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MEMORANDUM

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DECEMBER 1963

PROBABILITY AND SCATTER IN  
CUMULATIVE FATIGUE DAMAGE

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PREFACE

The proper size of the various parts of a structure is largely determined by the strength properties of the structural materials that go into it. But when these properties are measured, they are found to vary, because of variations in the manufacturing processes involved in production of the material. Structures meant to be identical turn out to have varying strengths, due to these material variations plus inevitable variation in fabrication techniques. This structural-strength variation, or scatter, has always been with us to complicate the job of the structural designer. In time, we learn how much strength we can count on with high certainty, and we then add some more material just to be sure. How much more depends on a balance between safety or price of structural failure and the cost or weight of the structure.

Fatigue strength is just one of many strengths that must be considered in design, undistinguished in many ways from any of the others, but a feeling has grown up about fatigue that surrounds it with some mysterious qualities. The main reason for this is probably the way that fatigue tests must be made. Strength cannot be measured directly; life is measured instead, and measurements of fatigue life exhibit large scatter. In fatigue tests in which seemingly identical specimens are exposed to identical load variations, it is common to find that one specimen endures over ten times as many cycles without failure as another. Only after many tests have been made is it possible to see how life and strength are related.

A large number of fatigue tests of complete airframes would be prohibitively expensive, and there is, of course, strong incentive to be as certain as possible that the airframe has adequate fatigue strength before production begins.

This study investigates the uncertainty factors that arise from scatter in fatigue life in order to discriminate the factors that are important in the design of aircraft, to see how uncertainties affect design procedures and results, and to find how much assurance of adequate fatigue strength is possible despite the wide experimental scatter in fatigue life.

SUMMARY

This Memorandum presents a study of the problems in aircraft design caused by scatter in fatigue life. Lives of similar test specimens under identical loading vary so greatly that fatigue is well known as a highly unpredictable phenomenon. Although attempts to predict the fatigue life of existing aircraft are greatly complicated by scatter, this unpredictability is a less severe problem in design, when the decision is made on how much material to put into a structure in order to provide satisfactory life.

A simple example (comparing design for static loads with design for cyclic loads) is given to provide a measure of the magnitude of the effect of fatigue scatter. Surprisingly enough, designing to account for scatter in static strength and in fatigue behavior can be very similar in terms of the tradeoff between increasing weight and decreasing failure probability.

Selecting the proper probability distribution to describe scatter in fatigue life is a source of considerable controversy. An example illustrates the point that different distributions may greatly affect estimates of failure probability and life, but that design effects may be far less pronounced, possibly even insignificant. This example emphasizes the importance of examining, in the design context, disputes about correct forms of probability distributions. The aim of proper design should be to turn uncertainties about probability distribution into an academic question of little importance during the service lifetime of a flight structure.

Important in fatigue, because of scatter and generally limited amounts of testing, is the degree of assurance possible from limited tests, which raises the question of confidence levels. A discussion of confidence levels is included in order to help clarify those concepts especially relevant to fatigue work.

Use of confidence levels in fatigue can benefit by application of acceptance-inspection-plan techniques to fatigue problems. The statistical theory behind this method of quality control is shown to be well suited to fatigue work and is probably the most efficient

method for obtaining low-failure-probability S - N curves. Application of this method is simplified by the quality-control tables already available.

Use of prescribed, constant scatter factors is then discussed. This approach, which is currently favored in Air Force specifications and requirements, has the unquestioned advantage of simplicity, but it is not well founded on present concepts of cumulative damage. Theoretically, a scatter factor should depend on the shape of the expected cyclic-load spectrum, but there has not been enough testing to establish a relationship. A limited analysis of spectrum-test results was made to investigate the validity of a constant scatter factor. It was found that the concept of a constant "composite" standard deviation, which leads to a constant scatter factor, agreed with the data studied. The data combined a variety of tests, making it difficult to suggest reasons for the agreement or to justify extension to other test conditions. When operational stress spectra are defined, it would be preferable to determine scatter factors by testing.

A straightforward approach to probability and scatter in spectrum loading is presented which is consistent with cumulative-damage theories. It uses constant-amplitude data and incorporates the determination of confidence levels as suggested in this Memorandum. Although spectrum testing is preferable, the method presented is reasonable when tests are not available and would also be useful as a basis for comparison in the analysis of spectrum-test results.

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## I. INTRODUCTION

The incorporation of probability effects in antifatigue design is similar in principle to the same process in designing and producing structures with adequate static strength. In static design, this may not be as conscious and explicit a process as is required for fatigue because of the benefits of accumulated experience and, in some sense, a tendency toward less scatter in important design parameters.

In static design, two fundamental questions must be answered: First, what are the maximum loads that the structure is expected to encounter? Second, what design (kind, amount, and distribution of material in the structures) is required to adequately withstand these loads? (Further consideration can then lead to selection of the "best" design in terms of weight, cost, etc., from the many possible designs that are structurally adequate.) It is, of course, necessary to be able to analyze conceived structures--i.e., to be able to predict their behavior when they are subjected to the predicted loads. While details of static failure such as plastic deformation, tensile instability, and initiation and propagation of cracks or failure areas are not fully understood, knowledge of this behavior is generally good enough to allow proceeding with confidence. A third important question remains: How can it be determined whether the design finally selected is indeed adequate? This question is answered by statically testing a representative structure, applying as nearly as possible the expected maximum loads. When the structure is shown to be adequate in these tests, flight tests are made, gradually approaching the expected maximum load conditions while observing stresses and deformations to certify that behavior was satisfactorily predicted.

Probability considerations may not be explicit, but it is of course recognized that there will be some scatter in load-carrying capacity of production airframes because of variations in material properties and in production techniques. Further, estimates of maximum loads contain some uncertainty, and there will certainly be some scatter among individual aircraft in the magnitude of maximum loads encountered.

There is always some unknowable probability of structural failure in service--an airplane may be subjected to a load greater than estimated, and/or an airplane may have built into it less strength than that of the static-test airframe. The experience and capability of designers, and the data available to them, are depended on to make this probability extremely low. History has generally justified this view.

One more important factor in static design is a "cushion" that is conventionally provided in aircraft structures through the possibility of inelastic deformation. The structure is designed so that the maximum loads that are reasonably expected (limit loads) can be exceeded by some amount (up to ultimate loads) without catastrophic failure. If loads in the range between limit and ultimate are encountered, some permanent deflection of the structure may be produced, but the capability for making a safe landing remains. This cushion thus allows for low-probability events and may also provide a warning that structural modification or changes in operation are required.

The fundamental questions in antifatigue design are analogous: What are the variations in loading (or load cycles) that the structure is expected to encounter in its lifetime? What kind, amount, and distribution of material is required in the structure to make the probability of serious structural-fatigue problems extremely low? (In addition to catastrophic failure in flight, the necessity for expensive repair of damaged structure is also considered to be a serious fatigue problem.) The need to demonstrate that the design selected is adequate is also analogous in static and fatigue design, while the important cushion mentioned for static design may or may not have its analog in antifatigue design.

In fatigue, however, greater difficulty attends this whole process. While it could never be ignored, this process has assumed heightened importance in recent years. Advances in structural design, including improved static-stress analyses, improved methods of load prediction, and introduction of higher-strength alloys, together with advances in propulsion, have all contributed to the high performance of today's aircraft. Unfortunately, these same factors have also had a direct

role in the heightened importance of fatigue. In trimming structural weight, stress levels are raised. Wings are likely to be more flexible--to flap more in turbulent air. Then, too, greater speed means flying through more gusts in a specified time. These factors all lead to increases in the number and size of the stress cycles that produce fatigue damage. Add to this the tendency of higher-strength alloys to behave relatively less well in fatigue than the lower-strength alloys they replace, and the link between improved aircraft performance and increased emphasis on fatigue becomes understandable. Difficulties in antifatigue design stem from uncertainties in predicting load variations and the fatigue damage they cause, and these difficulties are compounded by scatter in life. Thus, more explicit consideration of probabilities is required in fatigue than in static design. While it is not always the major problem, scatter in fatigue data usually gets first mention in discussion of fatigue problems. The large amount of scatter in life that is observed in fatigue tests is sometimes adduced to argue that difficulties in fatigue problems border on the hopeless (usually in the context of fatigue analysis or prediction of life). In the following, scatter is discussed in the design context, where this problem is more tractable.

II. SCATTER IN FATIGUE LIFE IN CONSTANT-  
STRESS-AMPLITUDE TESTS

The complexity of, and possible variations in, the load spectra experienced by aircraft make it impossible to obtain an adequate amount of directly applicable experimental data. Relatively simple, "basic" fatigue data are therefore used in conjunction with a cumulative-damage theory to make the extension to complex stress spectra. The basic data are obtained from constant, single-amplitude fatigue tests that provide the basic S - N curve. Here the problem of scatter is first encountered. Figure 1 shows some experimental fatigue data typical of those used to establish S - N curves. These data are the results of constant-amplitude axial-load fatigue tests on 7075-T6 aluminum-alloy sheet, with edge-cut notches providing a theoretical stress-concentration factor of 4.0.<sup>(1,2,3)</sup> The scatter in the measured life for tests at the same stress amplitude is clearly of considerable magnitude, with values of the ratio of longest to shortest life ranging up to 50 or more at low stress amplitudes. This indeed appears discouraging at first glance, and life prediction at low amplitudes would be a formidable task. However, when viewed in the light of the complete design procedure, the picture is less disturbing.

In order to include probability effects in cumulative-damage calculations, it is necessary to have some representation of probability distributions for the basic constant-amplitude fatigue data. This probability distribution is a fictitious or unknowable description of scatter in fatigue life. By making many tests on identical specimens at the same stress amplitude and recording the number of cycles at which failure occurred for each specimen, a histogram showing the fraction of the total specimens that failed at each number of cycles could be made. The shape of this histogram would approach the form of the probability distribution as the number of tests approached infinity.

It is commonly assumed that a normal distribution of the logarithm of cycles to failure provides an adequate representation, although familiarity with this distribution and its properties is certainly a factor in its selection. Compared with other suitable distributions,

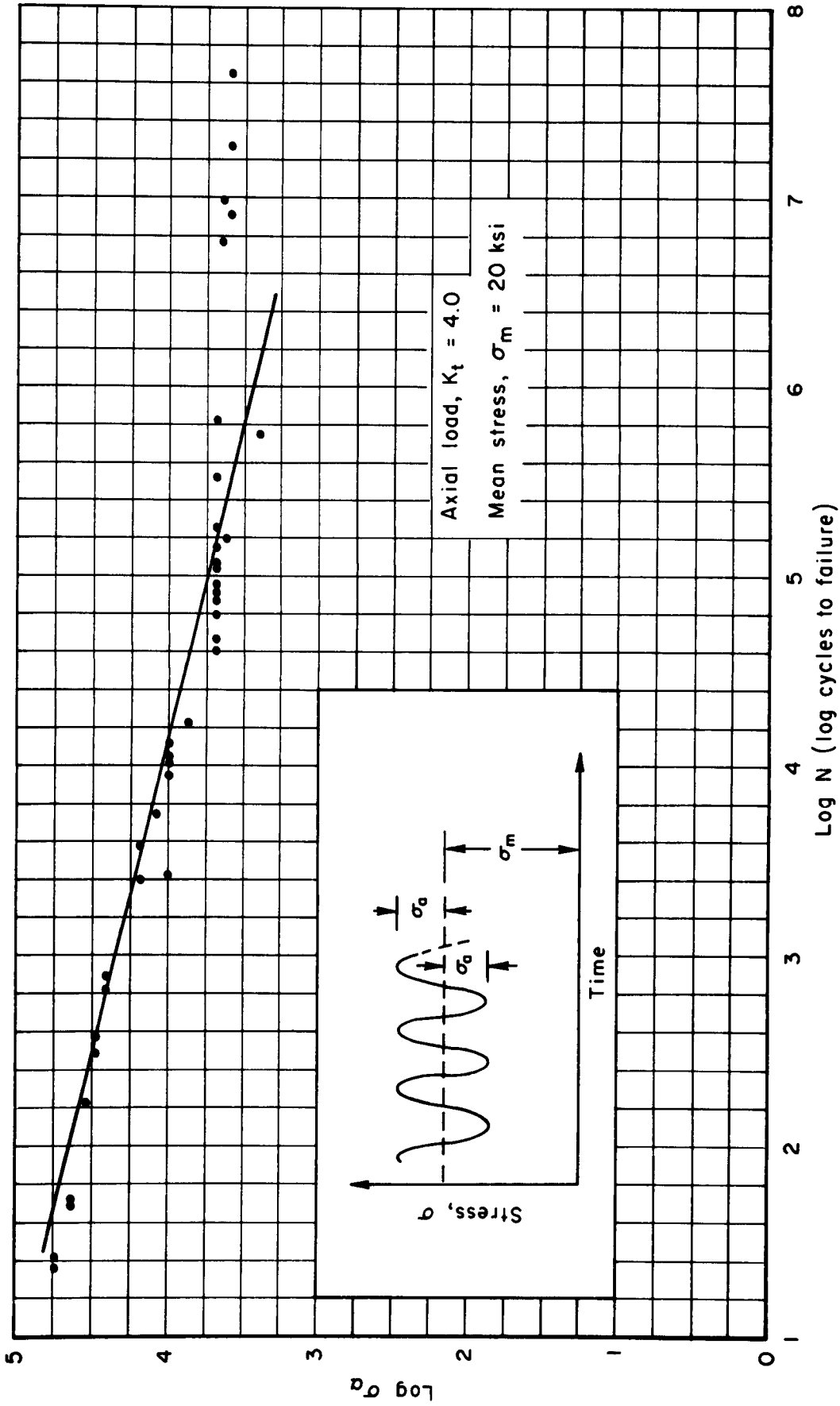


Fig. 1 — Fatigue-test data, 7075-T6 aluminum-alloy sheet

the normal distribution permits easier estimation of parameters of the distribution (and confidence intervals) from test data. Experimental values of the log of cycles to failure at a selected stress amplitude should, if this distribution is valid, approximate a straight line when plotted on normal probability paper. Results for two of the stress amplitudes in Fig. 1 are thus plotted in Figs. 2 and 3. If an extremely large number (approaching infinity) of tests had been made, and if the assumption of a log-normal distribution were correct, the large number of points would plot on a straight line, and the slope and position of the line would define the parameters of the distribution. For a relatively small number of tests, it is unlikely that the points will plot exactly on the straight line or that a straight line fitted to these points would fall on the straight line obtained from an infinite number of tests. If a plot of tests points shows no clear, consistent departure from the straight line of a log-normal distribution, this distribution can be considered adequate. Later, in discussion of the other distributions, it is shown that the exact form of the distribution may be relatively unimportant in design. That is, the amount of material required in the structure may be affected only slightly by alternative distributions, despite large differences in life prediction or failure probability associated with different distributions.

