v(n+1-i)\ cw_{\nu i}\right\}$$

$$= \underline{W}_\nu\, c^2\, \underline{W}'_\nu = \underline{W}_\nu Q \underline{W}'_\nu \qquad (A.4)$$

because of the independence of the v process. The second and third terms are not quite so obvious. First, write out a few terms in the summation of the second term

$$\underline{S}_\nu = w_{\nu 1}\underline{H}\underline{B}\,u(n) + w_{\nu 1}\underline{H}\underline{\phi}\underline{B}\,u(n-1) + \cdots + w_{\nu 1}\underline{H}\underline{\phi}^{\alpha-1}\underline{B}\,u(n+1-\alpha)$$

$$+ w_{\nu 2}\underline{H}\underline{B}\,u(n-1) + \cdots + w_{\nu 2}\underline{H}\underline{\phi}^{\alpha-2}\underline{B}\,u(n+1-\alpha) \qquad \text{(A.5)}$$

$$+ \cdots + w_{\nu\alpha}\underline{H}\underline{B}\,u(n+1-\alpha)$$

then rearrange and take expected values,

$$E\{\underline{S}_\nu\underline{S}'_\nu\} = [w_{\nu 1}\underline{H}]\,\underline{R}[w_{\nu 1}\underline{H}]' + [w_{\nu 1}\underline{H}\underline{\phi} + w_{\nu 2}\underline{H}]\,\underline{R}[w_{\nu 1}\underline{H}\underline{\phi} + w_{\nu 2}\underline{H}]' \qquad \text{(A.6)}$$

$$+ \cdots + [w_{\nu 1}\underline{H}\underline{\phi}^{\alpha-1} + \cdots + w_{\nu\alpha}\underline{H}]\,\underline{R}[w_{\nu 1}\underline{H}\underline{\phi}^{\alpha-1} + \cdots + w_{\nu\alpha}\underline{H}]'$$

$$= \underline{W}_\nu\underline{N}_1\underline{W}'_\nu$$

$$\underline{N}_1 = (n_{1ij}) \qquad i = 1, \ldots, \tau,\ j = 1, \ldots, \tau$$

$$n_{1ij} = \begin{cases} \sum_{k=\beta}^{\alpha-1} \underline{H}\underline{\phi}^{-i+k}\,\underline{R}\left(\underline{H}\underline{\phi}^{-j+k}\right)' & \text{if } i < \alpha+1,\ j < \alpha+1 \\ 0 & \text{if } i \geq \alpha+1,\ j \geq \alpha+1 \end{cases} \qquad \text{(A.7)}$$

where β is the larger of i and j. Likewise for the third term in $\sigma^2_{\nu\nu}$

$$\underline{T}_\nu = w_{\nu,\alpha+2}\underline{H}\underline{\phi}^{-1}\underline{B}\,u(n-\alpha) + w_{\nu,\alpha+3}\underline{H}\underline{\phi}^{-2}\underline{B}\,u(n-\alpha)$$

$$+ w_{\nu,\alpha+3}\underline{H}\underline{\phi}^{-1}\underline{B}\,u(n-\alpha-1) + \cdots + w_{\nu,\tau}\underline{H}\underline{\phi}^{-\tau+\alpha+1}\underline{B}\,u(n-\alpha) \qquad \text{(A.8)}$$

$$+ \cdots + w_{\nu,\tau}\underline{H}\underline{\phi}^{-1}\underline{B}\,u(n+2-\alpha)$$

Regrouping and taking expected values,

$$E\{\underline{T}_\nu \underline{T}_\nu'\} = [w_{\nu,\alpha+2}\underline{H\phi}^{-1} + \cdots + w_{\nu,\tau}\underline{H\phi}^{-\tau+\alpha+1}]\, \underline{R}\, [w_{\nu,\alpha+2}\underline{H\phi}^{-1} + \cdots + w_{\nu,\tau}\underline{H\phi}^{-\tau+\alpha+1}]'$$

$$+ [w_{\nu,\alpha+3}\underline{H\phi}^{-1} + \cdots + w_{\nu,\tau}\underline{H\phi}^{-\tau+\alpha+2}]\, \underline{R}\, [w_{\nu,\alpha+3}\underline{H\phi}^{-1} + \cdots + w_{\nu,\tau}\underline{H\phi}^{-\tau+\alpha+2}]'$$

$$+ \cdots + [w_{\nu,\tau}\underline{H\phi}^{-1}]\, \underline{R}\, [w_{\nu,\tau}\underline{H\phi}^{-1}]' \tag{A.9}$$

$$= \underline{W}_\nu \underline{N}_2 \underline{W}_\nu'$$

$$\underline{N}_2 = (n_{2ij}) \qquad i = 1, \ldots, \tau,\ j = 1, \ldots, \tau$$

$$n_{2ij} = \left\{ \begin{array}{ll} 0 & \text{if } i \le \alpha + 1,\ j \le \alpha + 1 \\ \displaystyle\sum_{k=\alpha+1}^{\gamma-1} \underline{H\phi}^{-i+k}\, \underline{R} \left(\underline{H\phi}^{-j+k} \right)' & \text{if } i > \alpha + 1,\ j > \alpha + 1 \end{array} \right\} \tag{A.10}$$

where γ is the smaller of i and j. The three terms are put together readily

$$\sigma_{\nu\nu}^2 = w_\nu Q w_\nu' + w_\nu \underline{N}_1 w_\nu' + w_\nu \underline{N}_2 w_\nu'$$

$$= \underline{W}_\nu \underline{N} \underline{W}_\nu' \tag{A.11}$$

where $\underline{N}$ is the same matrix defined in Eq. (2. 13).

