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THE SCATTERING OF ELECTROMAGNETIC WAVES FROM PLASMA CYLINDERS: PART I

Phyllis Greifinger

PREPARED FOR:
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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

This Memorandum is one result of a continuing investigation of the radar returns from missile wakes. The work is part of RAND's study of midcourse phenomenology, conducted under the Advanced Research Projects Agency's Defender program.

SUMMARY

This Memorandum reviews the theory of the scattering of electromagnetic radiation in the transverse magnetic mode by infinite plasma cylinders with radially varying electron-density distributions. Techniques for calculating the scattering cross section are described, with special emphasis placed on short wavelength scattering. Specific calculations of the back-scattering (radar) cross section are made in the geometrical-optics limit and in the Born-approximation limit for several monotonically decreasing electron-density distributions which have zero slope on the cylinder axis. The radar cross section in the region of transition from underdense to overdense plasma is programmed and computed on the IBM 7090 for a Gaussian and for a quadratic electron distribution for wavelengths equal to 1/10 and 1/100 of the cylinder radius.

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CONTENTS

SYMBOLS

a = geometrical-optics scattering radius; defined by $\alpha g(a/\rho_o) = 1$

b = impact parameter

e = electronic charge

E = electric field

$\mathcal{E}$ = complete elliptic integral of second kind

$f(\varphi)$ = scattering amplitude

$|f(\varphi)|^2$ = differential scattering cross section

$|f_B(\varphi)|^2$ = differential scattering cross section in Born approximation

$|f_G(\varphi)|^2$ = geometrical-optics differential scattering cross section

$g\left(\frac{\rho}{\rho_o}\right)$ = electron-density distribution; defined by

$$g\left(\frac{\rho}{\rho_o}\right) = \frac{\omega_p^2 - \omega_{p\infty}^2}{\omega_{po}^2 - \omega_{p\infty}^2}$$

$J_m(k\rho)$ = Bessel function; regular at origin

$$k = \frac{\sqrt{\omega^2 - \omega_{p\infty}^2}}{c}$$

$K_m(k\rho)$ = hyperbolic Bessel function; zero at infinity

$\mathcal{K}$ = complete elliptic integral of first kind

m = integer

m_e = electron mass

n = index of refraction

n_e = electron number density

P = integer

$u_m(\rho)$ = m^{th} partial wave function

V = potential

$w = 1/\rho$

W = energy

$x = \frac{\rho_c}{\rho}$

$y = \frac{\rho}{\rho_o}$

$\alpha = \frac{\omega_{po}^2 - \omega_{p\infty}^2}{\omega^2 - \omega_{p\infty}^2}$

Γ = gamma function

$\delta = \alpha - 1$ (α near one)

$\epsilon_m = 1$ $(m = 0)$

$\epsilon_m = 2$ $(m \neq 0)$

η_m = phase shift of m^{th} partial wave

ρ = perpendicular distance from cylinder axis

ρ_o = characteristic radius of scattering region

φ = scattering angle

ω = frequency of electromagnetic wave

ω_p = plasma frequency

ω_{po} = plasma frequency on cylinder axis $(\rho = 0)$

$\omega_{p\infty}$ = plasma frequency as $\rho \rightarrow \infty$

I. INTRODUCTION

The scattering of electromagnetic radiation by infinite plasma cylinders with radial variations in electron density has been of interest to those investigating meteor trails, and more recently, to those interested in the plasma trails generated by re-entering hypersonic vehicles.

Calculations which have been made in the past for the scattering from nonuniform cylinders have mostly made use of the Born approximation, which is generally valid for electron densities differing very little from the ambient density.

Recently, C. M. de Ridder and S. Edelberg[(1)] investigated the scattering of electromagnetic waves by nonuniform plasma cylinders for a range of wavelengths varying from values much larger than the cylinder radius to values one-quarter of the cylinder radius. The calculations were made for a few electron-density distributions in the underdense and overdense regions; all the distributions considered had the common characteristic of being identically zero outside the cylinder and having zero slope on the cylinder axis. In all cases, the incident wave was assumed to be a plane transverse-magnetic (T.M.) wave traveling perpendicular to the cylinder axis.

In this Memorandum, we present a review of the general theory for the scattering of electromagnetic radiation by radially varying electron distributions. As did de Ridder and Edelberg, we assume that the incident wave is a plane T.M. wave traveling perpendicular to the cylinder axis. Also, the electron-density distribution is considered monotonic but not necessarily confined to a finite-radius cylinder, and collisions are neglected. The theory is developed by analogy with the well-known theory for the scattering of particles or radiation by spherically symmetric scattering regions.

In particular, we will review the method of expanding the wave into partial waves and the familiar techniques for calculating the phase shifts of the partial waves. We will describe the very useful Born and geometrical-optics approximations to the scattering cross section and discuss the limits of their validity as descriptions of short-wavelength scattering. Finally, we will present specific calculations of the short-wavelength back-scattering cross section for several monotonically decreasing electron-density distributions which have zero slope on the cylinder axis.

II. FORMULATION OF PROBLEM

Let us consider a plane electromagnetic wave of angular frequency ω, polarized in the z direction, and traveling initially in the positive x direction, incident upon a region in which the electron density varies radially. Neglecting collisions and dropping the time-dependent factor $e^{-i\omega t}$, the wave equation in cylindrical coordinates, ρ, z, and φ, for $E = E_z$ is

$$\frac{\partial^2 E}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E}{\partial \varphi^2} + \left(\frac{\omega^2 - \omega_p^2}{c^2}\right) E = 0 \qquad (1)$$

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e}$$

where n_e and m_e are the electron-number density and mass, and e is the electronic charge.

In order to transform Eq. (1) into a form that is more familiar from scattering theory, let

$$k^2 = \frac{\omega^2 - \omega_{p\infty}^2}{c^2} \qquad (2)$$

$$\frac{\omega_p^2 - \omega_{p\infty}^2}{c^2} = k^2 \alpha\, g\left(\frac{\rho}{\rho_o}\right)$$

$$n^2 = \frac{c^2}{\omega^2} k^2 \left(1 - \alpha\, g\left(\frac{\rho}{\rho_o}\right)\right)$$

where $\omega_{p\infty}$ is the plasma frequency at $\rho \to \infty$, ρ_o is the characteristic radius of the scattering region, $g\left(\frac{\rho}{\rho_o}\right)$ is defined so that $g(0) = 1$, and n is the index of refraction of the medium. If the electron density goes to zero as $\rho \to \infty$, k is the ordinary wave number ($k = \omega/c$) and α is equal to ω_p^2/ω^2 at $\rho = 0$. We will consider only monotonic functions $g\left(\frac{\rho}{\rho_o}\right)$. For electron-density functions that decrease with increasing ρ, α will be positive, and the scattering problem will be analogous to the scattering of particles by repulsive potentials in particle-scattering theory. Increasing electron-density functions with increasing ρ (negative α) correspond to attractive potentials in particle-scattering theory. Plasma trails behind moderate-sized hypersonic vehicles moving below the ionosphere are positive-α media, whereas the partially evacuated region behind a hypersonic body moving through the ionosphere is a negative-α medium. In the literature, plasma trails are frequently referred to as "overdense" or "underdense." It should be pointed out that both terms apply to positive-α media: The former refers to $\alpha > 1$, and the latter, to $\alpha < 1$.

The total field E may be decomposed into an incident field E_i and a scattered field E_s

$$E = E_i + E_s \tag{3}$$

Taking the incident field to have unit amplitude, we have

$$E_i = e^{ikx} \tag{4}$$

The scattered field satisfies the condition

$$E_s \to f(\varphi) \frac{e^{ik\rho}}{\sqrt{\rho}} \text{ as } \rho \to \infty$$

where $f(\varphi)$ is the scattering amplitude and $|f(\varphi)|^2$ is the differential scattering cross section. The power scattered into the angular range $d\varphi$ per unit incident flux length is given by $|f(\varphi)|^2 d\varphi$. We will review a few of the methods for calculating $|f(\varphi)|^2$.