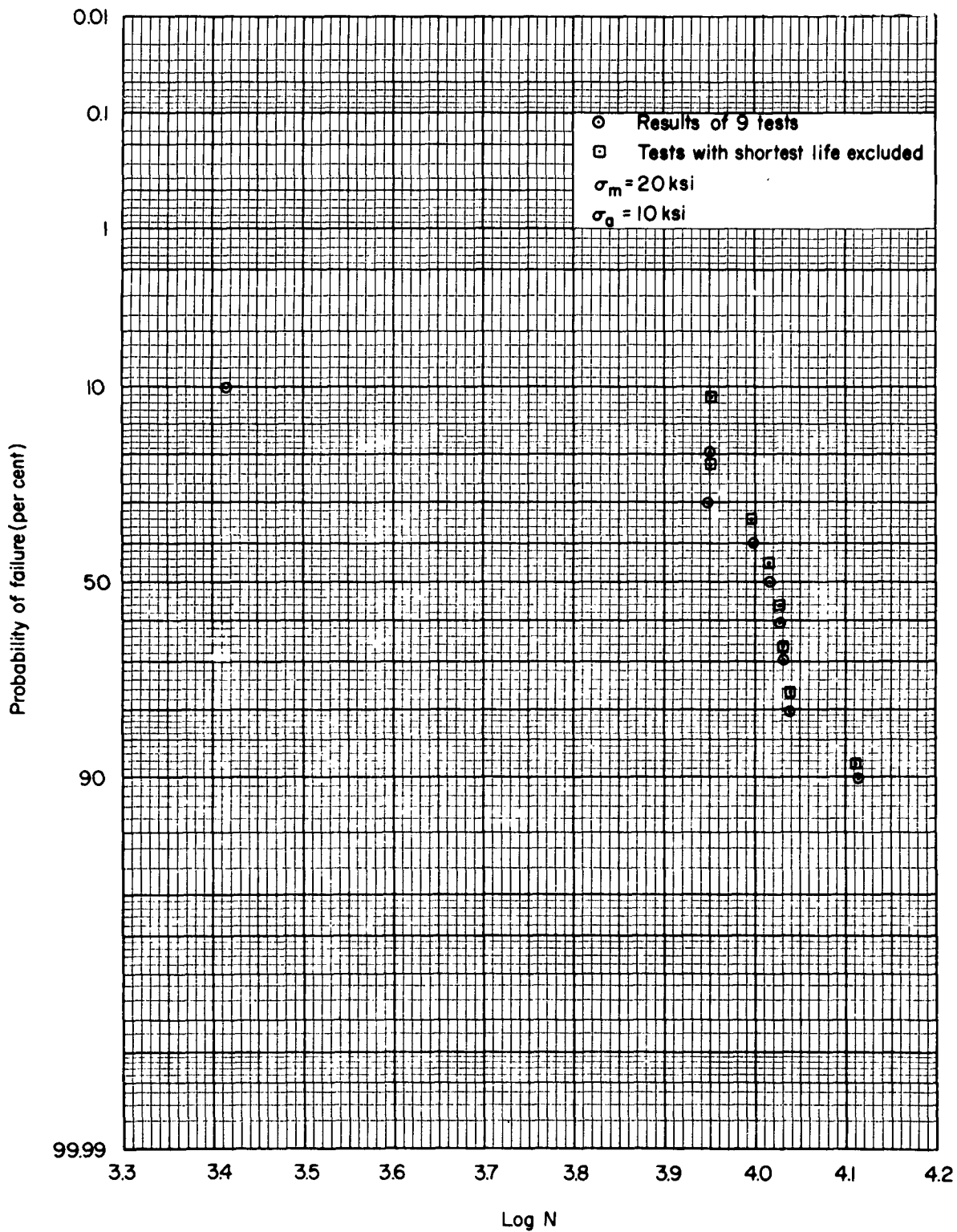


Fig. 2 — Log-normal probability plot of fatigue-test data

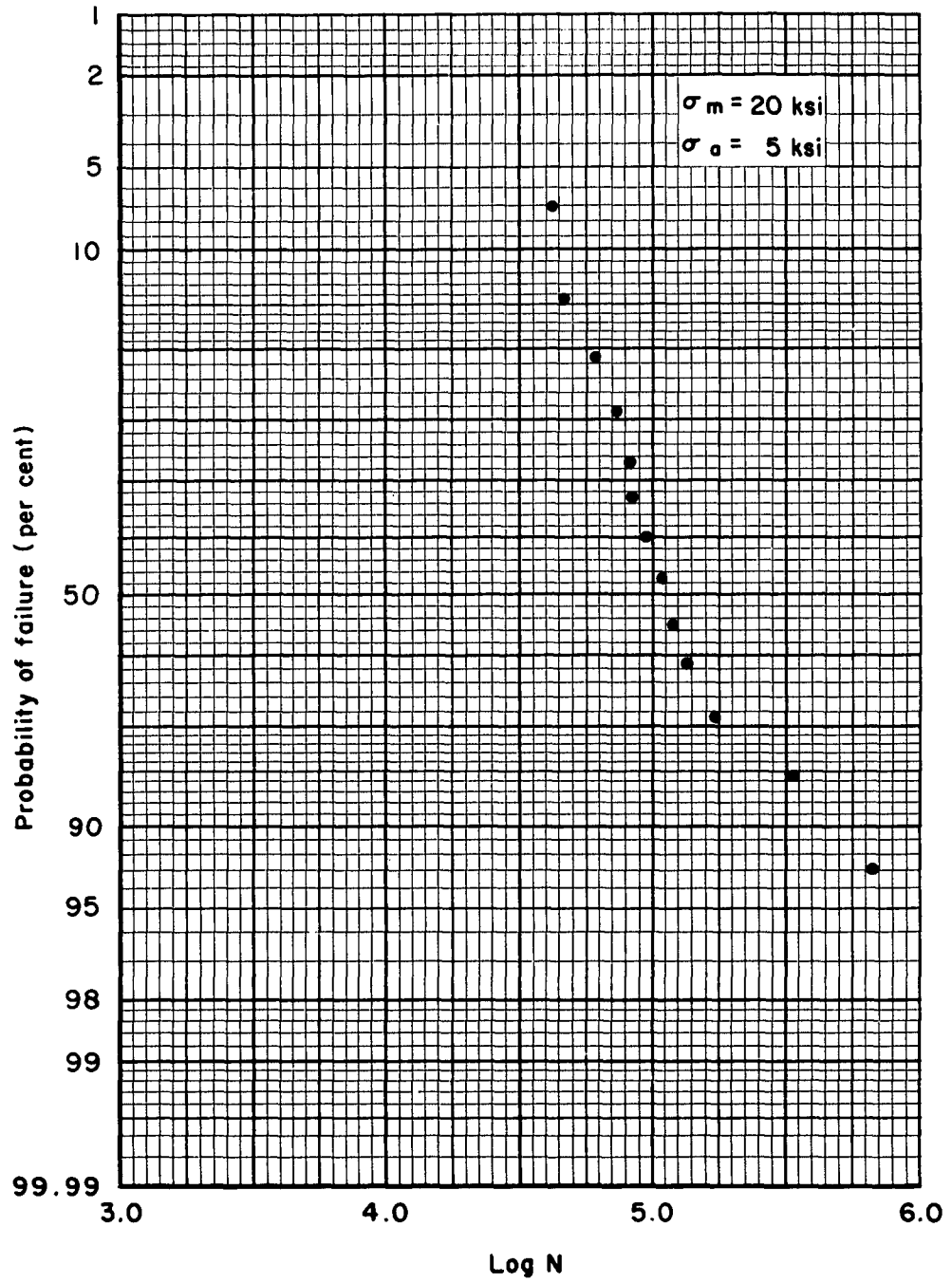


Fig. 3 — Log-normal probability plot of fatigue-test data

III. PROBABILITY EFFECTS IN STATIC  
AND FATIGUE DESIGN

Scatter in strength properties of materials raises the problem of choosing values to be used in design. In the following simple design example, experimental scatter in static and fatigue strength is used to compare the relationship between structural weight and failure probability in the two cases. The comparison shows that the large scatter in fatigue life does not show up in design weight relationships, which can be quite similar for static and fatigue design.

It is well known that the material-strength properties important in static design--the tensile yield stress and ultimate tensile strength--exhibit scatter. Figure 4 depicts the scatter in tensile yield stress as found in 4290 routine mill tests of 7075-T6 aluminum-alloy sheet.<sup>(4)</sup> The curve shown was obtained by fairing a curve through a histogram of test data. The range between the limits of the upper and lower 1 per cent in this distribution is about 10,000 psi, or about 5000 psi above and below the average value of 67,000 psi.

In designing a structure of this material to carry a specified tensile load, with failure defined as the occurrence of the plastic strain associated with yield stress, the amount of material required will depend on the choice of a value of yield stress. Choice of the average or median value would hardly be considered in any practical application, since this would result in failure of about half of the structures before the specified design load was reached. A lower value of yield stress would be chosen for design use, so that the probability that the material in any structure would yield at a lower stress would be only 1 per cent, or 0.1 per cent, for example.

A factor that is neglected for the moment is the uncertainty in the loads that the structure will be required to carry. There will usually be some kind of probability distribution, generally unknown, for loads. The probability of failure depends on the probability distributions for both the material property and the load. By use of a "factor of safety," the reasonably expected maximum load is increased

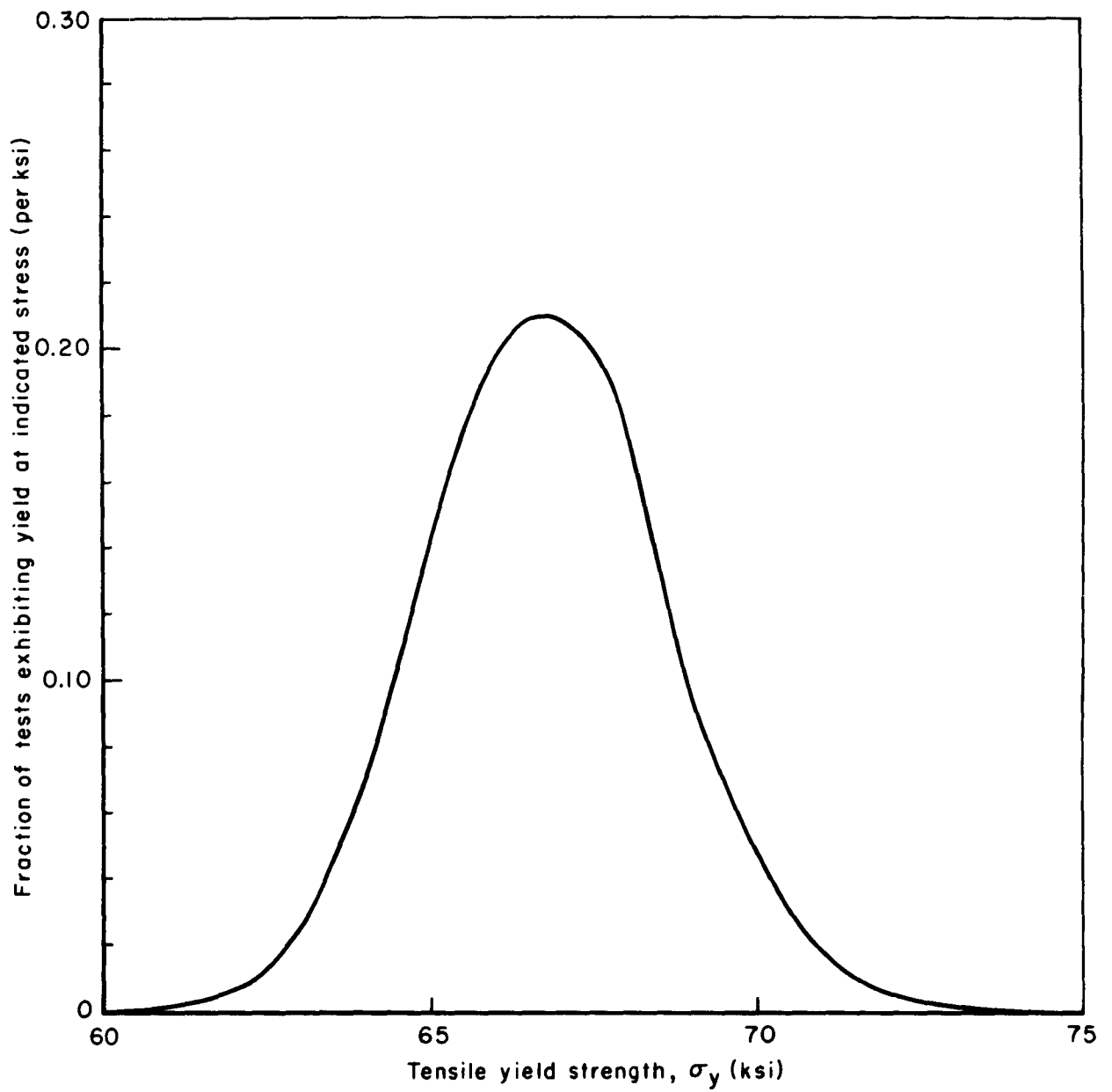


Fig. 4 — Distribution of tensile yield stress, static tests of 7075-T6 aluminum-alloy sheet

to a larger load, which is then used in design. This combination of designing for a load that is very unlikely to be exceeded in service and of using a value of yield stress for design that is very likely to be lower than any value found in the material used leads to acceptably low probability of failure. Also neglected are questions of the proper function for describing probability distributions and confidence in estimates of the parameters of these distributions.

While static material properties are practically never presented for design use in terms of average values, it is still usual for fatigue data to be presented this way. The S - N curves are generally plotted for average or median values of test data. In the comparison to follow, however, median values are needed in both cases. The median value of yield stress from the distribution of Fig. 4 is 66,500 psi.

Considering the design of a beam to carry a specified bending moment, M, with a plastic strain of 0.2 per cent at the outer fiber as the failure criterion, the outer fiber stress is approximately

$$\sigma = \frac{Mc}{I} = K \frac{M}{A^{3/2}} \quad (1)$$

and

$$A = \frac{(KM)^{2/3}}{\sigma_y} \quad (2)$$

where A is the required area of the cross section, K is a constant, and  $\sigma_y$  is the value of yield stress selected for use in design. If the median value of the distribution is selected, structural members with a cross section determined by this stress will have a 50 per cent probability of failure if one attempts to apply the specified moment. By selecting other, lower values for design yield stress, members with a larger cross section and a lower probability of failure would be designed. In this case it is simple to compute the additional weight required to lower the probability of failure from the value of 50 per cent associated with the median value of yield stress, since weight is directly proportional to area. The weight increase

required to provide lower failure probabilities is shown in Fig. 5 (using the distribution of Fig. 4).

A comparable curve for fatigue-probability effects can be obtained. In this case the specified load is cyclic in nature; an additional design requirement, the number of load cycles to be sustained, is specified. Uncertainty in the magnitude of load cycles and in the number of applications is neglected.

The results of 542 constant-amplitude rotating bending fatigue tests of 7075-T6 aluminum-alloy wire<sup>(5)</sup> provide the mean S - N curve and scatter data. The mean S - N curve is represented by a straight line on a log-log S - N plot, corresponding to the equation

$$N = \frac{10^{29.65}}{\sigma_a^{5.4}} \quad (3)$$

The stress amplitude,  $\sigma_a$ , is in pounds per square inch, and N is the mean number of cycles to failure. The log-normal distribution of life agrees with test results and will be used in the following. The mean and median of the distribution of log N are then equal.

The scatter in these test results as measured by standard deviation is clearly dependent on stress amplitude, as is shown in Fig. 6. The dashed curve in Fig. 6 shows an approximation to the relationship between stress amplitude and standard deviation that will be used in the present comparison.

For this example, the design requirement is for a specified moment to be applied for a specified number of cycles. The required number of cycles will be taken as 300,000. The median S - N relationship (Eq. (3)) shows that a stress amplitude of 30,000 psi will allow the required number of cycles, but with a probability of failure of 50 per cent. To reduce the failure probability, the area of the cross section must be increased in accordance with Eq. (2) to lower the stress amplitude. The required increase in weight can then be related to failure probability, as in the static case.

The procedure is to select a stress amplitude lower than 30,000 psi and to find the median value of log N corresponding to this stress

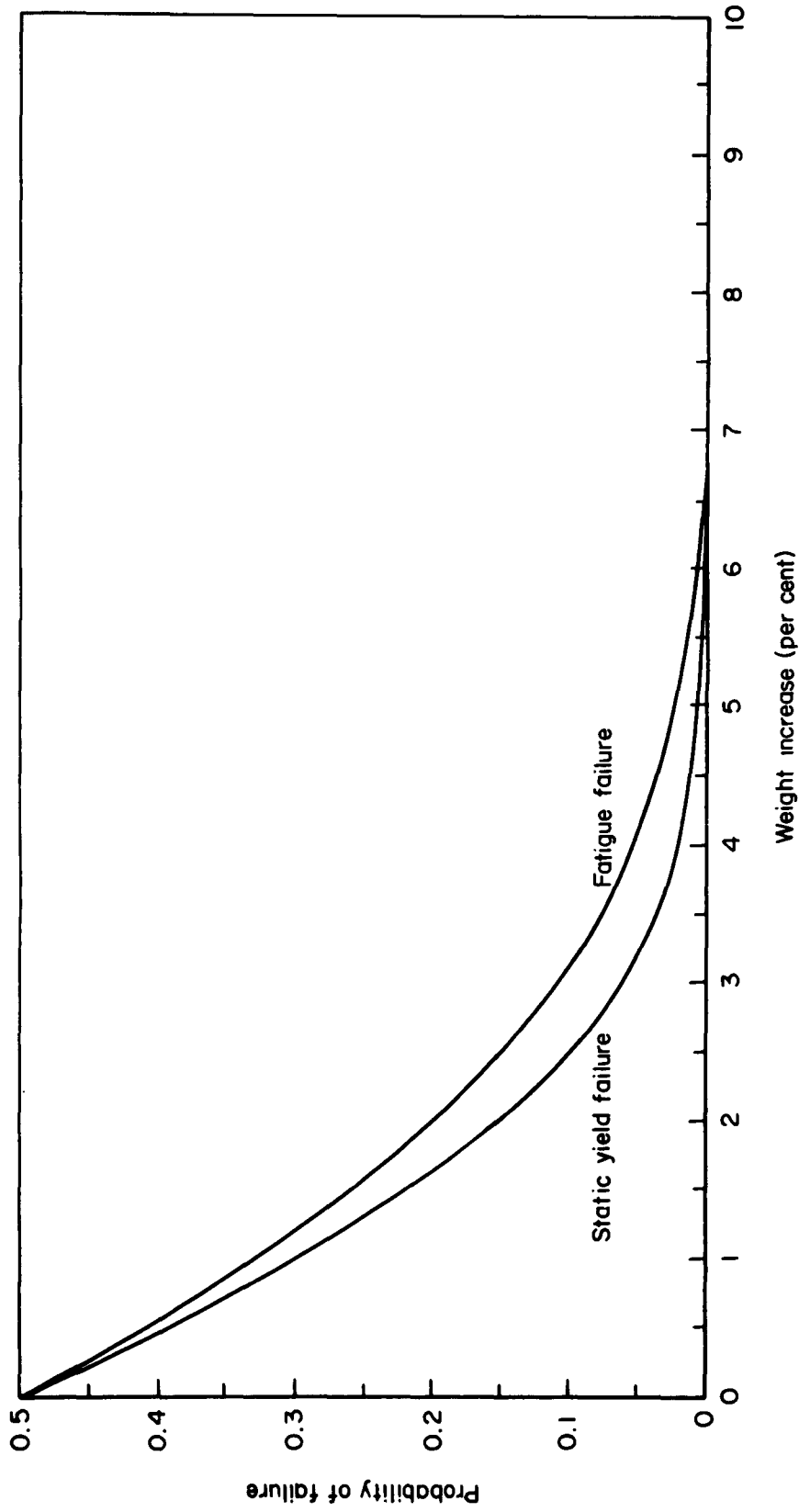


Fig. 5 — Comparison of weight increase (above weight determined by use of median values of strength properties) required to reduce probability of failure in static and fatigue design

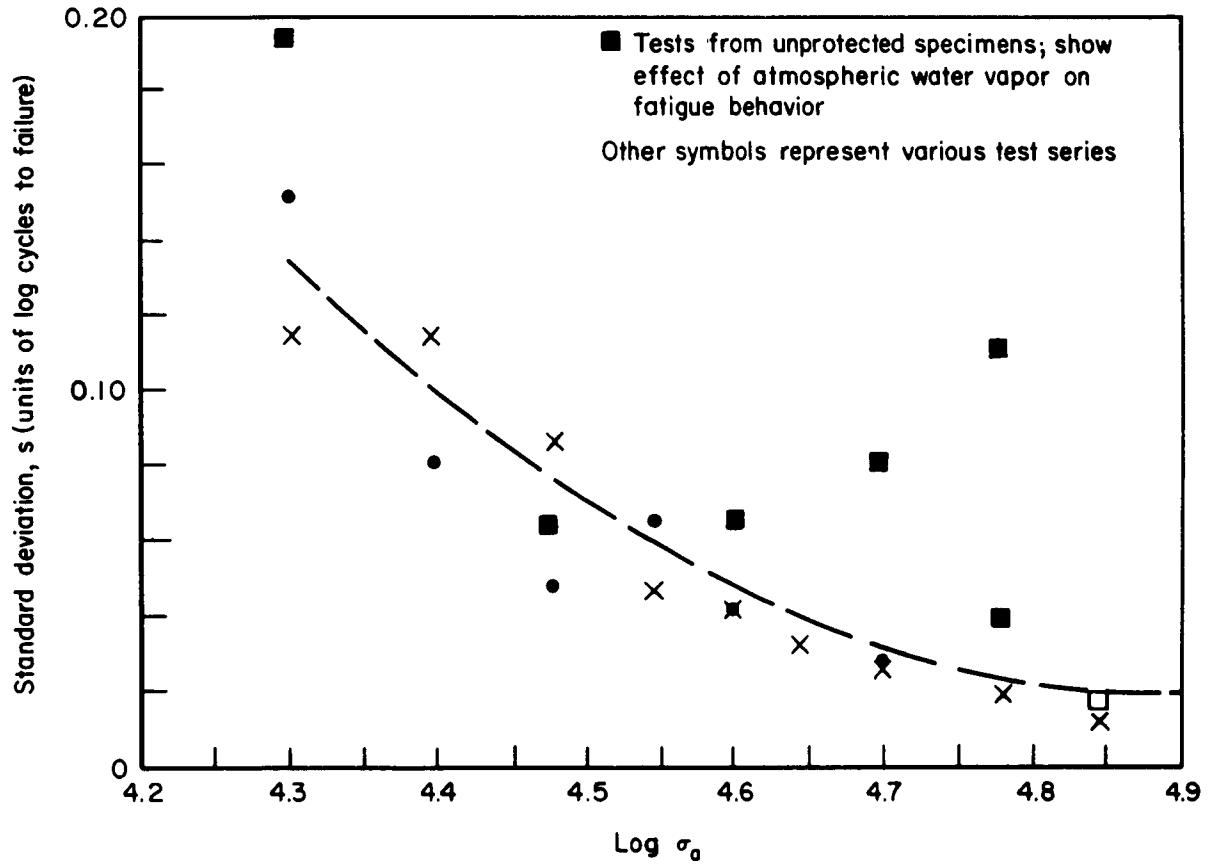


Fig. 6 — Experimental values of standard deviation, constant-amplitude rotating bending tests, 7075-T6 aluminum alloy

amplitude (Eq. (3)). The probability of failure at this stress amplitude at or before the design requirement of 300,000 cycles is found by computing the value of  $K_p$  from the following equation:

$$\log (N_D) = \log (N_M) - K_p s \quad (4)$$

In this equation,  $N_D$  is the number of cycles specified by the design requirement (300,000 in this example),  $N_M$  is the median number of cycles to failure at the selected stress amplitude,  $s$  is, from Fig. 6, the standard deviation at that stress amplitude and  $K_p$  is the normal deviate exceeded with probability  $p$ , which is the probability of failure. Thus, computing the failure probability for lower values of stress amplitude and the resulting increase in weight, the second curve in Fig. 5 can be plotted.

