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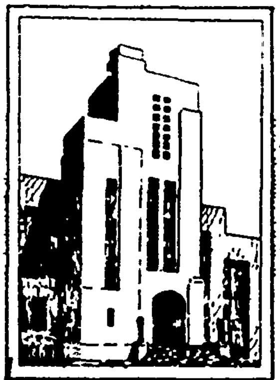

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SURVEY OF THERMAL, RADIATION, AND VISCOUS DAMPING OF PULSATING AIR BUBBLES IN WATER

by

Charles Devin, Jr.

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NOTATION

A	Attenuation
a_1	$R_0 P'/\rho s_{p_1}$
B	$b(\dot{v})^2/2$ is Rayleigh's dissipation function
b	Dissipation coefficient
C	Generalized driving function
c	Velocity of sound
D	Thermal diffusivity
d	Thickness of bubble screen
E_i	Incident sound energy
E_r	Reflected sound energy
F	Characteristic frequency of the gas bubble
f_M	Minnaert's resonant frequency
f_0	Resonant frequency
G	Universal gas constant
g	A factor which takes into account the effect of surface tension
h	Height above the bubble producers
I	Idem factor
K	Thermal conductivity
k	Restoring stiffness
L	Lagrangian function
m'	Mass of the gas contained in the volume v'
m_1	Mass of the gas in the bubble
m_2	Generalized mass
n_0	Average number of bubbles per unit volume
P'	Complex amplitude of sinusoidal pressure v'
P_2'	Instantaneous pressure on the bubble surface
P_i	Pressure inside the bubble
P_0	Static pressure

P_2	Instantaneous pressure in the undisturbed liquid
p	Sinusoidal pressure on the liquid surface
p'	Sinusoidal pressure on the bubble surface
p_a	Acoustic pressure on the bubble surface
p_i	Incident sound pressure
p_r	Reflected sound pressure
Q	Number of cycles required for the amplitude of motion to reduce to $e^{-\pi}$ of its original value
q	Amount of heat energy transferred
R	Radial distance
R'	Nonresonant bubble radius
R_0	Mean bubble radius
R_1	Instantaneous bubble radius
r	Change in radius from the mean bubble radius
S	Net stress dyadic
s_p	Specific heat at constant pressure
s_v	Specific heat at constant volume
T	Absolute temperature
T_0	Equilibrium absolute temperature
t	Time
U	Internal energy
V_0	Equilibrium bubble volume
V_1	Instantaneous bubble volume
v	Change in volume from the equilibrium bubble volume
v'	Infinitesimal element of volume in the gas bubble
W	Work done on the bubble
X	Rate of pure strain dyadic
y	Change in temperature of the gas as a function of the bubble radius
Z	Tube radius
α	A factor which describes the departure of the bubble stiffness from the adiabatic stiffness

β	$\left\{\left(\frac{R_0}{R'}\right)^2 - 1\right\}^2$
γ	Ratio of specific heats
δ	$1/Q$, the damping constant
δ_0	Resonant damping constant
ϵ	Angle between the incident sound ray and the normal to the bubble screen
η	Polytropic exponent
θ	Change in temperature from the equilibrium temperature
Λ	Natural logarithmic decrement
λ	Wavelength
μ	Coefficient of viscosity
ρ	Density
σ	Surface tension
ϕ	$(\omega/2D)^{1/2}$
ψ	$(j\omega/D)^{1/2}$
Ω	Velocity potential
ω	Circular frequency

ABSTRACT

A theoretical discussion of thermal, radiation, and viscous damping for resonant air bubbles in water is presented. An error in the derivation by Pfriem for the thermal damping constant is corrected. The experimental results verify that the damping constant at resonance is the sum of the thermal and radiation damping, and possibly viscous damping.

INTRODUCTION

The earliest reference to bubbles as sound sources was made by Bragg,[1] who attributed to entrained air bubbles the murmuring of a brook and the "plunk" of droplets falling into water. Minnaert[2] has since shown that the sound generated by gas bubbles in liquids is associated with simple volume pulsations of the bubble without change of shape. The bubble behaves as a simple damped oscillating system with one degree of freedom. Therefore, the differential equation of motion for the bubble system has the same form as the second-order linear differential equation for a mass fastened to a spring. As the bubble periodically expands and contracts, the surrounding liquid is the inert mass which is set into vibration, while the stiffness is due to the gas in the bubble. This zero-order radiator has a sharply defined resonance at the frequency:

$$f_M = \frac{1}{2\pi R_0}\sqrt{\frac{3\gamma P_0}{\rho_2}} \qquad [1]$$

where R_0 is the mean radius of the bubble,

P_0 is the static pressure at which the bubble has the mean radius R_0,

γ is the ratio of the specific heats of the gas enclosed in the bubble, and

ρ_2 is the density of the liquid. (In this report a subscript 1 will refer to properties of the gas while subscript 2 will refer to those properties of the liquid.)

Equation [1] will be derived later on page 5. This resonant frequency derived by Minnaert assumes an adiabatic equation of state for the gas in the bubble.

The volume pulsation frequency of nonspherical gas bubbles in liquids has been considered by Strasberg,[3] who used oblate spheroids to approximate the nonspherical shapes. This determination indicates that the frequency is only slightly dependent upon the ratio of the major to the minor axis of the spheroid. In fact, for a ratio of two, the volume pulsation frequency of an oblate spheroid differs by only 2 percent from that of a sphere with the same volume. Observations have shown that large bubbles are generally nonspherical whereas very small bubbles tend to be spherical.

[1]References are listed on page 34.

In addition to simple volume pulsations, there also may be oscillations in the shape of the bubble. The natural frequency for the higher modes of shape oscillation has been calculated by Lamb;[4] Strasberg[5] has used this analysis to demonstrate that shape oscillations do not seem to result in significant sound pressures except perhaps very close to the bubble. Physically, the case of a quadrupole demonstrates the reason for the feeble sound. The quadrupole represents two sets of sources and sinks for sound on the bubble surface. The distance between each source and sink is very small compared to the wavelength of the sound; therefore, on the bubble surface, almost all the sound from the source is fed back into the sink. The result is that, away from the bubble, only insignificant sound pressures occur.

The sound pressure resulting from excitation of volume pulsations by several mechanisms has recently been discussed in the literature.[6] The mechanisms, which cause bubbles to pulsate and radiate sound, are bubble formation, coalescence, or division; the motion of a free stream of liquid containing entrained gas bubbles past an obstacle, or the flow of liquid containing entrained bubbles through a pipe past a constriction; and an incident sound wave.

Experiments conducted by Sörensen[7] showed that liquids containing a gas possess higher sound damping characteristics than do those which are gas-free. Just a few widely dispersed bubbles which are so small as to be invisible can have an appreciable acoustic effect. When a large number of these small bubbles are present, the liquid will be nearly opaque acoustically. Small impurities in liquids, such as suspended particles, have negligible influence in comparison with the damping increase due to bubbles. Therefore, bubbles have a considerable importance in the transmission of underwater sound. In order to understand the attenuation of sound by gas bubbles in liquids, fundamental processes by which pulsating bubbles dissipate their energy must be known. This report will investigate the portion of the energy radiated in the form of spherical sound waves, the part which is transformed into heat during the polytropic compressions and expansions of the enclosed gas, and the part of the energy lost in viscous dissipation. It may be that these three processes completely account for the total damping of gas bubbles in liquids.

THEORY

Periodic enforced changes in the pressure on a bubble result in volume pulsations of the bubble. If the amplitude of the volume oscillation is small, the motion of the bubble system is described by a second-order linear differential equation. For this system possessing one degree of freedom, the condition of the bubble system is defined by the change in volume v from the equilibrium volume V_0. The instantaneous volume V_1 of the bubble is the algebraic sum of the mean volume V_0 and v. In a similar manner, the instantaneous radius R_1 of the bubble is the algebraic sum of the mean radius R_0 and the radial increment r. The bubble is assumed to be in an incompressible liquid; at the surface of the liquid, a sinusoidal pressure p is applied:

$$p = P e^{j\omega t} \qquad [2]$$

where P is a constant. Liquids are slightly compressible, but, as long as the bubble size is small compared to the wavelength of the pressure wave, the liquid is considered incompressible.[8] The instantaneous pressure P_2 in the undisturbed liquid is the sum of the sinusoidal pressure $P e^{j\omega t}$ and the static pressure P_0:

$$P_2 = P e^{j\omega t} + P_0 \qquad [3]$$

However, at the bubble surface, the instantaneous pressure P_2' is the instantaneous pressure P_2 in the undisturbed liquid minus the inertial reaction of the liquid in motion about the bubble. For the moment, until the inertial reaction of the liquid is determined, the instantaneous pressure P_2' *at the bubble surface* is defined as the sum of the sinusoidal pressure p' and the static pressure P_0 :

$$P_2' = p' + P_0 = P' e^{j\omega t} + P_0 \qquad [4]$$

where P' is the complex amplitude of the driving pressure p'. The bubble, which is in this uniform but alternating pressure field, cannot be in equilibrium with this oscillating pressure unless the bubble itself is pulsating. Uniform pressure in the gas bubble implies that the inertia of the gas is negligible. The liquid surrounding the bubble provides the inertia for the bubble system. The equation of motion for the bubble system is derived in terms of generalized coordinates by using Lagrange's equations. When there are no dissipation or forcing pressures present, Lagrange's equations are written in terms of the Lagrangian function L, which is defined as the kinetic energy minus the potential energy of the system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial v} = 0 \qquad [5]$$

When dissipation is present, the dissipation pressure is assumed to be proportional to the bubble volume velocity $\dot{v}$. Dissipation of this type may be derived in terms of a function B, known as Rayleigh's dissipation function, and defined as[9]

$$B = \frac{b(\dot{v})^2}{2} \qquad [6]$$

where b is the dissipation coefficient. The equation of motion for the bubble system when there are dissipation and a generalized driving function C, where neither arises from a potential, is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial v} + \frac{\partial B}{\partial v} = C \qquad [7]$$