In the following minimization of $\sigma_{\nu\nu}^2$, the diagonal terms in $\underline{\Sigma}$, the choice of $\underline{\phi}$ and the symmetry inherent in the formulation has caused the minimization of the entire $\underline{\Sigma}$ matrix. Using three 3-vector Lagrangian multipliers, define

$$\underline{L}_\nu = \sigma_{\nu\nu}^2 + 2\underline{F}_\nu \lambda_\nu \tag{A.12}$$

Defining a matrix of partial derivatives on L_ν as being

$$\frac{\partial \underline{L}_\nu}{\partial \underline{W}_\nu} = \begin{bmatrix} \frac{\partial L_\nu}{\partial w_{\nu 1}} \\ \frac{\partial L_\nu}{\partial w_{\nu 2}} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial L_\nu}{\partial w_{\nu \tau}} \end{bmatrix} \tag{A.13}$$

Then

$$\frac{1}{2}\frac{\partial L_\nu}{\partial w_\nu} = \underline{N}\,\underline{W}'_\nu + \underline{M}\,\underline{\lambda}_\nu = 0 \tag{A.14}$$

Eliminating $\underline{\lambda}_\nu$ between Eqs. (A.2)′ and (A.14) yields

$$\underline{W}_\nu = \underline{H}_\nu(\underline{M}'\underline{N}^{-1}\underline{M})^{-1}\underline{M}'\underline{N}^{-1} = \underline{H}_\nu \underline{\Sigma}\,\underline{M}'\underline{N}^{-1} \tag{A.15}$$

$$\sigma^2_{\nu\nu} = \underline{H}_\nu(\underline{M}'\underline{N}^{-1}\underline{M})^{-1}\underline{H}'_\nu = \underline{H}_\nu \underline{\Sigma}\,\underline{H}'_\nu \quad .$$

It is obvious that the $\underline{H}_\nu$ vector is serving to pick out a particular row of weights and a particular element in the diagonal of $\underline{\Sigma}$. Therefore, the complete covariance matrix Σ and weight matrix $\underline{W}$ are those given in Eqs. (2.14) and (2.15).

The preceding proof is self contained. However, the theory available in the literature can be used to obtain the same answers. One example is the theory of Ref. 7

or Ref. 9 which is directly applicable to the model of Eq. (2.1) and (2.2). Another example is to consider τ observations, $z(1), \ldots, z(\tau)$ as elements of a single $1 \times \tau$ column vector observation, $\underline{Z}$. Let $\underline{X}(\tau-\alpha)$ denote the column vector state of the signal which is to be estimated from the τ observations. Then, algebraic manipulations gives,

$$\underline{Z} = M\underline{X}(\tau-\alpha) + W$$

where M is a 3 by τ matrix of Eq. (2.7) and W is a zero mean, τ-dimensional random vector with covariance matrix $\underline{N}$. The formulas for linear regression analysis of Refs. 5 and 6 then give Eqs. (2.14) and (2.15) directly.

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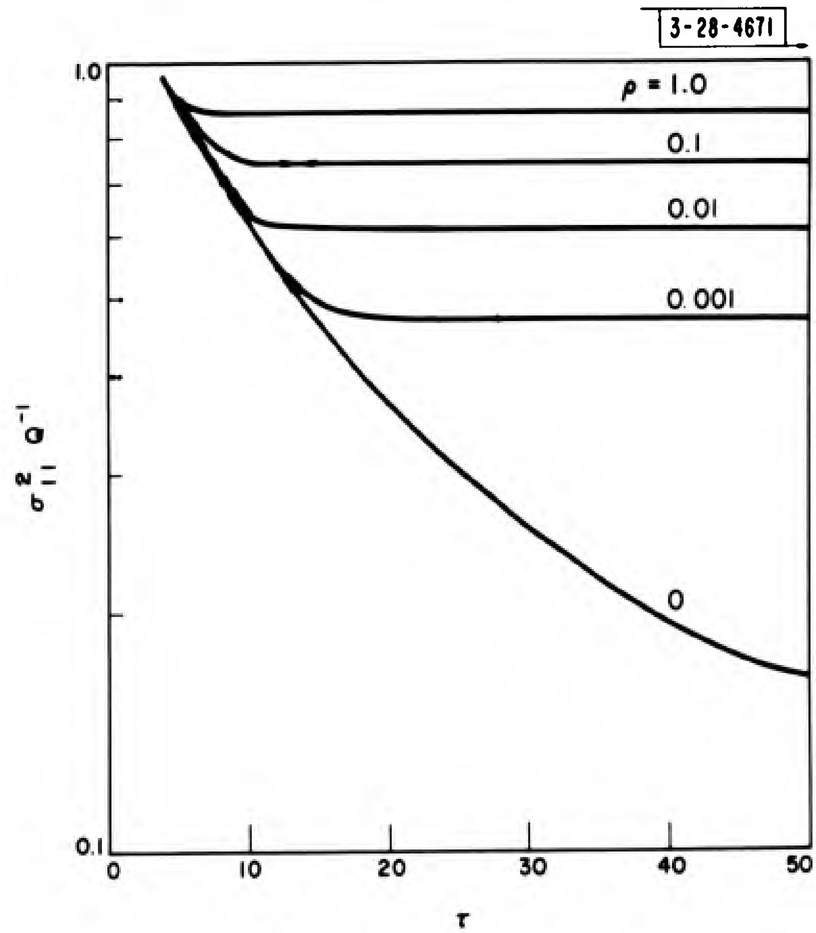

Fig. 1 Normalized variance in position versus filter length, smoothing filters.