III. PARTIAL WAVES

Returning to Eq. (1), let us perform a Fourier decomposition of E

$$E = \sum_{m=o}^{\infty} \epsilon_m \cos(m\varphi) \frac{u_m(\rho)}{\sqrt{k\rho}} \tag{6}$$

$$\epsilon_m = 1 \ (m = 0)$$

$$\epsilon_m = 2 \ (m \neq 0)$$

The functions $u_m(\rho)$ satisfy the equation

$$u_m''(\rho) + \left(k^2 - \frac{\left(m^2 - \frac{1}{4}\right)}{\rho^2} - k^2\alpha\ g\left(\frac{\rho}{\rho_o}\right)\right)u_m(\rho) = 0 \tag{7}$$

with the boundary conditions

$$u_m(0) = 0$$

$$u_{ms} = u_m - u_{mi} \rightarrow C_m e^{ik\rho} \quad \text{as } \rho \rightarrow \infty$$

Let us compare Eq. (7) with the radial equation in spherical coordinates familiar from the scattering of radiation (or particles) by spherically symmetric scattering regions

$$u_\ell''(r) + \left[k^2 - \frac{\ell(\ell + 1)}{r^2} - k^2\alpha\ g\left(\frac{r}{r_o}\right)\right] u_\ell(r) = 0 \tag{8}$$

with the boundary conditions

$$u_{\ell}(0) = 0$$

$$u_{\ell s} \to c_{\ell} e^{ikr} \quad \text{as } r \to \infty$$

Equations (7) and (8) are identical, with $(\ell + \frac{1}{2})^2$ replaced by m^2. This equivalence is extremely helpful in finding solutions to Eq. (7), since considerable work has been done with the scattering by spherically symmetric regions.

In particular, we may write the scattered field as a sum involving phase shifts. In order to do this, we shall first need the expansion of a plane wave in cylindrical coordinates[2]

$$e^{ikx} = \sum_{m=0}^{\infty} \epsilon_m \cos(m\varphi)\, e^{im\frac{\pi}{2}}\, J_m(k\rho) \tag{9}$$

The asymptotic form of $J_m(k\rho)$ is

$$J_m(k\rho) \sim \left(\frac{2}{\pi k\rho}\right)^{1/2} \cos\left(k\rho - \frac{m\pi}{2} - \frac{\pi}{4}\right) \tag{10}$$

for $k\rho \to \infty$. Since $g\left(\frac{\rho}{\rho_o}\right)$ goes to zero as $\rho \to \infty$, $u_m(k\rho)/\sqrt{k\rho}$ asymptotically satisfies the same equation as does $J_m(\rho)$. Therefore, it may be written as a linear combination of the two solutions $(J_m(\rho), N_m(\rho))$ of the $\alpha = 0$ equation. The asymptotic form of u_m can then be written as

$$u_m(\rho) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{im\frac{\pi}{2}}\, e^{-i\eta_m} \cos\left(k\rho - \frac{m\pi}{2} - \frac{\pi}{4} - \eta_m\right) \tag{11}$$

Noting that

$$u_{ms} = u_m - u_{mi} \sim c_m e^{ik\rho} = e^{im\frac{\pi}{2}}\left(\frac{2}{\pi}\right)^{1/2}$$

$$\times\left[e^{-i\eta_m}\cos\left(k\rho - \frac{m\pi}{2} - \frac{\pi}{4} - \eta_m\right) - \cos\left(k\rho - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right] \tag{12}$$

we get

$$C_m = \frac{1}{\sqrt{2\pi}} e^{-i\frac{\pi}{4}}\left(e^{-2i\eta_m} - 1\right) \tag{13}$$

and

$$E_s = \frac{e^{i(k\rho - \frac{\pi}{4})}}{\sqrt{2\pi k\rho}} \sum_{m=0}^{\infty} \epsilon_m \cos m\varphi \left(e^{-2i\eta_m} - 1\right) \tag{14}$$

$$|f(\varphi)| = \frac{1}{\sqrt{2\pi k}}\left|\sum_{m=0}^{\infty} \epsilon_m \cos(m\varphi)\left(1 - e^{-2i\eta_m}\right)\right|$$

The angle η_m is the phase shift of the m^{th} partial wave $u_m(\rho)$; it represents the difference in phase between the asymptotic forms of the actual radial function $u_m(\rho)$ and the radial function $\sqrt{k\rho}\, J_m(k\rho)$ for $\alpha = 0$. An integral for the phase shifts can be obtained by transforming Eq. (7) into an integral equation and employing a suitable Green's function. The result is[3]

$$\sin\eta_m = \alpha\frac{\pi}{2}k^2\int_0^{\infty}\rho g\left(\frac{\rho}{\rho_o}\right) J_m(k\rho)\, e^{-im\frac{\pi}{2}}\,\frac{u_m(\rho)}{\sqrt{k\rho}}\, d\rho \tag{15}$$

IV. CALCULATION OF PHASE SHIFTS

PHASE SHIFTS FOR LARGE m

If $g\left(\frac{\rho}{\rho_o}\right)$ goes to zero (faster than $\frac{1}{\rho}$) for large ρ, we can estimate η_m for large m. Let us consider the coefficient $F_m(\rho)$ of u_m in Eq. (7)

$$F_m = k^2 \left(1 - \alpha\, g\left(\frac{\rho}{\rho_o}\right)\right) - \frac{(m^2 - \frac{1}{4})}{\rho^2} \tag{16}$$

For $m \neq 0$, $F_m(\rho)$ is negative[(3)] for small ρ and positive for large ρ, and therefore, has at least one zero ρ_m. Let us assume that there is only one zero. (For positive α this will always be the case since $F'_m(\rho) \geq 0$.) For small ρ, $u_m(\rho)$ behaves like $\rho^{m+1/2}$ and then increases exponentially until $\rho = \rho_m$. For $\rho \geq \rho_m$, u_m oscillates with ρ. If

$$\left|\alpha\, g\left(\frac{\rho}{\rho_o}\right)\right| \ll \frac{m^2}{k^2\rho^2}$$

for ρ given by $k\rho \sim m$, the partial wave hardly penetrates the region where $\left|\alpha \hat{g}\left(\frac{\rho}{\rho_o}\right)\right|$ may be large, and the perturbation of the partial wave is small. The corresponding phase shift η_m is also small and may be calculated from Eq. (15) with $\sin \eta_m$ replaced by η_m, and $u_m(\rho)/\sqrt{k\rho}$ replaced by the unperturbed incident wave. This gives for η_m

$$\eta_m \approx k^2 \frac{\pi}{2} \alpha \int_0^\infty \rho\, g\left(\frac{\rho}{\rho_o}\right) J_m^2(k\rho)\, d\rho \tag{17}$$

For all functions $g\left(\frac{\rho}{\rho_o}\right)$ which fall off faster than $\frac{1}{\rho}$, the approximation given by Eq. (17) will be valid for sufficiently large m. The smaller the value of $k\rho_o$, the fewer the number of partial waves which contribute significantly to the scattering. For $k\rho_o \ll 1$, all phases except η_o will be negligible. For this case, the scattering is isotropic, $f(\varphi)$ being given by the first term in the series Eq. (14). The differential cross section becomes

$$|f(\varphi)|^2 = \frac{2 \sin^2 \eta_o}{\pi k} \tag{18}$$

where η_o is determined from the solution of Eq. (7) with $m = 0$.

APPROXIMATION FOR LARGE PHASE SHIFTS (WKB METHOD)

The WKB method for calculating phase shifts is useful whenever the phase shifts are large. It can be expected to apply (for m not too large) for large $k\rho_o$ and α if $g\left(\frac{\rho}{\rho_o}\right)$ does not vary much in a wavelength. The phase shifts as obtained by this method[2] are given by

$$\eta_m = \int_{m/k}^{\infty} \sqrt{k^2 - \frac{m^2}{\rho^2}}\, d\rho - \int_{\rho_1(m)}^{\infty} \sqrt{k^2 \left(1 - \alpha\, g\left(\frac{\rho}{\rho_o}\right)\right) - \frac{m^2}{\rho^2}}\, d\rho \tag{19}$$

$$k^2 - \frac{m^2}{\rho_1^2} - k^2 \alpha\, g\left(\frac{\rho_1}{\rho_o}\right) = 0$$

and we have assumed that there is only one such ρ_1. For $m = 0$, $\alpha < 1$, there is no ρ_1. Nevertheless, if η_o is large for this case, we may still use Eq. (19) to approximate η_o, with ρ_1 replaced by zero.