The potential energy of the bubble system is obtained by assuming that the gas in the bubble undergoes an adiabatic process during the volume pulsations of the bubble:

$$P_2' V_1^{\gamma} = P_0 V_0^{\gamma} \qquad [8]$$

$$dP_2' = -\frac{\gamma P_0}{V_0}\, \partial V_1 \text{, or} \qquad [9]$$

$$P_2' - P_0 = -\frac{\gamma P_0}{V_0}\, v \qquad [10]$$

Therefore, the potential energy is

$$\text{P.E.} = -\int_0^v (P_2' - P_0)\, dv = \frac{\gamma P_0}{2V_0}\, v^2 \qquad [11]$$

As the bubble periodically expands and contracts, the surrounding liquid is set into vibration. The maximum kinetic energy of the liquid particles occurs at the moment the bubble has again recovered its equilibrium volume V_0. The flow of the liquid is irrotational; therefore, a velocity potential exists. The velocity potential of a liquid particle at a distance R, due to a simple source, in a liquid at rest at infinity, is[10]

$$\Omega = \frac{\dot{v}}{4\pi R} \qquad [12]$$

and the velocity of this liquid particle is

$$\dot{R} = -\nabla\Omega = \frac{\dot{v}}{4\pi R^2} \qquad [13]$$

The kinetic energy of all the liquid volume elements of density ρ_2 is

$$\text{K.E.} = \frac{\rho_2}{2}\int_{R_0}^{\infty} (\dot{R})^2\, 4\pi R^2\, dR \qquad [14]$$

The integration is extended to infinity because the bubble is assumed to be surrounded by a very large liquid volume. Upon integration, the above expression yields the kinetic energy as

$$\text{K.E.} = \frac{\rho_2}{8\pi R_0}\, (\dot{v})^2 \qquad [15]$$

Accordingly, the Lagrangian L is

$$L = \frac{\rho_2}{8\pi R_0}\, (\dot{v})^2 - \frac{\gamma P_0}{2V_0}\, v^2, \qquad [16]$$

and the equation of motion for the bubble system, when a sinusoidal pressure $Pe^{j\omega t}$ is applied at the surface of the liquid and dissipation is present, is

$$\frac{\rho_2}{4\pi R_0}\ddot{v} + b\dot{v} + \frac{3\gamma P_0}{4\pi R_0^3} v = -Pe^{j\omega t} \qquad [17]$$

The forcing function $Pe^{j\omega t}$ is preceded by a minus sign as a decrease in pressure results in an increase in the bubble volume. The term $\rho_2/4\pi R_0$ is the generalized mass m_2 of the bubble system. The stiffness of the bubble system is defined as the change in pressure on the bubble surface associated with the change in bubble volume; therefore, the term $\gamma P_0/V_0$ is the adiabatic stiffness k_{ad}:

$$k_{ad} = -\frac{\partial P_2'}{\partial V_1} = \frac{\gamma P_0}{V_0} \qquad [18]$$

Therefore, the linear second-order differential equation of motion for the bubble system is written as

$$m_2\ddot{v} + b\dot{v} + k_{ad}\, v = -Pe^{j\omega t} \qquad [19]$$

When the bubble is slightly nonspherical, each term in Equation [19] is nearly independent of shape when the mean radius R_0 is taken as the radius of a sphere of the same volume.[11] Transient volume pulsations are given by the solution of Equation [19] when the right side of the equation is set equal to zero. Furthermore, if the dissipation is negligible, Equation [19] becomes

$$m_2\ddot{v} + k_{ad} v = 0 \qquad [20]$$

and the resonant frequency of the bubble system is

$$f_M = \frac{1}{2\pi}\sqrt{\frac{k_{ad}}{m_2}} = \frac{1}{2\pi R_0}\sqrt{\frac{3\gamma P_0}{\rho_2}} \qquad [21]$$

which is Minnaert's expression as given in Equation [1] on page 1.

When the right side of Equation [19] is zero, i.e., the driving pressure has been removed, the sound pulse from the bubble consists of a damped exponential sinusoidal oscillation. The number of cycles required for the amplitude of motion to reduce to $e^{-\pi}$ of its original value is the Q of the bubble system. When the dissipation is small, the difference between the frequency of the oscillation and the resonant frequency of the bubble system without dissipation is negligible; the Q of the bubble system is expressed as

$$Q = \frac{2\pi f_0 m_2}{b} \qquad [22]$$

where f_0 is the resonant frequency. The Q may also be defined for a driven system as

$$Q = \frac{f_0}{f_2 - f_1} \qquad [23]$$

where f_2 and f_1 are the two frequencies respectively above and below resonance at which the average sound power of the bubble has dropped to one-half its resonance value.[12] The total damping constant δ is now defined as the reciprocal Q of the bubble system or π^{-1} times the natural logarithmic decrement Λ:

$$\delta = \frac{1}{Q} = \frac{\Lambda}{\pi} \qquad [24]$$

In this report, the damping constant in all cases will refer to the reciprocal Q of the bubble. The total damping may be explained by losses originating from three processes:

1. Thermal damping δ_{th} due to the thermal conduction between the gas in the bubble and the surrounding liquid.

2. Sound radiation damping δ_{rad}.

3. Viscous damping δ_{vis} due to viscous forces at the gas-liquid interface.

The total damping, expressed as the sum of these three processes, is

$$\frac{1}{Q} = \delta = \delta_{th} + \delta_{rad} + \delta_{vis} \qquad [25]$$

THERMAL DAMPING

In the derivation of Equation [1] for the resonant frequency, the adiabatic equation of state was assumed. The pressure and volume changes are in phase with one another so that dP_2'/P_0 equals $-\gamma dV_1/V_0$. For the adiabatic case, there is no transfer of heat. In the other limiting case of a purely isothermal process in the gas space, the pressure and volume changes are again in phase; dP_2'/P_0 equals $-dV_1/V_0$. For this case, there is just as much heat flowing outward from the bubble during compression as flows inward during expansion. The work done by the driving pressure in compressing the gas space is just equal to the work done by the expanding gas in moving the surrounding liquid. However, for the case of a real bubble, the gas in contact with the liquid closely follows the isothermal equation of state since the liquid has a large specific heat and thermal conductivity. In the center of a real bubble away from a substance having a high specific heat, the gas nearly follows an adiabatic equation of state. Therefore, the thermal process is polytropic for a real bubble, and a phase difference exists between the increase in pressure per unit original pressure and the decrease in volume per unit original volume. This phase difference causes a hysteresis effect. The work done on the gas volume by the driving pressure during compression is more than the work done by the gas space in moving the surrounding liquid during expansion. This difference in the work done represents a net flow of heat into the liquid. The net flow of heat into the liquid is characterized by the thermal damping constant.

The subject of thermal damping has been investigated independently by Pfriem,[13] Willis,[14] and Saneyosi,[15] and all have obtained similar results. The results of both Willis and Saneyosi are available, but unfortunately their derivations are not easily accessible. Therefore, the derivation as outlined by Pfriem will be followed.

In deriving the expression for the thermal damping constant, the gas bubble is assumed to be in an incompressible liquid, and is excited to volume pulsations by a sinusoidal pressure $P'e^{j\omega t}$ applied at the surface of the bubble. The liquid has a large specific heat and thermal conductivity, and behaves as a heat reservoir. This very large mass of liquid is capable of absorbing or rejecting an unlimited supply of heat without suffering appreciable changes in its temperature. Consequently, in the liquid adjacent to the gas-liquid interface, it will be assumed that there are no changes in temperature. A temperature field in the gas will be found that satisfies this condition, and, the second condition, that the temperature in the center of the bubble is finite. The oscillations in the pressure, volume, and temperature of the gas in the bubble will be assumed small. Consequently, the equations relating these three thermodynamic coordinates are linear. In addition, the density and the specific heats of the gas are regarded as constant. In the gas, the pressure is not a function of position but only of time. Therefore, the gas is in a uniform but alternating pressure field; the inertia of the gas in the bubble is negligible. The heat transfer process is conduction. Convection is unimportant as the time factor for establishment of this process is considerably larger than the time consumed during a half-cycle compression of the bubble.

In the subsequent discussion, the thermal damping constant is found by first finding an expression for the change in bubble volume v. During a compression of the bubble by the forcing pressure $P'e^{j\omega t}$ applied at the surface of the bubble, work is done on an infinitesimal element of volume in the gas. The internal energy of the gas in this volume element is increased, and there is a flow of heat from this volume. A differential equation for conservation of energy is formed, and the solution of this equation, subject to boundary conditions, yields the temperature field for all points in the gas space. Since the temperature field is known, the change in bubble volume v is calculated for a change in the temperature and a change in the driving pressure at the bubble surface. The expression for the change in bubble volume is then substituted into the differential equation of motion for the total gas space within the bubble. This operation yields the thermal damping constant and the stiffness of the gas. Finally, the thermal damping constant is determined as a function of the resonant frequency of the bubble system.