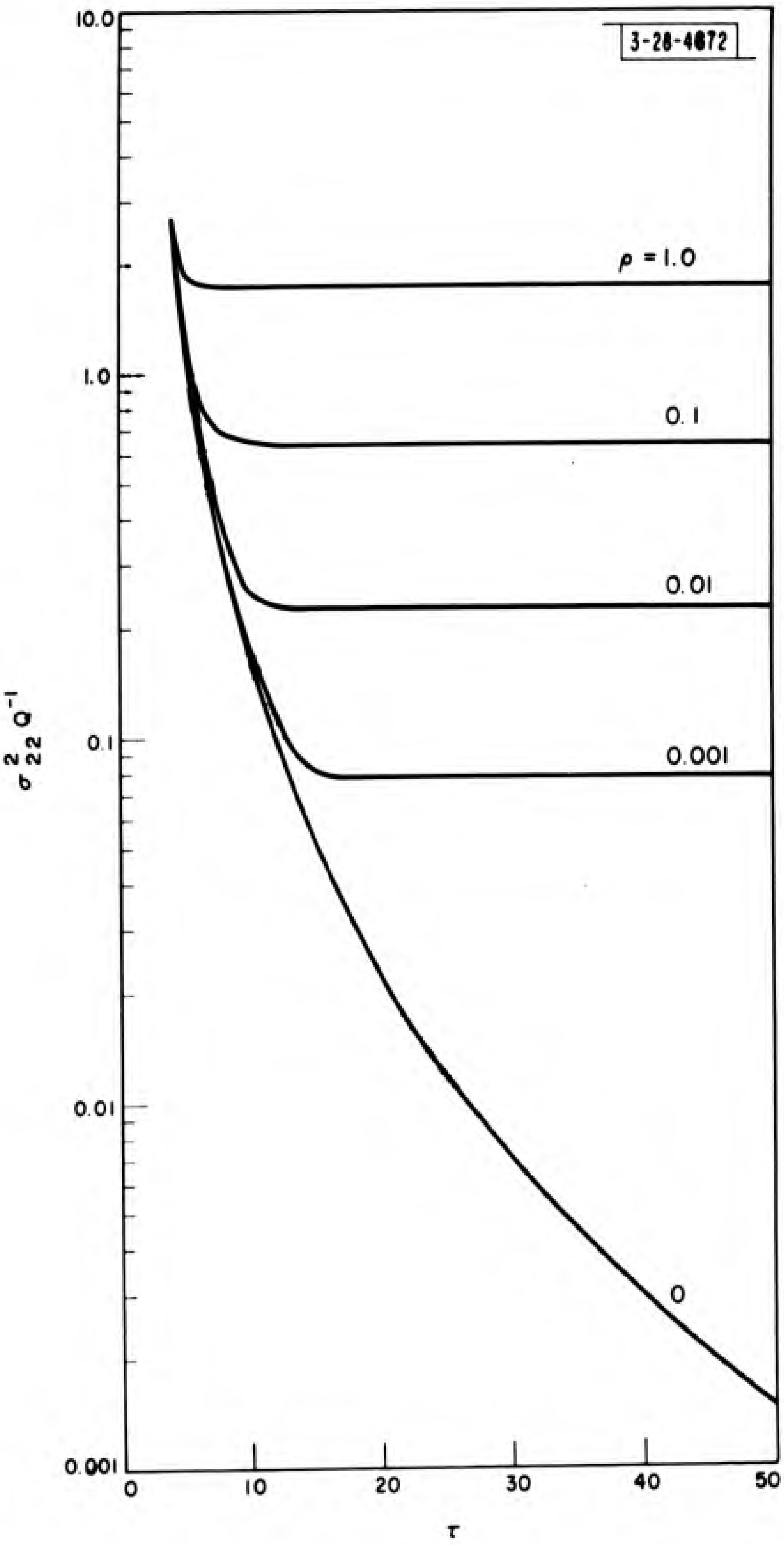

Fig. 2 Normalized variance in velocity versus filter length, smoothing filters.

