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AN EXACT ANALYSIS OF LAMINAR FLOW IN THE ENTRANCE REGION OF AN ANNULAR PIPE

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ABSTRACT

Non-linear equations of laminar flow of a viscous incompressible fluid in the entrance region of an annular-pipe have been solved by an exact method. It is assumed that the velocity of the fluid is uniform at the entrance section. Velocity profiles, pressure distribution and inlet lengths have been obtained for various values of the ratio of inner and outer radii of the pipe.

AN EXACT ANALYSIS OF LAMINAR FLOW IN THE ENTRANCE REGION OF AN ANNULAR PIPE

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1. Introduction

The problem of laminar flow of a viscous incompressible fluid in the entrance region of an annulus of a pipe has been investigated by two different approximate methods by Chang and Atabek [1] and Sugino [2]. The fluid is assumed to enter the annular space with a uniform velocity parallel to the axis of the coaxial cylinders and the development of the velocity profiles and the pressure distribution downstream are calculated. Based on a method due to Targ (see Slezkin [3]), Chang and Atabek use the initial velocity at the entrance to linearize the equations of motion, while Sugino has applied the technique of Langhaar [4] to linearize the same equations. Considerable difference exists between the results of these workers and hence an exact analysis of the problem has been undertaken here. The Navier-Stokes equations are first simplified under the usual boundary layer type assumptions and the resulting non-linear differential equations are solved by a method which was first used by Hartree [5] for some problems in boundary layer theory. In view of the requirements of the present day high-speed computers, Hartree's method was modified by Leigh [6] and we shall use this modified form of the method for our problem.

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2. The Equations and Boundary Conditions

We shall take the origin of the coordinates at the center of the entrance section and the axis of x' along the common axis of the cylinders of radii R_1 and R_2 $(R_1 < R_2)$. It is assumed that the cylinders are long enough to permit full development of the velocity which is assumed to be uniform $(= u_o)$ at the entrance section. Let r' denote the radial distance, u' and v' the components of velocity in the direction of x' and r' and p' be the static pressure above that at the entrance section. The flow is considered to be steady, laminar and axial symmetric. Assuming that the pressure p' is a function of x' only and that $\partial^2 u/\partial x'^2 << \partial^2 u'/\partial r'^2$ (see Goldstein [7] pp. 304), the equations of motion and continuity are

$$u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial r'} = -\frac{1}{\rho} \frac{dp'}{dx'} + \frac{\nu}{r'} \frac{\partial}{\partial r'}\left(r' \frac{\partial u'}{\partial r'}\right), \tag{2.1}$$

$$\frac{\partial u'}{\partial x'} + \frac{1}{r'} \frac{\partial (r'v')}{\partial r'} = 0 . \tag{2.2}$$

The boundary conditions are

$$u' = v' = 0 \text{ at } r' = R_1 \text{ and } r' = R_2 \text{ for } x' > 0$$

and

$$u' = u_o \text{ at } x' = 0 \text{ for } R_1 < r' < R_2 . \tag{2.3}$$

The equation of continuity (2.2) can be put in the form

$$2\pi \int_{R_1}^{R_2} u'r'dr' = u_o \pi (R_2^2 - R_1^2) \tag{2.4}$$

Introducing non-dimensional quantities

$$x = \frac{x'}{R_2 Re} \quad , \quad r = \frac{r'}{R_2} \quad , \quad u = \frac{u'}{u_o} \quad , \quad v = \frac{v'Re}{u_o} \quad \text{and} \quad p = \frac{p'}{\frac{1}{2}\rho u_o^2}$$

where $Re = u_o R_2/\nu$, in the equations (2.1 - 2.4) we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{1}{2}\frac{dp}{dx} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) , \tag{2.5}$$

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0 \ , \tag{2.6}$$

and

$$\int_{\lambda}^{1} r u dr = \frac{1}{2}(1 - \lambda^2) \tag{2.7}$$

The boundary conditions are

$$u = v = 0 \ \text{at} \ r = \lambda \ \text{and} \ r = 1 \ \text{for} \ x > 0,$$

and

$$u = 1 \ \text{at} \ x = \wedge \ \text{for} \ \lambda < r < 1 . \tag{2.8}$$

Replacing v from (2.6) in (2.5) we get

$$u\frac{\partial u}{\partial x} - \frac{1}{r}\frac{\partial u}{\partial r}\int_{\lambda}^{r} r\frac{\partial u}{\partial r}\,dr = -\frac{1}{2}\frac{dp}{dx} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right). \qquad (2.9)$$

We shall now solve equation (2.9) and obtain u and p satisfying (2.7) and (2.8).