In order to calculate the thermal contributions to the total damping of the bubble system, the change in bubble volume must first be determined. When the driving pressure at the bubble surface compresses the bubble, work is done on the gas space. This work done on the gas space increases the internal energy of the gas, and also results in a transfer of heat energy through the gas. The added heat is transferred by conduction from the gas bubble into the surrounding liquid. The compression process must obey the conservation of energy principle as stated in the first law of thermodynamics:

$$\Delta U = \Delta q + \Delta W \qquad [26]$$

where ΔU is the increase in internal energy of the gas space, Δq is the heat added to the gas space, and ΔW is the work done on the gas space. When each term in Equation [26] is divided by an infinitesimal time Δt and Δt is allowed to approach zero as a limit, the rate of increase in internal energy is given as

$$\frac{dU}{dt} = \frac{đq}{dt} + \frac{đW}{dt} \qquad [27]$$

(A line drawn through the differential sign indicates an inexact differential.) At a point in the gas space, the rate at which work is done per unit volume by the driving pressure on an infinitesimal element of volume v' of the gas is

$$\frac{đW}{dt} = - \frac{P_2'}{v'} \; \frac{\partial v'}{\partial t} \qquad [28]$$

Since the volume decreases during compression, the term on the right side of Equation [28] is preceded by a minus sign as the rate of work done per unit volume is positive. For this small element of volume at a point, the rate of increase in internal energy per unit volume is

$$\frac{dU}{dt} = \rho_1 s_{v_1} \frac{d\theta_1}{dt} \qquad [29]$$

where s_{v_1} is the specific heat of the gas at constant volume, and θ_1 is the change in gas temperature from the equilibrium absolute temperature T_0. The rate of transfer of heat energy per unit volume for an infinitesimal volume at a point as a result of conduction is proportional to the divergence of the temperature gradient; the proportionality constant is the thermal conductivity K_1 of the gas:

$$\frac{\partial q}{\partial t} = K_1 \; \nabla^2 \theta_1 \qquad [30]$$

Since the temperature is a function of time and radial distance, Equation [30] is rewritten as

$$\frac{\partial q}{\partial t} = \frac{K_1}{R} \; \frac{\partial^2 (R\theta_1)}{\partial R^2} \qquad [31]$$

where the spherical coordinate system originates at the center of the bubble. When the expressions for the rate of increase in internal energy per unit volume at a point, the rate at which heat is added per unit volume at a point, and the rate of work done per unit volume at a point are substituted into Equation [27], the differential equation for the temperature field within the gas space is obtained:

$$\frac{\rho_1 s_{v_1}}{R}\frac{\partial(R\theta_1)}{\partial t} = \frac{K_1}{R}\frac{\partial^2(R\theta_1)}{\partial R^2} - \frac{P_2'}{v'}\frac{\partial v'}{\partial t} \qquad [32]$$

A substitution can be made for the second term on the right side of the above equation by considering the ideal gas equation:

$$P_2' v' = m'G(T_0 + \theta_1) \qquad [33]$$

where m' is the mass of the gas contained in the volume element v', and G is the universal gas constant. Equation [33] may be differentiated with respect to time to yield:

$$P_2' \frac{\partial v'}{\partial t} = \frac{m'G}{R}\frac{\partial(R\theta_1)}{\partial t} - v'\frac{\partial P_2'}{\partial t} \qquad [34]$$

or

$$\frac{P_2'}{v'}\frac{\partial v'}{\partial t} = \frac{\rho_1 G}{R}\frac{\partial(R\theta_1)}{\partial t} - \frac{\partial P_2'}{\partial t} \qquad [35]$$

As the universal gas constant G is the specific heat at constant pressure, s_{p_1}, minus the specific heat at constant volume and

$$P_2' = P_0 + P'e^{j\omega t}, \qquad [4]$$

Equation [35] may be written as

$$\frac{P_2'}{v'}\frac{\partial v'}{\partial t} = \frac{\rho_1 s_{p_1}}{R}\frac{\partial(R\theta_1)}{\partial t} - \frac{\rho_1 s_{v_1}}{R}\frac{\partial(R\theta_1)}{\partial t} - j\omega P'e^{j\omega t} \qquad [36]$$

Consequently, when this expression is introduced into Equation [32], the differential equation becomes

$$\frac{\partial(R\theta_1)}{\partial t} = \frac{K_1}{\rho_1 s_{p_1}}\frac{\partial^2(R\theta_1)}{\partial R^2} + j\frac{\omega R}{\rho_1 s_{p_1}} P'e^{j\omega t} \qquad [37]$$

Equation [37] is a linear differential equation describing the temperature field within the gas bubble. The term $K_1/\rho_1 s_{p_1}$ is a parameter of the gas which Kelvin called the thermal diffusivity D_1 of the gas. Thermal diffusivity is a measure of the rate of heat propogation due to the thermal conduction. A solution of this differential equation for the temperature field inside the gas bubble must satisfy certain boundary conditions. At the center of the bubble, the change in temperature θ_1 must be finite, and the gradient of the change in temperature must be zero. Finally, the change in temperature must be zero at the gas-liquid interface and the gradient of the change in temperature must be finite. The solution of Equation [37] may be obtained by several methods. One method is to assume that the change in temperature θ_1 is

$$\theta_1 = y\,e^{j\omega t} \qquad [38]$$

where y is a function of R only.* Therefore, Equation [37] becomes

$$j\omega\,(Ry) = D_1 \frac{\partial^2}{\partial R^2}(Ry) + j\frac{\omega R P'}{\rho_1 s_{p_1}} \qquad [39]$$

A possible solution of Equation [39] is

$$Ry = a_1 \left[\frac{R}{R_0} - \frac{\sinh \psi_1 R}{\sinh \psi_1 R_0}\right] \qquad [40]$$

Differentiation and substitution of this expression in Equation [39] shows that it is a solution if we identify a_1 with

$$a_1 = \frac{R_0 P'}{\rho_1 s_{p_1}}$$

and ψ_1 with

$$\psi_1 = \sqrt{j\frac{\omega}{D_1}}$$

Equation [40] is a solution for the temperature field providing the boundary conditions are satisfied. When R equals the mean bubble radius R_0, Equation [40] becomes

$$R_0\,y = 0, \quad y = 0 \qquad [41]$$

and the gradient of y is finite:

---

*It is evident that Equation [38] is not an exact solution of the physical problem, although it satisfies all the conditions of the mathematical equations. The physical reasoning indicates that the mean temperature inside the bubble must increase toward the center of the bubble. However, according to Equation [38], at the bubble surface the temperature gradient is $(dy/dR)\,e^{j\omega t}$, with a time average value of zero, whereas its time average value should be negative for a net outward flow of heat. This discrepancy comes about in neglecting certain second-order terms in order to obtain a differential equation with linear coefficients, and then in assuming that θ_1 and v are sinusoidal. The need for an increase in the mean value of θ_1 indicates also the existence of higher even harmonics. However, the subsequent treatment of the solution of Equation [38] leads to results that are quite good. It is possible to go back and correct the equations by the method of successive approximations.

$$\frac{\partial y}{\partial R} = \frac{a_1}{R_0}\left[\frac{1}{R_0} - \psi_1 \coth \psi_1 R_0\right] \qquad [42]$$

When R is very small, Ry is

$$Ry \simeq a_1 \left[\frac{R}{R_0} - \frac{\psi_1 R}{\sinh \psi_1 R_0}\right] \qquad [43]$$

Consequently, when R is zero, y is finite:

$$y \simeq a_1 \left[\frac{1}{R_0} - \frac{\psi_1}{\sinh \psi_1 R_0}\right] \qquad [44]$$

and the gradient of y is

$$\frac{\partial y}{\partial R} = 0 \qquad [45]$$

Therefore, the boundary conditions are satisfied; the temperature field within the whole gas space is now known. The change in bubble volume v can now be determined for a change in the driving pressure at the bubble surface and a change in the temperature θ_1. The total change in volume of the gas space is the sum of the changes in volume of all concentric shells whose radius is R and thickness dR. A shell of thickness dR has a volume:

$$v_0 = 4\,\pi\,R^2\,dR \qquad [46]$$

In accordance with the ideal gas law, two different states of the gas are expressed as

$$P_2' v = \frac{P_0 v_0}{T_0} T \qquad [47]$$

where T is the absolute temperature of the compressed gas. When Equation [47] is differentiated, the result is

$$dv = \frac{v_0}{T_0}\,dT - \frac{v_0}{P_0}\,dP_2' \qquad [48]$$

Since

$$dT = \theta_1 = y\,e^{j\omega t} \qquad [38]$$

and

$$P_2' = P_0 + P'e^{j\omega t}, \qquad [4]$$

Equation [18] is rewritten as

$$dv = 4\pi e^{j\omega t}\left[\frac{R^2 y}{T_0} - \frac{R^2 P'}{P_0}\right] dR \qquad [19]$$

The expression for $R^2 y$ is obtained from Equation [40]:

$$Ry = \frac{R_0 P'}{\rho_1 s_{p_1}}\left[\frac{R}{R_0} - \frac{\sinh \psi_1 R}{\sinh \psi_1 R_0}\right] \qquad [40]$$

Therefore, the total change in the bubble volume is

$$v = 4\pi e^{j\omega t}\int_0^{R_0}\left\{\frac{R_0 P'}{\rho_1 s_{p_1} T_0}\left[\frac{R^2}{R_0} - R\frac{\sinh \psi_1 R}{\sinh \psi_1 R_0}\right] - \frac{P' R^2}{P_0}\right\} dR \qquad [50]$$

or

$$v = \frac{V_0 P' e^{j\omega t}}{\rho_1 s_{p_1} T_0}\left\{1 - \frac{\rho_1 s_{p_1} T_0}{P_0} - \frac{3}{\psi_1^2 R_0^2}\left[\left(\psi_1 R_0 \coth \psi_1 R_0\right) - 1\right]\right\} \qquad [51]$$

Equation [51] is further simplified by noting that

$$T_0 = \frac{P_0 V_0}{m_1 G} = \frac{P_0 V_0}{m_1 (s_{p_1} - s_{v_1})} = \frac{P_0}{\rho_1 (s_{p_1} - s_{v_1})}$$

therefore,

$$v = -\frac{V_0 P' e^{j\omega t}}{\gamma P_0}\left\{1 + \frac{3(\gamma - 1)}{\psi_1^2 R_0^2}\left[\left(\psi_1 R_0 \coth \psi_1 R_0\right) - 1\right]\right\} \qquad [52]$$

or

$$-\frac{v}{V_0} = \frac{V_0 - V_1}{V_0} = \frac{P_2' - P_0}{\gamma P_0}\left\{1 + \frac{3(\gamma - 1)}{\psi_1^2 R_0^2}\left[\left(\psi_1 R_0 \coth \psi_1 R_0\right) - 1\right]\right\} \qquad [53]$$

When the change in volume per unit original volume $(V_0 - V_1)/V_0$ is plotted against the change in pressure per unit original pressure $(P_2' - P_0)/P_0$ on a pressure-volume graph for the real components of Equation [53], the area enclosed by the compression and expansion curves represents the net loss of energy by heat conduction. The work done in compressing the gas bubble is more than the work done by the gas in expanding. The change in bubble volume is now known. There remains now the task of relating the change in the bubble volume and the assumed harmonic excitation pressure at the surface of the bubble to the vibrational properties of the gas bubble; i.e., the stiffness and damping attributes.