Experience concerning the scattering from spherically symmetric regions has shown that Eq. (19) is not a bad approximation for phases as small as 1/2.[2,3] Therefore, we can obtain an estimate of all the phase shifts for $k\rho_o \gg 1$, by using Eq. (19) for $|\eta_m| \gtrsim 1/2$ and by using Eq. (17) for the small phase shifts involving large m.

We should be able to show that Eqs. (17) and (19) are roughly equivalent for $|\eta_m| \sim 1/2$. Let us consider the case where $|\alpha k\rho_o| \gg 1$, so that for $|\eta_m|$ as given by Eq. (19) to be as small as 1/2, m will be large enough for $\rho_1(m)$ to be given approximately by m/k with $|\alpha\ g(\rho_1/\rho_o)| << 1$. With this the case, Eq. (19) becomes approximately

$$\eta_m \approx \frac{k^2\alpha}{2} \int_{m/k}^{\infty} \frac{g\left(\frac{\rho}{\rho_o}\right) \rho\ d\rho}{\sqrt{k^2\rho^2 - m^2}} \tag{20}$$

Let us compare this with Eq. (17). For $k\rho < m$, $J_m^2(k\rho)$ is very small.[2] For $k\rho > m$, we have

$$J_m^2(k\rho) \sim \frac{2}{\pi} \frac{\sin^2\left[m(q - \tan^{-1} q) + \frac{\pi}{4}\right]}{\sqrt{k^2\rho^2 - m^2}} \tag{21}$$

$$mq = \sqrt{k^2\rho^2 - m^2}$$

Replacing $J_m^2(k\rho)$ by zero for $k\rho < m$, and by its asymptotic value (Eq. (21)) for $k\rho > m$, Eq. (17) becomes

$$\eta_m \approx k^2\alpha \int_{m/k}^{\infty} \frac{\sin^2\left[m(q - \tan^{-1} q) + \frac{\pi}{4}\right]}{\sqrt{k^2\rho^2 - m^2}} \rho g\left(\frac{\rho}{\rho_o}\right) d\rho \tag{22}$$

Finally, writing

$$\sin^2\left[m(q - \tan^{-1}q) + \frac{\pi}{4}\right]$$

as

$$\frac{1}{2} + \frac{1}{2} \sin\left[2m(q - \tan^{-1}q)\right]$$

and noting that the contribution to the integral from the sine term for large $k\rho_o$ will be small due to the rapid oscillations of the integrand, we see that Eqs. (20) and (22) are equivalent.

For larger values of m so that $|\eta_m|$ becomes very small, this equivalence can be expected to break down, since the error involved in neglecting the integral from o to m/k in Eq. (17) becomes large. For these large values of m ($|\eta_m| \lesssim 1/2$), the full Eq. (17) should be used if η_m is to be calculated.

V. SMALL-PERTURBATION THEORY (BORN APPROXIMATION)

The simplest and most useful approximation to the scattering from nonuniform regions is the Born approximation. If $|\alpha|$ is small enough so that the incident wave is distorted very little by the scattering region, the scattering cross section can be calculated by assuming that each part of the medium scatters the incident wave independent of the other parts, the contributions being summed. The scattering amplitude $f_B(\varphi)$ [2] as obtained by this method represents the first approximation to $f(\varphi)$ resulting from an iteration-perturbation method of solution to the integral equation for the problem.

We can immediately obtain $f_B(\varphi)$ by a combination of Eqs. (14) and (17). If $|\alpha|$ is sufficiently small so that all the η_m's are small and the wave can be replaced by the incident wave, then Eq. (17) can be used for all m. Equation (14) then becomes

$$|f_B(\varphi)| = \sqrt{\frac{2}{\pi k}}\left|\sum_{m=0}^{\infty} \epsilon_m \eta_m \cos(m\varphi)\right| \tag{23}$$

$$= \alpha \frac{\pi}{2} k^2 \sqrt{\frac{2}{\pi k}} \int_0^{\infty} \rho g\left(\frac{\rho}{\rho_o}\right) d\rho \sum_{m=0}^{\infty} \epsilon_m \cos(m\varphi)\, J_m^2(k\rho)$$

$$= k^2 \alpha \sqrt{\frac{\pi}{2k}} \int_0^{\infty} \rho g\left(\frac{\rho}{\rho_o}\right) J_o\left(2k\rho \sin\frac{\varphi}{2}\right) d\rho$$

Since this approximation will apply if all the phase shifts are small, and the largest $|\eta_m|$ is generally $|\eta_o|$, we can apply the Born approximation if

$$|\eta_o| \approx \frac{\pi}{2} k^2 |\alpha| \int_0^{\infty} \rho g\left(\frac{\rho}{\rho_o}\right) J_o^2(k\rho)\, d\rho < 1 \tag{24}$$

If we are considering values of $k\rho_o \gg 1$, we can use the asymptotic form of the Bessel function, and we get

$$|\alpha| \, k\rho_o \int_0^\infty g(y) \cos^2 (k\rho_o y - \frac{\pi}{4}) \, dy < 1 \tag{25}$$

For large $k\rho_o$, we can replace $\cos^2 (k\rho_o y - \frac{\pi}{4})$ by 1/2, and we have

$$|\alpha| < \frac{2}{k\rho_o \int_0^\infty g(y) \, dy} \tag{26}$$

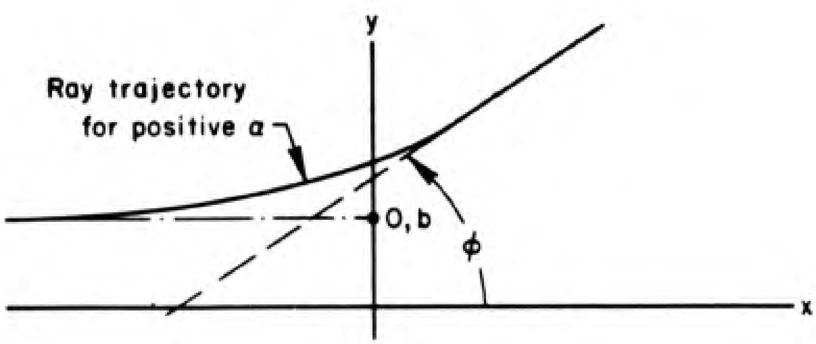
as our criterion for application of the Born approximation for large $k\rho_o$. (It is assumed that $g(y)$ falls off faster than $1/y$ as $y \to \infty$.) This requirement on $|\alpha|$ may be too stringent, but we can regard it as sufficient.

VI. GEOMETRICAL-OPTICS APPROXIMATION

If $k\rho_o \gg 1$ and α is large enough so that many phases are large, Eq. (19) can be expected to represent a good approximation to the phase shifts up to large values of m. The sum (Eq. (14)) over phase shifts for $\varphi \neq 0$ or 2π may then be estimated by a stationary-phase approximation, provided that there is a value of m for which $2\eta_m \pm m\varphi$ is stationary. If this procedure is carried out, with η_m given by Eq. (19), it can be shown[3] that the result for the differential scattering cross section is identical with that obtained by geometrical optics (ray tracing).

The geometrical-optics scattering cross section is most simply derived by analogy with classical particle mechanics, since geometrical optics as a limit of a wave theory corresponds, in particle mechanics, to the classical limit of wave mechanics. In particular, the trajectory of a particle of energy W and impact parameter b in a potential $V(\rho)$, as calculated in classical mechanics, is identical with the optical path of a ray of impact parameter b in a medium with an index of refraction n, where the function $n^2(\rho)$ in optics is equivalent to the function $1 - V/W$ in mechanics.

Let us consider a particle of energy W (or an electromagnetic ray of frequency ω) as shown in the sketch below, incident from $x = -\infty$ and $y = b$ on a region where the potential energy function is $V(\rho)$ (or index of refraction is $n(\rho)$).



