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A NOTE ON THE COMPUTATIONAL SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH VARYING TIME-LAGS

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

Part of the RAND research program consists of basic supporting studies in mathematics. The mathematical research presented here concerns techniques for the solution of differential equations with time-lags. This method is of importance in connection with the study of more realistic models of chemotherapy, of the type being studied under GM-09608.

SUMMARY

In this paper we briefly indicate how a technique for the reduction of the solution of differential-difference equations with one time-lag to the solution of systems of ordinary differential equations can be extended to the more complex situation involving different time-lags.

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A NOTE ON THE COMPUTATIONAL SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH VARYING TIME-LAGS

1. INTRODUCTION

In this paper we wish briefly to indicate how a technique for the reduction of the solution of differential-difference equations with one time-lag to the solution of systems of ordinary differential equations can be extended to the more complex situation involving different time-lags. This method is of some importance in connection with the study of more realistic models of chemotherapy, of the type studied in [5], [6].

2. DESCRIPTION OF METHOD

Consider a system of two differential-difference equations with different, but commensurable, time-lags, as and bs, where $s > 0$ and a and b are positive integers:

$$(2.1) \qquad y'(t) = f(t,y(t),y(t-as),z(t),z(t-bs)),$$

$$z'(t) = g(t,y(t),y(t-as),z(t),z(t-bs)),$$

with the initial conditions $y(t) = z(t) = 0, \quad t \leq 0.$

Let us now present an algorithm which obtains the numerical solution of (2.1) by means of the solution of sequences of ordinary differential equations of increasing order. The case of one time-lag was treated in [1], [2].

Introduce the functions

$$y_n(t) = y(t + ns), \qquad (2.2)$$

$$z_n(t) = z(t + ns),$$

with $n = -b, -b + 1, -b + 2, \ldots, 0, \ldots, N$, where N is determined by the range of integration, and $0 \leq t \leq s$.

To begin with, we solve numerically the following system of ordinary differential equations:

$$\begin{aligned}
(2.3) \qquad & y'_{-b} = 0, & z'_{-b} = 0, \\
& y'_{-b+1} = 0, & z'_{-b+1} = 0, \\
& \vdots & \vdots \\
& y'_{-1} = 0, & z'_{-1} = 0, \\
& y'_0 = f(t, y_0, y_{-a}, z_0, z_{-b}), & z'_0 = g(t, y_0, y_{-a}, z_0, z_{-b}),
\end{aligned}$$

in the range $0 \leq t \leq s$, with the initial values $y_i(0) = z_i(0)$ for all i.

Note that

$$y_1(0) = y_0(s), \quad z_1(0) = z_0(s). \qquad (2.4)$$

In the next round, we adjoin to the system in (2.3) the equations

(2.5) $$y_1' = f(t + s, y_1, y_{-a+1}, \ldots, z_1, z_{-b+1}),$$

$$y_1(0) = y_0(s),$$

$$z_1' = g(t + s, y_1, y_{-a+1}, \ldots, z_1, z_{-b+1}),$$

$$z_1(0) = z_0(s).$$

Integrating this larger system over the interval $[0,s]$, we obtain two more initial values

(2.6) $$y_2(0) = y_1(s), \quad z_2(0) = z_1(s).$$

Generally, in the $(i + 1)$-st round of iteration, we add to the i-th system the equations

(2.7) $$y_i' = f(t + is, y_i, y_{-a+i}, z_i, z_{-b+i}), \quad y_i(0) = y_{i-1}(s),$$

$$z_i' = f(t + is, y_i, y_{-a+i}, z_i, z_{-b+i}), \quad z_i(0) = z_{i-1}(s).$$

3. DISCUSSION

In many cases the time-lags, λ_1 and λ_2, will not be commensurable. If we do not wish to use the usual storage technique for the numerical integration of (2.1), we can approximate to λ_1 and λ_2 by rational fractions p_1/q, p_2/q in such a way that

(3.1) $$\left(\lambda_1 - \frac{p_1}{q}\right)^2 + \left(\lambda_2 - \frac{p_2}{q}\right)^2$$

is as small as possible, with $q \leq N$. If q is too large, the systems of equations described in Sec. 2 become unwieldy.

In general, we can cut down on the dimension of these differential equations by applying the preceding technique over an interval $[0,T_1]$ and then using the technique of differential approximation [3], [4] to replace storage of the "initial values" over $[T_1,T_1 + s]$.

REFERENCES

1. Bellman, R., "On the Computational Solution of Differential-difference Equations," J. Math. Anal. and Appl., Vol. 2, 1961, pp. 108-110.

2. Bellman, R., and B. Kotkin, "On the Numerical Solution of a Differential-difference Equation Arising in Analytic Number Theory," Math. of Comp., Vol. 16, 1962, pp. 473-475.

3. Bellman, R., and J. M. Richardson, "On Some Questions Arising in the Approximate Solution of Nonlinear Differential Equations," Q. Appl. Math., Vol. 20, 1963, pp. 333-339.

4. Bellman, R., and B. Kotkin, A Numerical Approach to the Convolution Equations of a Mathematical Model of Chemotherapy, The RAND Corporation, RM-3716-NIH, July 1963.

5. Bellman, R., J. Jacquez, and R. Kalaba, "Some Mathematical Aspects of Chemotherapy—I: One-organ Models," Bull. Math. Biophys., Vol. 22, 1960, pp. 181-198.

6. ———, "Some Mathematical Aspects of Chemotherapy—II: The Distribution of a Drug in the Body," Bull. Math. Biophys., Vol. 22, 1960, pp. 309-322.