In order to determine the stiffness and damping, the differential equation of motion for the bubble system, which was given in Equation [19] on page 5, is considered:

$$m_2 \ddot{v} + b_{th} \dot{v} + kv = - P e^{j\omega t} \qquad [19]$$

where b_{th} is the thermal dissipation coefficient and the sinusoidal forcing pressure $P e^{j\omega t}$ is applied at the surface of the liquid. Since the inertia of the gas in the bubble is negligible, Equation [19] can be rewritten for the differential equation of motion for the total gas space in the bubble:

$$b_{th} \dot{v} + kv = - (P e^{j\omega t} + m_2 \ddot{v}) = - P' e^{j\omega t} \qquad [54]$$

where $P' e^{j\omega t}$ is the sinusoidal excitation pressure at the surface of the bubble. When the expression for the change in bubble volume v:

$$v = - \frac{V_0 P' e^{j\omega t}}{\gamma P_0} \left\{ 1 + \frac{3(\gamma - 1)}{\psi_1^2 R_0^2} \left[\left(\psi_1 R_0 \coth \psi_1 R_0 \right) - 1 \right] \right\} \qquad [52]$$

is differentiated and substituted into Equation [54], the following expression is obtained:

$$\frac{1}{k + j\omega b_{th}} = \frac{V_0}{\gamma P_0} \left\{ 1 + \frac{3(\gamma - 1)}{\psi_1^2 R_0^2} \left[\left(\psi_1 R_0 \coth \psi_1 R_0 \right) - 1 \right] \right\} \qquad [55]$$

Since the parameter ψ_1 has the symbol j under the square root, which is undesirable, and there is the need to separate Equation [55] into real and imaginary components, a substitution for ψ_1 is introduced:

$$\psi_1^2 = (1 + j)^2 \phi_1^2 = 2j\phi_1^2 = j \frac{\omega}{D_1}$$

$$\psi_1 = (1 + j) \phi_1 = (1 + j) \sqrt{\frac{\omega}{2D_1}}$$

Accordingly, the parameter $\phi_1 R_0$ is

$$\phi_1 R_0 = R_0 \sqrt{\frac{\omega}{2D_1}} = R_0 \sqrt{\frac{\pi \rho_1 s_{p_1} f}{K_1}}$$

When the frequency and the radius of the bubble are kept constant, the quantity $\phi_1 R_0$ varies as the square root of the density of the gas, or alternatively as the square root of the average pressure inside the bubble since the specific heat at constant pressure and the thermal conductivity are independent of density:

$$\phi_1 R_0 \sim \sqrt{\rho_1} \sim \sqrt{P_0}$$

$(\omega, R_0$ are constant)

Another condition exists for $\phi_1 R_0$ when the excitation frequency is constant and the radius of the bubble satisfies Equation [1]:

$$R_0 = \frac{1}{\omega_M}\sqrt{\frac{3\gamma P_0}{\rho_2}} \qquad [1]$$

for a resonant bubble. Then for the resonant case, the parameter $\phi_1 R_0$ varies as the average pressure inside the bubble:

$$\phi_1 R_0 \sim P_0$$

$(\omega = \omega_M = \text{constant})$

By introducing the substitution for ψ_1 and noting the identities:

$$\sinh(\phi_1 R_0 + j\phi_1 R_0) = \sinh(\phi_1 R_0)\cos(\phi_1 R_0) + j\cosh(\phi_1 R_0)\sin(\phi_1 R_0)$$

$$\cosh(\phi_1 R_0 + j\phi_1 R_0) = \cosh(\phi_1 R_0)\cos(\phi_1 R_0) + j\sinh(\phi_1 R_0)\sin(\phi_1 R_0),$$

Equation [55] becomes

$$\frac{k - j\omega b_{th}}{k^2 + (\omega b_{th})^2} = \frac{V_0}{\gamma P_0}\left\{1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\left[\frac{\sinh(2\phi_1 R_0) - \sin(2\phi_1 R_0)}{\cosh(2\phi_1 R_0) - \sin(2\phi_1 R_0)} - j\left(\frac{\sinh(2\phi_1 R_0) + \sin(2\phi_1 R_0)}{\cosh(2\phi_1 R_0) - \cos(2\phi_1 R_0)} - \frac{1}{\phi_1 R_0}\right)\right]\right\} \qquad [56]$$

Even though Equation [56] is separated into real and imaginary terms, the form is still not suitable for determing the thermal damping constant at resonance. In Equation [22]:

$$\frac{1}{Q_{th}} = \frac{b_{th}}{\omega_0 m_2}, \qquad [22]$$

the thermal damping constant is given in terms of the thermal dissipation coefficient b_{th}, resonant circular frequency ω_0, and the generalized mass m_2. However, the generalized mass is simply the stiffness k divided by the square of the resonant circular frequency. Therefore, the dimensionless thermal damping constant, which at resonance is the reciprocal

Q or π^{-1} times the natural logarithmic decrement, is $\omega b_{th}/k$. At resonance, the maximum value of the thermal damping constant is about one-tenth; therefore, $(\omega b_{th}/k)^2$ is very small and can be neglected with respect to unity:

$$\frac{\frac{1}{k}\left(1 - j\,\frac{\omega b_{th}}{k}\right)}{1 + \left(\frac{\omega b_{th}}{k}\right)^2} \simeq \frac{1}{k}\left(1 - j\,\frac{\omega b_{th}}{k}\right) \tag{57}$$

Accordingly, Equation [56] becomes

$$\frac{1}{k}\left(1 - j\,\frac{\omega b_{th}}{k}\right) \simeq \frac{V_0}{\gamma P_0}\left\{1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\left(\frac{\sinh(2\phi_1 R_0) - \sin(2\phi_1 R_0)}{\cosh(2\phi_1 R_0) - \cos(2\phi_1 R_0)}\right)\right\} \left[1 - j\left(\frac{\dfrac{\sinh(2\phi_1 R_0) + \sin(2\phi_1 R_0)}{\cosh(2\phi_1 R_0) - \cos(2\phi_1 R_0)} - \dfrac{1}{\phi_1 R_0}}{\dfrac{2\phi_1 R_0}{3(\gamma - 1)} + \dfrac{\sinh(2\phi_1 R_0) - \sin(2\phi_1 R_0)}{\cosh(2\phi_1 R_0) - \cos(2\phi_1 R_0)}}\right)\right] \tag{58}$$

Therefore, the dimensionless thermal damping constant is

$$\frac{\omega b_{th}}{k} \simeq \left[\frac{\dfrac{\sinh(2\phi_1 R_0) + \sin(2\phi_1 R_0)}{\cosh(2\phi_1 R_0) - \cos(2\phi_1 R_0)} - \dfrac{1}{\phi_1 R_0}}{\dfrac{2\phi_1 R_0}{3(\gamma - 1)} + \dfrac{\sinh(2\phi_1 R_0) - \sin(2\phi_1 R_0)}{\cosh(2\phi_1 R_0) - \cos(2\phi_1 R_0)}}\right] \tag{59}$$

For large values of $2\phi_1 R_0$, i.e., equal to or greater than 5, the thermal damping constant is given to within 1 percent by

$$\frac{\omega b_{th}}{k} \simeq \frac{1 - \dfrac{1}{\phi_1 R_0}}{1 + \dfrac{2\phi_1 R_0}{3(\gamma - 1)}} \simeq \frac{3(\gamma - 1)}{2\phi_1 R_0} \tag{60}$$

Large values of $2\phi_1 R_0$ correspond to large bubbles or the adiabatic case. The dissipation arising from heat conduction vanishes. The thermal damping constant for very small values of $2\phi_1 R_0$, i.e., equal to or less than 2, is given to within 1 percent by

$$\frac{\omega b_{th}}{k} \simeq \frac{(\gamma - 1)}{\gamma} \frac{(2\phi_1 R_0)^2}{30} \tag{61}$$

Very small values of $2\phi_1 R_0$ correspond to small bubbles or the isothermal case, and again dissipation arising from heat conduction vanishes. The amount of heat that flows from the gas space into the liquid during compression is just equal to the amount that flows from the liquid into the gas during expansion; the net flow of heat is zero. However, there is a transition region between small and large values of $2\phi_1 R_0$ where the thermal dissipation is a maximum. In this transition region, the relation between the pressure and the total gas volume can be expressed as $P_2' V_1^{\eta}$ equals a constant, where the exponent η varies from unity to γ; the state is polytropic. Figure 1, a plot of the thermal damping constant $\omega\, b_{th} / k$ *vs* $2\phi_1 R_0$, clearly illustrates this transition region between isothermal and adiabatic states.

The stiffness of gas bubbles is also important since the resonant frequency for the bubble system is directly proportional to the square root of the restoring stiffness. By comparing the real terms on both sides of Equation [56], the stiffness is expressed as

$$\frac{kV_0}{\gamma P_0} = \frac{1}{\left\{1 + \left(\frac{\omega\, b_{th}}{k}\right)^2\right\} \left\{1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\left(\frac{\sinh\,(2\phi_1 R_0) - \sin\,(2\phi_1 R_0)}{\cosh\,(2\phi_1 R_0) - \cos\,(2\phi_1 R_0)}\right)\right\}} \tag{62}$$

For large values of $2\phi_1 R_0$, i.e., for large bubbles and the stiffness approaching the adiabatic stiffness, the dimensionless stiffness $kV_0/\gamma P_0$ is given to within one percent by

$$\frac{k_{ad} V_0}{\gamma P_0} \simeq \frac{1}{\left\{1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\left(1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\right)\right\}} \tag{63}$$