In this simple case of designing for a constant-amplitude cyclic load, the effect of scatter in fatigue is seen to be nearly comparable to the effect in static design of scatter in yield strength, the latter being so commonplace and accepted as to barely merit the designer's awareness as he uses minimum guaranteed values.

IV. PROBABILITY DISTRIBUTIONS IN CONSTANT-  
AMPLITUDE FATIGUE

In the previous example the logarithm of cycles to failure in constant-amplitude fatigue tests was assumed to be adequately represented by the normal distribution. Concern has been expressed that this distribution may have a general tendency to predict lower values of failure probability in the low region of cycles to failure than indicated by the test results. If this is true, straightforward application of the log-normal distribution could be on the unsafe side when the aim is a design with low failure probability.

An indication of this tendency for experimental data to pull away from the log-normal distribution at the lower tail of the distribution is shown in Fig. 7. This is a composite plot of experimental results from tests with 2024 aluminum-alloy sheet. Tests include notched specimens at one stress level,<sup>(6)</sup> riveted joints at two different stress levels,<sup>(6)</sup> and notched flat specimens at another stress level.<sup>(7)</sup> In the first three cases, 20 specimens were tested, and 103 specimens were tested in the last case.

In the original presentation<sup>(7)</sup> of the latter test series, curves for three other probability distributions in addition to log-normal are compared with test results. The other distributions all tend to fit better at the low-probability end. One is simply normal distribution of the number of cycles to failure. The second is a form of the Weibull distribution, modified so that the probability of failure below a lower limiting number of cycles is equal to zero. The probability of failure in this case is given by

$$P(\text{failure}) = 1 - \exp [-\lambda(N - N_0)^\alpha] \quad (5)$$

for  $N > N_0$  and

$$P(\text{failure}) = 0, N < N_0 \quad (6)$$

where  $\lambda$  and  $\alpha$  are parameters of the distribution, determined from test results. The third distribution also leads to Eq. (5) with  $N$

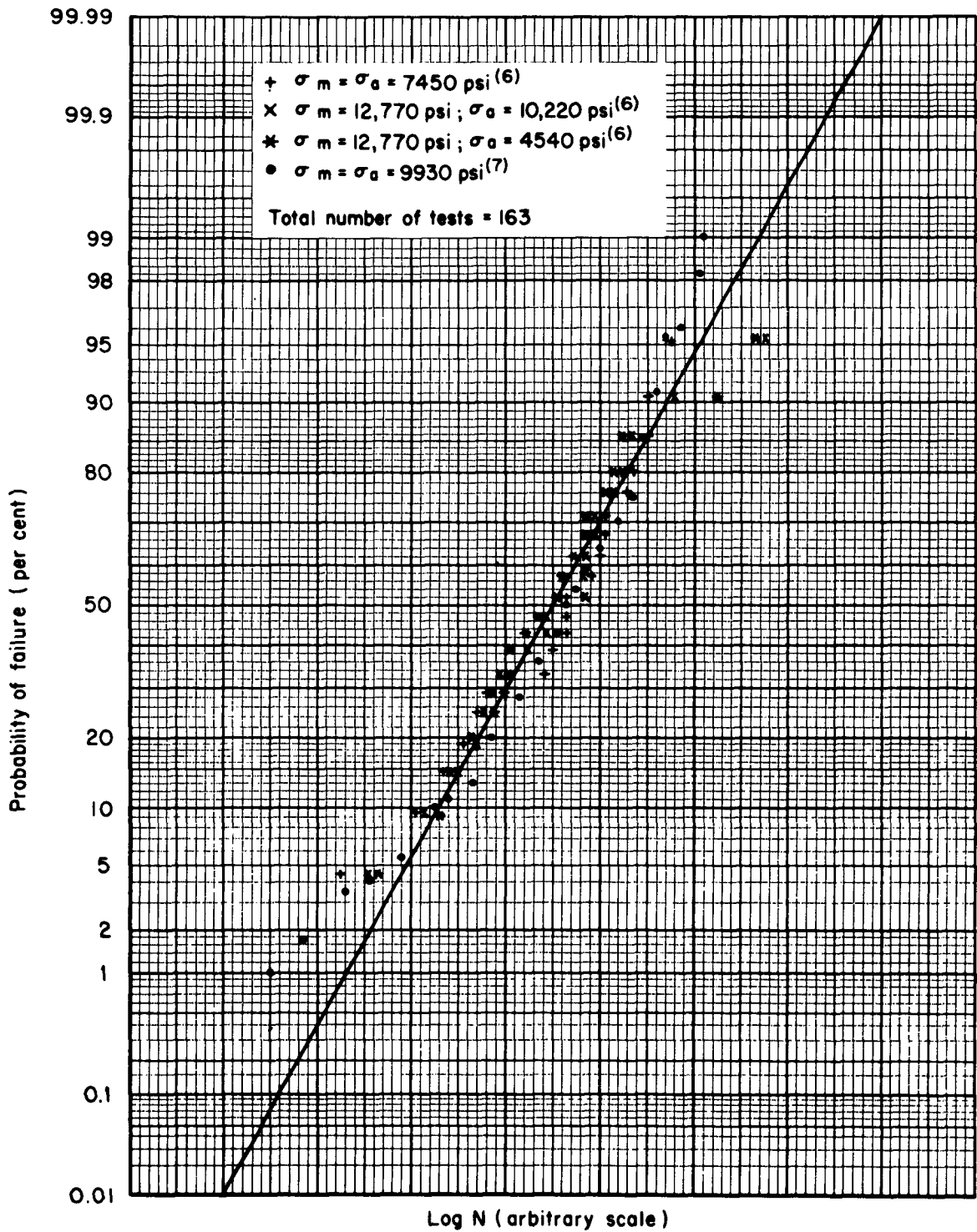


Fig. 7—Comparison of fatigue-test results (notched and riveted 2024 aluminum specimens) with log-normal failure probability distribution

replaced by  $\log N$ . (In the first three test series,<sup>(6)</sup> normal distribution of cycles to failure and  $\log$  cycles to failure were found to be nearly equivalent.)

The improved agreement of the distribution of Eq. (5) with test results in the low-probability region has prompted a considerable amount of speculation, both mathematical and philosophical. An extensive discussion of the characteristics and estimation of parameters of this distribution also presents an application to test results.<sup>(8)</sup> While the tests in question are reversed-torsion tests on nickel wire, it is instructive to examine the results. Choice of the Weibull distribution with a minimum-value cutoff can lead to estimates of higher probability of failure than  $\log$  normal in the low-probability end of the distribution, which is the region of primary interest in fatigue.

However, it is clear that at even lower failure probabilities the situation must change, since the  $\log$ -normal distribution provides finite failure probabilities down to zero cycles, while the modified Weibull distribution leads to zero failure probability between zero cycles and the minimum cycle cutoff. (Finite failure probability down to zero cycles is another objection that has been raised to the  $\log$ -normal distribution.)

As indicated previously, the chief aspect of probability distributions of interest here is the design effect--i.e., determining the amount of material required in a structure to permit application of a specified number of load cycles with a specified low probability of failure. Thus, while the choice of different probability distributions may lead to large differences in the results of analysis (the prediction of the number of cycles to failure at various probability levels or estimation of failure probability at various numbers of cycles), it does not follow that design effects will be pronounced. As a matter of fact, the general form of S - N relationships leads to the expectation that design effects may indeed be small.

In the analysis of Freudenthal and Gumbel<sup>(8)</sup> of data from another source (reversed-torsion tests of nickel wire), the estimated parameters of the modified Weibull distribution are presented. The estimates of minimum-cycle cutoff values vary with stress amplitudes,

decreasing with increasing stress amplitude and becoming zero for the higher stress amplitudes. For one of the lower amplitudes ( $\pm 21.5$  kg/mm<sup>2</sup> or 30,500 psi) for which a minimum cycle cutoff is postulated, parameters of the failure probability in the form of Eq. (5) are obtained. In order to compare this approach with the assumption of log-normal distribution, the same test data (from 20 tests) were used to obtain the mean and standard deviation of the logarithms of cycles to failure.

The cumulative probability of failure found from the two distributions is presented in Fig. 8. This shows the generally higher failure probability predicted by the modified Weibull distribution, but it also shows the crossover near the minimum cycle cutoff of this distribution. The region of greatest interest in fatigue, the low-failure-probability region, is shown in Fig. 9. These figures show that in the interesting range of failure probability, the log-normal distribution actually shows higher failure probabilities than does the modified Weibull distribution.

Either choice of probability distribution leads to about the same predicted cycles to failure at the same failure probability over much of the range, although estimates of failure probability may vary markedly at a specified number of cycles. More important from the design viewpoint, however, is the conclusion that the required amounts of material for a specified life and low failure probability, as determined by the two distributions presented, would not be significantly different.

For example, from Fig. 9, a failure probability of one in 10,000 is reached at about 175,000 cycles, according to the log-normal distribution, and at about 189,000 cycles, according to the modified Weibull distribution. Taking a failure probability of  $10^{-4}$  and a life requirement of 189,000 cycles as a design requirement, a design based on the modified Weibull distribution would have a certain weight, determined by the magnitude of the alternating torque. A design based on the log-normal distribution would have to be somewhat heavier to decrease the stress enough to increase the life from 175,000 to 189,000 cycles (an increase of 8 per cent). Since life

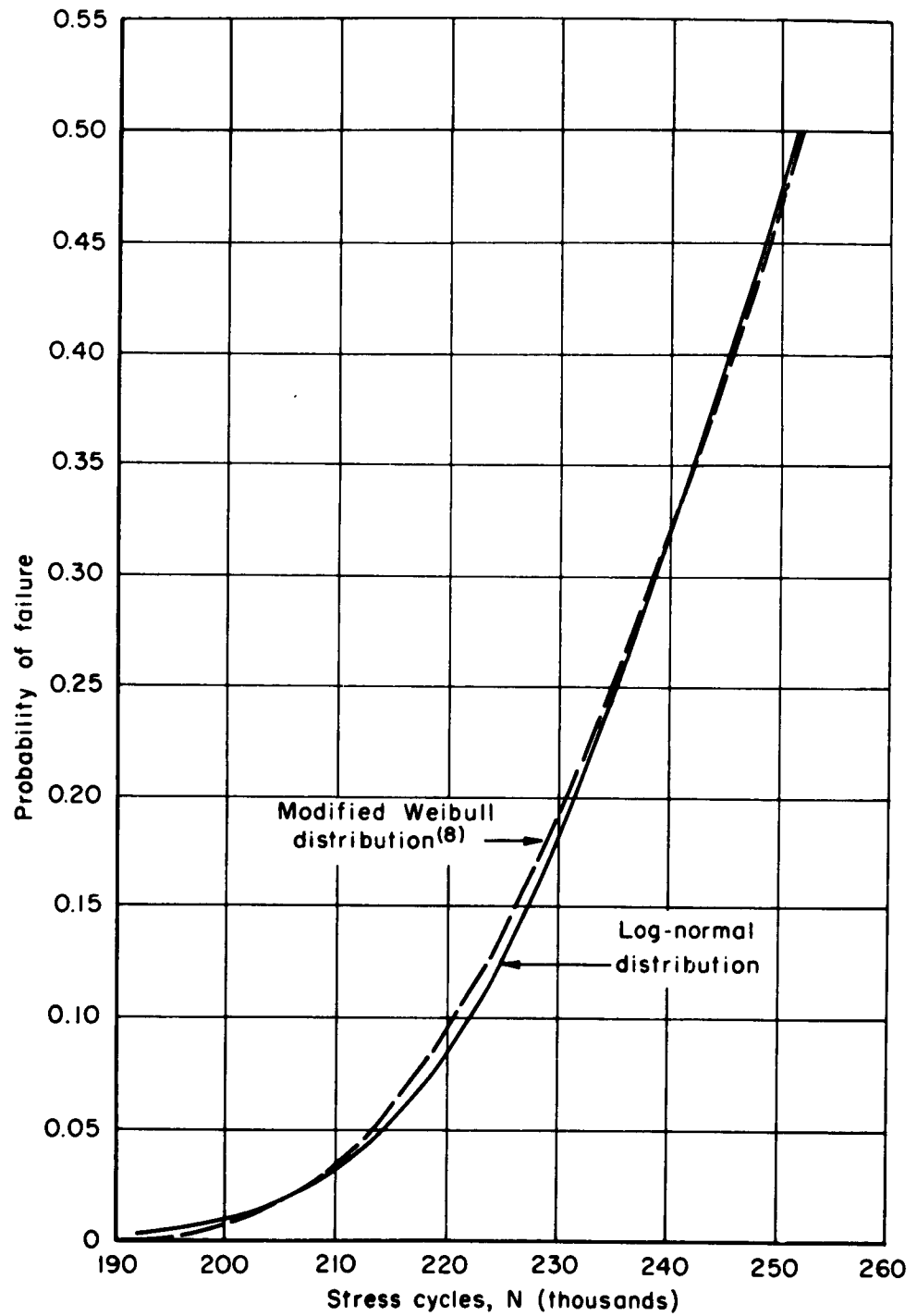


Fig. 8 — Comparison of failure-probability distributions



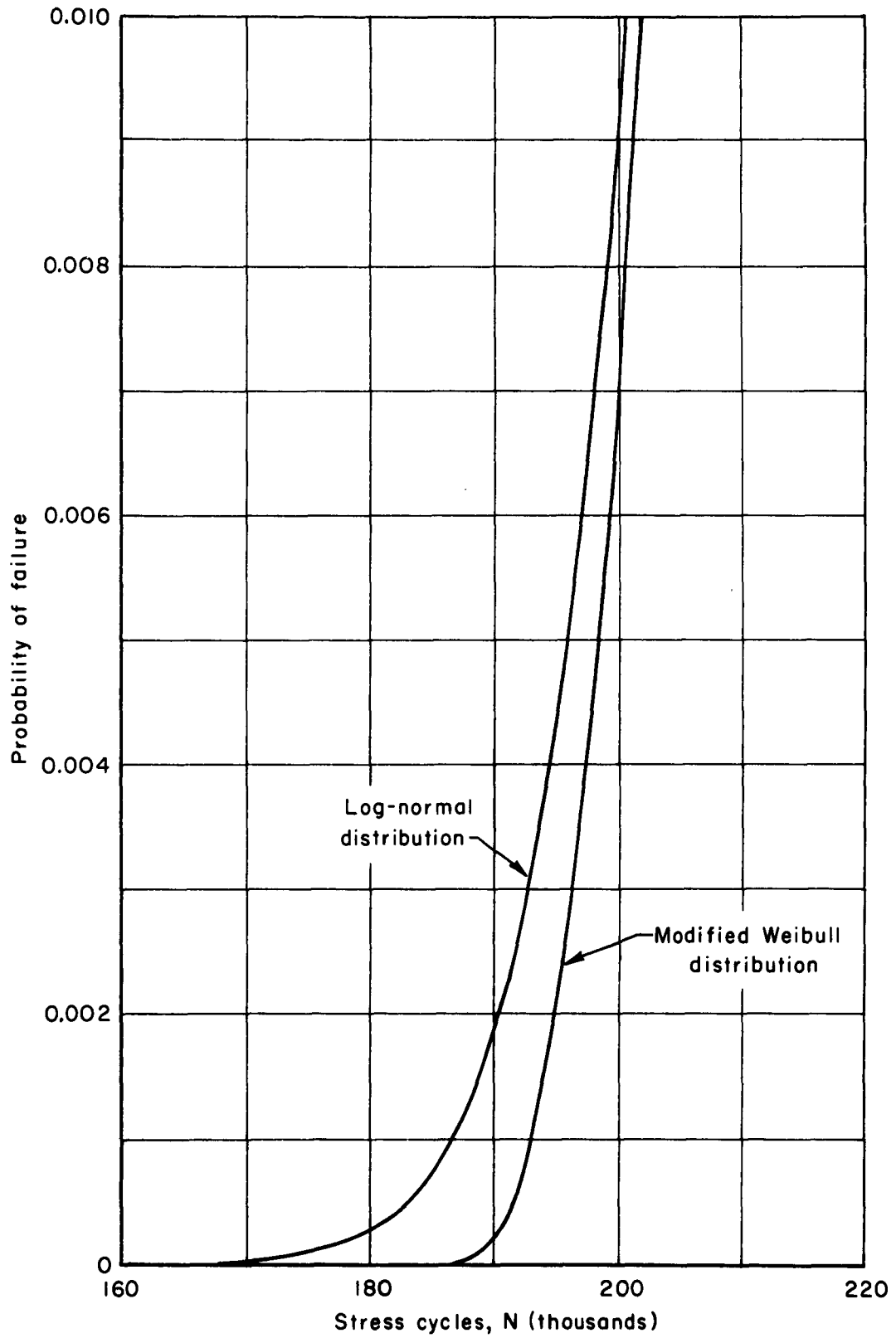


Fig. 9 — Comparison of failure-probability distributions at low failure probability

varies approximately with the negative 2.75 power of stress (from the test data), and area (or weight) varies with the negative 2/3 power of stress, area will vary as life to the 8/27 power. Thus, the required 8 per cent increase in life is obtained by only a little over 2 per cent increase in weight.