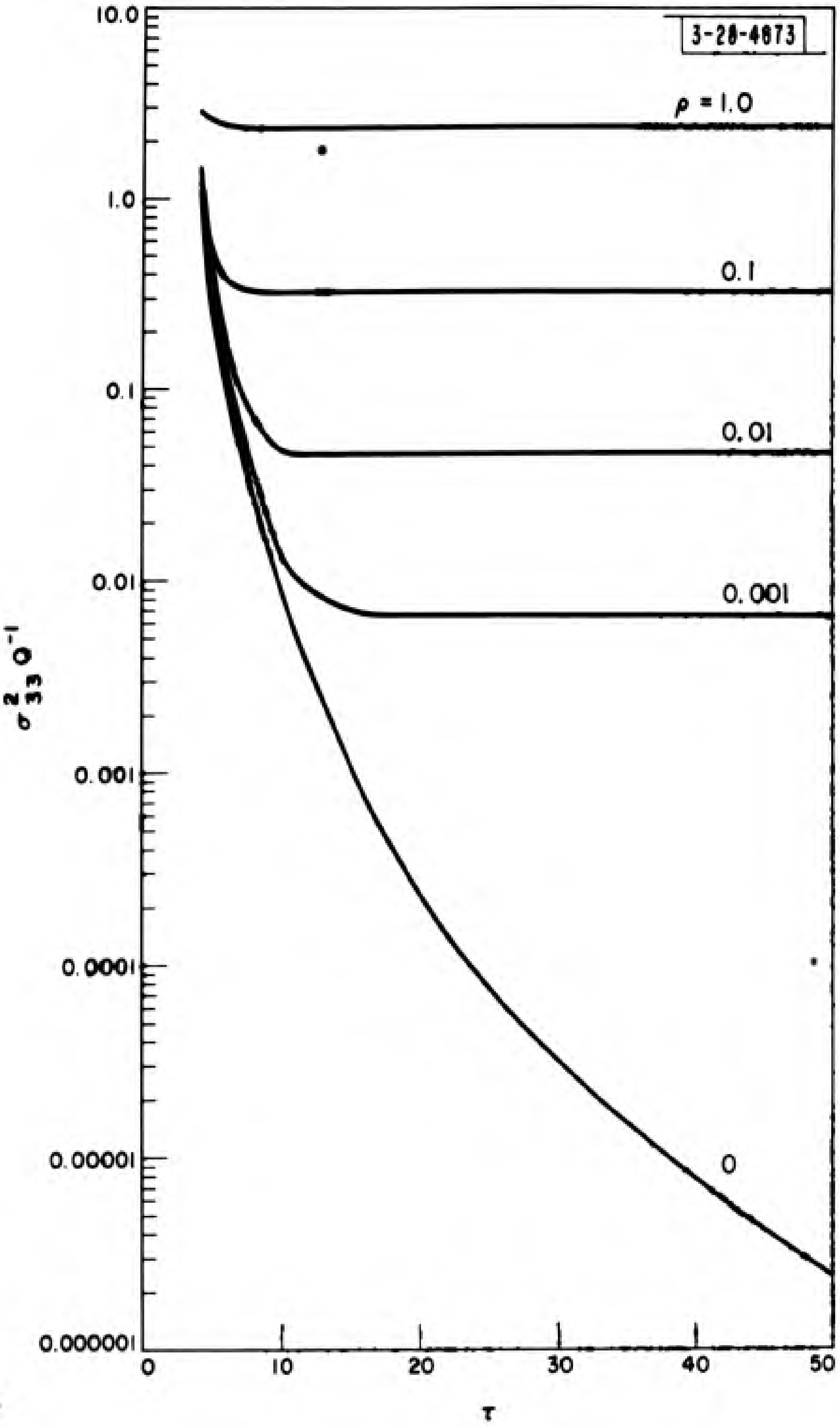

Fig. 3 Normalized variance in acceleration versus filter length, smoothing filters.

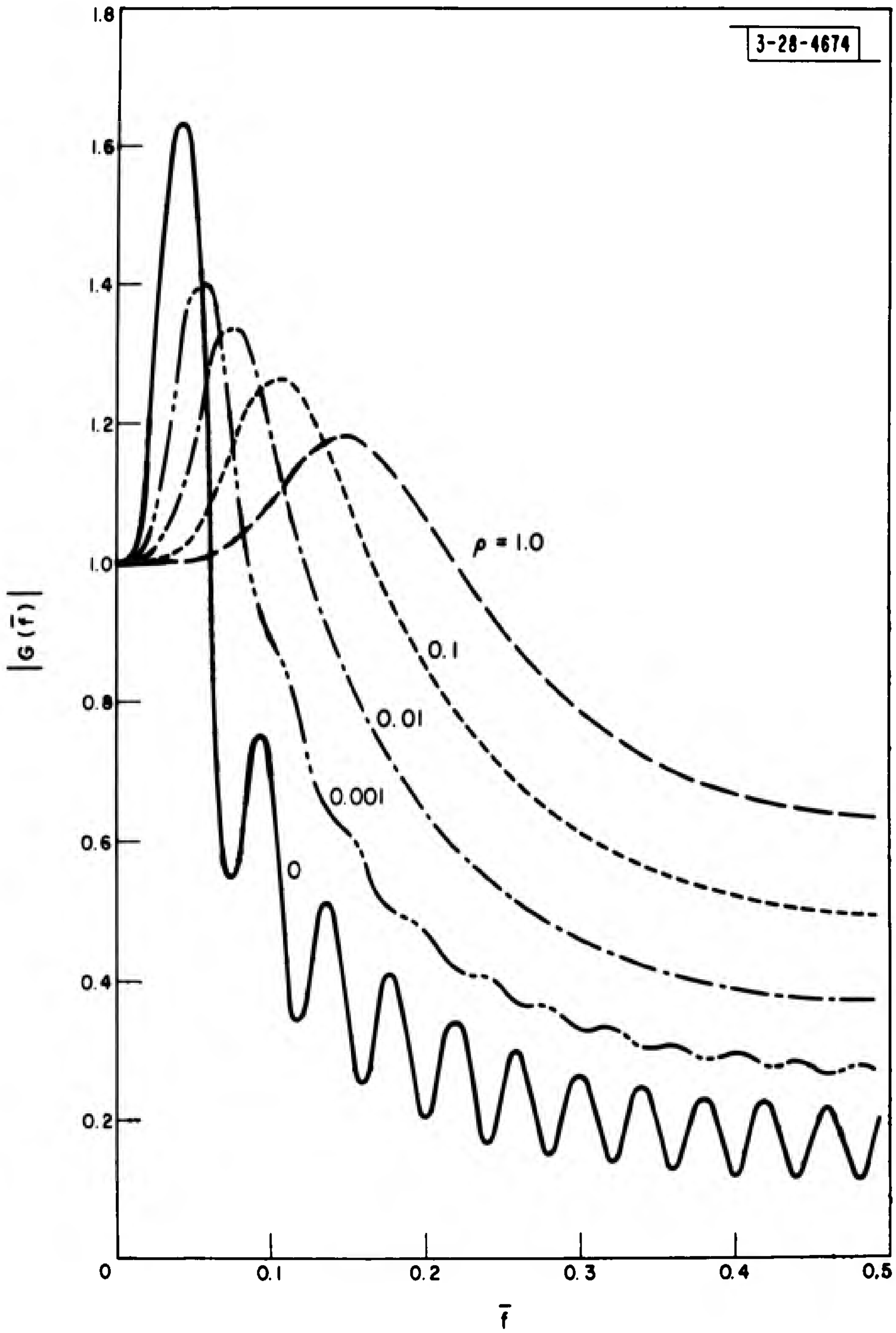

Fig. 4 Smoothing filter transfer function magnitude versus normalized frequency, $\tau = 25$.