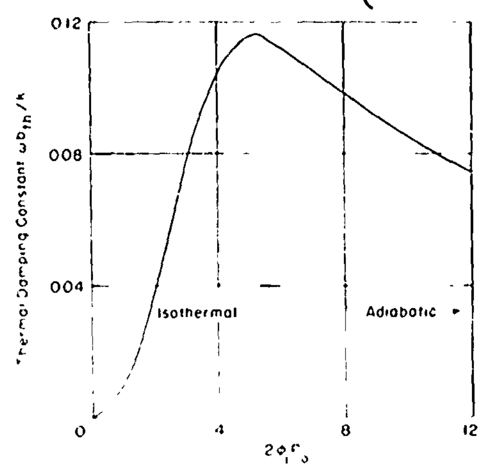

Figure 1 – Thermal Damping Constant versus Dimensionless Parameter $2\phi_1 R_0$

Pfriem, due to an oversight, used only the first power of $\omega\delta_{th}/k$ in obtaining Equation [63], and, therefore, derived the erroneous result:

$$\frac{k_{ad}V_0}{\gamma P_0} \simeq \frac{1}{\left\{1 + \frac{3(\gamma - 1)}{\phi_1 R_0}\left(1 - \frac{1}{2\phi_1 R_0}\right)\right\}} \qquad [64]$$

for the dimensionless stiffness. The stiffness as given by Equation [63] is used to determine the thermal damping constant at resonance. Therefore, the result obtained by Pfriem yields an incorrect thermal damping constant at resonance. For small values of $2\phi_1 R_0$, i.e., for small bubbles and the stiffness approaching the isothermal stiffness, the dimensionless stiffness is given to within one percent by

$$\frac{k_{iso}V_0}{\gamma P_0} \simeq \frac{1}{\left\{\gamma - \frac{(2\phi_1 R_0)^4}{1390}\left(1 - \frac{2.1(\gamma - 1)^2}{\gamma}\right)\right\}} \qquad [65]$$

When the dimensionless stiffness $kV_0 / \gamma P_0$ is plotted as a function of $2\phi_1 R_0$, as in Figure 2, the dimensionless stiffness approaches γ^{-1} for small values of $2\phi_1 R_0$ and approaches unity for large values of $2\phi_1 R_0$. The restoring stiffness of the gas and the thermal damping constant are now known. The only remaining task is to introduce the correct expression for the stiffness into the equation for the resonant frequency, and then determine the thermal damping constant δ_{th} at resonance.

The correct expression for the stiffness will now be introduced into the equation for the resonant frequency. Minnaert derived the equation for the resonant frequency of pulsating gas bubbles in liquids by considering an adiabatic equation of state:

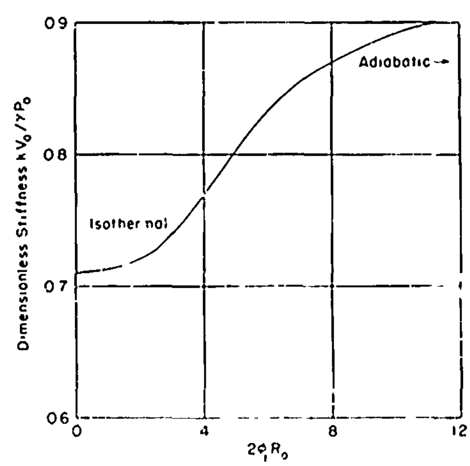

Figure 2 – Dimensionless Stiffness versus Dimensionless Parameter $2\phi_1 R_0$

$$f_M = \frac{1}{2\pi}\sqrt{\frac{k_{ad}}{m_2}} = \frac{1}{2\pi R_0}\sqrt{\frac{3\gamma P_0}{\rho_2}} \qquad [1]$$

However, the state is polytropic; the stiffness constant k for large bubbles, which is of practical importance in discussing underwater sound transmission, is given by Equation [63].

$$k = \frac{\eta P_0}{V_0} \simeq \frac{\gamma P_0}{V_0}\left\{\frac{1}{1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\left(1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\right)}\right\} \qquad [63a]$$

$$k = \frac{\eta P_0}{V_0} \simeq \frac{\gamma P_0}{V_0 \alpha} \qquad [66]$$

where

$$\alpha = \frac{\gamma}{\eta} = 1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\left(1 + \frac{3(\gamma - 1)}{2\phi_1 R_0}\right)$$

The factor α describes the departure of the bubble stiffness from the adiabatic stiffness. Consequently, Equation [1] becomes

$$f = \frac{1}{2\pi R_0}\sqrt{\frac{3\gamma P_0}{\rho_2 \alpha}} = \frac{f_M}{\sqrt{\alpha}} \qquad [67]$$

In the discussion so far, the instantaneous pressure inside the bubble has been considered the same as the instantaneous pressure on the surface of the bubble. However, when the bubble is small, the surface tension pressure increases the pressure inside the bubble; consequently, the instantaneous pressure inside the bubble is greater than the instantaneous pressure on the bubble surface. Smith;[16] Briggs, Johnson, and Mason;[17] Spitzer;[18] and Robinson and Buchanan[19] are among some of the investigators who have discussed the effect of surface tension on the bubble stiffness. The problem is to relate the pressure on the surface of the bubble, which is associated with the change in bubble volume, to the pressure inside the bubble. The stiffness k' is defined as

$$k' = -\frac{\partial P_2'}{\partial V_1} \qquad [18a]$$

The pressure P_i inside the bubble is

$$P_i = P_2' + \frac{2\sigma}{R_1} \qquad [68]$$

where P'_2 is the pressure on the bubble surface, σ is the surface tension, and R_1 is the instantaneous bubble radius. The polytropic equation of state for the gas inside the bubble is

$$P_i = \left(P_0 + \frac{2\sigma}{R_0}\right)\left(\frac{V_0}{V_1}\right)^{\eta} \qquad [69]$$

Therefore,

$$P'_2 = P_i - \frac{2\sigma}{R_1} = \left(P_0 + \frac{2\sigma}{R_0}\right)\left(\frac{V_0}{V_1}\right)^{\eta} - \frac{2\sigma}{R_1} \qquad [70]$$

and

$$k' = -\frac{\partial P'_2}{\partial V_1} = \eta\left(P_0 + \frac{2\sigma}{R_0}\right)\left(\frac{V_0}{V_1}\right)^{\eta} \frac{1}{V_1} - \frac{2\sigma}{3R_1V_1} \qquad [71]$$

$$k' = \frac{\eta P_0}{V_0}\left\{1 + \frac{2\sigma}{P_0R_0} - \frac{2\sigma}{3\eta P_0 R_0}\right\} \qquad [71a]$$

$$k' = \frac{\eta P_0}{V_0}\, g = \frac{\gamma P_0 g}{V_0 \alpha} \qquad [71b]$$

where

$$g = 1 + \frac{2\sigma}{P_0R_0} - \frac{2\sigma}{3\eta P_0 R_0}$$

Therefore, the correct expression for the resonant frequency f_0 is

$$f_0 = \frac{1}{2\pi R_0}\sqrt{\frac{3\gamma P_0 g}{\rho_2 \alpha}} = f_M\sqrt{\frac{g}{\alpha}} \qquad [72]$$

When the bubbles are very large, the stiffness is the adiabatic stiffness and also surface tension effects are negligible; consequently, the ratio g/α is unity and the resonant frequency is given exactly by Minnaert's equation.

Since damping is of prime importance at resonance, the thermal damping constant will now be determined as a function of the resonant frequency. The dimensionless parameter $\phi_1 R_0$ is written as

$$\phi_1 R_0 = R_0\sqrt{\frac{2\pi f_0}{2D_1}} = \frac{1}{2\pi f_0}\sqrt{\frac{3\gamma P_0 g}{\rho_2\alpha}\left(\frac{\pi f_0}{D_1}\right)} \qquad [73]$$

$$\phi_1 R_0 = \sqrt{\frac{3\gamma P_0}{4\pi\rho_2 D_1}\left(\frac{g}{f_0\alpha}\right)} = \sqrt{\frac{Fg}{f_0\alpha}} \qquad [73a]$$

where

$$F = \frac{3\gamma P_0}{4\pi\rho_2 D_1}$$

The quantity F, for a given pressure, is a constant for the gas. Since the parameter F has the dimensions of reciprocal time, it is sometimes called the characteristic frequency of the gas bubble. When Equation [73a] is squared and the expression for α substituted, a quadratic equation for $\phi_1 R_0$ results in terms of the resonant frequency f_0, characteristic frequency F, and g. The solution of this quadratic equation is

$$\phi_1 R_0 = \frac{3}{4}(\gamma - 1)\left(\sqrt{\frac{16}{9(\gamma-1)^2}\frac{Fg}{f_0} - 3} - 1\right) \qquad [74]$$

By substituting the value of $\phi_1 R_0$, as given by Equation [74], into Equation [60]:

$$\frac{\omega b_{th}}{k} \simeq \frac{1 - \dfrac{1}{\phi_1 R_0}}{1 + \dfrac{2\phi_1 R_0}{3(\gamma - 1)}} \qquad [60]$$

the thermal damping constant δ_{th} at resonance is found:

$$\delta_{th} = \frac{\omega_0 b_{th}}{k} = 2 \cdot \left[\frac{\sqrt{\dfrac{16}{9(\gamma-1)^2}\dfrac{Fg}{f_0} - 3} - \dfrac{(3\gamma - 1)}{3(\gamma - 1)}}{\dfrac{16}{9(\gamma-1)^2}\dfrac{Fg}{f_0} - 4}\right] \qquad [75]$$

As Pfriem used an incorrect expression for α, he derived an incorrect thermal damping constant; his expression is

$$\delta_{th} = \frac{1}{\sqrt{\dfrac{3\gamma - 1}{3(\gamma-1)} - \dfrac{Fg}{f_0}}}\left\{1 - \frac{2}{3(\gamma - 1)\sqrt{\dfrac{3\gamma - 1}{3(\gamma - 1)} - \dfrac{Fg}{f_0}} - 1}\right\} \qquad [76]$$

When the following values are used:

$P_0 = 1 \times 10^6$ dynes/cm^2,

$\sigma = 75$ dynes/cm,

$K_1 = 5.6 \times 10^{-5}$ cal/cm-sec-deg c,

$\gamma = 1.40$

s_{p_1} 0.24 cal/gm,

ρ_2 1.00 gm/cm^3,

ρ_1 = 1.29 · 10^{-3} gm/cm^3,

the thermal damping constant δ_{th} calculated using Equation [75] and that using Pfriem's equation are plotted in Figure 3 as a function of the resonant frequency. Figure 3 clearly shows that the result obtained by Pfriem is 50 to 65 percent too high. The thermal damping constant determined in this report, using the method outlined by Pfriem, agrees exactly with the curve of Willis as given in the report by Spitzer. The thermal damping constant δ_{th} for air bubbles in water larger than 15 microns radius and with resonant frequencies less than 240 kilocycles per second is determined to within 1 percent by using Equation [75]. When the resonant frequency is small, Equation [75] for the thermal damping constant is replaced by the simple equation:

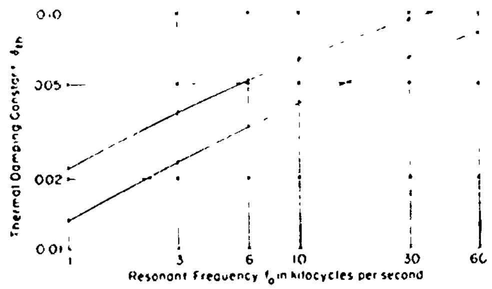

Figure 3 – Thermal Damping Constant for Resonant Air Bubbles in Water

The upper curve is the incorrect theoretical result obtained by Pfriem.

$$\delta_{th} = \frac{1}{2}\sqrt{\frac{9(\gamma - 1)}{F}\, f_0} = 4.41 \times 10^{-4} \sqrt{f_0} \text{ (seconds)} \qquad [77]$$

where f_0 is the resonant frequency in cycles per second. This simple relationship gives the thermal damping constant to within 1 percent for air bubbles in water larger than 0.05 cm; i.e., for air bubbles with resonant frequencies less than about 7 kilocycles per second. Equation [77] reveals that the thermal damping constant at low frequencies is proportional to the square root of the resonant frequency.

