64-8

ESD-TDR-64-77

W-6637

POSITION AND VELOCITY ERRORS RESULTING FROM IMPERFECT ESTIMATION OF NEARLY CIRCULAR SATELLITE ORBITS

TECHNICAL DOCUMENTARY REPORT NO. ESD-TDR-64-77

JANUARY 1964

R. Manasse

Prepared for

DIRECTORATE OF SYSTEMS PLANS
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 602.0
Prepared by

THE MITRE CORPORATION
Bedford, Massachusetts
Contract AF19(628)-2390

ESD-TDR-64-77 W-6637

POSITION AND VELOCITY ERRORS RESULTING FROM IMPERFECT ESTIMATION OF NEARLY CIRCULAR SATELLITE ORBITS

TECHNICAL DOCUMENTARY REPORT NO. ESD-TDR-64-77

JANUARY 1964

R. Manasse

Prepared for

DIRECTORATE OF SYSTEMS PLANS
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 602.0
Prepared by

THE MITRE CORPORATION
Bedford, Massachusetts
Contract AF19(628)-2390

FOREWORD

The author is indebted to Mr. William Riordan for a critical review of this paper.

ABSTRACT

An error analysis is presented for the problem of prediction of position and velocity of a satellite with a nearly circular orbit from initial measurements of position and velocity. Relatively simple expressions are obtained for these errors which should prove useful both for obtaining numerical results and for the physical interpretation of the effect of errors on prediction. The error formulas are in agreement with results obtained earlier by Schweppe using a more general method of analysis.

REVIEW AND APPROVAL

Publication of this technical documentary report does not constitute Air Force approval of the reports findings or conclusions. It is published only for the exchange and stimulation of ideas.

Jack Segal

Jack Segal
Project Officer, 602.0

POSITION AND VELOCITY ERRORS RESULTING FROM IMPERFECT ESTIMATION OF NEARLY CIRCULAR SATELLITE ORBITS

The orbit of a satellite can be determined from an initial measurement of vector position and vector velocity. Errors in the initial measurement will lead to errors in the prediction of object position and velocity at some future time. The form of these prediction errors for the general case appears to be rather complicated.[1] For nearly circular orbits, however, closed form expressions for these prediction errors can be derived which can be used to obtain simple numerical results and which can be subjected to useful physical interpretation. It is the purpose of this paper to derive these expressions.

The following definitions are used:

t = time

t_o = time of perigee

t_m = time of measurement

T_p = orbit period

ω = $2\pi/T_p$

θ = center of earth angle in plane of the ellipse

θ_o = angle of perigee

θ_m = value of θ at time t_m

a = semimajor axis

e = eccentricity (assumed small).

v = satellite speed

ρ = radius from earth center to satellite at time t

$\dot{\rho}$ = $d\rho/dt$

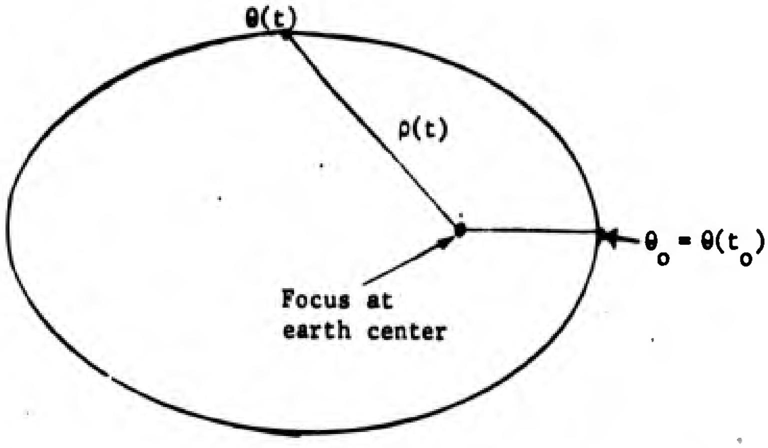
$\dot{\theta}$ = $d\theta/dt$

K = earth gravitational constant.