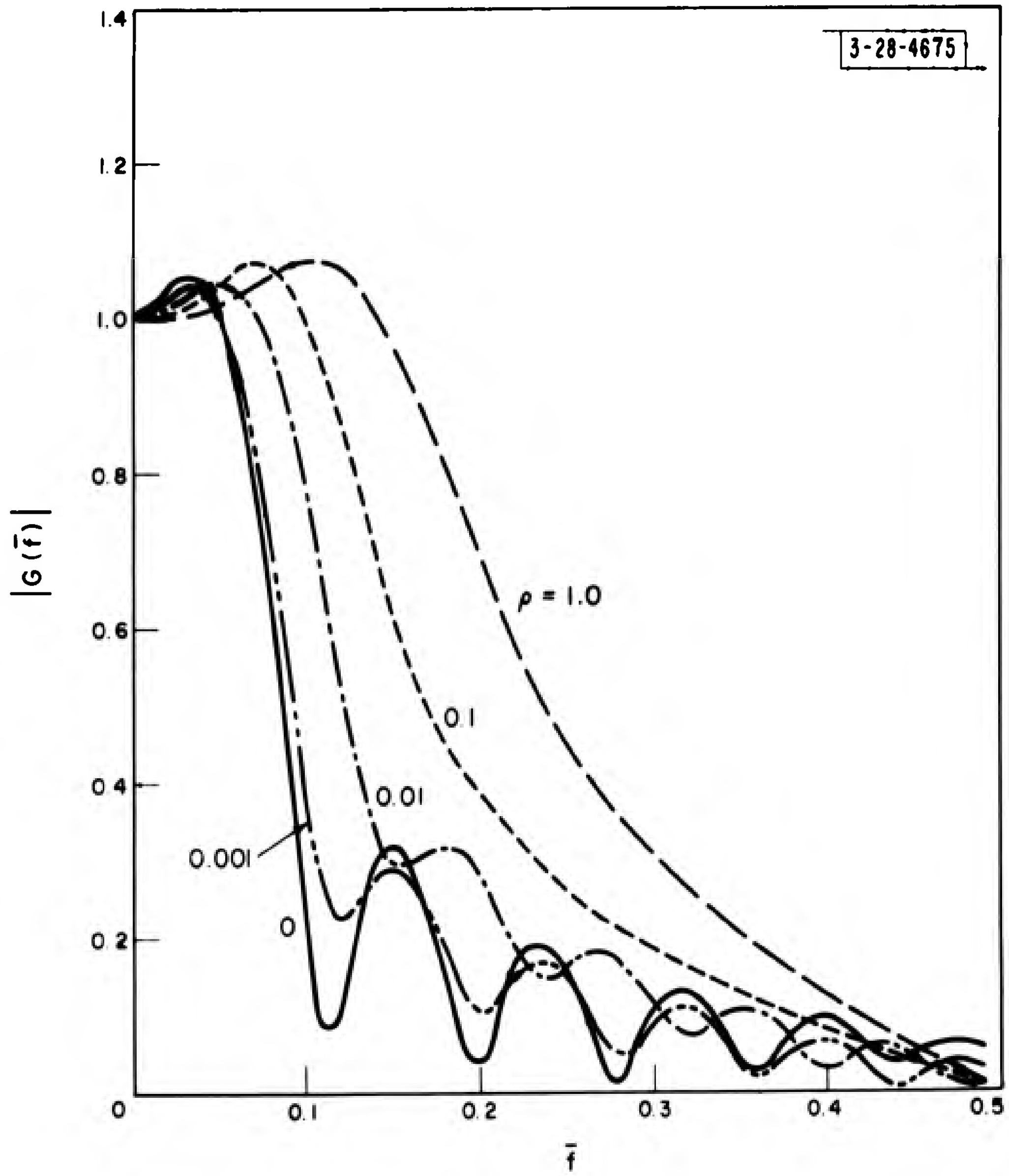

Fig. 5 Midpoint, position filter transfer function magnitude versus normalized frequency, $\tau = 25$.