Solving the equations of motion in plane polar coordinates,[4] we find that the angle φ of the asymptote of the orbit after the particle has passed the scattering center is given by

$$\frac{\varphi - \pi}{2} = \pm \int_{0}^{w_o} \frac{b}{\sqrt{1 - \frac{V(w)}{W} - b^2 w^2}} \, dw = \pm \int_{0}^{w_o} \frac{b}{\sqrt{n^2(w) - b^2 w^2}} \, dw \tag{27}$$

where

$$w = 1/\rho$$

$$n^2(w_o) = b^2 w_o^2$$

The plus sign refers to negative V (or $\alpha < 0$) and the minus sign, to positive V (or $\alpha > 0$). If we consider a uniform beam of particles (or a plane wave) incident on the scatterer, the particles (or rays) having impact parameters between b and b + db will be scattered into angles lying between φ and $\varphi + d\varphi$. From the definition of the scattering cross section $|f(\varphi)|^2$

$$|f_G(\varphi)|^2 = -\frac{db}{d\varphi} \tag{28}$$

where the subscript G will indicate the geometrical-optics cross section.

Back-scattering ($\varphi = \pi$, $b = 0$) for positive α can occur in this limit if $\alpha > 1$, and we have

$$|f_G(\pi)|^2 = \frac{1}{2\int_0^{w_o} \frac{dw}{n(w)}} \tag{29}$$

For $\omega_{p\infty} = 0$, this gives

$$|f_G(\pi)|^2 = \frac{\rho_o}{2\int_0^{x_o} \frac{dx}{\sqrt{1 - \alpha g(x)}}} \tag{30}$$

where

$$x = \frac{\rho_o}{\rho}$$

$$g(x_o) = \frac{1}{\alpha}$$

Generally speaking, this very useful approximation may apply, except at very small angles of scattering, when a large number of phases, many of which are large, are required to represent the scattering.

To be more precise, if we are considering $\alpha > 0$ and scattering at an angle $\varphi \approx \pi$, the geometrical-optics approximation will certainly not apply if $\alpha < 1$. Let us see how much larger than one α must be in order for the method to be applicable. Let us suppose that $\alpha = 1 + \delta$, where δ is small, and $g'(0) = 0$. (We are mostly interested in electron distributions that have zero slope on the cylinder axis.) The geometrical-optics scattering occurs mostly at a radial distance, a, given by

$$\alpha g\left(\frac{a}{\rho_o}\right) \sim 1 \tag{31}$$

This gives

$$(1 + \delta)\ g\left(\frac{a}{\rho_o}\right) \sim 1 \sim (1 + \delta)\left(1 + \frac{g''(0)a^2}{2\rho_o^2}\right) \tag{32}$$

so that

$$a^2 \approx -\frac{2\rho_o^2\delta}{g''(0)} \tag{33}$$

The geometrical-optics approximation will not be valid unless[3] $ka \gg 1$, so that we have

$$\delta \gg -\frac{g''(0)}{2k^2\rho_o^2} \qquad (34)$$

as a criterion for application of the approximation to 180-deg scattering.

We can also use the fact that the geometrical-optics back-scattering arises from a radial distance, a, given by Eq. (31) to obtain a crude estimate of the dependence of $|f_G(\pi)|^2$ on α for different functional forms of $g\left(\frac{\rho}{\rho_o}\right)$. For α not much larger than 1, $|f_G(\pi)|^2$ will depend mostly on the behavior of $g\left(\frac{\rho}{\rho_o}\right)$ near $\rho = 0$, and not on the way it falls off at large distances. On the other hand, for $\alpha \gg 1$, the scattering will occur at large distances, a, and the behavior of $g\left(\frac{\rho}{\rho_o}\right)$ for large $\frac{\rho}{\rho_o}$ will be very important. Thus, we can conclude, for example, that all functions $g\left(\frac{\rho}{\rho_o}\right)$ that vanish for $\frac{\rho}{\rho_o} \geq 1$ will give essentially the same back-scattering cross section for $\alpha \gg 1$ as will an impenetrable cylinder of radius ρ_o. For functions $g\left(\frac{\rho}{\rho_o}\right)$ that extend to infinity, the radar cross section for $\alpha \gg 1$ will be larger than the radar cross section of an impenetrable cylinder of radius ρ_o.

VII. EXAMPLES

We will now calculate $|f(\varphi)|^2$ for a few specific functions $g\left(\frac{\rho}{\rho_o}\right)$, with special emphasis on $\varphi = \pi$. We will consider only wavelengths that are small compared with the radius of the scattering region ($k\rho_o \gg 1$), positive values of α, and functions $g\left(\frac{\rho}{\rho_o}\right)$ with zero slope on the cylinder axis.

(a) $$g\left(\frac{\rho}{\rho_o}\right) = \left(1 - \frac{\rho^2}{\rho_o^2}\right)^P \qquad \rho \le \rho_o$$

$$g\left(\frac{\rho}{\rho_o}\right) = 0 \qquad \rho \ge \rho_o$$

Electron-density distributions that fall off rapidly for $\rho > \rho_o$ are frequently approximated by functions of this form.[1] Functions $g\left(\frac{\rho}{\rho_o}\right)$ with large values of P will give a better match to free space than those with small values of P but may fall off too rapidly in the region $\rho < \rho_o$. The Born-approximation scattering amplitude (small α) can readily be calculated

$$\begin{aligned} |f_B(\varphi)| &= \alpha k^2 \rho_o^2 \sqrt{\frac{\pi}{2k}} \int_0^1 dy(1 - y^2)^P \, y J_o(2k\rho_o \sin\frac{\varphi}{2} y) \\ &= \alpha k^2 \rho_o^2 \sqrt{\frac{\pi}{2k}} \frac{\Gamma(P+1)}{2(k\rho_o \sin\frac{\varphi}{2})^{P+1}} J_{P+1}(2k\rho_o \sin\frac{\varphi}{2}) \end{aligned} \tag{35}$$

For $P + 1 \ll 2k\rho_o$, the cross section oscillates with angle, the amplitude of the oscillations decreasing with increasing φ. These oscillations are a consequence of the sharpness of the boundary at $\rho = \rho_o$ (discontinuous derivatives of the electron distribution). For $P + 1 > 2k\rho_o$, the transition between the plasma and free space becomes smoother, and the oscillations with angle of the cross section vanish.

The back-scattering cross section $|f_B(\pi)|^2$ is given by

$$|f_B(\pi)|^2 = \alpha^2 \frac{\pi}{8k} (k\rho_o)^4 \frac{(\Gamma(P+1))^2}{(k\rho_o)^{2(P+1)}} J^2_{P+1}(2k\rho_o) \tag{36}$$

and obviously depends critically on the choice of P. For

$$\alpha \gg 1 + \frac{-g''(0)}{2k^2\rho_o^2} = 1 + \frac{P}{k^2\rho_o^2}$$

we can use the geometrical-optics approximation, and we get from Eq. (27)

$$\frac{\varphi - \pi}{2} = -\sin^{-1}\frac{b}{\rho_o} - \frac{b}{\rho_o}\int_1^{x_o} \frac{dx}{\sqrt{1 - \alpha\left(1 - \frac{1}{x^2}\right)^P - \frac{b^2}{\rho_o^2}x^2}} \tag{37}$$

For P = 0 (homogeneous cylinder), the result is independent of α, and the cross section is the same as that for the scattering from an impenetrable cylinder

$$|f_G(\varphi)|^2 = \frac{\rho_o}{2}\sin\frac{\varphi}{2} \tag{38}$$

For $P \neq 0$, large α, and P small enough so that $\left(\frac{1}{\alpha}\right)^{1/P} \ll 1$, x_o will be near 1, and the cross section will differ very little from that for the scattering from an impenetrable cylinder.