While it may not be valid to apply this conclusion to the variety of materials, stress spectra, and stress distributions represented in flight structures, this example is illustrative of the approach required to evaluate effects of probability distributions in design terms.

A feeling for the effect of the nonvanishing failure probability in the log-normal distribution can be gotten by considering an extremely low failure probability,  $10^{-12}$  for example. The log-normal distribution of the present example would predict this failure probability at a life about 30 per cent shorter than the life predicted by the modified Weibull distribution. This great a difference in life would lead to a difference of only 12 to 15 per cent in design weight for equal failure probability and life. For an idea of the magnitude of this failure probability, consider a fatigue-testing program using specimens 0.5 in. round and 4 in. long. A sample size for which one failure corresponds to a failure probability of  $10^{-12}$  would require the entire 1960 U.S. steel production.

These examples are not intended to be conclusive evidence that effects of different failure-probability distributions are negligible in fatigue, but rather to emphasize the importance of examining various suggested distributions in terms of design effects in order to place these effects in proper perspective relative to the other uncertainties that affect antifatigue design.

## V. CONFIDENCE LEVELS

Confidence levels play an especially important part in fatigue because in the important problems the complexity and cost of real or simulated structures and structural elements severely limit the number of tests that can be made.

In this and the following two sections, the problem of confidence is discussed. Confidence in this context refers to the question of how certain or confident we can be of the validity of applying the results of a limited number of fatigue tests to a large fleet of airplanes. This section discusses the sources and meaning of confidence concepts.

Any consideration of confidence levels soon discloses a strong stimulus for assuming (or hoping) that normal distributions adequately represent fatigue behavior. Many of the techniques for assigning confidence values to estimates of the parameters of a normal distribution do not have counterparts for other distributions. This fact, together with the tendency for lesser importance of distribution effects in the design process and the possibility of transforming nonnormal into normal distributions, justifies the basis of normal distribution in considering confidence levels.

The following or a similar quotation is always in order when confidence intervals or regions are included in fatigue work: "Confidence intervals enable one to obtain a useful type of information about population parameters without the necessity of treating such parameters as statistical variables. It should be clearly understood that one is merely betting on the correctness of the rule of procedure when applying the confidence-interval technique to a given experiment."<sup>(9)</sup> This warning contained in this quotation becomes clearer when the application of confidence techniques to fatigue data is investigated.

The basic problem is, of course, the prediction of failure probabilities applicable to a large number of structures or structural elements from a small number of tests in order to be able to design structures with an acceptably low failure probability. It is practically never possible to make enough tests to avoid having to extrapolate test results down to the low-probability region.

Thus, for example, the mean value of the logarithm of cycles of failure can be computed from the constant-amplitude tests on several similar specimens. The question is, How can this value, the sample mean, be utilized for computing probabilities for a large number of specimens or structures? At this point the impetus arises for assuming that the probability density distribution for the extremely large population (theoretically infinite) is normal. Usually neither the mean nor the standard deviation of the large population is known, but it is possible to use the Student's t distribution to find a confidence interval for the mean, if the basic distribution is normal.

For example, if from a test series the mean value of log cycles to failure is 5.647, it is possible by using only the test data and the tabulated Student's t distribution (with the assumption of log-normal population) to make a statement such as this: The probability is 0.95 that the range 5.647 (sample mean)  $\pm$  0.231 includes the true mean value of the total population. Referring to the earlier warning, the real meaning of this statement is this: If the test series were repeated a large number of times, and if each time an interval were computed in the same way on either side of each (different) sample mean, 95 per cent of these intervals would include the true population mean. In other words, the probability is 0.95 that the results of any one test series will be such that the confidence interval computed by using these results will include the true population mean.

A similar procedure involving use of another probability distribution, the  $\chi^2$  distribution, permits similar statements to be made about the standard deviation of the total population.

Because neither the population mean nor standard deviation are known beforehand, study of confidence regions rather than confidence intervals is required in fatigue work. That is, an area in the  $\mu$ -s plane can be found (where  $\mu$  is the true population mean and s is the true population standard deviation) such that the same kind of confidence statement is allowed--e.g., the probability is 0.95 that the results of a test series will be such that the confidence region constructed by using these results will include the true population mean and standard deviation.

It is not permissible to construct this confidence region by determining confidence intervals for the mean and standard deviation with the Student's  $t$  and  $\chi^2$  distributions because they are not independent distributions. The region can be constructed by using the distribution of the sample mean in place of the Student's  $t$  distribution.<sup>(10)</sup>

VI. CONSTRUCTION OF CONFIDENCE REGIONS  
IN FATIGUE

The purpose of this section is to present some discussion and examples to help illuminate the meaning of confidence and the means for evaluating confidence. Understanding of the procedure of the examples is not a necessary step toward proper use of confidence concepts--a better method is discussed in the following section. This section is only intended to provide a graphic aid for grasping the fundamental concepts of confidence.

There is a particular feature of fatigue effects in design that affects the construction of confidence regions. This is the almost exclusive interest in the low-failure-probability region, or those failures at the lowest number of cycles. The effect is to make the confidence intervals of interest on the mean and standard deviation one-sided. Instead of confidence intervals stated as the sample values of mean and standard deviation plus or minus some quantity, intervals between some lower limit and infinity for the mean and between zero and some upper limit for standard deviation are appropriate in fatigue work.

To construct a confidence region of this form, we first consider the quantity

$$\frac{\bar{x} - \bar{\mu}_0}{\bar{s}_0 / \sqrt{n}} \quad (7)$$

In this expression,  $\bar{x}$  is the mean value of the test results for  $n$  tests and would be the mean value of log cycles to failure if the distribution is assumed log normal. The unknown, true population mean and standard deviation are expressed by  $\bar{\mu}_0$  and  $\bar{s}_0$ . If test specimens are chosen from a normal population, the quantity (7) is also normally distributed with zero mean and standard deviation of unity. This permits the probability statement

$$P\left(\frac{\bar{x} - \bar{\mu}_0}{\bar{s}_0 / \sqrt{n}} < K_\alpha\right) = 1 - \alpha \quad (8)$$

where probability is used in the restricted sense appropriate to confidence intervals. The normal deviate exceeded with probability  $\alpha$  is expressed by  $K_\alpha$ . In similar fashion, one can obtain the probability statement

$$P\left(\bar{s}_o < \sqrt{\frac{\sum(x_i - \bar{x})^2}{\chi_\beta^2}}\right) = \beta \quad (9)$$

where  $x_i$  are the individual results of the  $n$  tests and  $\chi_\beta^2$  is the value in the  $\chi^2$  distribution with  $n - 1$  degrees of freedom that is exceeded with probability  $\beta$ .

The confidence region of interest can then be constructed in a plane with  $\mu_o$  and  $s_o$ , the population mean and standard deviation, as coordinates. The region is between the two straight lines  $s_o$  equal to zero and  $s_o$  equal to the value obtained by changing the inequality in Eq. (9) to an equality. It is bounded on the left by the straight line obtained by changing the inequality in Eq. (8) to an equality, and it extends to infinity on the right. The confidence level is the joint probability  $(1 - \alpha)\beta$ . The true population values,  $\bar{\mu}_o$  and  $\bar{s}_o$ , are represented by a single but unknown point on the plane. The confidence level,  $(1 - \alpha)\beta$ , is the probability that the results of our tests will, when treated in the foregoing manner, lead to the specification of an area in the  $\mu_o, s_o$  plane that covers the true point  $\bar{\mu}_o, \bar{s}_o$ .

The construction and use of a confidence region can be illustrated by using some of the test data displayed in Fig. 1. The results of thirteen tests at an alternating stress of 5000 psi are shown there and also on a normal probability plot in Fig. 3. The mean value of log cycles to failure,  $\bar{x}$ , is found to be 5.0498, and the unbiased estimate of standard deviation,  $\hat{s}$ , is 0.33506 (in units of log cycles).

An approximate 0.95 confidence region can be constructed from these data by selecting a value of  $\alpha$  equal to 0.025 and  $\beta$  equal to 0.975 so that

$$\text{Confidence level} = (1 - \alpha)\beta = 0.9506 \quad (10)$$

From Eq. (8)

$$\mu_o = \bar{x} - K_{\alpha} \frac{s_o}{\sqrt{n}} \quad (11)$$

$$= 5.0498 - \frac{1.96}{3.6056} s_o$$

$$\mu_o = 5.0498 - 0.5436 s_o \quad (12)$$

where  $K_{\alpha}$  is 1.96, the normal deviate exceeded with probability 0.025. Equation (12) is plotted in Fig. 10 with the confidence region shown as the hatched area to the right of the straight line from this equation.

From Eq. (9), the upper bound for  $s_o$  is found:

$$s_o = \sqrt{\frac{\sum(x_i - \bar{x})^2}{\chi_{\beta}^2}} \quad (13)$$

$$= \frac{\sqrt{n-1} \hat{s}}{\sqrt{\chi_{\beta}^2}} \quad (14)$$

$$= \frac{0.33506 \sqrt{12}}{\sqrt{4.40}}$$

$$s_o = 0.5532 \quad (15)$$

where 4.40 is the value of the variable in the  $\chi^2$  distribution with 12 degrees of freedom exceeded with probability 0.975. The line resulting from Eq. (15) is also shown in Fig. 10, and the hatched region below it is the corresponding confidence region.

The doubly hatched area in Fig. 10 is thus the confidence region sought. To review the meaning of this region, if the underlying distribution, from which the thirteen test specimens were selected, has

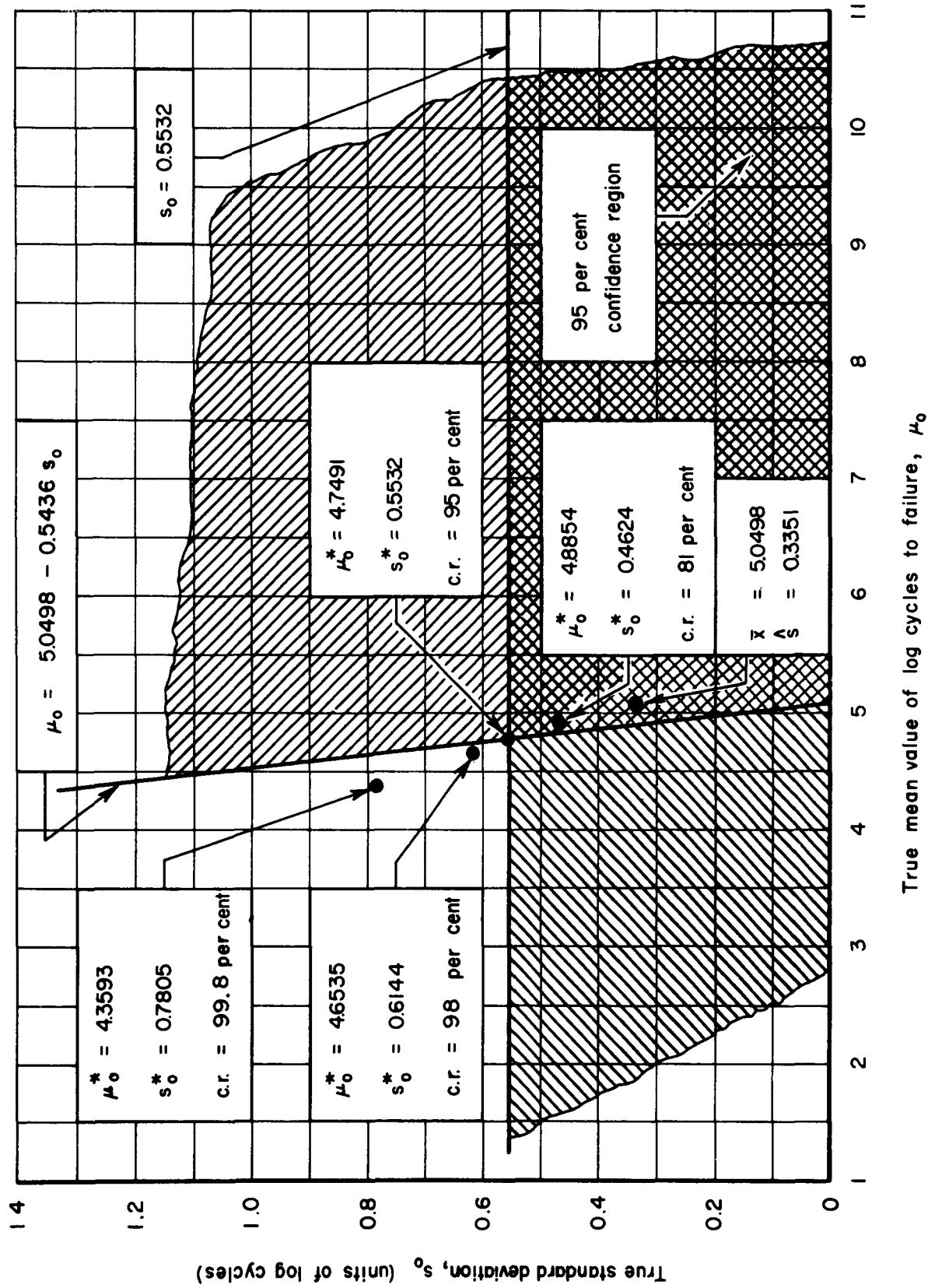


Fig. 10—Construction of confidence region from fatigue-test data for estimating low-probability values of cycles to failure

a log-normal distribution, and if the above procedure for constructing the confidence region is followed, the a priori probability that the test results will lead to a confidence region that covers the point  $\bar{\mu}_0, \bar{s}_0$  corresponding to the true mean and standard deviation of the underlying distribution is 0.95.

The reason for considering confidence regions is to enable us to have confidence (and state it quantitatively) when considering the low failure probabilities of main interest in fatigue. In the preceding example, we could have proceeded by straightforward application of values of mean and standard deviation (estimated from the thirteen tests) to a normal distribution to estimate the number of cycles at which the failure probability is, say, 0.001. In this case, the estimated value of log cycles to failure would be

$$x_{e.p.}^* = \bar{x} - K_{\alpha} \hat{s} \quad (16)$$

$$= 5.0498 - 3.09(0.33506)$$

$$x_{e.p.}^* = 4.0145 \quad (17)$$

The subscript e.p. refers to the estimated parameters of the distribution, from test results, which are used to obtain the value  $x^*$ . Again,  $K_{\alpha}$  is the normal deviate exceeded with probability  $\alpha$ , where  $\alpha$  is the failure probability, 0.001.

While this procedure is simple, there is not much that can be said about the adequacy of the estimates of distribution parameters from test results--how near the estimates are to the true population parameters. It is clear that running another thirteen tests would undoubtedly lead to different estimates, and it is quite unlikely that the estimates would correspond exactly to the true parameters in either case.

Returning to the confidence region in Fig. 10, we recall that the point  $\bar{\mu}_0, \bar{s}_0$  corresponding to the true parameters may lie anywhere

in the  $\mu_o, s_o$  plane, but by following the prescribed procedure, we have a 95 per cent chance that the doubly hatched area contains  $\bar{\mu}_o, \bar{s}_o$ .

Of all points in the confidence region, the worst point, the one that would give us the lowest estimate of cycles to failure at any probability level, is the upper left corner of the region. This point, noted as  $\mu_o^*, s_o^*$  in Fig. 10, represents the lowest value of the mean and the highest value of the standard deviation in the confidence region. The value of  $s_o^*$  is found from Eq. (13) or Eq. (14), and then that of  $\mu_o^*$  from Eq. (11).

Then, making use of the confidence region to estimate the value of log cycles to failure below which failure probability is  $\alpha$

$$x_{c.r.}^* = \mu_o^* - K_{\alpha} s_o^* \quad (18)$$

Again,  $K_{\alpha}$  is the normal deviate exceeded with probability  $\alpha$ , the failure probability. The subscript c.r. denotes the value estimated through use of the confidence region.

(That the point  $\mu_o^*, s_o^*$  leads to the lowest value of  $x^*$  for a specified failure probability may be seen by writing the general form of Eq. (18)

$$x_{c.r.}^* = \mu_o - K_{\alpha} s_o \quad (19)$$

At a specified probability,  $K_{\alpha}$  is constant, and Eq. (19) plots as a straight line in the  $\mu_o, s_o$  plane of Fig. 10. These lines slope upward to the right and intercept the  $\mu_o$  axis at  $x_{c.r.}^*$ . Thus, it is clear that for a specified probability, of all lines that pass through points in the confidence region, the line passing through  $\mu_o^*, s_o^*$  will result in the lowest value of  $x_{c.r.}^*$ .)