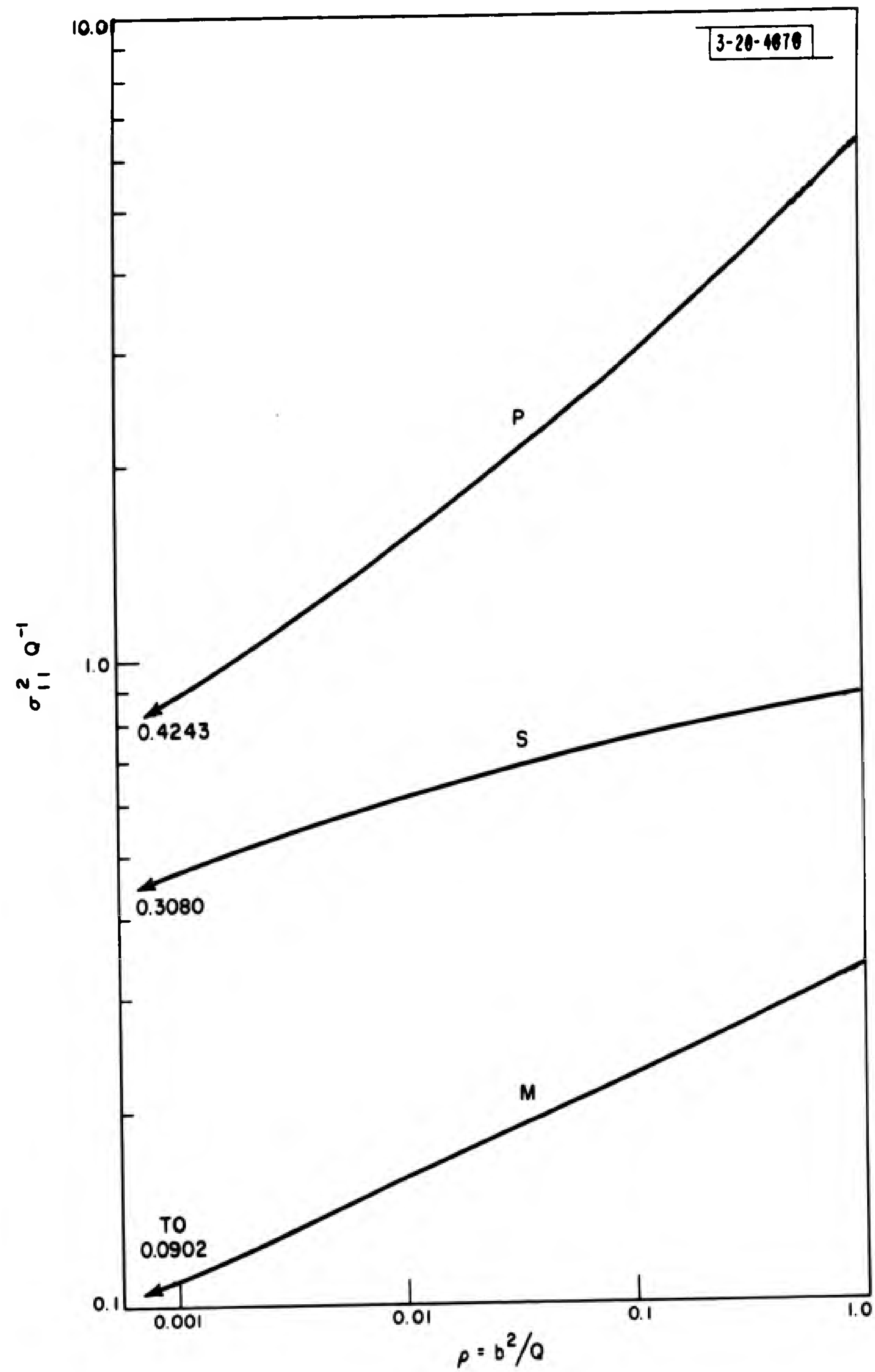

Fig. 6 Normalized variance in position versus ρ τ = 25, prediction, smoothing, and midpoint filters.

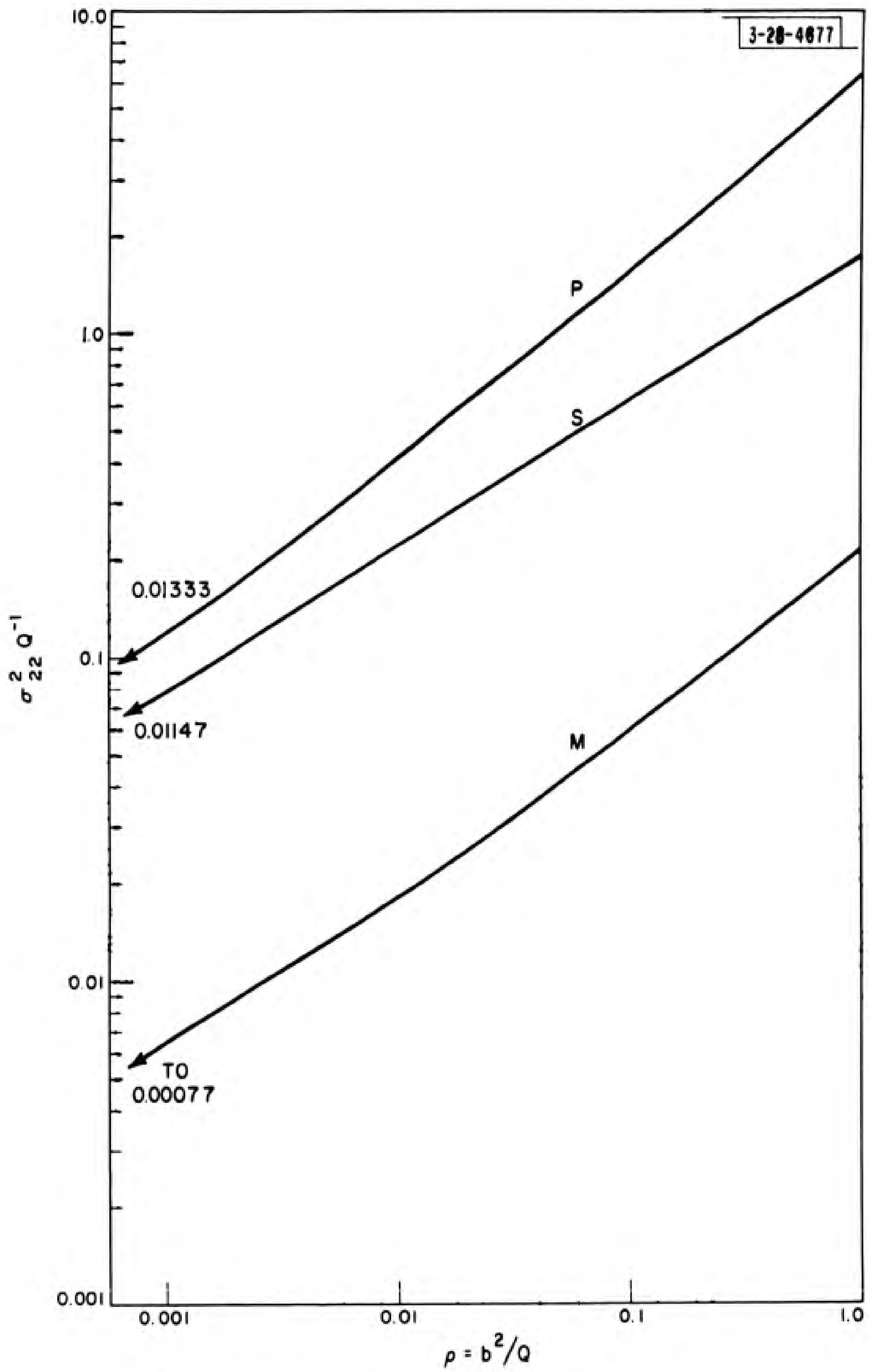

Fig. 7 Normalized variance in velocity versus ρ τ = 25, prediction, smoothing, and midpoint filters.