The geometrical-optics radar cross section $|f_G(\pi)|^2$ is given by

$$|f_G(\pi)|^2 = \frac{\rho_o}{2\left[1 + \int_1^{x_o} \frac{dx}{\sqrt{1 - \alpha\left(1 - \frac{1}{x^2}\right)^P}}\right]} \tag{39}$$

This can be easily calculated for any P and $\alpha > 1$. For very large α, $(\frac{1}{\alpha})^{1/P} \ll 1$, $|f_G(\pi)|^2$ is approximately

$$|f_G(\pi)|^2 \simeq \frac{\rho_o}{2\left[1 + \frac{\sqrt{\pi}}{2}\alpha^{-1/P}\frac{1}{P}\frac{\Gamma\left(\frac{1}{P}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{P}\right)}\right]} \tag{40}$$

For P = 1 (quadratic distribution), the integral in Eq. (39) is trivial, and $|f_G(\pi)|^2$ for all $\alpha > 1$ is given by

$$|f_G(\pi)|^2 = \frac{\rho_o}{2}\left(\frac{\alpha - 1}{\alpha}\right) \tag{41}$$

The phase shifts as given by Eq. (19) are also especially simple to calculate for this case because the integrals can be evaluated analytically. The sum over phase shifts (Eq. (14)) for $k\rho_o = 10$ and 100 for several values of α between 0.9 and 2 was programmed and computed on the IBM 7090. The cross sections obtained by this method, together with the geometrical-optics cross sections (for $\alpha > 1$), are shown in Figs. 1 and 2. We can see from Fig. 1 (in which $k\rho_o = 10$) that for α as small as 1.1, the error involved in the geometrical-optics approximation is only about 25 per cent. For larger values of α, the error becomes much smaller. We can see from Fig. 2 that for $k\rho_o = 100$ the geometrical-optics approximation is valid down to small values of $\alpha - 1$; the error for $\alpha = 1.01$ is only 23 per cent.

It happens that an exact solution to Eq. (7) can be found for the quadratic distribution for all α in terms of hypergeometric functions. These solutions are rather cumbersome to work with except in the special case of $\alpha = 1$, for which the hypergeometric functions reduce to Bessel functions. The exact value of $|f(\pi)|^2$ was calculated for $\alpha = 1$ and $k\rho_o = 10$ and was found to be only 10 per cent higher than the cross section as calculated from the approximate phase shifts as given by Eq. (19).

(b) $$g\left(\frac{\rho}{\rho_o}\right) = \frac{1}{\left(1 + \frac{\rho^2}{\rho_o^2}\right)^{P+1}}$$

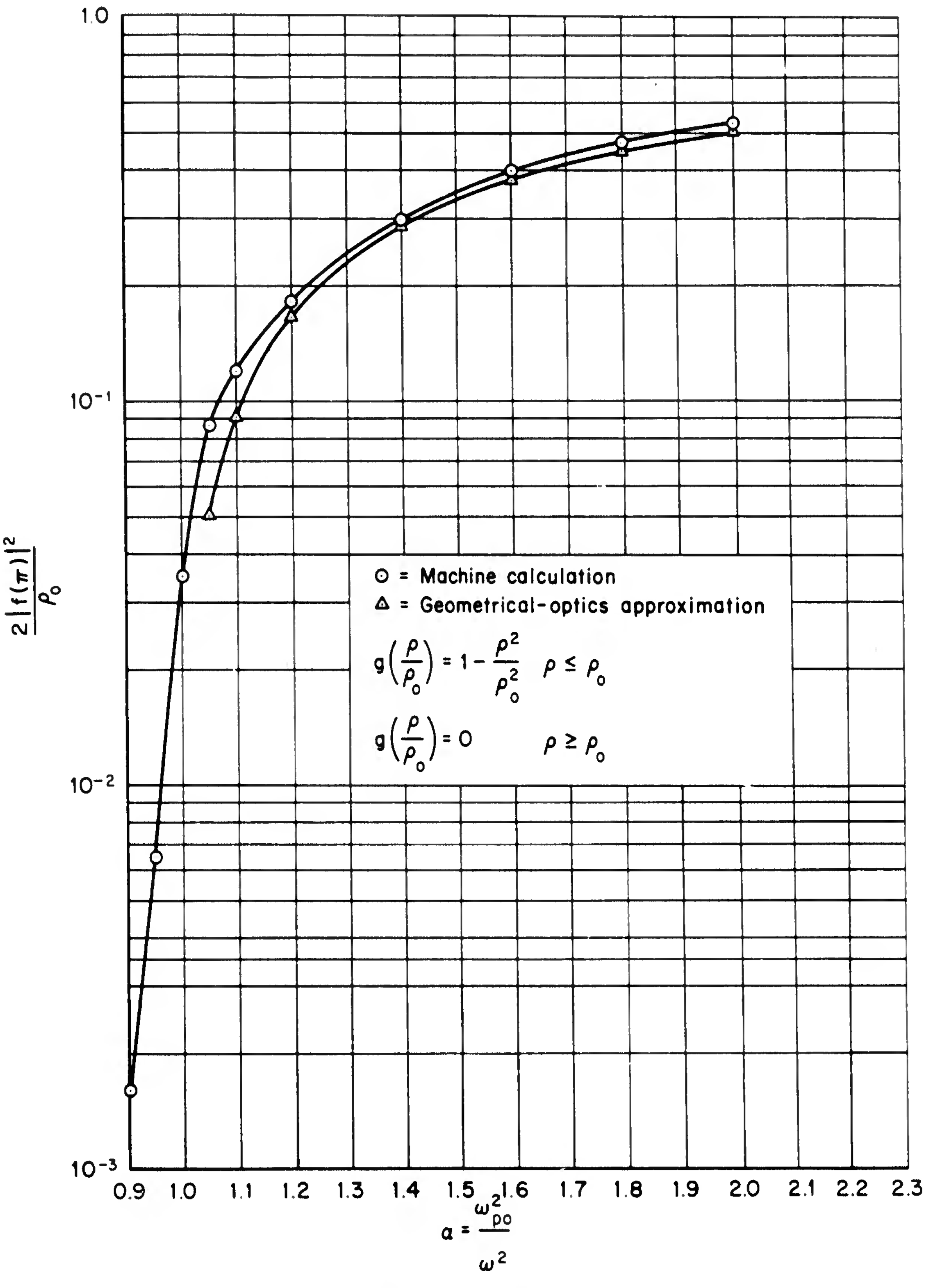


Fig. 1—Back-scattering cross section in transition region for quadratic electron distribution ($k\rho_o = 10$)