Use of this confidence-region approach leads to values of  $x^*$  which can be compared with values found from use of estimated parameters. For the same failure probability, 0.001, used in the estimated parameter example, the value of log cycles to failure is, from Eq. (19)

$$x_{c.r.}^* = 4.7491 - 3.09(0.5532) = 3.0397 \quad (20)$$

This value gives a much lower value of cycles to failure--only about 1/9 of that obtained by use of estimated parameters. However, it is now possible to make a confidence statement--i.e., to say with 95 per cent confidence that for a test specimen from the same population as the original thirteen specimens the probability of failure before 1096 cycles (antilog of 3.0397) is not greater than 0.001. (The comparable number of cycles from the estimated-parameter approach is antilog 4.0145, or about 10,340 cycles. The mean for the sample, 5.0498, is equivalent to 112,150 cycles.)

Other  $\mu_o^*$ ,  $s_o^*$  points corresponding to other confidence levels are also shown in Fig. 10. Values of  $x^*$  are found to be larger or smaller as the confidence level is decreased or increased. It is clear that the general effect on design is a requirement for more material (and greater weight) to withstand a repeated number of cycles of a specified load--the price that must be paid to be able to make a confidence statement about the design when test data are limited.

The points  $\mu_o^*$ ,  $s_o^*$  in Fig. 10 were all obtained by equal confidence levels on the mean and standard deviation. It is possible to obtain confidence regions with different boundaries but the same confidence level by using different confidence levels for the mean and standard deviation while keeping the product  $(1 - \alpha)\beta$  constant. This could lead to higher values of  $x^*$  (which is desirable) at the same failure probability and confidence level than when  $\beta$  is kept equal to  $(1 - \alpha)$ . Further, the confidence region as constructed is not a minimum-area region, and this too is a possible means for increasing  $x^*$ . Determination of the confidence level associated with the area to the right of the lines defined by Eq. (19) and above  $s_o = 0$  might also lead to a more efficient confidence region.

The preceding discussion leads to a valid confidence region and also provides a picture of the method for establishing this confidence region. There is, however, a different statistical approach to this problem, which leads to higher values of  $x^*$ . This approach (described in the next section) would thus lead to a lighter structure, meeting the same design requirements and providing the same confidence.

VII. APPLICATION OF ACCEPTANCE-INSPECTION-PLAN TECHNIQUES  
TO FATIGUE--THE k FACTOR

In acceptance inspection, a number of sample items, drawn from a lot of manufactured items, are tested. The simplest method, attribute testing, involves a yes-no test on each of the sample items. For example, if the manufactured items were solar cells, the test might be exposure to a standard light source and measurement of the output. The answer to the test question might be yes, the cell generates 23 or more milliwatts (acceptance) or no, the cell does not generate 23 or more milliwatts (rejection). By fairly simple and well-defined statistical theory, the number of rejects in the sample leads to the decision of whether or not the lot is of acceptable quality. These methods have little to contribute to fatigue problems, however.

Acceptance-inspection plans based on variables testing do have a framework of statistical theory that lends itself in important ways to probability problems in fatigue. In the example of solar cells, instead of the yes-no test, the output of each cell in the sample would be measured and recorded. The decision of whether to accept or reject the lot would now be based on statistical properties of the test results--the mean value and standard deviation of the output of the cells in the sample. Most of the theory behind this procedure is, as one might expect, based on a normal distribution of the variable in the parent population.

Reference 11 reports the work of a group assembled during World War II to assist the armed services with their statistical problems. An important problem was the design of acceptance-testing programs. Simply stated, the problem was to specify the proper number of items in the test sample taken from a manufactured lot and a criterion, using the results of the tests, for accepting or rejecting the lot. Two requirements had to be satisfied: First, the probability of accepting lots with a defective proportion greater than a specified (low) value had to be not more than a specified (also low) value. Second, the probability of rejecting a lot with a defective proportion less than a specified (lower than in the first case) value also had

to be not more than a specified, low value. The first requirement makes it unlikely that the purchaser will receive poor-quality lots, and the second protects the manufacturer, making it unlikely that good-quality lots will be rejected.

The acceptance criterion, continuing with the solar-cell example, would be expressed as

$$\bar{x} - k\hat{s} > x^* \quad (21)$$

where  $x^*$  is the lower limit of acceptable performance of the cells (e.g., 23 milliwatts), and  $\bar{x}$  and  $s$  are the mean and standard deviation of the sample. At the core of this kind of inspection is the parameter  $k$ . The proper value of the  $k$  factor depends upon the specified defective proportions and the acceptance and rejection probabilities previously mentioned. It is possible to find the required value of  $k$  and also the required number of items in the sample to provide the specified probabilities at the specified values of defective proportion.

If we could know that our estimates of mean and standard deviation coincided exactly with the true values of mean and standard deviation, the value of  $k$  in Eq. (21) would simply be  $K_\alpha$ , the normal deviate exceeded with probability  $\alpha$ , where  $\alpha$  would be the magnitude of defective proportion below which the lot is acceptable. Then too, there would be no risk of either accepting lots that were defective or rejecting acceptable lots.

We know, of course, that the estimates of mean and standard deviation are only estimates, and that therefore a value of  $k$  greater than  $K_\alpha$  must be used. Further, there is some chance that we shall occasionally guess incorrectly whether a lot is acceptable or not.

Calculation of the proper value of  $k$  and number of test items is a complicated procedure in which the non-central  $t$  distribution is used. (11)

Extension of these techniques to fatigue problems can be seen by analogy. The aims are different, but the underlying theory is the same. Generally, we have available the test results from a (usually

small) number of constant-amplitude fatigue tests. The test results are the number of cycles to failure of the individual test specimens. The problem is how to be able to say something about low failure probabilities at numbers of cycles considerably lower than would be likely to be observed in our relatively small number of test specimens. The previous discussion of confidence regions showed one way of doing this. The method under discussion (which is also sometimes referred to as that of statistical-tolerance limits) is another way, one which proves to be more efficient. We mean by this that if both methods are used (with the same test data) to determine the number of cycles below which the failure probability is, for example, 0.001 with 95 per cent confidence, the present method yields a higher number of cycles than does the confidence-region method. This is obviously desirable, since it permits, with equal confidence, a lighter-weight design for the same failure probability.

If we assume that the underlying probability distribution in fatigue is a normal distribution of the log of cycles to failure, we can treat  $x^*$  in Eq. (21) as the log of cycles to failure below which the failure probability is a specified value (corresponding to the defective proportion in acceptance inspection) with a specified confidence level (corresponding to one minus the probability of accepting lots with a greater defective proportion than the value specified in acceptance inspection).

In other words, we can take test results, find  $\bar{x}$  and  $\hat{s}$  from these results, and then find a value of  $k$  from available tables<sup>(12,13)</sup> or, for parameters not tabulated, by a procedure<sup>(11)</sup> utilizing the non-central  $t$  distribution. (The value of  $k$  will depend on (1) the specified failure probability, (2) the specified confidence level, and (3) the number of test specimens for which data are available. The tables<sup>(12,13)</sup> directly provide values of  $k$ , denoted as "tolerance factors for normal distribution.")

To illustrate, the test results shown in Fig. 3 will be used again. For these thirteen tests, the values of  $\bar{x}$  and  $\hat{s}$  are 5.0498 and 0.33506, as given in the earlier discussion of confidence regions.

From the above-mentioned tables, for a 95 per cent confidence level, failure probability of 0.001, and results from thirteen tests, the value of  $k$  is found to be 4.787. We can thus find

$$\begin{aligned}x_{tol}^* &= 5.0498 - 4.787(0.33506) \\x_{tol}^* &= 3.4459\end{aligned}\tag{22}$$

The quantity  $x^*$  is the value of log cycles to failure analogous to the values previously obtained by the confidence-region method and the straightforward use of estimated parameters. The subscript refers to the determination of the value by tolerance tables. The value of cycles to failure,  $N$ , corresponding to Eq. (22) is antilog 3.4459 or about 2790 cycles--about 2-1/2 times greater than the value obtained from the confidence-region approach for the same failure probability and confidence level. This illustrates the improved efficiency of the present method. It also indicates the inadequacy of the straightforward use of estimated parameters, by which the present data yielded a value of 10,340 cycles, for a failure probability of 0.001. The highest number of cycles to failure for the same probability but with a confidence of 95 per cent is thus only about 27 per cent of this value.

To review the meaning of confidence: In this example, if the thirteen test specimens can be considered to have been selected at random from a population characterized by a log-normal distribution, we have a 95 per cent chance that the results of these tests when used in the above computations will lead to a value of  $N$  such that a fraction 0.001 or less of the whole population will fail before this number of cycles has been applied. A final illustration of confidence-level effects appears in Fig. 11. Shown there are the test results of Fig. 3, with the straight line determined by estimates of the mean and standard deviation plotted through the test points. Also shown is the straight line determined by the confidence-region approach of the previous section for a 95 per cent confidence level.

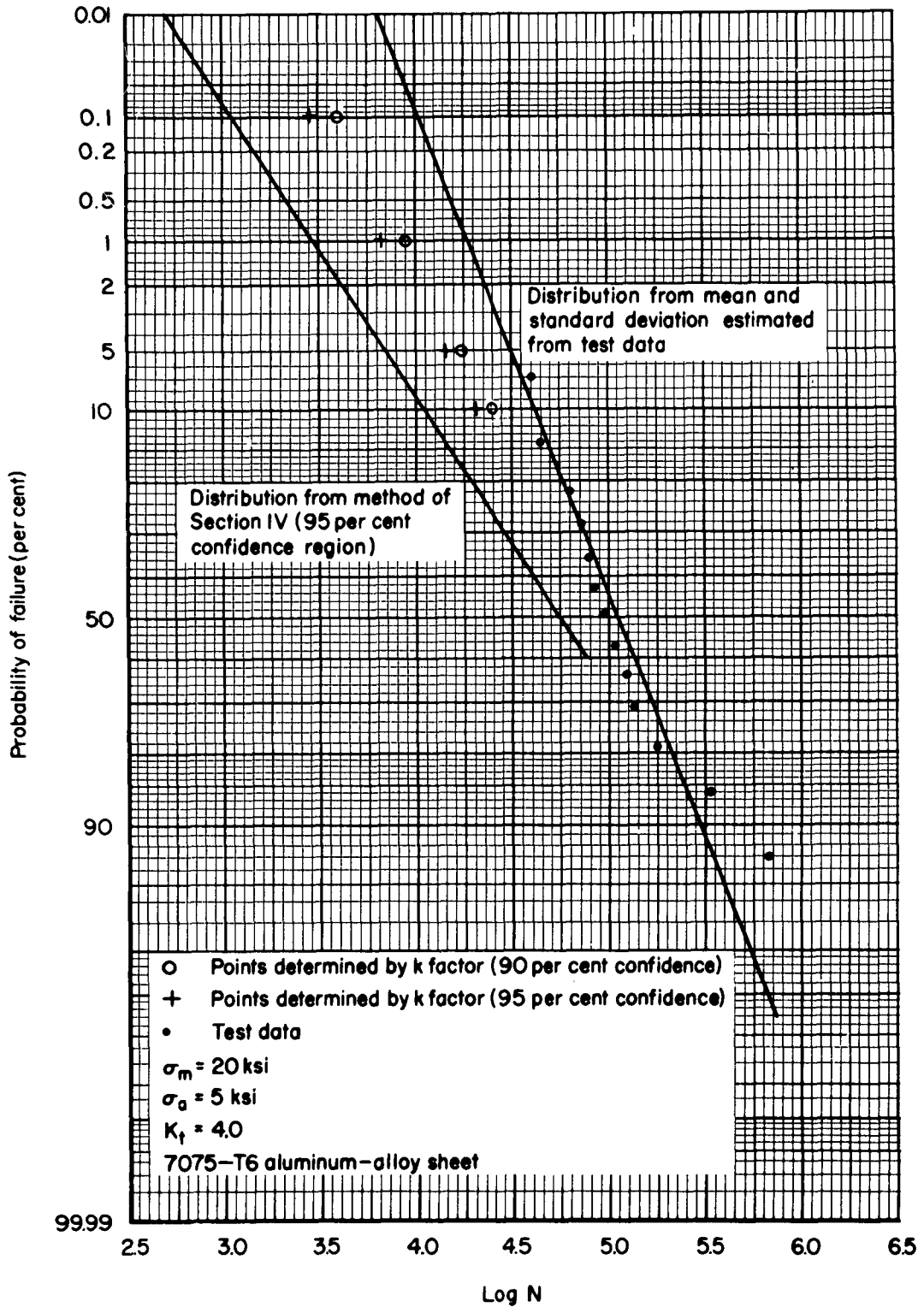


Fig. 11—Confidence-level effects on failure probability (confidence-region, k-factor, and estimated-parameter methods)

Finally, some points determined by the method of the present section are also shown. Points are plotted for several failure probabilities at two confidence levels, 90 and 95 per cent. Figure 11 shows graphically some of the conclusions drawn in the preceding discussion; namely, the possible danger of straightforward use of estimates of mean and standard deviation for probability estimates and the improvement possible by application of acceptance-inspection theory compared with that of the confidence-region method developed previously.

It is interesting to note that the points plotted in Fig. 11 for the 90 and 95 per cent confidence levels fall nearly on straight lines. Therefore these points can be closely approximated by a normal distribution with a fictitious mean (lower than the sample mean) and a fictitious standard deviation (larger than that estimated from the sample).

VIII. PROBABILITY AND SAFETY FACTORS  
(CONSTANT AMPLITUDE)

For constant-amplitude loading, it is natural and usually acceptable to associate fatigue failure with fracture, the physical breaking of the test specimen or structure. In the infrequent case of design requirements for a cyclic load with constant specified amplitude to be repeated for a specified number of cycles, S - N curves for any desired failure probability can be obtained from test data by the methods of the previous section. These curves can then be used in design by simply finding the stress amplitude corresponding to the specified number of cycles and the specified failure probability and then using this stress and the magnitude of the load amplitude to determine the required cross-sectional area. (If the load does not cycle between tension and compression loads of equal magnitude, effects of mean stress may complicate the design procedure slightly unless S - N curves for ratios of maximum to minimum stress identical with the ratio of maximum to minimum design load are available.)

Even in this simple case, it is interesting and instructive to consider the problem of factors of safety. It would generally be unlikely that the exact magnitude and number of applications of the cyclic load would be known beforehand. Consequently, the design procedure should incorporate the possibility of higher loads and more cycles than expected. (More accurately, the concern is with higher stress amplitudes than expected, which might result from manufacturing variations in cross-sectional area, fit of parts, etc., as well as from higher loads.)

To illustrate the meaning of factors of safety, the low-failure-probability curve shown in Fig. 12 will be used. This curve was obtained from the same data used for Fig. 5. This particular S - N curve represents a 0.001 failure probability, with 95 per cent confidence as determined by the use of tolerance factors as demonstrated in the previous section using the standard-deviation curve of Fig. 6 and assuming twenty tests at each test stress amplitude. (It should be noted that Fig. 12 does not have log scales.)

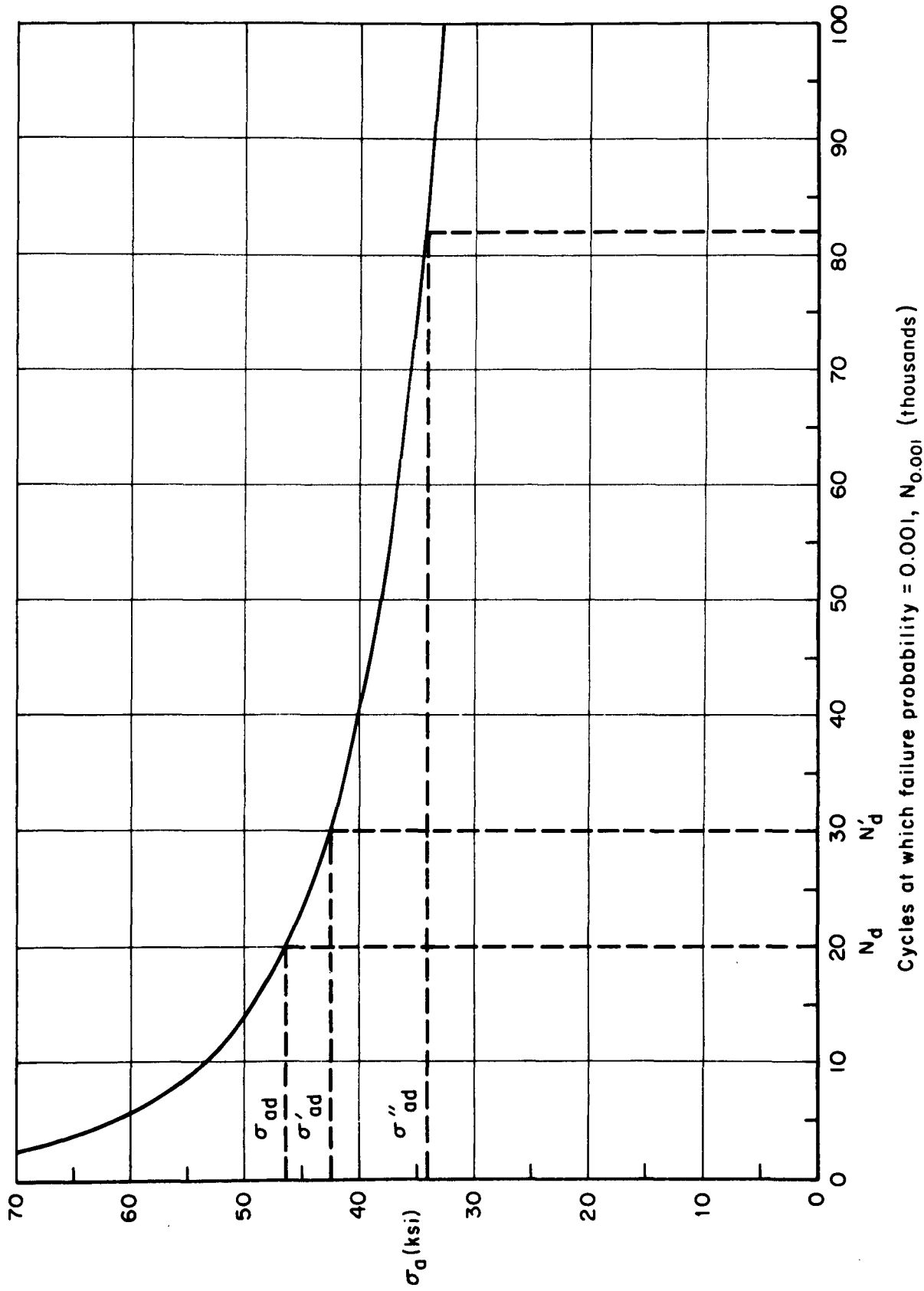


Fig. 12 — Incorporation of factors of safety on load and on life  
in antifatigue design

Figure 12 shows an example in which the desired design life,  $N_d$ , is 20,000 cycles. The stress amplitude at which the failure probability is 0.001 with 95 per cent confidence at the desired number of cycles is thus 46,600 psi. This is called  $\sigma_{ad}$ , the design stress amplitude. Together with  $M_d$ , the magnitude of the cyclic bending moment,  $\sigma_{ad}$  determines the required cross-sectional area.