RADIATION DAMPING

In a compressible liquid, a bubble excited into volume pulsations expends a portion of its energy by radiating spherical sound waves. The bubble is considered as a simple sound source; the bubble radius R_0 is considered small compared to the wavelength λ of the radiated sound. Smith[20] has calculated the radiation damping for gas bubbles in liquids. In order to derive the expression for the radiation damping constant, the velocity potential for a simple sinusoidal source in a compressible liquid is stated:

$$\Omega_0 = j\frac{\omega v_1}{4\pi R} e^{j\omega(t - R/c_2)} \qquad [78]$$

where c_2 is the velocity of sound in the liquid,

R is the radial distance, and

v_1 is the complex amplitude of the change in bubble volume v.

Equation [52]:

$$v = -\frac{V_0 P' e^{j\omega t}}{\gamma P_0}\left\{1 + \frac{3(\gamma - 1)}{\psi_1^2 R_0^2}\left(\psi_1 R_0 \coth(\psi_1 R_0) - 1\right)\right\} \qquad [52]$$

$$v = v_1 e^{j\omega t} \qquad [79]$$

defines v_1 as

$$v_1 = -\frac{V_0 P'}{\gamma P_0}\left\{1 + \frac{3(\gamma - 1)}{\psi_1^2 R_0^2}\left(\psi_1 R_0 \coth(\psi_1 R_0) - 1\right)\right\} \qquad [80]$$

The acoustic pressure is determined from the velocity potential by the following equation:

$$p_a = \rho_2 \frac{\partial \Omega_0}{\partial t} = -\frac{\rho_2 \omega^2 v_1}{4\pi R} e^{j\omega(t - R/c_2)} \qquad [81]$$

On the bubble surface, the acoustic pressure is

$$p_a = -\frac{\rho_2 \omega^2 v_1}{4\pi R_0} e^{j\omega t} e^{-j\omega R_0/c_2} \qquad [82]$$

or

$$p_a = -\frac{\rho_2 \omega^2 v_1}{4\pi R_0} e^{j\omega t}\left\{1 - j\frac{\omega R_0}{c_2} - \frac{\omega^2 R_0^2}{c_2^2 2!} + j\frac{\omega^3 R_0^3}{c_2^3 3!}\right\} \qquad [82a]$$

where only the first four terms are kept in the expansion. The acoustic pressure on the bubble surface is just the difference between the driving pressure on the surface of the liquid and the change in pressure on the bubble surface associated with the change in bubble volume. The change in pressure on the bubble surface is $k'v$. Therefore,

$$p_a + k'v = p_a + k'v_1 e^{j\omega t} = -P e^{j\omega t} \qquad [83]$$

$$
-\frac{\rho_2 \omega^2 v_1 e^{j\omega t}}{4\pi R_0}\left(1-\frac{\omega^2 R_0^2}{c_2^2\, 2!}\right)+j\frac{\rho_2 \omega^2 v_1 e^{j\omega t}}{4\pi R_0}\left(\frac{\omega R_0}{c_2}\right.
$$

$$
\left.-\frac{\omega^3 R_0^3}{c_2^3\, 3!}\right)+k' v_1 e^{j\omega t}=-P e^{j\omega t} \qquad [84]
$$

Since

$$
\dot{v}=j\omega\, v_1\, e^{j\omega t} \qquad [85]
$$

and

$$
\ddot{v}=-\omega^2 v_1 e^{j\omega t}, \qquad [86]
$$

then

$$
\frac{\rho_2}{4\pi R_0}\left(1-\frac{\omega^2 R_0^2}{c_2^2\, 2!}\right)\ddot{v}+\frac{\rho_2 \omega}{4\pi R_0}\left(\frac{\omega R_0}{c_2}-\frac{\omega^3 R_0^3}{c_2^3\, 3!}\right)\dot{v}+k' v=-P e^{j\omega t} \qquad [87]
$$

For a large resonant air bubble in water with a resonant frequency of one kilocycle per second and a radius of 0.33 cm, the term $\omega_0 R_0/c_2$ is

$$
\frac{\omega_0 R_0}{c_2}=\frac{2\pi R_0}{\lambda_2}\simeq\frac{1}{75}
$$

Therefore, the terms of higher order than $\omega_0 R_0/c_2$ will be neglected and Equation [87] becomes

$$
\frac{\rho_2}{4\pi R_0}\ddot{v}+\frac{\rho_2 \omega^2}{4\pi c_2}\dot{v}+k' v=-P e^{j\omega t} \qquad [88]
$$

As long as the bubble radius is small compared to the wavelength of the radiated sound, it is seen that the generalized mass term for the case of a compressible liquid is the same as the corresponding term in Equation [17] where the liquid was considered incompressible.

$$
\frac{\rho_2}{4\pi R_0}\ddot{v}+b\dot{v}+k' v=-P e^{j\omega t} \qquad [17a]
$$

The radiation dissipation coefficient b_{rad} is

$$
b_{rad}=\frac{\rho_2 \omega^2}{4\pi c_2} \qquad [89]
$$

Therefore, the radiation damping constant at resonance is

$$\delta_{rad} = \frac{1}{Q_{rad}} = \frac{b_{rad}}{\omega_0 m_2} = \frac{2\pi R_0 f_0}{c_2} = \frac{2\pi R_0 f_g}{c_2} \sqrt{\frac{g}{\alpha}} \qquad [90]$$

where m_2 is the generalized mass. The factor g takes into account the effect of surface tension whereas α describes the departure of the bubble stiffness from the adiabatic stiffness. For large bubbles, the ratio g/α is unity, and, since the quantity $2\pi R_0 f_g / c_2$ is a constant, the radiation damping constant at resonance is independent of frequency.

VISCOUS DAMPING

The problem of a pulsating spherical bubble in a viscous, incompressible liquid will now be treated; the viscous damping constant at resonance will be derived. Mallock[21] in 1910, and, later, Spitzer[22] and Poritsky[23] investigated this problem. For a pulsating bubble, the effect of viscosity is perhaps difficult to visualize. Lamb[24] states, "The only condition under which a liquid can be in motion without dissipation of energy by viscosity is that there must be nowhere any extensions or contraction of linear elements; in other words, the motion must consist of a translation and a rotation of the mass as a whole, as in the case of a rigid body." Mallock gives us a physical picture of the effect of viscosity on a pulsating bubble in an incompressible, viscous liquid by considering a small element of a spherical shell of liquid at the bubble surface. This element has definite radial and lateral dimensions at the instant the bubble radius is at its mean position. When the bubble expands, the small liquid element is distorted; the radial thickness decreases while the lateral dimension increases. Likewise, when the bubble contracts, the liquid element is again distorted; this time the radial thickness increases and the lateral dimension decreases. Since the liquid is incompressible, the distortion is not caused by a change in the volume of the liquid element but by viscous stresses. Consequently, more energy is required to compress the bubble than is regained in the subsequent expansion.

In the presence of viscosity, momentum is transmitted from one region of the liquid to another moving at a different velocity. An element of liquid moving rapidly in a particular direction tends to transmit its momentum to other elements of the liquid. The Navier-Stokes equation of motion describes the force per unit volume acting on an infinitesimal element of volume, at a point in a viscous liquid. The force per unit volume is due to the instantaneous pressure distribution of the surrounding liquid as in the case of a nonviscous liquid, and is also due to the rate of change of momentum caused by the presence of viscosity. When there are no external forces acting on each unit mass of the liquid, the equation of motion is

$$\rho_2 \left(\frac{\partial}{\partial t} + \vec{\dot{R}} \cdot \nabla \right) \vec{\dot{R}} = - \nabla p_0 + \frac{\mu}{3} \nabla (\nabla \cdot \vec{\dot{R}}) + \mu \nabla^2 \vec{\dot{R}} \qquad [91]$$

where

$$\dot{\vec{R}} = \text{radial velocity vector,}$$

$$\left(\frac{\partial}{\partial t} + \dot{\vec{R}} \cdot \nabla\right) \dot{\vec{R}} = \text{acceleration vector,}$$

$$p_0 = \text{mean pressure, and}$$

$$\mu = \text{coefficient of viscosity.}$$

As the liquid is considered to be incompressible, the divergence of the velocity $\nabla \cdot \dot{\vec{R}}$ vanishes so that the second term on the right side of Equation [91] disappears. The only remaining viscous term is $\mu \nabla^2 \dot{\vec{R}}$. Since the motion of the liquid is irrotational, the velocity can be expressed as the gradient of a scalar velocity potential Ω, and the last term on the right side of Equation [91] is written:

$$\mu \nabla^2 \dot{\vec{R}} = \mu \nabla^2 (-\nabla\Omega) = -\mu \nabla (\nabla \cdot \nabla \Omega)$$

$$\mu \nabla^2 \dot{\vec{R}} = \mu \nabla \; (\nabla \cdot \dot{\vec{R}}) = 0$$

Therefore, there are no net viscous forces acting inside the liquid for the case of a pulsating spherical bubble in an incompressible, viscous liquid. Due to the presence of viscosity, momentum is transmitted through the liquid, but each infinitesimal element of liquid volume receives just as much as it loses; therefore, there is no net viscous force acting on any element of volume internal to the liquid. The Navier-Stokes equation is not applicable for discussing the effect of viscosity for pulsating spherical bubbles.