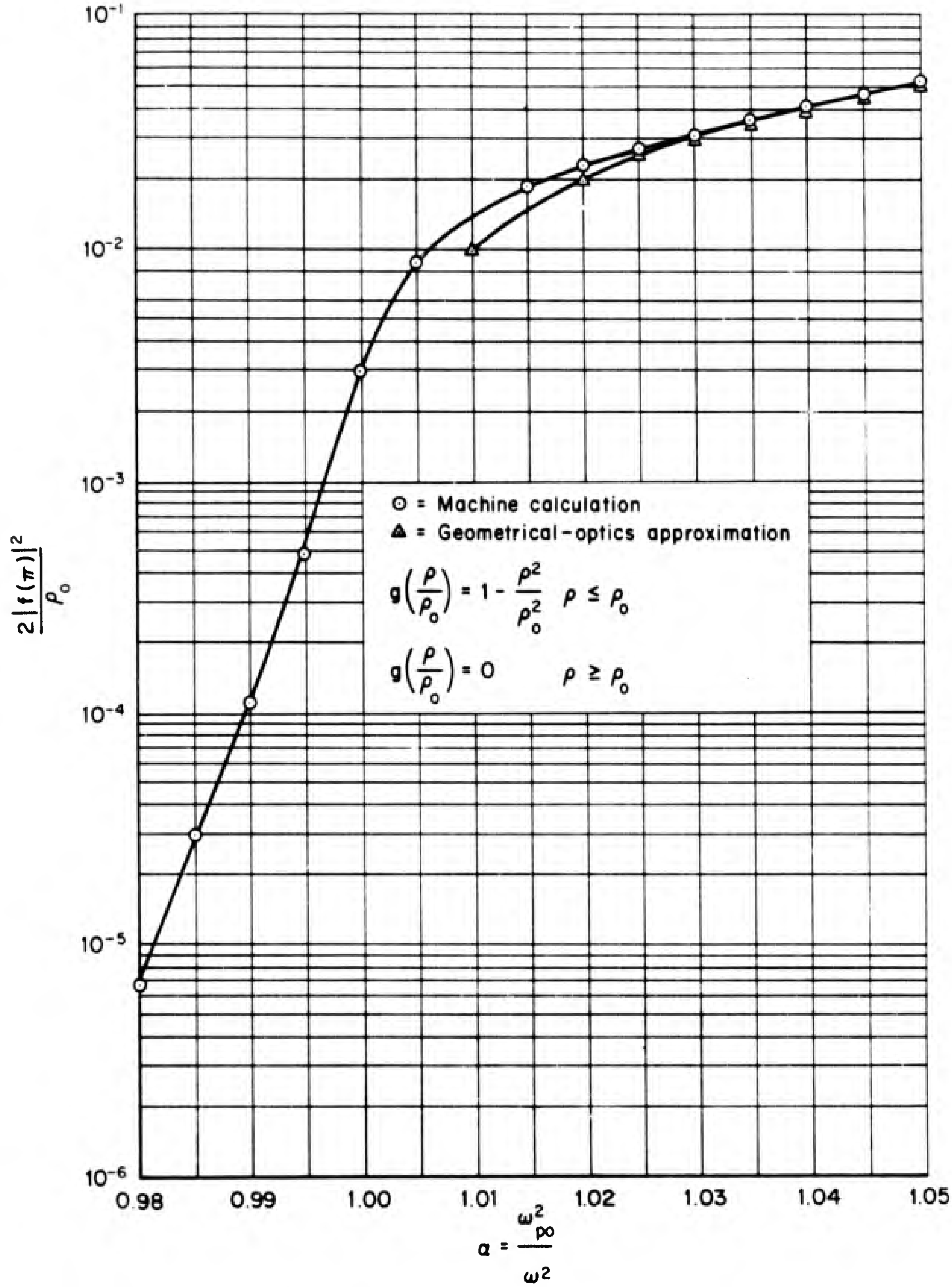


Fig. 2 — Back-scattering cross section in transition region for quadratic electron distribution ($k\rho_o = 100$)

Functions of this form represent smooth, long-range electron-density distributions. The Born-approximation cross section can be calculated immediately.

$$|f_B(\varphi)|^2 = \frac{\pi}{2k}\,\alpha^2 (k\rho_o)^4 \left[\int_o^\infty \frac{y}{(1+y^2)^{P+1}} J_o(2k\rho_o \sin\tfrac{\varphi}{2}\, y)\,dy\right]^2$$

$$= \frac{\pi}{2k}\,\alpha^2 (k\rho_o)^4 \frac{(k\rho_o \sin\frac{\varphi}{2})^{2P}}{[\Gamma(P+1)]^2}\left[K_P(2k\rho_o \sin\tfrac{\varphi}{2})\right]^2 \qquad (42)$$

For $P \ll 2k\rho_o$, $|f_B(\pi)|^2$ is approximately

$$|f_B(\pi)|^2 \approx \frac{\pi^2}{8k}\,\alpha^2 (k\rho_o)^{2P-1} e^{-4k\rho_o} \qquad (43)$$

To calculate $|f_G(\varphi)|^2$, we have

$$\frac{\varphi - \pi}{2} = -\frac{b}{\rho_o}\int_o^{x_o} \frac{dx}{\sqrt{1 - \alpha\left(\frac{x^2}{1+x^2}\right)^{P+1} - \frac{b^2}{\rho_o^2}x^2}} \qquad (44)$$

For very large α, and P small enough so that $\left(\frac{1}{\alpha}\right)^{\frac{1}{2(P+1)}} \ll 1$, x_o will be small, and we have approximately

$$\frac{\varphi - \pi}{2} \cong -\frac{b}{\rho_o}\int_o^{x_o} \frac{dx}{\sqrt{1 - \alpha x^{2(P+1)} - \frac{b^2}{\rho_o^2}x^2}} \qquad (45)$$

For P = 0 and $\alpha \gg 1$, we can obtain a solution in terms of simple functions, and we get

$$|f_G(\varphi)|^2 = \sqrt{\alpha}\,\frac{\rho_o}{\pi\left[1 - (1 - \frac{\varphi}{\pi})^2\right]^{3/2}} \qquad (46)$$

The cross section is smallest for $\varphi = \pi$ and varies very little with angle for large-angle scattering. For $P = 0$ and $P = 1$, $|f_G(\pi)|^2$ can be evaluated analytically for arbitrary $\alpha > 1$. In the case of $P = 0$ and $\alpha > 1$, we find

$$|f_G(\pi)|^2 = \frac{\rho_o}{2} \frac{(\alpha - 1)}{\sqrt{\alpha}\,\mathcal{E}\left(\frac{1}{\sqrt{\alpha}}\right)} \tag{47}$$

and in the case of $P = 1$ and $\alpha > 1$

$$|f_G(\pi)|^2 = \frac{\rho_o}{2} \frac{(\alpha - 1)\sqrt{2}}{\alpha^{1/4}} \frac{1}{\left[2\mathcal{E}\left(\left(\frac{1 + \sqrt{\alpha}}{2\sqrt{\alpha}}\right)^{1/2}\right) + (\sqrt{\alpha} - 1)K\left(\frac{1 + \sqrt{\alpha}}{2\sqrt{\alpha}}\right)^{1/2}\right]} \tag{48}$$

The K and $\mathcal{E}$ functions are the complete elliptic integrals of the first and second kind, respectively.

For arbitrary $P > 0$ and $\alpha > 1$, $|f_G(\pi)|^2$ can be easily evaluated numerically and $|f_G(\pi)|^2$ is given by

$$|f_G(\pi)|^2 = \frac{\rho_o}{2\int_0^{x_o} \frac{dx}{\sqrt{1 - \alpha\left(\frac{x^2}{1 + x^2}\right)^{P+1}}}} \tag{49}$$

For large α, and P small enough so that $\left(\frac{1}{\alpha}\right)^{\frac{1}{2(P+1)}} \ll 1$, we have approximately

$$|f_G(\pi)|^2 \simeq \rho_o(\alpha)^{\frac{1}{2(P+1)}} \frac{(P + 1)}{\pi} \frac{\Gamma\left(\frac{1}{2}\left(1 + \frac{1}{P + 1}\right)\right)}{\Gamma\left(\frac{1}{2(P + 1)}\right)} \tag{50}$$

As would be expected, the more rapidly the electron density falls off with distance, the slower is the increase of cross section with increasing α.

$$\text{(c)} \qquad g\left(\frac{\rho}{\rho_o}\right) = e^{-\rho^2/\rho_o^2}$$

For the Gaussian electron distribution, the cross section in the Born approximation is given by

$$|f_B(\varphi)|^2 = \frac{(k\rho_o)^4\pi}{8k} e^{-2k^2\rho_o^2 \sin^2\frac{\varphi}{2}} \tag{51}$$

and $|f_G(\pi)|^2$ is given by

$$|f_G(\pi)|^2 = \frac{\rho_o}{2\int_0^{x_o} \frac{dx}{\sqrt{1 - \alpha e^{-1/x^2}}}} = \frac{\sqrt{2}\rho_o}{\int_0^{\infty} \frac{dy}{[\ln(\sqrt{\alpha}\cosh y)]^{3/2}}} \tag{52}$$

We can obtain a rough estimate of $|f_G(\pi)|^2$ for large α by replacing $\cosh y$ by $\frac{e^y}{2}$, and for $\frac{\alpha}{4} >> 1$ we get

$$|f_G(\pi)|^2 = \frac{\rho_o}{2}\left(\ln\left(\frac{\alpha}{4}\right)\right)^{1/2} \tag{53}$$

The integral in Eq. (52) was calculated numerically for several values of α between 1 and 100. The phase shifts as given by Eq. (19) and the sum (Eq. (14)) over phase shifts for $\varphi = \pi$, and $k\rho_o = 10$ and 100 were programmed and computed on the IBM 7090 for several values of α between 0.9 and 1.6. The results are shown in Figs. 3 and 4.