Now consider the application of factors of safety. If, for example, a factor of safety of 1.5 is applied to the required number of cycles, the design life becomes

$$\begin{aligned} N_d' &= F.S._N \times N_d \\ &= 1.5 \times 20,000 \\ N_d' &= 30,000 \text{ cycles} \end{aligned}$$

As indicated in Fig. 12, this determines a new design stress,  $\sigma_{ad}' = 42,600$  psi and, consequently, a different, larger cross section. This cross section would provide the required factor of safety on life for failure probability of 0.001 with 95 per cent confidence if the magnitude of the load were known.

To apply a factor of safety to the load of, for example, 1.25, the design bending moment becomes

$$\begin{aligned} M_d' &= F.S._M \times M_d \\ &= 1.25 M_d \end{aligned}$$

This requires a still larger cross section, so that if  $M_d'$  is applied, the stress will still be  $\sigma_{ad}'$ . From the relationship between bending moment and stress, a new design stress is then determined:

$$\sigma_{ad}'' = \frac{\sigma_{ad}'}{F.S._M} = \frac{42,600}{1.25} = 34,100 \text{ psi}$$

The design stress determined in the last step and the design load,  $M_d$ , determine the cross section required, so that the combination of experiencing a cyclic load 1.25 times the expected load,  $M_d$ , and a number of cycles 1.5 times the expected number,  $N_d$ , will result in a failure probability of 0.001 with 95 per cent confidence. (The preceding step-by-step procedure brings out relationships between stress and life that would be obscured by a more direct procedure. Multiplying both life and load by the specified factors of safety gives  $N'_d$  and  $M'_d$ . Using the stress determined by  $N'_d$  (from Fig. 12) together with  $M'_d$  determines the same cross section as above.)

It is clear from Fig. 12 that the factors of safety on load and life are interrelated. For the final design with cross section determined by  $M_d$  and  $\sigma''_{ad}$ , other factors of safety than those used are equally applicable. Lower factors on life result in higher factors on load, and vice versa. This relationship between the two factors of safety is shown for the above example in Fig. 13. While by no means as simple, some manner of similar relationship also applies to cumulative damage from spectrum loading. In particular, the trend evident in Fig. 13 for much greater sensitivity to stress than life is generally characteristic of fatigue.

Briefly examining the above concepts in more detail, still for the simple case of constant-amplitude loading, makes the meaning of these factors of safety clearer. They really represent someone's judgment that the load and number of cycles obtained by multiplying the reasonably expected values by factors of safety will not ever be exceeded, or if pressed further, that it is extremely unlikely that they will be exceeded. In most real situations there is actually a probability distribution (probably unknown) for the magnitude of the cyclic load and number of applications of that load. (There can be situations in which the number of cycles expected is known exactly, by counting cycles and replacing the item after a specified number.)

For a specified cross section, the probability distribution for load can provide a probability distribution for stress amplitude. Let this be  $p(\sigma_a)$ , and let the probability density-distribution function for the required number of applied cycles be  $p(N)$ . There can

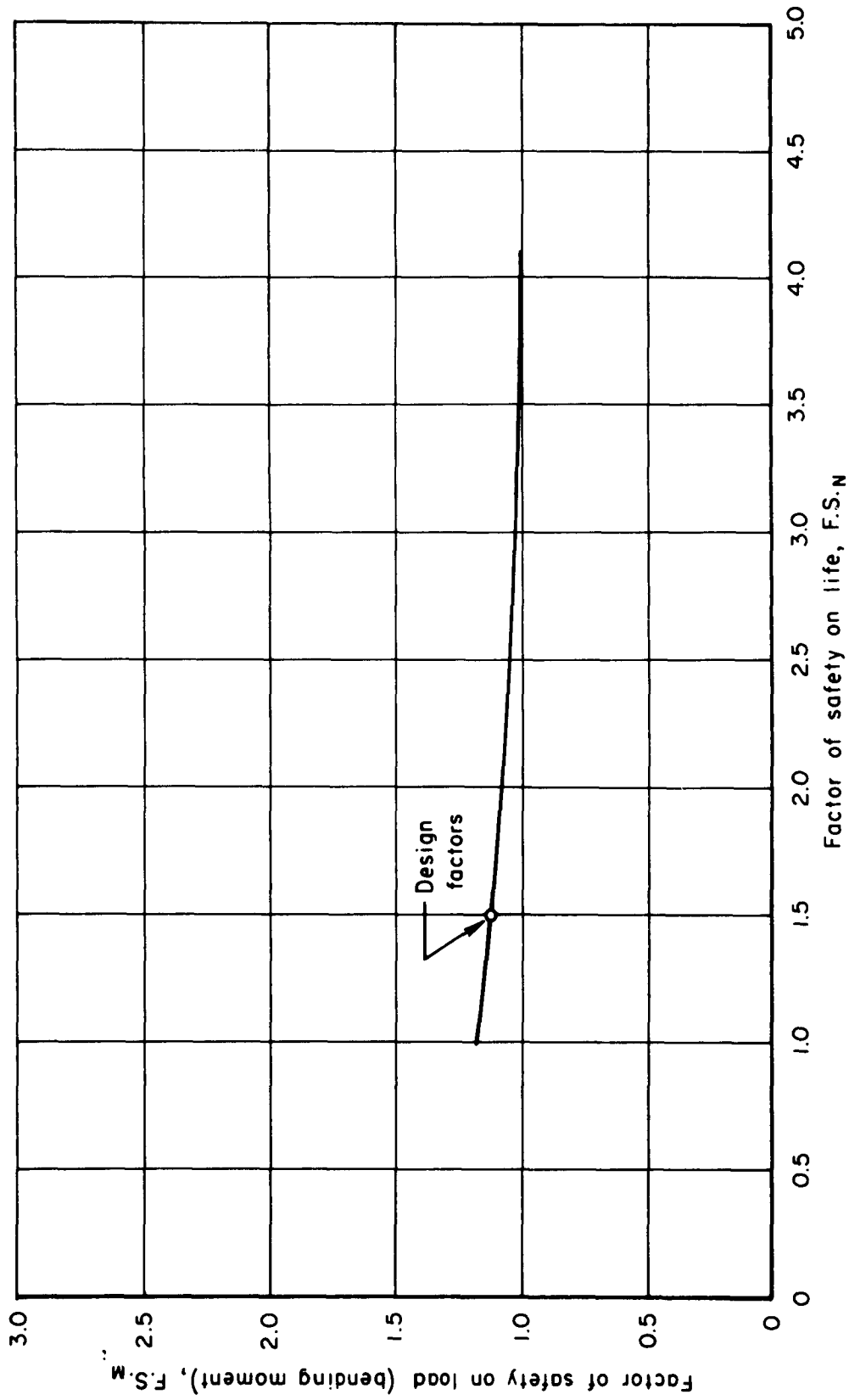


Fig. 13 — Relationship between factors of safety on load and on life

be found (with difficulty) an expression for the cumulative probability that at a stress  $\sigma_a$ , failure will occur by the time  $N$  cycles have been applied. Let this be called  $P(\sigma_a, N)$ . Then it is theoretically possible to find the failure probability for the design with the previously specified cross section. If the distributions are independent, this is

$$P(\text{failure}) = \int_0^{\infty} \int_0^{\infty} p(\sigma_a) p(N) P(\sigma_a, N) d\sigma_a dN$$

Repeating this process for a range of sizes of cross section would finally lead to a relationship between failure probability and design stress. Thus, even in this simplest kind of fatigue problem, it is possible to sink very quickly into a morass of probability problems. Since probability distributions depend on guesses, judgment must play an extremely important part in developing criteria for designing structures that provide assurance of freedom from fatigue problems.

### IX. THE PROBLEM OF PROBABILITY IN SPECTRUM LOADING

The preceding discussion of methods for obtaining low-failure-probability S - N curves from limited test data is directly applicable only to design for single-level, constant-amplitude cyclic stress. There are some important concepts that must be examined before applying these methods to study of cumulative damage under spectrum loading. When spectrum loading is considered, additional problems arise. As noted earlier, one problem is the concept of damage at failure. One aspect of this problem has already been discussed,<sup>(14)</sup> and it was shown that it is implicit in Miner's theory that the S - N curve used for cumulative-damage computations be determined for equal damage at all stress levels in the spectrum. If, as is conventional, the S - N curve is determined by counting cycles until the specimen breaks, the assumption is implicit that the rate of increase of damage with cycles as the breaking point is approached is so large that little error is introduced by counting cycles to breaking at all stress levels. If this assumption is not valid, an S - N curve, different from the conventional curve, which does represent cycles to equal damage is required for cumulative-damage computations. S - N curves for cycles to first visible crack are sometimes used to satisfy this requirement.

Another aspect of interest is the possibility of establishing S - N curves for use in cumulative-damage calculations on a different conceptual basis. In a discussion of the place of damage concepts in cumulative-damage theories,<sup>(14)</sup> it was pointed out that in many cases much of the fatigue life is characterized by very little damage in the conventional sense of crack length or crack area or reduction in strength. Physically, this part of fatigue life is probably associated with a more nebulous damage concept--intrusion and extrusion phenomena, progressive unbonding, appearance of fissures, etc., primarily at the intragranular level. Cumulative damage at this level is far from understood as far as any direct observation of stress dependence or interaction effects is concerned. On the other hand, it is fairly clear that in later stages, characterized by large observable

cracks, some kind of interaction between cycles of different amplitude is important. At least in the one-step type of tests, marked effects on crack propagation have been noted.<sup>(15)</sup> When the step in stress amplitude is from a higher to a lower stress, progress of damage as measured by crack length is halted for a considerable period, while the tests with an upward step in stress level show little effect on the progress of damage. (It must be remembered that one-step tests explore one extremity of cumulative damage and that results from this kind of test are not directly applicable to random spectrum loading.)

The important consequence of these facts is the possible advantage of changing the damage level associated with S - N curves to an earlier stage, perhaps characterized by a transition from the primarily intragranular effects noted to the period during which intergranular effects are predominant. The potential advantages of this concept include the possibility that cumulative-damage theories based on relatively simple hypotheses may prove adequate for damage in this region; the obvious desirability of basing designs on fatigue data to assure that cracks large enough for propagation rates to be of concern will not appear during operational lifetime; and, finally, the possibility that scatter in the number of cycles in spectrum loading producing this damage level may be less than the scatter observed based on conventional damage concepts.

These attractive possibilities must of course be balanced against any undesirable consequences, which might be a heavier structure than would be required were another approach for antifatigue design followed. A rough measure of this effect is possible by considering the number of cycles at which this transition may occur relative to the number of cycles to conventional failure. For some materials, depending on stress level and configuration, this transition occurs at somewhere around half of the fatigue life. Depending on the shape of the S - N curve, this factor of 1/2 on cycles translates into roughly a 10 to 15 per cent difference in stress. In terms of a complete structure, adding material at a limited number of locations to produce a change in stress of this magnitude would generally result in a minor increase in structural weight. Thus, further study to determine whether the

possible advantages of this change in damage concept do in fact exist appears worthwhile.

Finally, a somewhat philosophical aspect of low-failure-probability S - N curves and cumulative damage needs to be examined. The reason for discussing it here is not that the problem will be treated differently as a consequence, but merely to make explicit what is usually implicit.

Considering an S - N curve for failure probability of 0.001, for example, a reasonable procedure is to use this curve for cumulative-damage calculations and to assume that spectrum tests of a great number of specimens of structures would result in failure of about one in a thousand at or before a total number of cycles calculated from an appropriate cumulative-damage theory using that S - N curve. The implicit assumption is that there is one chance in a thousand that a specimen under spectrum loading will behave in accordance with this S - N curve. In fact, however, there is no way to determine an S - N curve for a single specimen, as it is obviously impossible to make more than one test to failure of a specimen.

Low-probability (shorter-than-average life) S - N curves must be built up from tests of different specimens. It is assumed that a specimen that fails at a relatively low number of cycles at one stress amplitude would also have demonstrated the same behavior at other stress amplitudes, and, more important, that this behavior can be projected to behavior under spectrum loading. This seems intuitively reasonable, but that is all that can be said, as there is no means of verification.

X. PROBABILITY IN CUMULATIVE DAMAGE, CONSTANT SCATTER  
FACTORS, AND "COMPOSITE" STANDARD DEVIATION

It is clear that use of a single S - N curve in cumulative-damage computations must be qualified by probability statements as well as by the assumptions about interaction postulated in the cumulative-damage theory under consideration. The existence of probability S - N curves also means that there must be a probability distribution for the total number of cycles to failure predicted by a cumulative-damage theory for spectrum loading. This fact is, of course, observed experimentally, and the same problem--estimating low-probability events from a limited number of tests--arises.

Although it is known that scatter in fatigue data as measured by the standard deviation of log number of cycles to failure at constant stress amplitude varies with stress amplitude (see, e.g., Fig. 6), the effect of scatter is frequently accounted for by applying a constant "scatter factor" to an average or median S - N curve in order to obtain an S - N curve for use in cumulative-damage computations. All values of N (some average number of cycles to failure) on the S - N curve are divided by the scatter factor, or, equivalently, all values of  $n_i$  (the expected number of cycles to be applied at each stress amplitude during service lifetime) are multiplied by the scatter factor.

The assumptions underlying this approach must be either that scatter is independent of stress level or that under spectrum loading some kind of interaction effectively makes scatter independent of stress level; otherwise, this approach just represents a good approximate method for assuring that failure probabilities are satisfactorily low.

We might assume that some kind of an averaging process operates to justify this approach, although caution is certainly called for. A rather large amount of averaged data for exploring this assumption is available. For example, estimated values of standard deviation of log cycles to failure have been computed from a large number of spectrum tests.<sup>(16)</sup> A histogram is shown for the frequency of

occurrence of values of unbiased estimates of standard deviation in units of log cycles (by increments of 0.05) as obtained from 357 test series with 2 to 5 specimens, 60 test series with 6 to 15 specimens, and 43 test series with 16 or more specimens. Specimens included joints, notched specimens, and unnotched specimens, of both aluminum and steel. In combining all these data the relationship of scatter to stress level, geometry, and material is, of course, obscured.

The theoretical probability distribution for values of standard deviation estimated from test results can be compared with the histogram mentioned above.

The quantity

$$u = \sum_1^n \left( \frac{x_i - \bar{x}}{s_o} \right)^2 \quad (23)$$

has the  $\chi^2$  distribution with  $n - 1$  degrees of freedom. In the present case,  $s_o$  is the true, unknown, standard deviation of the normal population from which the test items are drawn. While  $s_o$  quite clearly varies with stress level, geometry, etc., the purpose of the following is to see whether the assumption of a constant  $s_o$  may be a useful simplification. The quantities  $x_i$  and  $\bar{x}$  are the log cycles to failure of each test item and the average value of log cycles to failure, determined from the results of  $n$  (presumably identical) items tested at a specified constant stress amplitude.