$\perp$ = subscript denoting an out of plane error

Δv_m, $\Delta \rho_m$, Δa_m, $\Delta \theta_m$ = errors in v, ρ, a, θ respectively at $t = t_m$

The geometry of the problem is shown below:



The motion of a ballistic object in the presence of earth's gravitational field takes the form of an ellipse with one focus at the earth's center, as shown above. The solution of the equations of motion can be found in any elementary text on the subject.* The equation of the ellipse is given by

$$\rho(\theta) = \frac{a(1-e^2)}{1+e\cos(\theta-\theta_o)} \tag{1}$$

The rate of change of θ is given by

$$\dot{\theta} = \omega(1-e^2)^{-3/2}(1+e\cdot\cos(\theta-\theta_o))^2 \tag{2}$$

The quantity ω is the average angular rate of the object, and is given by

$$\omega = \frac{2\pi}{T_p} = K^{\frac{1}{2}} a^{-3/2} \tag{3}$$

where K is the earth's gravitational constant.

---

* See, for example, "Celestial Mechanics," W. M. Smart (Longmans, Green and Co., 1953).

In the error analysis which follows, the errors will be assumed small so that second and higher order errors can be neglected. It is also assumed that the orbit is nearly circular so that terms in e^2 and higher powers of e can be neglected. Terms involving products of errors and e will likewise be neglected.

With these approximations (1) and (2) become

$$\rho \approx a\left[1 - e\cos(\theta - \theta_o)\right] \tag{4}$$

$$\dot{\theta} \approx \omega\left[1 + 2e\cos(\theta - \theta_o)\right] \tag{5}$$

From (5) it follows approximately that

$$\theta - \theta_o \approx \omega(t - t_o) \tag{6}$$

Substituting this expression in (4) and (5) gives

$$\rho \approx a\left[1 - e\cos\omega(t - t_o)\right] \tag{7}$$

$$\dot{\theta} \approx \omega\left[1 + 2e\cos\omega(t - t_o)\right] \tag{8}$$

Differentiating (7) and integrating (8) gives

$$\dot{\rho} \approx a\omega e\sin\omega(t - t_o) \tag{9}$$

$$\theta \approx \theta_o + \omega(t - t_o) + 2e\sin\omega(t - t_o) \tag{10}$$

Equations (7), (8), (9) and (10) are the approximate motion equations which will be used in an error analysis of the problem. It is seen that the four parameters θ_o, t_o, e and a determine the object motion in the orbit plane (note that ω is determined by a in accordance with (3)). Out of plane errors will be considered later.

Equations (7 - 10) taken at time t_m, the time of the measurement, are

$$\theta_m \approx \theta_o - \omega t_o + \omega t_m + 2e\sin\omega(t_m - t_o) \tag{11}$$

$$\rho_m \approx a - a e \cos\omega (t_m - t_o) \tag{12}$$

$$\dot{\theta}_m \approx \omega + 2\omega e \cos\omega (t_m - t_o) \tag{13}$$

$$\dot{\rho}_m \approx a\omega e \sin\omega(t_m - t_o) \tag{14}$$

Taking the differential of both sides yields the following equations,

$$\Delta \theta_m \approx \Delta(\theta_o - \omega t_o) + \Delta\omega t_m + 2\Delta [e \sin\omega(t_m - t_o)] \tag{15}$$

$$\Delta \rho_m \approx \Delta a - a\Delta [e \cos\omega(t_m - t_o)] \tag{16}$$

$$\Delta \dot{\theta}_m \approx \Delta\omega + 2\omega\Delta [e \cos\omega (t_m - t_o)] \tag{17}$$

$$\Delta \dot{\rho}_m \approx a\omega\Delta [e \sin\omega(t_m - t_o)] \tag{18}$$

where certain differential terms of higher order have been neglected. When $\Delta\omega$ is expressed in terms of Δa using

$$\Delta\omega = -\frac{3\omega}{2a}\Delta a \tag{19}$$

we recognize that (15-18) represent four linear equations in four unknowns. Solving for these unknowns yields

$$\Delta(\theta_o - \omega t_o) \approx \Delta\theta_m + \left(\frac{3\Delta\dot{\theta}_m}{\omega} + \frac{6\Delta\rho_m}{a}\right)\omega t_m - \frac{2\Delta\dot{\rho}_m}{a\omega} \tag{20}$$

$$\Delta a \approx \frac{2a\Delta\dot{\theta}_m}{\omega} + 4\Delta\rho_m \tag{21}$$

$$\Delta[e \sin\omega(t_m - t_o)] \approx \frac{\Delta\dot{\rho}_m}{a\omega} \tag{22}$$

$$\Delta[e \cos\omega(t_m - t_o)] \approx \frac{2\Delta\dot{\theta}_m}{\omega} + \frac{3\Delta\rho_m}{a} \tag{23}$$