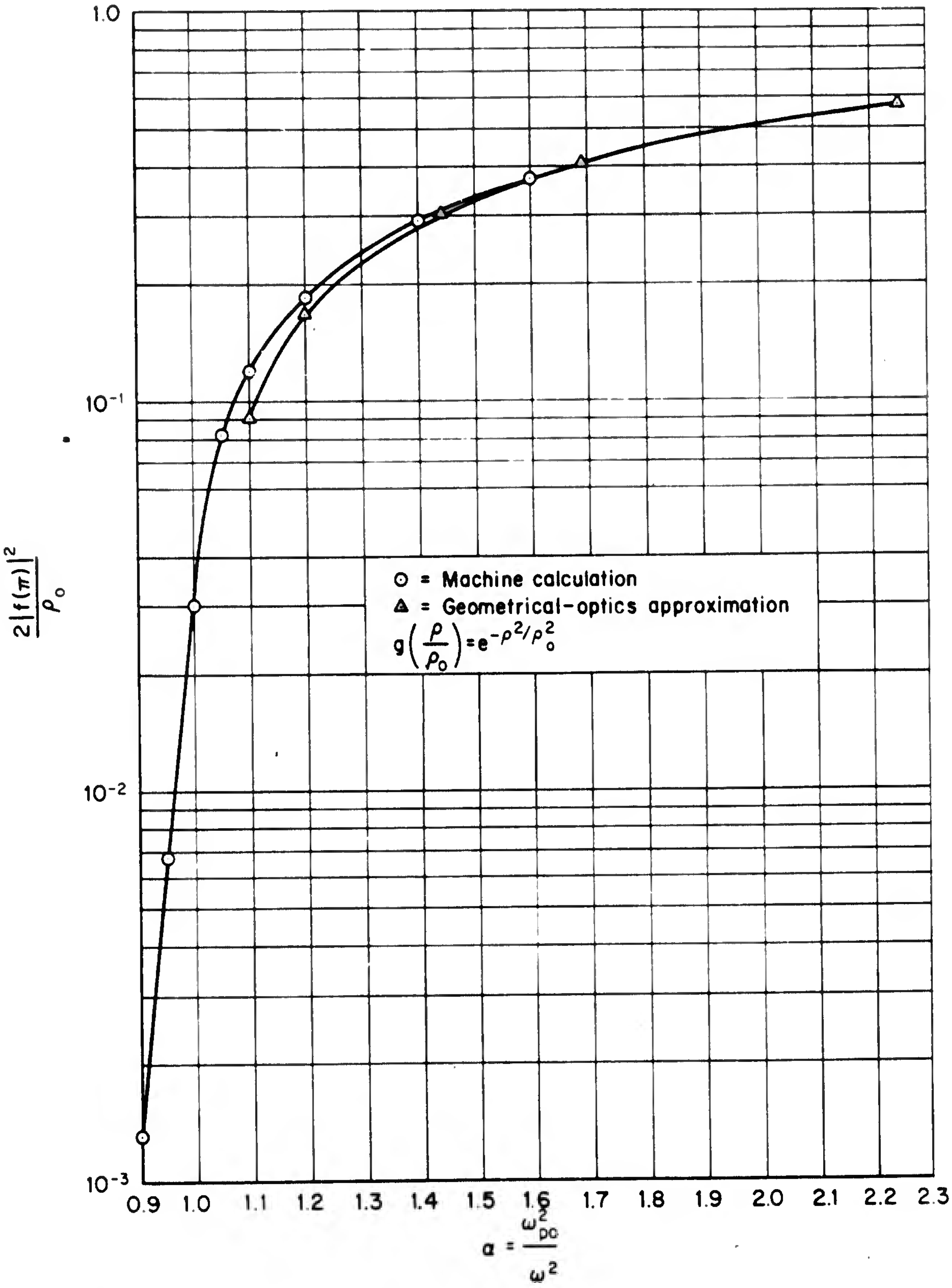


Fig. 3 — Back-scattering cross section in transition region for Gaussian electron distribution ($k\rho_0 = 10$)

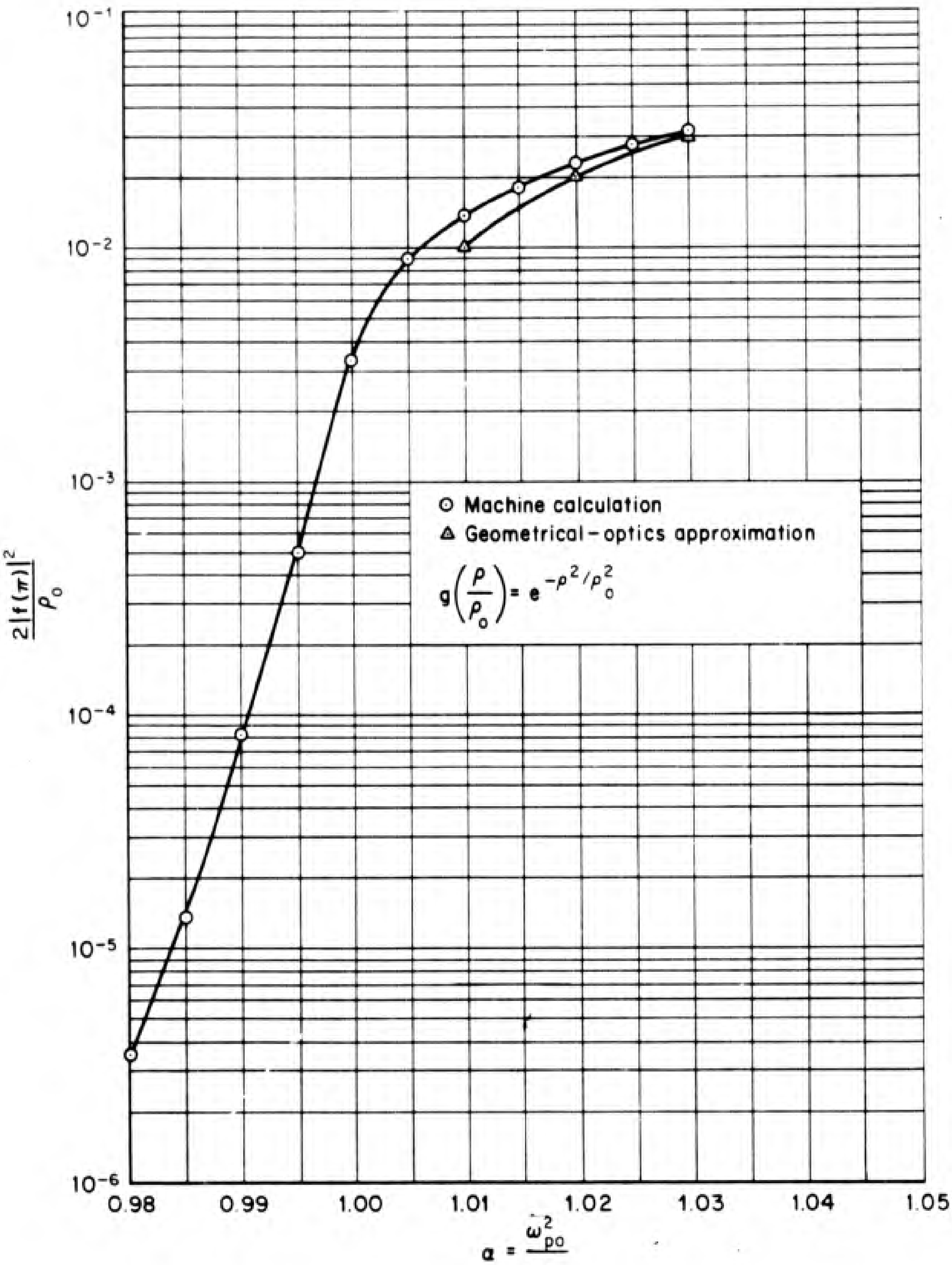


Fig. 4 — Back-scattering cross section in transition region for Gaussian electron distribution ($k\rho_o = 100$)