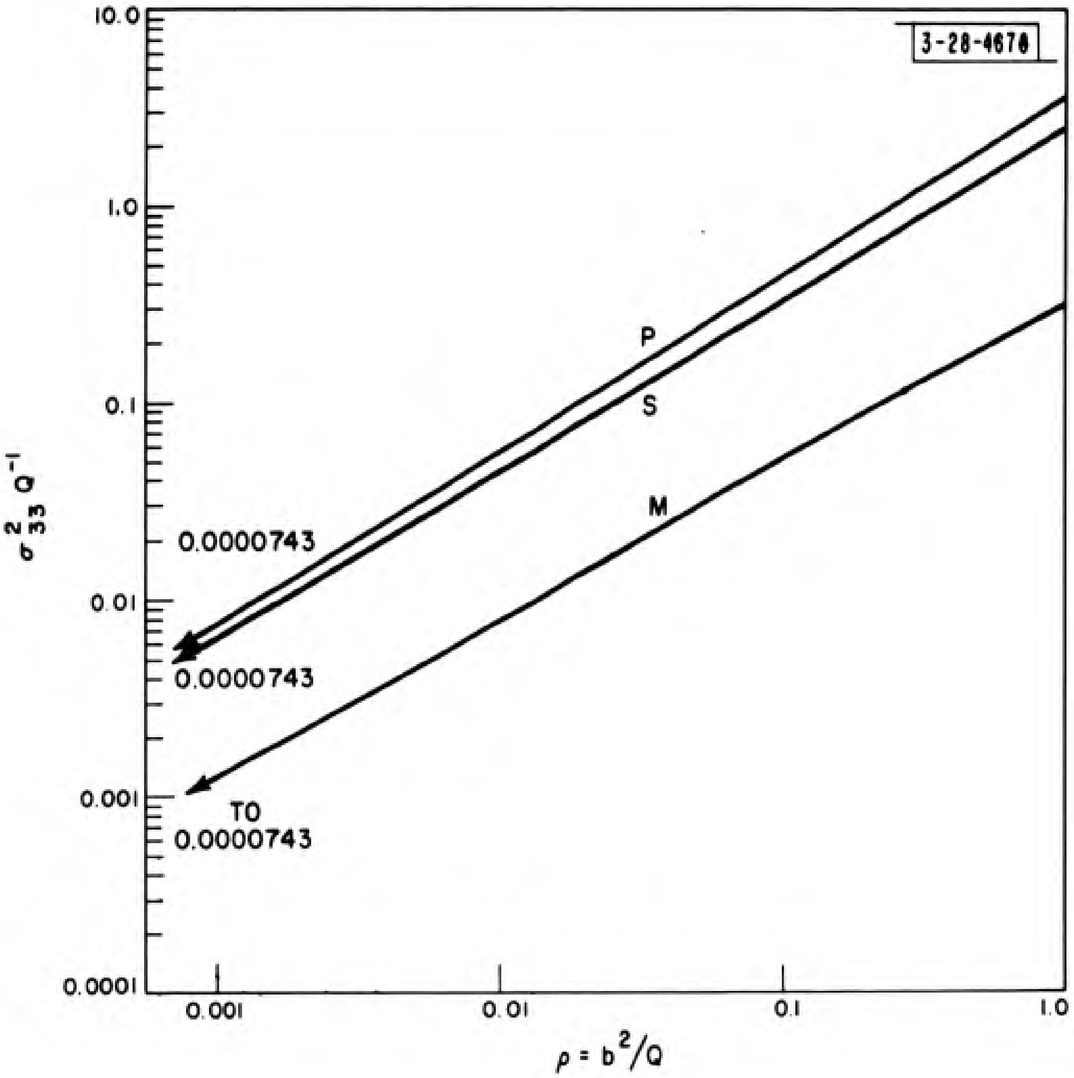

Fig. 8 Normalized variance in acceleration versus ρ T = 25, prediction, smoothing, and midpoint filters.