Since the unbiased estimator of standard deviation is

$$\hat{s}^2 = \frac{\sum_1^n (x_i - \bar{x})^2}{n - 1} \quad (24)$$

we can write Eq. (23) as

$$u = (n - 1) \frac{\hat{s}^2}{s_o^2} \quad (25)$$

The probability-density-distribution function for  $u$  ( $\chi^2$  distribution with  $n - 1$  degrees of freedom) is

$$p(u) = \frac{1}{\left(\frac{n-3}{2}\right)!} \frac{1}{2^{[(n-1)/2]}} u^{\left(\frac{n-3}{2}\right)} e^{-u/2} \quad (26)$$

For comparison with the histogram, the distribution of  $\hat{s}$  is needed, and the distribution of Eq. (26) can be transformed by the relationship

$$p(\hat{s}) = p\left(u(\hat{s})\right) \frac{du}{d\hat{s}} \quad (27)$$

The function  $u(\hat{s})$  is given in Eq. (25), and also, from this function

$$\frac{du}{d\hat{s}} = \frac{2(n-1)\hat{s}}{s_0^2} \quad (28)$$

Thus the desired distribution of  $\hat{s}$  is

$$p(\hat{s}) = \frac{1}{\left(\frac{n-3}{2}\right)!} \frac{1}{2^{[(n-3)/2]}} \frac{(n-1)}{s_0^{n-1}} \hat{s}^{n-2} e^{-\left(\frac{n-1}{2} \frac{\hat{s}^2}{s_0^2}\right)} \quad (29)$$

This distribution depends on  $n$ , the number of items in a test series, but the data combine many different values of  $n$ . It also depends on the value of  $s_0$ , which of course is not known and which in fact exists only as an idealization. Thus it was to investigate this idealization that the distribution of Eq. (29) was computed for several values of  $n$  and  $s_0$  for comparison with the histogram. The original histogram and some theoretical distributions are shown in Figs. 14, 15, 16, and 17 for several values of size of test series and composite

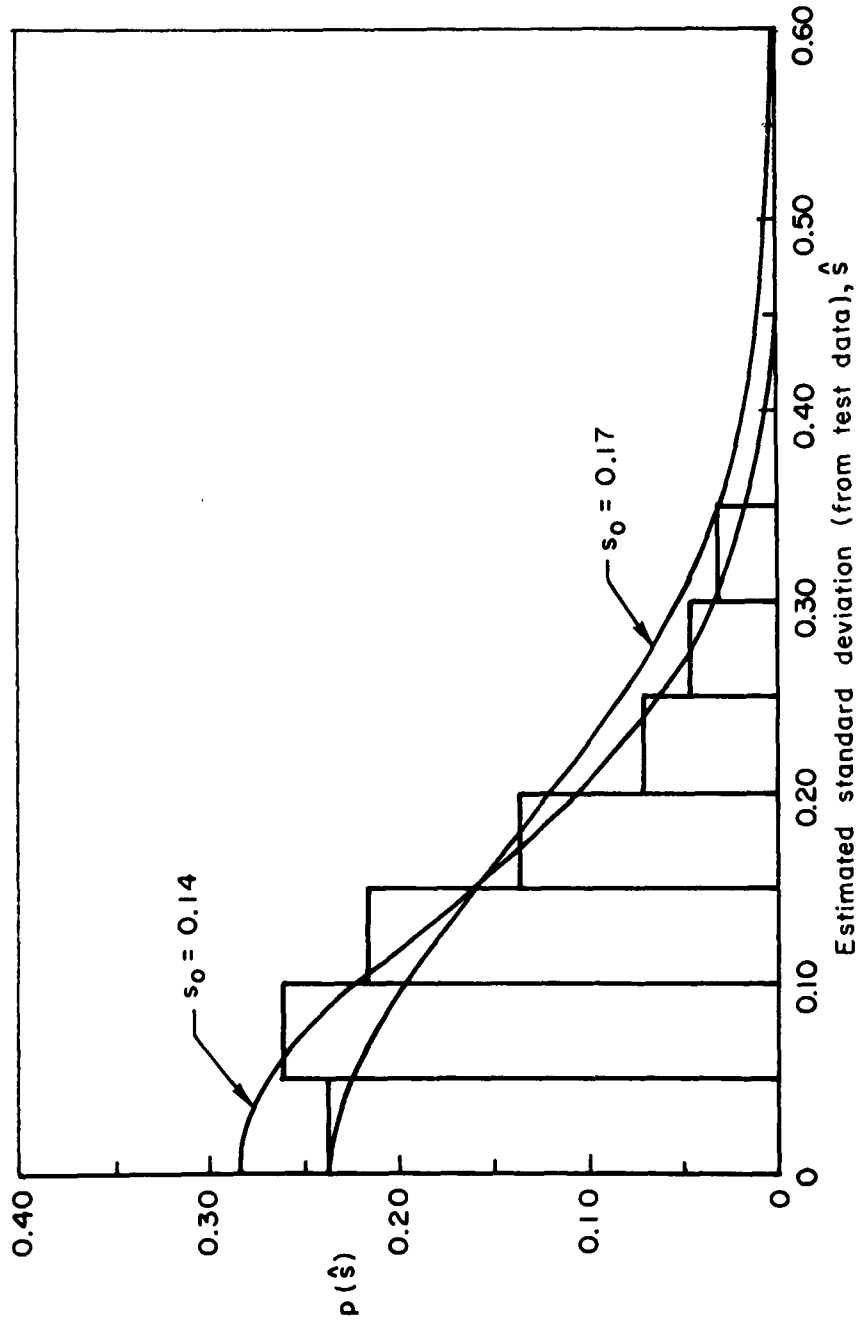


Fig. 14 — Investigation of composite-standard-deviation concept by comparison with distribution of standard deviation observed from tests (2 specimens per test series)

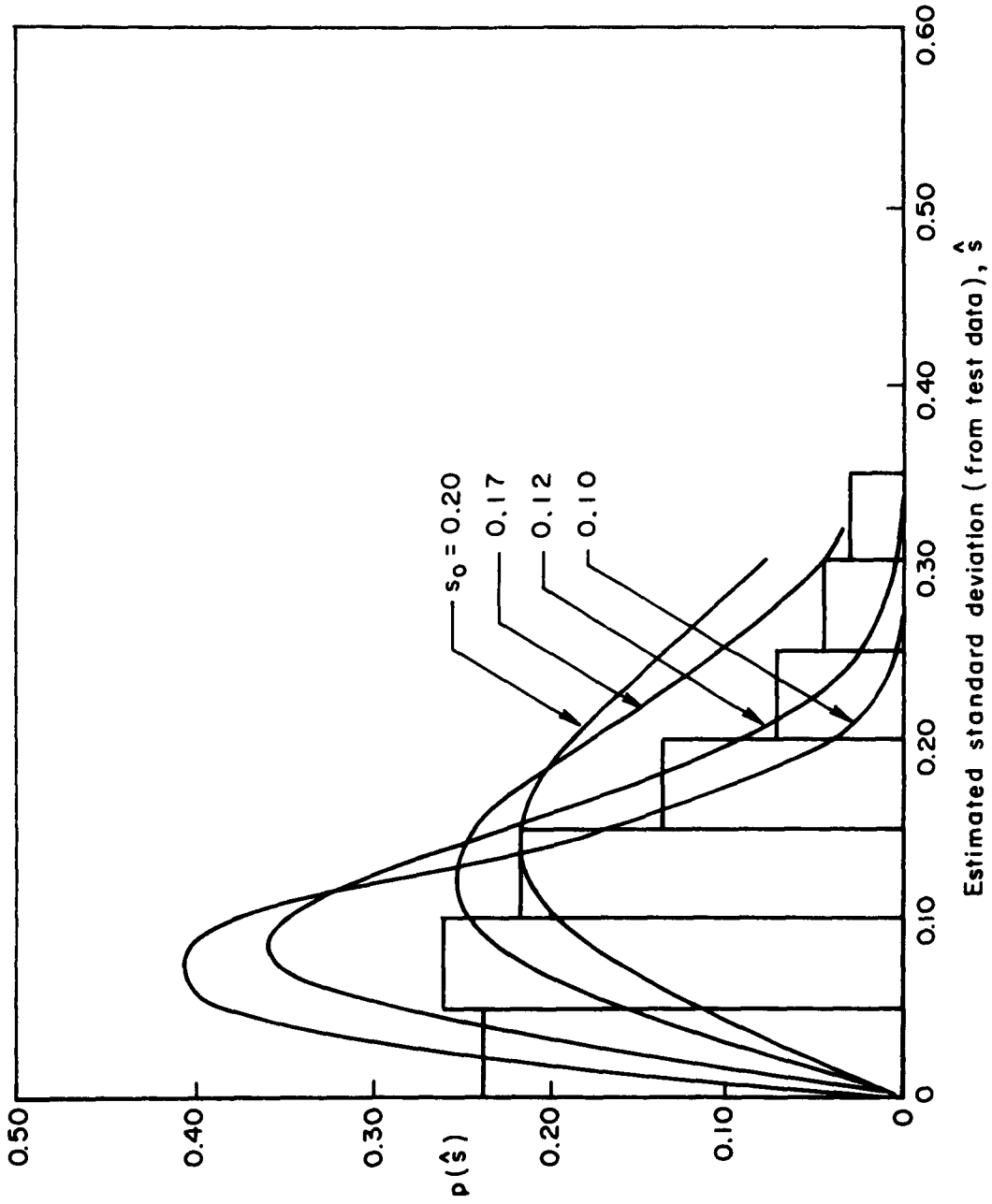


Fig. 15 — Investigation of composite-standard-deviation concept by comparison with distribution of standard deviation observed from tests (3 specimens per test series)

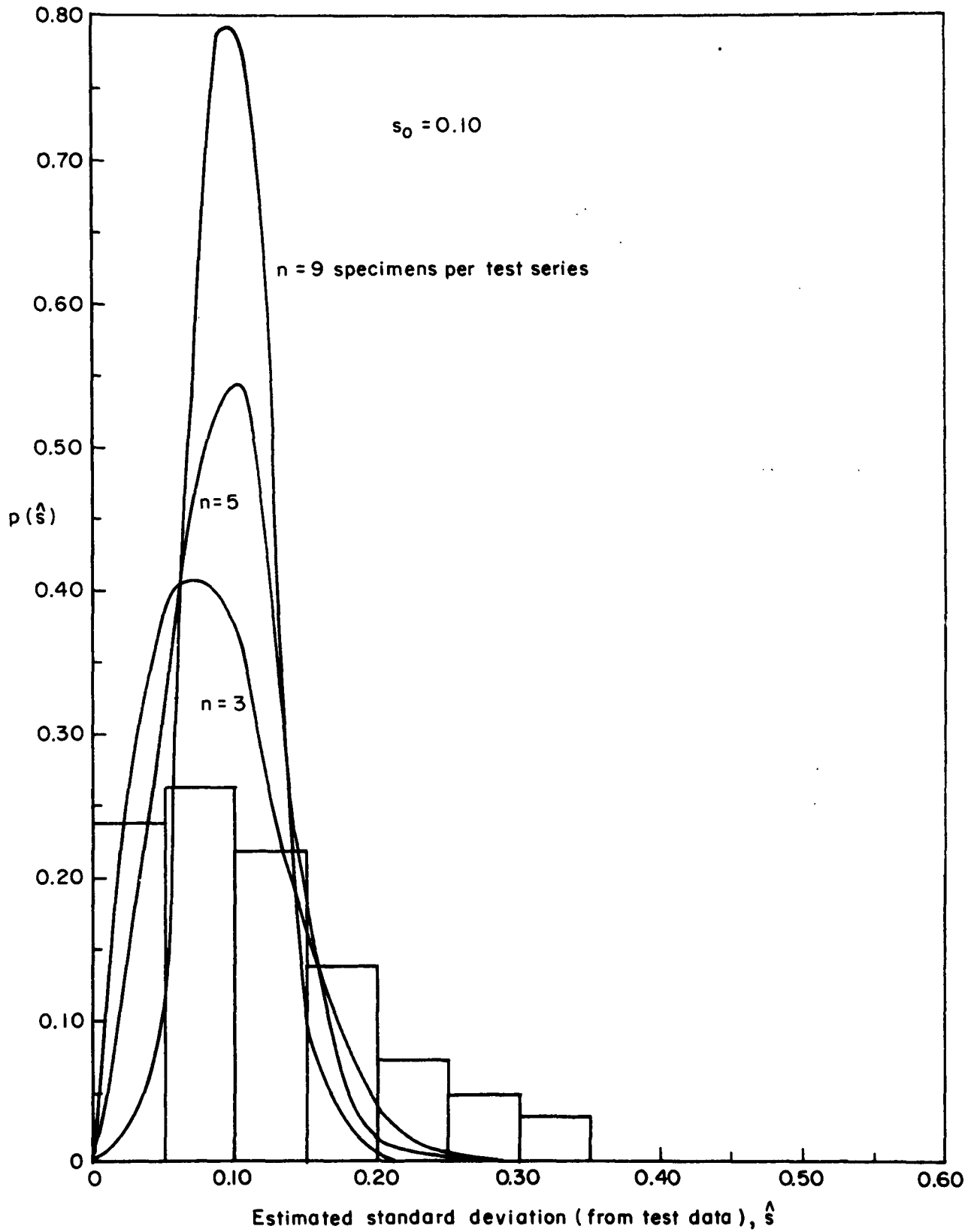


Fig. 16 — Investigation of composite-standard-deviation concept by comparison with distribution of standard deviation observed from tests (9, 5, and 3 specimens per test series)

standard deviation,  $s_o$ . (The distribution of Eq. (29) is multiplied by 0.05 to correspond with the histogram interval.)

Because the histogram is a compound of different kinds of tests, it should not be expected that a well-fitted theoretical distribution exists. Despite this fact, the distribution for  $n = 2$  or 3 and  $s_o = 0.17$  fits fairly well. Since tests with small numbers of test items predominate in the data used, this value of  $n$  appears reasonable. The conclusion follows that if better data are not available, a value of standard deviation of from 0.15 to 0.20 may be suitable for initial cumulative-fatigue-damage calculations.

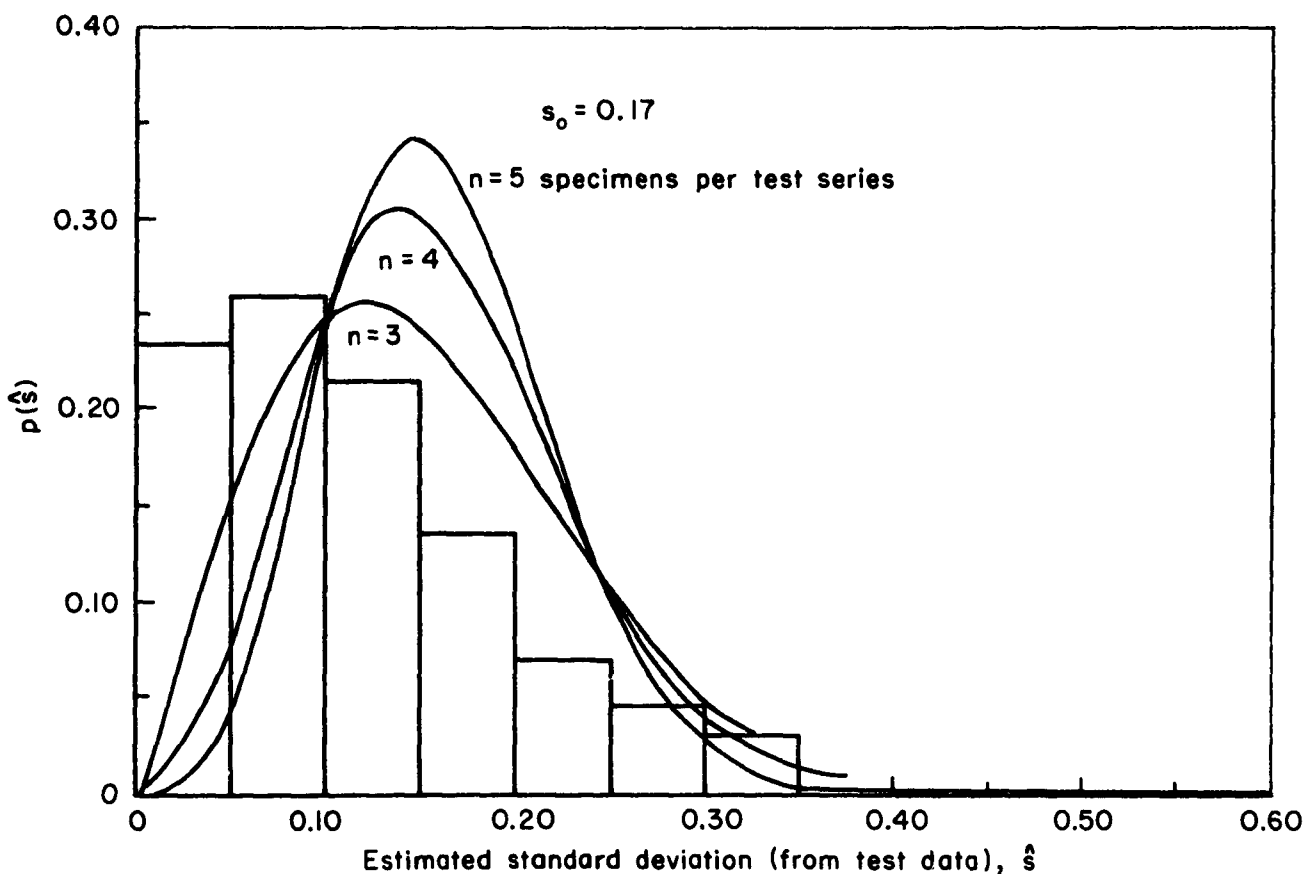


Fig. 17 — Investigation of composite-standard-deviation concept by comparison with distribution of standard deviation observed from tests (5, 4, and 3 specimens per test series)

XI. DISTRIBUTION OF  $\Sigma \frac{n}{N}$  WITH A "COMPOSITE"  
STANDARD DEVIATION

The real test of the value of the composite standard deviation determined in the previous section is its use in cumulative-damage calculations. Reference 16 also presents a histogram that shows experimental values of  $\Sigma \frac{n}{N}$  for 217 axial-load spectrum tests on steel and aluminum specimens at two or more stress levels.\* (It was noted previously that two-level tests, if of the one-step variety, are at one extreme of cumulative-damage investigations and generally of very limited usefulness for flight applications.) Again, effects of material, geometry, stress spectrum, and mean stress are obscured in the compilation, but in the same vein as the investigation of standard deviation, it is interesting to see whether average parameters are useful.

The use of a single value of standard deviation in calculating cumulative damage implies, as previously noted, that scatter can be considered to be independent of stress level. On a log-log S - N plot, curves for various failure probabilities would thus be obtained by translating the median curve in a direction parallel to the log N axis. This means that for a specified failure probability, the values of cycles to failure, N, are proportional to  $\bar{N}$  (the antilog of the mean of log cycles to failure). The same constant of proportionality then applies at all stress levels.

For a log-normal distribution of cycles to failure, the distribution is

$$p(\log N) = \frac{1}{\sqrt{2\pi} s_0} e^{-\frac{[\log N - \log \bar{N}]^2}{2s_0^2}} \quad (31a)$$

which can also be written as

$$p\left(\log \frac{N}{\bar{N}}\right) = \frac{1}{\sqrt{2\pi} s_0} e^{-\frac{[\log(\frac{N}{\bar{N}})]^2}{2s_0^2}} \quad (31b)$$

( $\bar{N}$  is the antilog of the mean value of log cycles to failure.)

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\*In this section, n is the number of stress cycles applied.

If a well-mixed spectrum is applied to a specimen that happens to be an average specimen, i.e., if its S - N curve is the median curve or the S -  $\bar{N}$  curve, the Miner cumulative-damage criterion for failure would be

$$\sum \frac{n}{N} = \sum \frac{n}{\bar{N}} = 1 \quad (32)$$

For another specimen, behaving according to a different S - N curve but with the same spectrum applied, the failure criterion would still be

$$\sum \frac{n}{N} = 1 \quad (33)$$

but the values of N are now different. This can be written as

$$\sum \frac{n}{N} = \sum \frac{n}{\bar{N}} \cdot \frac{\bar{N}}{N} = \sum \frac{\left(\frac{\bar{N}}{N}\right)^n}{\bar{N}} = 1 \quad (34)$$

Comparing Eq. (32) with Eq. (34) shows that the number of cycles, n, that can be applied at each stress amplitude before failure must be different in the latter case. At each stress amplitude the number of cycles that can be applied is equal to  $\frac{N}{\bar{N}}$  times the number of cycles that could be applied to the specimen characterized by the mean or S -  $\bar{N}$  curve. Thus the total applied cycles to failure will be  $\frac{N}{\bar{N}}$  times the number of cycles that produce failure in the average specimen. Therefore the ratio of applied cycles to failure for any specimen to the applied cycles to failure for the average specimen is

$$\frac{\sum n}{\sum \bar{n}} = \frac{N}{\bar{N}} \quad (35a)$$

Alternatively, from Eq. (34), the summation of cycle ratios for the nonaverage specimens (if  $\bar{N}$ , the average cycles to failure, is used in computing the cycle ratios) is

$$\sum \frac{n}{\bar{N}} = \frac{N}{\bar{N}} \quad (35b)$$

Thus, by finding the distribution of  $\frac{N}{\bar{N}}$  we will have the distribution of the ratio of cycles to failure for any specimen to cycles to failure for the average specimen as well as the distribution of experimental values of the Miner summation based on mean or S -  $\bar{N}$  curves.