Finally, in Fig. 5, $\frac{2}{\rho_o}|f_G(\pi)|^2$ for $1.2 \leq \alpha \leq 100$ is shown for the four functions

$$(1)\quad g\left(\frac{\rho}{\rho_o}\right) = 1 - \frac{\rho^2}{\rho_o^2} \qquad \rho \leq \rho_o$$

$$= 0 \qquad \rho \geq \rho_o$$

$$(2)\quad g\left(\frac{\rho}{\rho_o}\right) = e^{-\rho^2/\rho_o^2}$$

$$(3)\quad g\left(\frac{\rho}{\rho_o}\right) = \frac{1}{1 + \frac{\rho^2}{\rho_o^2}}$$

$$(4)\quad g\left(\frac{\rho}{\rho_o}\right) = \frac{1}{\left(1 + \frac{\rho^2}{\rho_o^2}\right)^2}$$

The radar cross sections for these four distributions behave as predicted in the discussion of Section VI. For α not much larger than one, the radar cross section should depend mainly on the behavior of the functions $g\left(\frac{\rho}{\rho_o}\right)$ for small $\frac{\rho}{\rho_o}$. Thus, the first three functions $g\left(\frac{\rho}{\rho_o}\right)$, which all behave as $1 - \rho^2/\rho_o^2$ for small $\left(\frac{\rho}{\rho_o}\right)$, yield radar cross sections which are roughly equivalent for α not much larger than one. The fourth function $g\left(\frac{\rho}{\rho_o}\right)$ differs from the other three in its radar cross section for α near one because it differs in its dependence upon ρ for small $\frac{\rho}{\rho_o}$.

The dependence of the cross section on α for large α is quite different for all four distributions. For large α, $|f_G(\pi)|^2$ as a function of α for the four functions $g\left(\frac{\rho}{\rho_o}\right)$ being considered is as follows (c.f. Eqs. (41), (47), (48), and (53)):

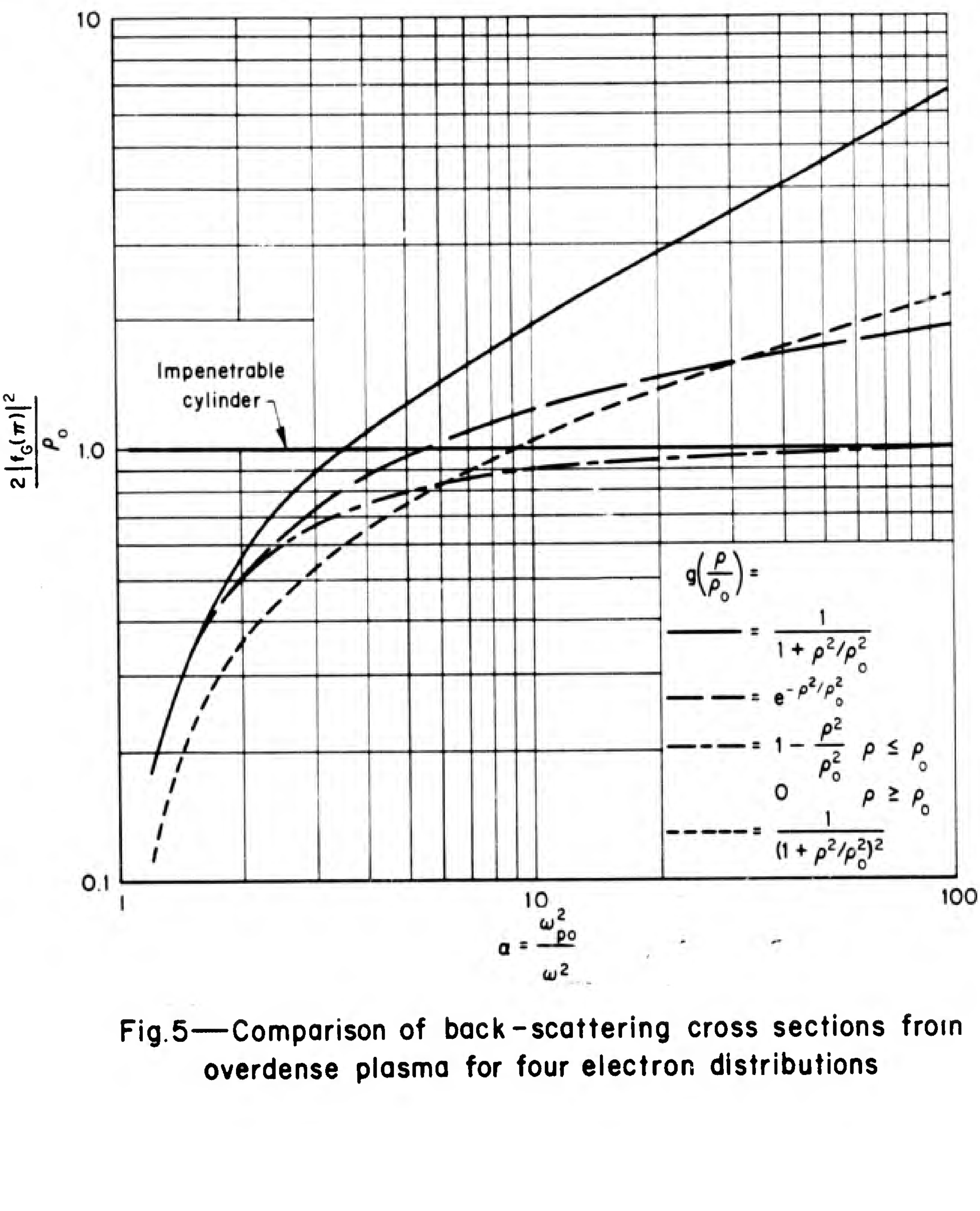


Fig. 5—Comparison of back-scattering cross sections from overdense plasma for four electron distributions

(1) $\frac{2}{\rho_o} |f_G(\pi)|^2 \to 1$

$$\text{for } g\left(\frac{\rho}{\rho_o}\right) = 1 - \frac{\rho^2}{\rho_o^2} \qquad \rho \le \rho_o$$

$$= 0 \qquad \rho \ge \rho_o$$

(2) $\frac{2}{\rho_o} |f_G(\pi)|^2 \to \sqrt{\ln \alpha}$

$$\text{for } g\left(\frac{\rho}{\rho_o}\right) = e^{-\rho^2/\rho_o^2}$$

(3) $\frac{2}{\rho_o} |f_G(\pi)|^2 \to \frac{2}{\pi}\sqrt{\alpha}$

$$\text{for } g\left(\frac{\rho}{\rho_o}\right) = \frac{1}{1 + \frac{\rho^2}{\rho_o^2}}$$

(4) $\frac{2}{\rho_o} |f_G(\pi)|^2 \to \frac{\sqrt{2}}{1.85} \alpha^{1/4}$

$$\text{for } g\left(\frac{\rho}{\rho_o}\right) = \frac{1}{\left(1 + \frac{\rho^2}{\rho_o^2}\right)^2}$$

For back scattering, the quadratic distribution for large α is indistinguishable from an impenetrable cylinder of radius ρ_o; this is as expected. The three long-range distributions (the last three functions) yield radar cross sections for very large α that can be estimated by assuming that the distributions are equivalent to impenetrable cylinders of radius, a, the distance, a, being given by Eq. (31). This method of estimating the cross section gives, for α very large,

$$\frac{2}{\rho_o} |f_G(\pi)|^2 \to \sqrt{\alpha}$$

$$\text{for } g\left(\frac{\rho}{\rho_o}\right) = \frac{1}{1 + \frac{\rho^2}{\rho_o^2}}$$

$$\frac{2}{\rho_o} |f_G(\pi)|^2 \to \alpha^{1/4}$$

$$\text{for } g\left(\frac{\rho}{\rho_o}\right) = \frac{1}{\left(1 + \frac{\rho^2}{\rho_o^2}\right)^2}$$

$$\frac{2}{\rho_o} |f_G(\pi)|^2 \to \sqrt{\ln \alpha}$$

$$\text{for } g\left(\frac{\rho}{\rho_o}\right) = e^{-\rho^2/\rho_o^2}$$

We can see that this quick and convenient way of estimating the radar cross section for very large α gives correct results except for a numerical factor of the order of unity.

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