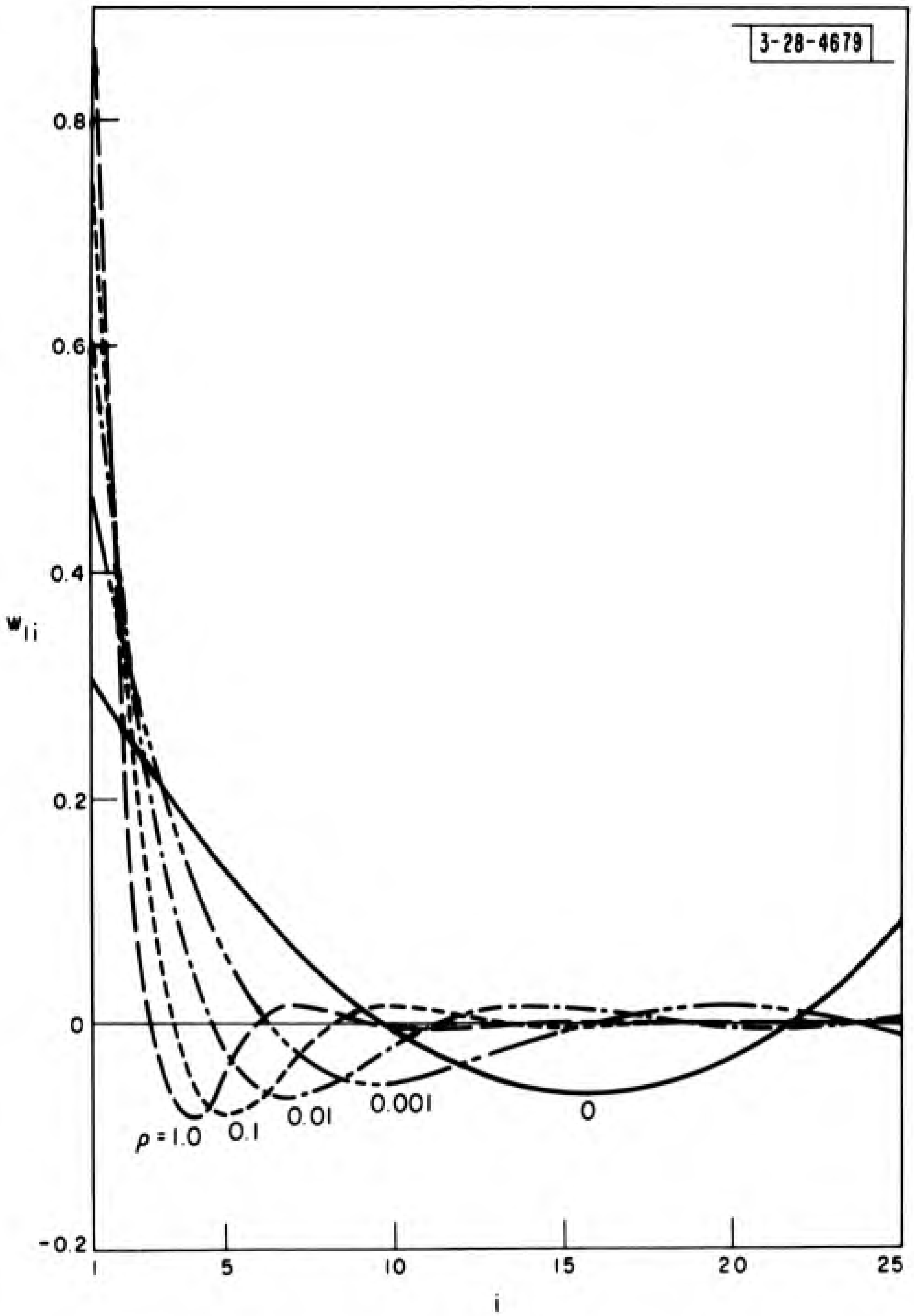

Fig. 9 Smoothing position filter versus i τ = 25.

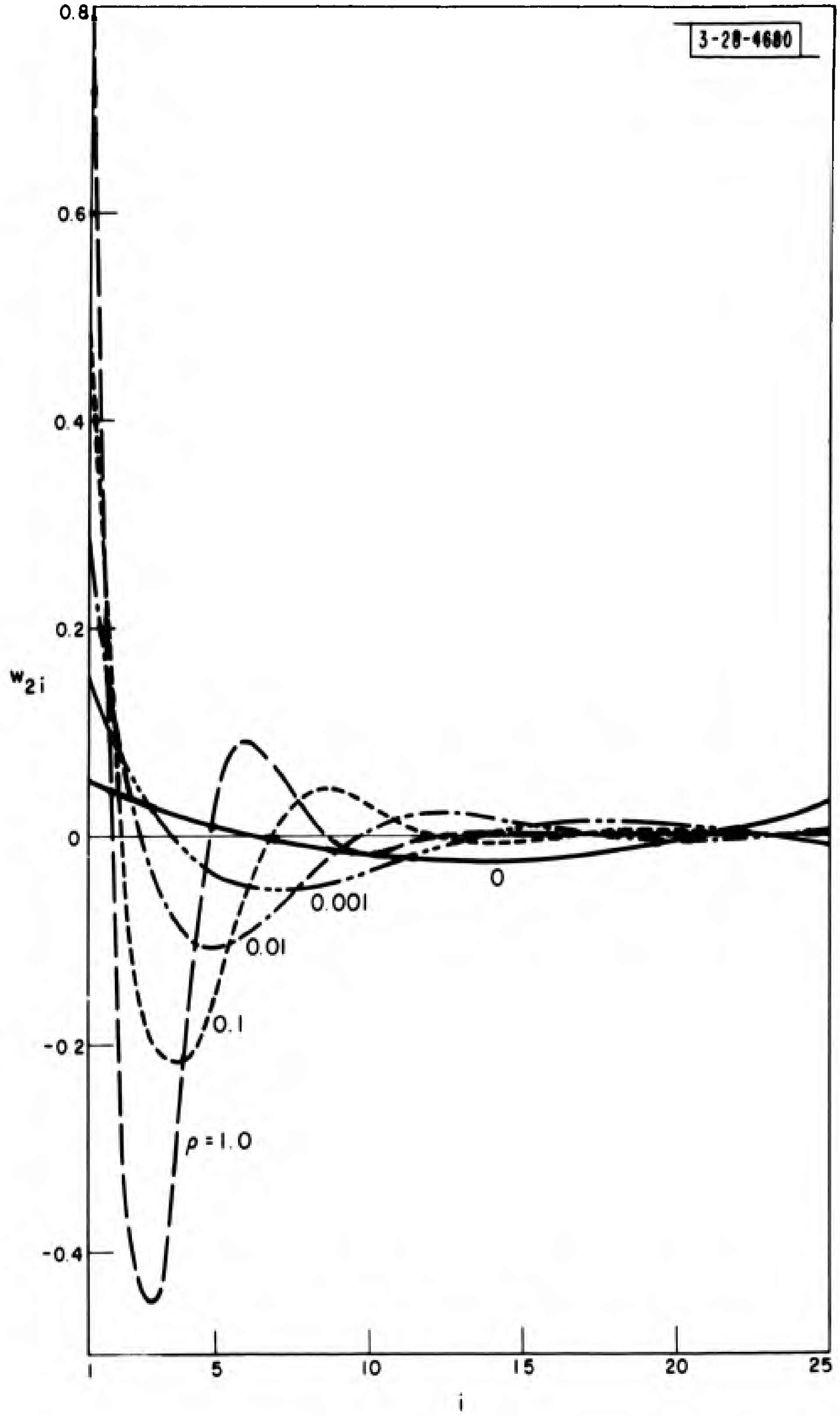

Fig. 10 Smoothing velocity filter versus i $\tau = 25$.

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