If we let

$$u = \log \left( \frac{N}{\bar{N}} \right)$$

$$v = \frac{N}{\bar{N}}$$

the distribution of u is given by Eq. (31b)

$$p(u) = \frac{1}{\sqrt{2\pi} s_o} e^{-\frac{u^2}{2s_o^2}}$$

To obtain the distribution of v

$$p(v) = p(u(v)) \frac{du}{dv}$$

$$u = \log v$$

$$\frac{du}{dv} = \frac{\log e}{v}$$

$$p(v) = \frac{\log e}{\sqrt{2\pi} s_o v} e^{-\frac{(\log v)^2}{2s_o^2}}$$

Therefore

$$p\left(\frac{\sum n}{\bar{\sum n}}\right) = \frac{\log e}{\sqrt{2\pi} s_o \frac{\sum n}{\bar{\sum n}}} e^{-\frac{[\log \left(\frac{\sum n}{\bar{\sum n}}\right)]^2}{2s_o^2}} \quad (36)$$

or

$$p\left(\sum \frac{n}{\bar{N}}\right) = \frac{\log e}{\sqrt{2\pi} s_o \sum \frac{n}{\bar{N}}} e^{-\frac{[\log(\sum \frac{n}{\bar{N}})]^2}{2s_o^2}} \quad (37)$$

The histogram of Ref. 16 is shown in Fig. 18. Superimposed on it is the distribution of Eq. (37), with various values of  $s_o$ . (The distribution is adjusted to conform to the increments used in the histogram.)

Once again it appears that the use of a composite standard deviation leads to an adequate approximation of the experimental scatter of data, in this case the observed value of the cycle-ratio summation. The magnitude of the composite standard deviation for good fit is also about that found in the preceding section through evaluation of the distribution of estimates of standard deviation from experiments-- about 0.17. This value corresponds to a factor of about 2/3 per standard deviation for the reduction from  $\bar{N}$  to  $N$ . An S - N curve for failure probability of 0.001 (3.09 standard deviations below  $\bar{N}$ ) would be drawn through values of  $N$  found by

$$N = \frac{\bar{N}}{\text{antilog}[3.09(0.17)]} \quad (38)$$

$$N \approx 0.3 \bar{N}$$

This would also mean that a cycle-ratio summation of 0.3 or less at failure, based on a mean S -  $\bar{N}$  curve, would be found with a probability of 0.001, and thus the use of an S - N curve based on Eq. (38) for design purposes would lead to a structure with failure probability of 0.001. The scatter factor for this failure probability would thus be the reciprocal of 0.3, or about 3.3.

It must be emphasized that the above statements are subject to severe limitations and provide no more than a roughly justifiable procedure to be used when the fatigue data really needed are lacking. The lumping of data from many kinds of tests obscures the salient

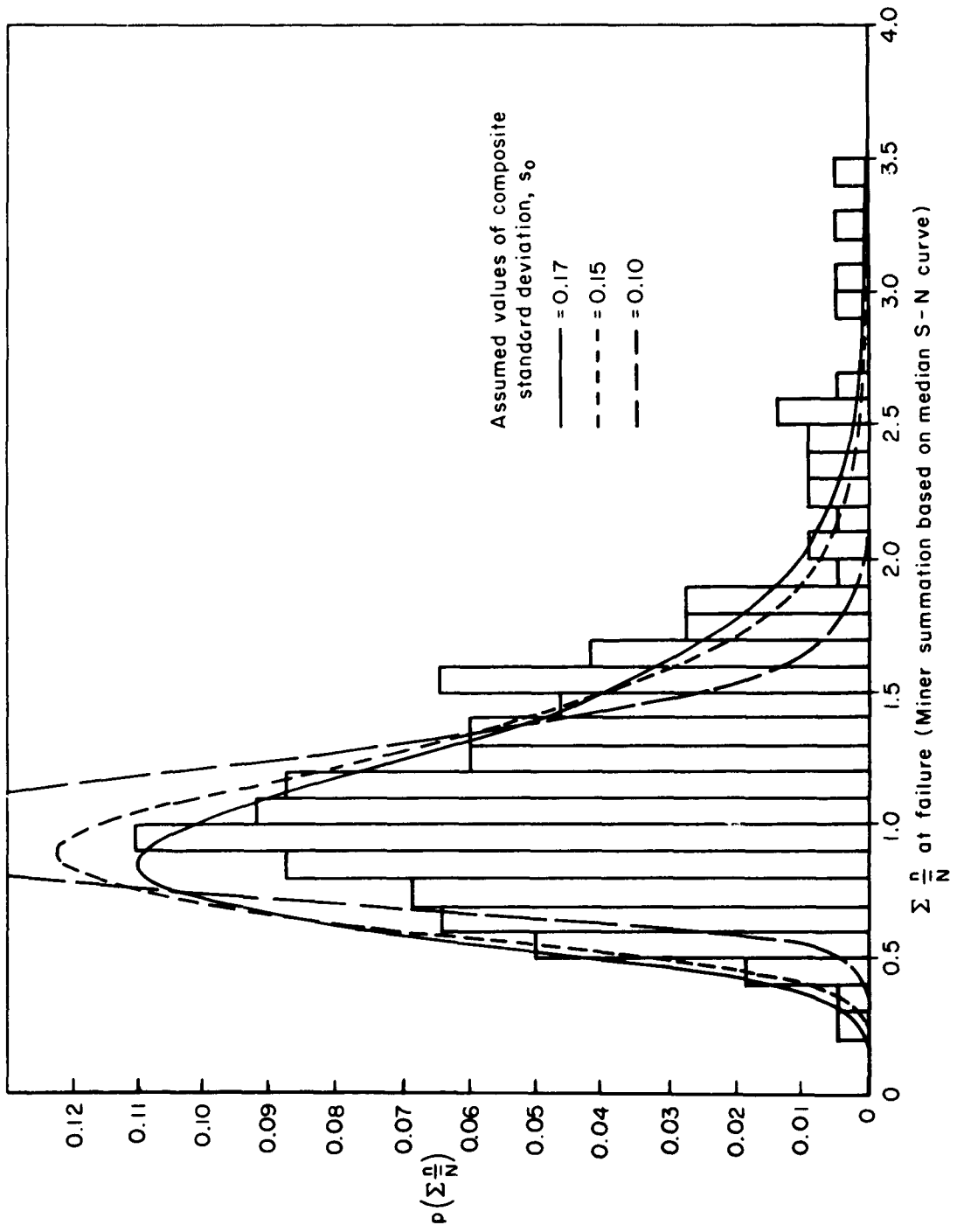


Fig. 18 — Comparison of distribution of life in spectrum tests with distribution based on composite-standard-deviation concept

features and character of specific test programs and may in fact contribute to the shape of the distribution of test results shown.

A warning example showing quite different results is obtained by making a histogram from the results given for 220 rotating bending-fatigue tests.<sup>(17)</sup> Here again, differences in test series at different stress amplitudes are lost when all tests are grouped. However, the resulting histogram, shown in Fig. 19, is significantly different from the one in Fig. 18. (The cycle-ratio summation in Fig. 19 is based on  $N_r$ , the number of cycles by which about 63 per cent of specimens have failed at constant amplitude, instead of the number of cycles by which 50 per cent fail, as was the case in the previous figure.) It should be noted that these tests were made on smooth specimens at zero mean stress, conditions which generally show the poorest agreement with Miner's theory.

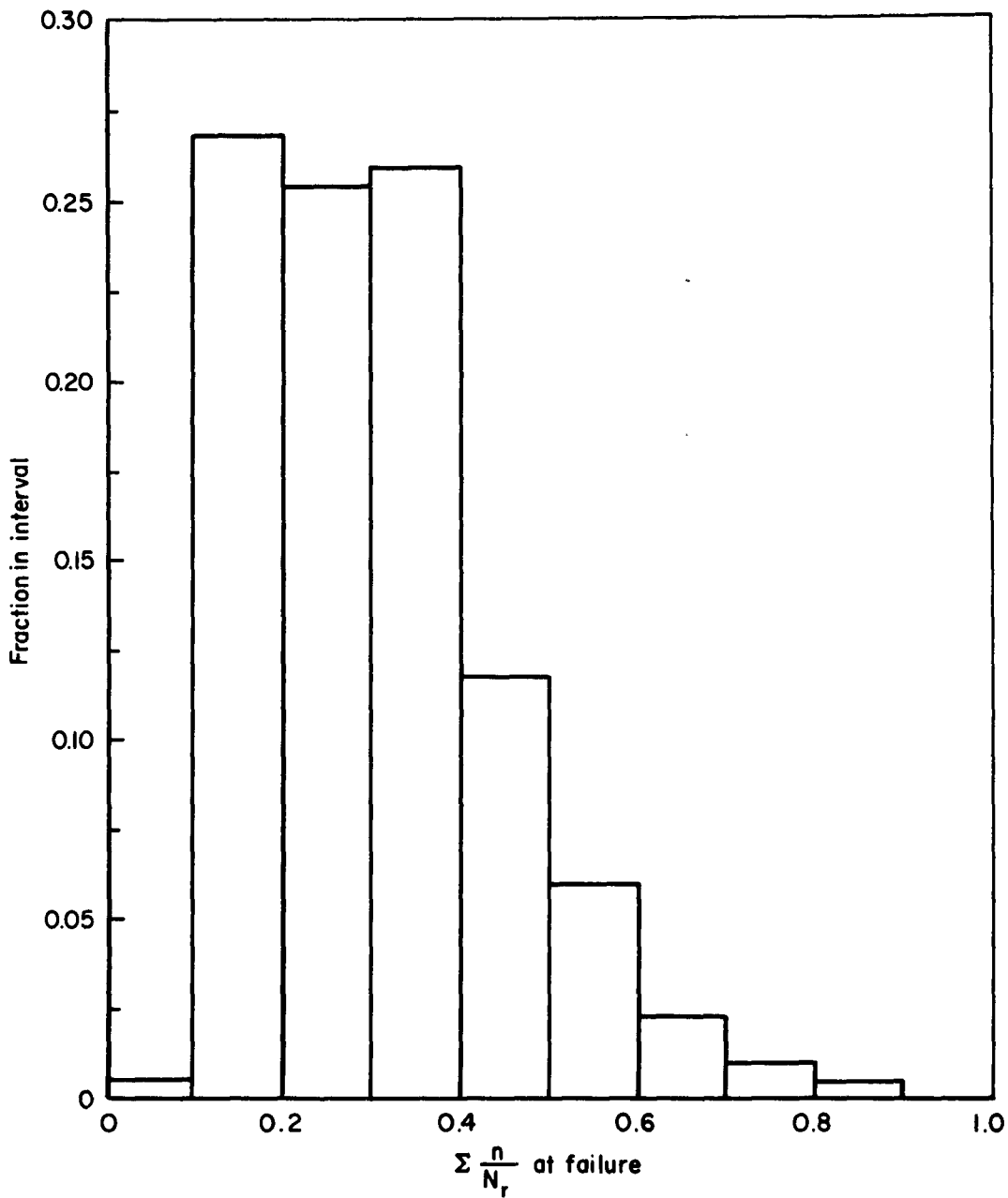


Fig. 19 — Histogram of observed life in spectrum tests  
(220 rotating-bending specimens)

XII. PROBABILITY IN CUMULATIVE DAMAGE WITH  
STRESS-DEPENDENT SCATTER

In the design of a structure to withstand repeated loads, the most desirable fatigue data would be the results of spectrum tests in which the test load spectrum duplicated the spectrum to be encountered in service and the test specimens duplicated the structure to be placed in service, and in which the number of tests was sufficient to establish failure-time distributions with high confidence over an adequate range of cross-section sizes.

In a real design situation, such availability of fatigue data is hardly conceivable. At present and in the foreseeable future, basic fatigue data in the form of constant-amplitude-test results appear necessary, although it should be possible through properly designed spectrum-test programs to obtain more directly relevant design data for fatigue.

In general, the basic fatigue data needed are test results from constant-amplitude tests covering a range of stress amplitude and mean stress, ideally with test specimens that closely simulate the structural areas where fatigue effects are to be investigated. The basic data can then be converted into S - N curves for design use (at desired failure-probability and confidence levels) through application of the k factor described previously. In particular cases, if the S - N curve for mean log cycles to failure can be fitted by

$$\bar{N} = \left( \frac{K}{\sigma_a} \right)^x \quad (39)$$

(a straight line on a log-log plot), and the estimate of standard deviation from tests by

$$\hat{s} = \frac{\log \bar{N}}{K_1} \quad (40)$$

design S - N curves for various probability and confidence levels can be expressed quite simply.

The values of standard deviation in Fig. 20 are from the results of 542 constant-amplitude tests of 7075-T6 aluminum specimens. (5) (These tests, zero-mean-stress rotating bending tests with smooth specimens, were also used in Fig. 6. Tests made on different machines are shown by the different symbols, and flagged symbols are from tests with uncoated specimens showing the effect on results of water vapor in the air. The rest were made with vaseline-coated specimens.) The simple approximation of Eq. (40) for standard deviation appears to be fairly good, and Eq. (39) is also a good approximation for these tests.

An approximation for standard deviation proportional to the mean log cycles to failure at any stress level may simplify cumulative-damage computations. Depending upon the number of tests made at any stress amplitude and the desired confidence level, a k factor can be found as discussed previously so that the value of N determined by

$$\log N = \log \bar{N} - k\hat{s} \quad (41)$$

is the number of cycles below which failure probability is any desired value. If the same number of tests is made at each stress amplitude for establishing the basic S - N curve (for a specified mean stress), k would be the same for all stress amplitudes for a specified confidence level and failure probability.

Combining Eqs. (39), (40), and (41)

$$\begin{aligned} \log N &= \log \bar{N} - \frac{k}{K_1} \log \bar{N} \\ &= \left(1 - \frac{k}{K_1}\right) \log \bar{N} \\ &= \left(1 - \frac{k}{K_1}\right) \times \log \left(\frac{K}{\sigma_a}\right) \end{aligned}$$



$$N = \left( \frac{K}{\sigma_a} \right) \left( 1 - \frac{k}{K_1} \right)^x \quad (42)$$

or

$$N = \left( \frac{K}{\sigma_a} \right)^{x'} \quad (43)$$

where

$$x' = \left( 1 - \frac{k}{K_1} \right) x$$

The value of  $x'$  depends on the number of tests at each stress amplitude, the confidence level selected, and the desired failure probability. Thus a low-failure-probability  $S - N$  curve is still a straight line on a log-log plot, but it has a different (steeper) slope than the  $S - N$  curve for mean  $\log N$ .

With low-failure-probability curves thus generated, cumulative-damage computations can be carried out. However, in this case it is not possible to obtain a general probability distribution for total cycles to failure such as shown in Fig. 18, because now the distribution depends specifically on the shape of the stress spectrum and the  $S - N$  relationship. Further, the distribution, due to the introduction of the  $k$  factor, is extremely complicated in any case.

In practice, the details of the probability distribution in itself are not necessarily of great interest because the important aspect is the integral of the distribution, or cumulative probability. This is included directly in the method for obtaining the low-probability  $S - N$  curves. Illustration of the use of these methods in the design context, where choice of appropriate stress spectra and cumulative-damage theories must also be considered, is reserved for a later study.

### XIII. CONCLUSIONS

While fatigue scatter raises serious problems for the aircraft designer, the outlook is not as bleak as sometimes portrayed. There are many useful analogies between designing structures to preclude static failure and designing structures to preclude major fatigue problems. Scatter in static-strength properties has always existed, but it has been met and essentially eliminated as a cause of structural failure. When fatigue scatter is included in design, the effects are not appreciably different from comparable static-design effects.

A significant beneficial factor is the sensitivity of fatigue life to stress level. In many structures, adequate design for static loads and proper attention to details of design that are important in fatigue result in a structure with practically zero probability of fatigue failure all through its lifetime. In other cases, statically adequate structures may be marginal or completely inadequate as far as fatigue is concerned. Here, the fact that slight additions of material produce large increments in fatigue life makes it possible to produce an antifatigue design with little weight penalty.

The design goal should be to insure that questions of and uncertainties in the exact form of probability distributions, in estimating failure probability, etc., do not become important during the design lifetime of an aircraft. A critical look at alternative probability distributions showed that, for the data used, design would be little affected by one choice or another. This example supports the conclusion that it is important to examine such questions in terms of design effects.

It is unlikely that the amount of fatigue testing to verify a design will ever approach the number of tests of static properties. Thus the confidence with which test results can be applied to a fleet of aircraft becomes critical in fatigue. The approach outlined in this Memorandum--use of the k factor from quality-control statistics--merits consideration as a standard method for assigning confidence levels to flight structures in which fatigue is a design consideration.

The use of scatter factors and, in general, the scatter associated with spectrum loading needs to be critically examined. The idea of a constant scatter factor, independent of the form of the applied-stress spectrum, was shown to be consistent with an analysis of spectrum-test data, but this agreement is not sufficient reason for concluding that the concept is generally valid. On the other hand, the straightforward application of cumulative-damage theories, using low-failure-probability S - N curves at confidence levels determined by the proposed procedure, can provide theoretical distributions of fatigue life under spectrum loading. However, theoretical values of the mean life and of the scatter (or standard deviation) may disagree with spectrum-test results. Physical processes that occur during cumulative damage sometimes alter both the shape and position of the distribution of cycles to failure under spectrum loading. Much more work is needed before these changes can be associated with the shape and order of application of the stress spectrum and with "basic" fatigue properties of a material and geometry. The approach described here can supply a reference distribution that can be used in preliminary fatigue designs and also in studying and identifying factors that change observed life under spectrum loading.

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