However, even though the net viscous forces in the liquid vanish, there are viscous forces acting at the surface of the bubble where they exert an excess pressure. The net stress dyadic S is written as[25]

$$S = -\frac{2}{3} \mu \nabla \cdot \dot{\vec{R}} I + 2 \mu X \qquad [92]$$

where I is the idem factor and X is the rate of pure strain dyadic. (The algebraic signs conform with the practice in elasticity of denoting a tension as positive and a pressure as negative.) As the liquid is incompressible, the divergence of the velocity is zero, and Equation [92] simplifies to:

$$S = 2 \mu X \qquad [93]$$

Due to the radial motion and spherical symmetry, the principal directions of stress and rate of strain must be radial; this will be chosen to correspond to S_R and X_R, respectively:

$$S_R = 2\mu\lambda_R \qquad [94]$$

The rate of pure strain λ_R is the gradient of the radial velocity.

$$\lambda_R = \nabla\vec{R} = \frac{\partial}{\partial R}\left(\frac{\dot{v}}{4\pi R^2}\right) \qquad [95]$$

Therefore, the radial stress S_R at the bubble surface is

$$S_R = 2\mu\lambda_R = -\frac{\mu\dot{v}}{\pi R_0^3} \qquad [96]$$

and the equation of motion for the bubble system, when the effect of viscosity is included, is

$$m_2\ddot{v} + \frac{\mu}{\pi R_0^3}\dot{v} + k'v = -Pe^{j\omega t} \qquad [97]$$

where

$$\frac{\mu}{\pi R_0^3} = b_{vis}$$

and

$$m_2 = \frac{\rho_2}{4\pi R_0}$$

Therefore, the viscous damping constant at resonance is

$$\delta_{vis} = \frac{1}{Q_{vis}} = \frac{b_{vis}}{\omega_0 m_2} = \frac{8\pi\mu\alpha}{3\gamma P_0 g} f_0 = \frac{8\pi\mu}{3\gamma P_0} f_0\sqrt{\frac{\alpha}{g}} \qquad [98]$$

where g is the factor which takes into account the effect of surface tension, and α is a factor which describes the departure of the bubble stiffness from the adiabatic stiffness. The viscous damping constant δ_{vis} is directly proportional to the resonant frequency f_0.

The effect of viscosity is realized only through the boundary condition at the surface of the bubble, rather than through the Navier-Stokes equation where the resultant of the viscosity stresses per unit volume at any point internal to the liquid vanishes.

TOTAL DAMPING CONSTANT AT RESONANCE

The total damping for resonant air bubbles in water may be explained by losses originating from the energy which is transformed into heat during the polytropic compressions and expansions of the air in the bubble, the energy radiated in the form of spherical sound waves, and the energy lost in viscous dissipation. The total damping constant δ_0 at resonance is

$$\delta_0 = \frac{1}{Q_0} = \delta_{th} + \delta_{rad} + \delta_{vis} \qquad [25]$$

or

$$\delta_0 = 2\left[\frac{\sqrt{\frac{16}{9(\gamma - 1)^2}\frac{Fg}{f_0} - 3} - \frac{(3\gamma - 1)}{3(\gamma - 1)}}{\frac{16}{9(\gamma - 1)^2}\frac{Fg}{f_0} - 1}\right] + \frac{2\pi f_0 R_0}{c_2} + \frac{8\pi\mu\alpha}{3\gamma P_0 g}F_0 \qquad [99]$$

Figure 4 is a plot of the thermal, radiation, viscous, and total damping constants as a function of the resonant frequency f_0 for air bubbles in water. Damping constants for air bubbles in water ranging in radius from 3 microns to 3 millimeters are displayed in Figure 4. The thermal damping constant δ_{th} reaches a maximum around 200 kilocycles per second while the radiation damping constant δ_{rad} is nearly constant up to 600 kilocycles per second and then very slowly begins to increase. The viscous damping constant δ_{vis} is of the same order of magnitude as the radiation damping constant at 180 kilocycles per second; just above 1000 kilocycles per second, the viscous damping constant and the thermal damping constant have the same value.

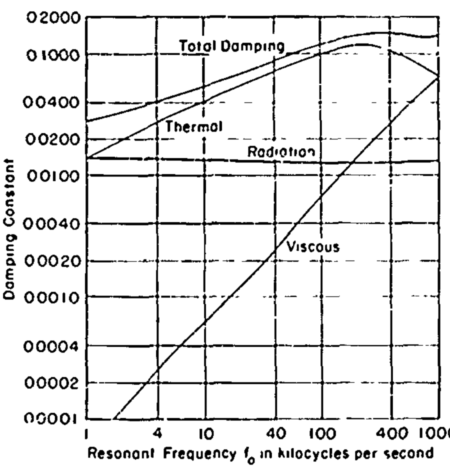

Figure 4 – Theoretical Thermal, Radiation, Viscous, and Total Damping Constants for Resonant Air Bubbles in Water

EXPERIMENTAL METHODS

There are essentially four methods by which the damping constant δ_0 can be determined experimentally. Most of these methods are indirect ones involving the calculation of the damping constant from certain measured acoustical properties of the bubbles.

SUCCESSIVE OSCILLATIONS[26]

The method of successive oscillations is a direct process for determining the damping constant. The signal from a hydrophone, which is placed close to an oscillating bubble, is amplified and applied to the input terminals of a cathode-ray oscilloscope; the bubble pulse appears on the screen as a damped sine wave. If the amplitude of successive oscillation is plotted on semi-log paper as a function of the cycle number of oscillation, the logarithmic decrement, and, therefore, the reciprocal Q can be determined from the slope. The resonant frequency of the pulsating bubble is determined by applying a signal of known frequency to the input of the oscilloscope and measuring the time scale across the screen.

WIDTH OF THE RESONANCE RESPONSE[27,28,29]

On page 5, the Q of the bubble system was defined in terms of the resonant frequency and the two frequencies above and below at which the average sound power of the bubble had decreased to one-half its resonance value. Since the power is proportional to the square of either the radial velocity or the radial displacement, the damping constant δ_0 can be found by plotting the square of either of these parameters as a function of the frequency. The amplitude of oscillation of large gas bubbles can be found using a photoelectric method, and, for small bubbles, the radial velocity can be measured using a kind of velocity-ribbon microphone.

1. Photoelectric Method. A single gas bubble oscillating to a sonic excitation is illuminated optically and the scattered light measured by a photoelectric cell. The change in cross section of the bubble image modulates the quantity of light received at the photocell. The alternating current generated in the circuit of the photocell, by the change in the bubble cross section, is amplified and recorded on a suitable recorder. By varying the sonic excitation frequency and noting the changes in the bubble cross section, the band width and the resonant frequency can be determined.

2. Ribbon Microphone Method. A single bubble is caught on a small wax sphere fastened to a platinum thread which is placed between the poles of an electromagnet. The bubble oscillations are produced by a constant frequency magnetostriction projector. As the bubble oscillates, the platinum thread is carried along with the oscillations and this motion of the platinum thread produces an alternating EMF which is proportional to the radial velocity of the gas bubble. This arrangement represents a sort of ribbon microphone. Since the generalized mass is greater than the vibrating part of the mass of the thread or the wax, the platinum wire and the wax are assumed to exert negligible influences on the resonant frequency. The bubble is allowed to grow slowly and its diameter measured with a microscope; the voltage produced by the ribbon microphone traverses a maximum as the diameter of the bubble increases. Therefore, the resonant frequency and damping constant can be determined.

STANDING-WAVE RATIOS[30,31,32]

A single gas bubble is allowed to rise freely in a liquid-filled tube and oscillate under the influence of a plane progressive sound wave. The diameter of the tube is less than half a wavelength so that the sound pressure is constant over the cross section of the tube. Disturbances by reflection of this sound wave from the free surface are prevented by an absorption device or by using a pulse technique. The sound energy E_t, which is radiated by a transducer at the lower end of the tube, is partly reflected by the bubble E_r and recorded by a probe hydrophone arranged between the transducer and the bubble. The damping constant δ_0 can be measured from the relative reflection coefficient $(E_r/E_i)^{1/2}$ of a bubble oscillating at its resonant frequency:

$$\delta_0 = \sqrt{2}\,\frac{R_0}{Z}\sqrt{\frac{E_r}{E_i}} = \sqrt{2}\,\frac{R_0}{Z}\left(\frac{p_r}{p_i}\right) \qquad [100]$$

where Z is the radius of the tube, and p_i and p_r are the sound pressures of the incident and reflected waves, respectively. However, the energy of the reflected signal cannot be referred to that of the direct signal because of friction losses occuring in the tube. In order to obtain the corresponding energy of the direct signal, the energy of the reflected pulse is measured when the bubble is replaced by an obstacle covering the entire cross section of the tube, a large bubble, for instance. Therefore, the damping constant can be measured from the standing-wave ratio, and the resonant frequency is that frequency of the plane progressive sound wave which produces the maximum oscillation for the bubble.