The speed v of the object is approximately

$$v \approx \rho \dot{\theta} \tag{24}$$

The contribution of $\dot{\rho}$ to v is of second order and can be ignored, as can any velocity error out of the ellipse plane. The logarithmic derivative of (24) is

$$\frac{\Delta v}{v} \approx \frac{\Delta \rho}{\rho} + \frac{\Delta \dot{\theta}}{\dot{\theta}} \approx \frac{\Delta \rho}{\rho} + \frac{\Delta \dot{\theta}}{\omega} \tag{25}$$

Equations (20 - 23) can be rewritten with the aid of (25) and the fact that $a \approx \rho$.

$$\Delta(\theta_o - \omega t) \approx \Delta \theta_m + \left(\frac{3 \Delta v_m}{v} + \frac{3 \Delta \rho_m}{\rho}\right) \omega t_m - \frac{2 \Delta \dot{\rho}_m}{\rho \omega} \tag{26}$$

$$\frac{\Delta a}{a} \approx \frac{2 \Delta v_m}{v} + \frac{2 \Delta \rho_m}{\rho} \tag{27}$$

$$\Delta \left[e \sin \omega (t_m - t_o) \right] \approx \frac{\Delta \dot{\rho}_m}{\rho \omega} \approx \frac{\Delta \dot{\rho}_m}{v} \tag{28}$$

$$\Delta \left[e \cos \omega (t_m - t_o) \right] \approx \frac{2 \Delta v_m}{v} + \frac{\Delta \rho_m}{\rho} \tag{29}$$

Combining (19) and (27) gives also

$$\frac{\Delta \omega}{\omega} \approx -3 \left(\frac{\Delta v_m}{v} + \frac{\Delta \rho_m}{\rho} \right) \tag{30}$$

The equations for θ and ρ can be rewritten

$$\theta \approx [\theta_o - \omega t_o] + \omega t + 2e \sin \omega (t - t_m + t_m - t_o)$$

$$\approx [\theta_o - \omega t_o] + \omega t + 2 \sin \omega (t - t_m) [e \cos \omega (t_m - t_o)] \tag{31}$$

$$+ 2 \cos \omega (t - t_m) [e \sin \omega (t_m - t_o)]$$

$$\rho \approx a - ae \cos \omega(t - t_m + t_m - t_o) \tag{32}$$

$$\approx a - ae \cos \omega(t - t_m) \cos \omega(t_m - t_o) + ae \sin \omega(t - t_m) \sin \omega(t_m - t_o)$$

Taking the differential of both sides,

$$\begin{aligned} \Delta\theta \approx\ & \Delta[\theta_o - \omega t] + \Delta\omega t + 2 \sin \omega(t - t_m) \Delta[e \cos \omega(t_m - t_o)] \\ & + 2 \cos \omega(t - t_m) \Delta[e \sin \omega(t_m - t_o)] \end{aligned} \tag{33}$$

$$\begin{aligned} \Delta\rho \approx\ & \Delta a - a \cos \omega(t - t_m) \Delta[e \cos \omega(t_m - t_o) \\ & + a \sin \omega(t - t_m) \Delta[e \sin \omega(t - t_m)] \end{aligned} \tag{34}$$

Substituting for the Δ's from equations (26 - 30),

$$\begin{aligned} \Delta\theta \approx\ & \Delta\theta_m - \left(\frac{3\Delta v_m}{v} + \frac{3\Delta\rho_m}{\rho}\right) \omega(t - t_m) - 4\left(\frac{\Delta\dot{\rho}_m}{v}\right) \sin^2 \frac{\omega}{2}(t - t_m) \\ & + \left(\frac{4\Delta v_m}{v} + \frac{2\Delta\rho_m}{\rho}\right) \sin \omega(t - t_m) \end{aligned} \tag{35}$$

$$\begin{aligned} \frac{\Delta\rho}{\rho} \approx\ & \frac{\Delta\rho_m}{\rho} + \left(\frac{4\Delta v_m}{v} + \frac{2\Delta\rho_m}{\rho}\right) \sin^2 \frac{\omega}{2}(t - t_m) \\ & + \left(\frac{\Delta\dot{\rho}_m}{v}\right) \sin \omega(t - t_m) \end{aligned} \tag{36}$$