RESONANCE ABSORPTION[33,34]

This method of determining the damping constant depends upon measuring the attenuation of sound by a screen of bubbles. For bubbles of a single size, the attenuation through a bubble screen is a maximum at the resonant frequency of the bubbles. A projector and transmission hydrophone are located on opposite sides of the bubble screen. Each instrument is faced toward the other, and the line joining them, at the point of intersection with the bubble screen, forms an angle ϵ with the normal to the screen. In order to obtain data as to the distribution of bubbles according to size, the rate of rise of bubbles, which is a function of the bubble radius, is determined. If only a very short burst of bubbles is allowed to escape from the bubble producers and the resultant screen is observed at a height h and time t later, the screen contains only those bubbles whose rate of rise is h/t. When the bubbles are allowed to rise freely but in these definite bursts or pulses, attenuation measurements versus time elapsed after the initiation of the pulse bubble screen are made using the transmission hydrophone. This method is repeated for several projector frequencies. Carstensen and Foldy[35] give the resonant damping constant δ_0 in terms of the attenuation A:

$$1 \quad 1.31\, n_0 \left[\frac{\dfrac{1\pi R_0^4 \delta_0}{(R')^2 \delta_{\mathrm{rad}}}}{\beta + (1 \pm \beta^{1/2})^3\, \delta_0^2} \right] d \sec \epsilon \qquad [101]$$

where n_0 is the average number of bubbles per unit volume,

d is the thickness of the bubble screen,

R' is the off-resonant bubble radius, and

$$\beta = \left(\frac{R_0^2}{(R')^2} - 1\right)^2$$

In deriving Equation [101], Carstensen and Foldy made some assumptions about the off-resonance behavior of the damping constant. They assumed, for a bubble screen containing bubbles of essentially uniform size, that the off-resonance damping constant equals $(R_0/R')^3\delta_0$ where δ_0 is the resonant damping constant. The distribution in size, space, and number of the bubbles must be known to determine the resonant damping constant.

COMPARISON OF THEORY AND EXPERIMENTAL RESULTS

The theoretical damping constant and the experimental values for the damping constants are plotted as a function of the resonant frequency in Figure 5. Therefore, Figure 5 gives an indication as to how well the experimental results agree with the theory of damping.

Meyer and Tamm[36] have used the width of the resonance response method to obtain the damping constant; these results are extremely high. This high damping constant may be due to the particular conditions of the experiment. In using the ribbon microphone, considerable damping may have been due to the oscillation of the platinum thread in the magnetic field. In addition, the experimenters themselves state the bubbles appeared dull and blurred near the resonance point; consequently, the diameters of the bubbles could not have been measured accurately with a microscope, which would affect the determination of the resonant frequency. The damping constant for large bubbles was determined using the photoelectric method. For this procedure, Meyer and Tamm, and later Lauer,[37] used a thin wire annulus to hold the bubbles and prevent them from rising to the surface while the measurements were being made. Indeed, the high damping constants found by Meyer and Tamm may be due to the wire annuli adding to the damping of the bubble system. At low frequencies, the damping constants measured by Lauer are about 25 percent higher than the theoretical prediction.

Bauer, formerly of the David Taylor Model Basin and now at Reeves Instrument Company, used the successive oscillation method for determining the damping constant. In this experiment, the damping constant for a free bubble was measured; therefore, there is no additional damping due to a bubble holder. The bubble was formed at a nozzle; the volume pulsations started just as the bubble closed and separated from the nozzle. The unpublished damping constant measurements of Bauer are about 20 percent higher than the theoretical prediction.

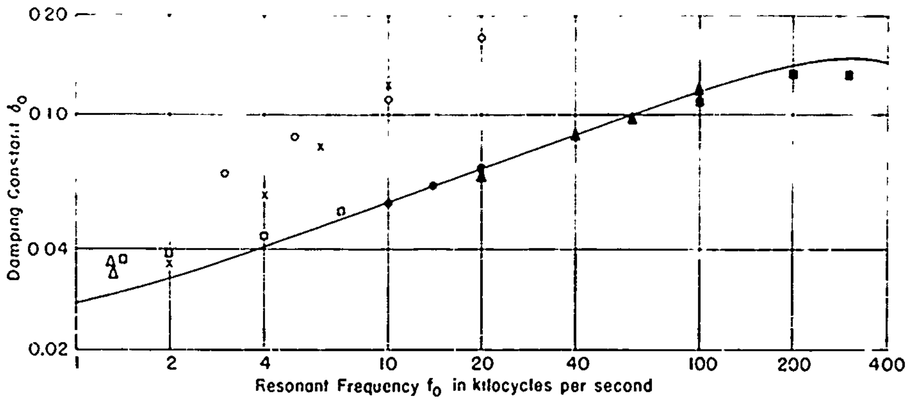

Figure 5 – Theoretical and Experimental Values of the Damping Constant for Resonant Air Bubbles in Water

Points are from faired curves through the experimenter's data.

Symbol	Experimenter	Method
X	Meyer and Tamm	Width of Resonance Response
o	Carstensen and Foldy	Resonance Absorption
Δ	Bauer	Successive Oscillations
□	Lauer	Width of Resonance Response
●	Exner	Standing-Wave Ratios
▲	Exner and Hampe	Standing-Wave Ratios
■	Haeske	Standing-Wave Ratios

The method of standing-wave ratios was used by Exner,[38] Exner and Hampe,[39] and Haeske,[40] to determine the resonant damping constant; the results of Exner, and Exner and Hampe agree very well with the theoretical curve. According to the theory, the viscous damping becomes important around 200 kilocycles per second; Haeske has measured the damping constant in this frequency range. At resonant frequencies of 200 and 300 kilocycles per second, the damping constants determined by Haeske are 4 to 8 percent lower than the theoretical curve. When the theoretical damping constant curve does not include the viscous damping constant, but only the thermal and radiation damping constants, the experimental results of Haeske are 4 to 8 percent higher than the theoretical curve. However, the measurements by Haeske are only accurate to within 10 percent. Therefore, a definite conclusion cannot be formed as to whether viscous damping contributes or does not contribute to the total damping. Also, the value for the coefficient of viscosity, which is used in determining the theoretical viscous damping constant, was obtained experimentally for steady flow. At high frequencies, the value for the coefficient of viscosity may be considerably smaller than for the steady-flow case. This subject is now being investigated. Some additional damping experiments in this frequency range using a different experimental method may also decide this dilemma. Above 40 kilocycles per second, Exner and Hampe very often found "anomalous" bubbles with much lower damping constants than the regular bubbles. The measured resonant frequency did not agree with the frequency calculated from the measured diameter of the bubble when Equation [72],

$$f_0 = \frac{1}{2\pi R_0}\sqrt{\frac{3\gamma P_0 q}{\rho_2 \alpha}} \qquad [72]$$

was used. The "anomalous" bubbles have higher frequencies than this equation predicts. It was noted that in almost all cases the "anomalous" bubbles had dust particles on their surfaces. This increase in resonant frequency could not be explained by a decrease in the generalized mass as the dust particles would add to this mass, and there does not seem to be a logical explanation for a possible increase in the stiffness. Strasberg[41] tentatively suggested that this behavior may perhaps be associated with surface oscillations of the bubble since for very small bubbles the frequency of surface oscillations may be of the same order as the frequency of ordinary volume pulsations. The excitation of surface oscillations by sonic excitation may require some nonsymmetry supplied by the dust particles. The damping associated with surface oscillations is not the same as the damping associated with volume pulsations, and this would account for the different damping constant measured for "anomalous" bubbles. When Haeske performed his experiment, he took extreme care to obtain clean experimental conditions, and found no trace of "anomalous" bubbles in the 100-300 kilocycles per second range.

Carstensen and Foldy[42] used the resonance absorption method to determine the damping constant. The damping constant results are very high. The authors admit they have only a small amount of evidence to indicate that the off-resonance damping constant equals $(R_0/R')^3\,\delta_0$. The off-resonance damping constant was used in deriving an expression for the resonant damping constant in terms of the attenuation. In this method, a large number of bubbles are present, and the exact distribution in size, space, and number is difficult to determine. Also, there may be interaction between individual bubbles; these interactions are surely complex and difficult to determine. The curve for the resonance damping constant as a function of the resonant frequency is certainly broadened by the presence of bubbles having different resonant frequencies. However, this broadening is difficult to calculate since the exact bubble distribution in size, space, and number is not known to a sufficient degree.

Excluding the results obtained by Meyer and Tamm, and Carstensen and Foldy, the experimental damping constants agree very well with the theoretical curve. The damping constants at low resonant frequencies obtained by Bauer and Lauer, using different experimental methods, agree quite well with each other, but their results are higher than the theory predicts. As mentioned on page 1, the resonant frequency is only slightly different for nonspherical bubbles. However, the viscous damping theory assumed spherical bubbles where the motion is completely radial. Perhaps there is considerably more viscous damping when the bubble is slightly nonspherical. In addition to volume pulsations, there are also oscillations in the shape of the bubble. Recent work* indicates that for large amplitude radial pulsations, there is some coupling between the radial motion and the shape oscillation This may result in the removal of some of the energy associated with the pulsation. At high frequencies, the experimental data of Haeske do not confirm whether or not viscous damping is important.

SUMMARY AND CONCLUSION

Bubbles excited to volume pulsations have a polytropic equation of state for the gas which results in a phase difference between the change in pressure per unit original pressure and the change in volume per unit original volume. Therefore, the work done in compressing the bubble is more than the work done by the bubble in expanding; this difference in the work done represents a net flow of heat energy into the liquid. When an error in the derivation by Pfriem for this thermal damping is corrected, the curve for the theoretical thermal damping constant at resonance agrees exactly with Willis' theoretical curve as given in the report by Spitzer.

Pulsating bubbles expend a portion of their energy in the form of spherical sound waves. The radiation damping is just this loss of energy in the form of sound waves.

The effect of viscosity on pulsating bubbles in an incompressible, viscous liquid is understood through the stress equations and the boundary conditions, rather than the Navier-Stokes equation of motion. At the bubble surface, there are viscous forces acting which exert an excess pressure; this results in the dissipation of energy.

Experimental results verify that the damping at resonance is due to thermal and radiation, and possibly viscous damping. The discrepancies between theory and experimental results found in measurements by Meyer and Tamm, and Carstensen and Foldy are due to the particular conditions of the experiments. The small discrepancy between the theory and the results of Bauer and Lauer may be due to an increase in viscous damping caused by the non-spherical shape of the gas bubble, and the possibility that there is some coupling between the radial motion and the shape oscillations which may remove some of the energy associated with the pulsation.

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*Unpublished communication from Dr. M. Strasberg, David Taylor Model Basin, and Dr. T.B. Benjamin, King's College, Cambridge, England.

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