Note that the errors in θ and ρ reduce to the proper values at $t = t_m$. It is seen the errors in θ and ρ are of several types. There are constant errors associated with an initial error in position. The second term in the expression for Δθ corresponds to a period error and it increases linearly with time. Eventually this period error will predominate over all other error terms. There are two types of periodic error terms. One of these is maximum at the point antipodal to the measurement ($\theta - \theta_m = 180^\circ$), while the other is zero at the antipodal point and reaches a maximum at $\theta - \theta_m = 90^\circ$ and $\theta - \theta_m = 270^\circ$.

The errors $\Delta\dot{\theta}$ and $\Delta\dot{\rho}$ can be found simply by taking the derivatives of (35) and (36).

$$\Delta\dot{\theta} \approx - \left(\frac{3\Delta v_m}{v} + \frac{3\Delta\rho_m}{\rho}\right) \omega - \frac{2\Delta\dot{\rho}_m}{\rho} \sin \omega(t - t_m) \qquad (37)$$

$$+ \omega \left(\frac{4\Delta v_m}{v} + \frac{2\Delta\rho_m}{\rho}\right) \cos \omega (t - t_m)$$

$$\frac{\Delta\dot{\rho}}{\rho} \approx \omega \left(\frac{2\Delta v_m}{v} + \frac{\Delta\rho_m}{\rho}\right) \sin\omega (t - t_m) + \frac{\Delta\dot{\rho}_m}{\rho} \cos \omega (t - t_m) \qquad (38)$$

Out of plane errors have been neglected up to this point in the analysis. The effect of small out-of-plane errors on the in-plane measurements will be of second order and hence their effect on the in-plane measurements can be ignored. The effect of these out-of-plane errors is simply to tilt the plane of the predicted orbit by a small amount. The out-of-plane position and velocity errors, $\Delta\rho_\perp$ and $\Delta v_\perp$, must be of the form

$$\Delta\rho_\perp \approx A \sin[\omega(t - t_m) + B] \qquad (39)$$

$$\Delta v_\perp \approx A\, \omega \cos [\omega (t - t_m) + B] \qquad (40)$$

The constants A and B are found by setting

$$\Delta\rho_{\perp m} = \Delta\rho_\perp (t_m) \approx A \sin B \qquad (41)$$

$$\Delta v_{\perp m} = \Delta v_\perp (t_m) \approx A\, \omega \cos B \qquad (42)$$

Solving for A and B

$$A = [\Delta\rho_{\perp m}^2 + (\Delta v_{\perp m}/\omega)^2]^{\frac{1}{2}} \tag{43}$$

$$B = \tan^{-1}\left(\frac{\omega\Delta\rho_{\perp m}}{\Delta v_{\perp m}}\right) \tag{44}$$

Using (41) and (42) in (39) and (40), one obtains

$$\Delta\rho_{\perp} \approx \Delta\rho_{\perp m} \cos\omega(t - t_m) + \frac{\Delta v_{\perp m}}{\omega} \sin\omega (t-t_m) \tag{45}$$

$$\Delta v_{\perp} \approx - \Delta\rho_{\perp m}\omega \sin\omega (t-t_m) + \Delta v_{\perp m} \cos\omega (t-t_m) \tag{46}$$

Equations (35-38), (45-46) are in agreement with results derived by Schweppe[2] (See Eq. 2.5 of Ref. 2), who specialized the general formulation of the orbit error analysis problem by Levin to the case of nearly circular orbits. Expression of the results in other coordinate systems is also contained in Schweppe's paper.

R. Manassee

R. Manassee

BIBLIOGRAPHY

1. J. J. Levin, "The Local Behavior of Solutions to the Two-Body Problem and Related Problems," Lincoln Lab. T.R. No. 182, June, 1958.

2. F. C. Schweppe, "The Local Behavior of Circular Orbits," Lincoln Lab. Group Report 22 G-6, February, 1962.