3. Method of Solution

The derivatives in the x-direction are replaced by finite differences while the other quantities are replaced by averages. Let us suppose that the solution $u = u_1$, $p = p_1$ at $x = x_1$ is known and we wish to find the solution $u = u_2$, $p = p_2$ at $x = x_2$. After introducing necessary finite difference approximations in (2.9) we get

$$\frac{u_1+u_2}{2}\cdot\frac{u_2-u_1}{\ell} - \frac{1}{2r}(u_1'+u_2')\int_{\lambda}^{r} r\frac{u_2-u_1}{\ell}\,dr = -\frac{1}{2}\frac{p_2-p_1}{\ell} + \frac{1}{2r}(u_1'+u_2') + \frac{1}{2}(u_1''+u_2'') \qquad (3.1)$$

where dashes denote differentation with respect to r and $\ell = x_2 - x_1$. Substituting $w = u_1 + u_2$ and rearranging the terms in (3.1) we get

$$\ell w'' + \frac{w'}{r}\int_{\lambda}^{r} rw\,dr - w^2 - p_2 = -p_1 - \frac{\ell w'}{r} - 2u_1 w + \frac{2w'}{r}\int_{\lambda}^{r} ru_1\,dr. \qquad (3.2)$$

Equation (2.7) can be written as

$$\int_{\lambda}^{1} wr\,dr = (1-\lambda^2) = 2\int_{\lambda}^{1} u_1 r\,dr; \qquad (3.3)$$

and the boundary conditions for w from (2.8) are

$$w = 0 \quad \text{at} \quad r = \lambda \quad \text{and} \quad r = 1. \qquad (3.4)$$

Once w and p_2 are determined as solutions of (3.2 - 3.4) we can easily obtain $u_2 = w - u_1$.

Let w_m, $p_{2,m}$ be an approximate solution of (3.1 - 3.4). We shall use this solution to linearize the equation (3.2) and obtain w_{m+1}, $p_{2,m+1}$ as the solution satisfying (4.3), (4.4) and the equation

$$\ell w''_{m+1} + \frac{w'_m}{r}\int_\lambda^r r w_{m+1} dr - w_m w_{m+1} - p_{2,m+1} = -p_1 - \frac{\ell w'_m}{r} - 2u_1 w_m + \frac{2w'_m}{r}\int_\lambda^r r u_1 dr . \qquad (3.5)$$

Equation (3.5) defines an iterative process for w and p_2. This process is repeated until

$$|w_{m+1} - w_m| < \epsilon ,$$

where ϵ depends upon the accuracy desired. We shall now make the following substitutions in (3.5). (Here the second subscript k indicates the value at $r = \lambda + kh$, where h is the mesh size in the radial direction).

$$w''_{m+1} = (w_{m+1,k+1} - 2w_{m+1,k} + w_{m+1,k-1})/h^2$$

$$\int_\lambda^{r=\lambda+kh} r w_{m+1} dr = h[(\lambda+h)w_{m+1,1} + (\lambda+2h)w_{m+1,2} + \ldots + (\lambda+\overline{k-1}h)w_{m+1,k-1} + (\lambda+kh)w_{m+1,k}]$$

$$\frac{w'_m}{r} = (w_{m,k+1} - w_{m,k-1})/2h(\lambda+kh) = \alpha_{m,k}/h$$

$$\int_\lambda^{r=\lambda+kh} r u_1 dr = h[(\lambda+h)u_{1,1} + (\lambda+2h)u_{1,2} + \ldots + (\lambda+\overline{k-1}h)u_{1,k-1} + (\lambda+kh)u_{1,k}] = h\gamma_k$$

$$\frac{\ell}{h^2} = \beta .$$

Further writing

$$A_{m,k} = \alpha_{m,k}(\lambda + \overline{k-1}h) + \beta$$

$$B_{m,k} = \frac{1}{2}\alpha_{m,k}(\lambda + kh) - 2\beta - w_{m,k}$$

$$C_{m,k} = -p_1 - 2u_{1,k}w_{m,k} + 2\alpha_{m,k}\gamma_k - \beta h \alpha_{m,k}$$

Rearranging the terms after making the above substitutions in (3.5) we get the equations at $x = x_2$ and $r = \lambda + kh$, given by

$$-p_{2,m+1} + \alpha_{m,k}(\lambda+h)w_{m+1,1} + \alpha_{m,k}(\lambda + 2h)w_{m+1,2} + \dots + \alpha_{m,k}(\lambda + \overline{k-2}h)w_{m+1,k-2}$$

$$+ A_{m,k}w_{m+1,k-1} + B_{m,k}w_{m+1,k} + \beta w_{m+1,k+1} = C_{m,k}\,. \qquad (3.6)$$

Taking h such that $\frac{1-\lambda}{h} = n$, we have $w_{m,o} = w_{m,n} = w_{m+1,o} = w_{m+1,n} = 0$ from (3.4). Equations for $k = 1$ and $k = n-1$ are

$$-p_{2,m+1} + B_1 w_{m+1,1} + \beta w_{m+1,2} = C_{m,1}\,, \qquad (3.7)$$

and

$$-p_{2,m+1} + \alpha_{m,n-1}(\lambda+h)w_{m+1,1} + \alpha_{m,n-1}(\lambda+2h)w_{m+1,2} + \dots + \alpha_{m,n-1}(\lambda+\overline{n-3}h)w_{m+1,n-3}$$

$$+ A_{m,n-1}w_{m+1,n-2} + B_{m,n-1}w_{m+1,n-1} = C_{m,n-1}\,. \qquad (3.8)$$

Equation (3.3) may be replaced by

$$(\lambda+h)w_{m+1,1} + (\lambda+2h)w_{m+1,2} + \dots + (\lambda+\overline{n-1}h)w_{m+1,n-1} = 2\gamma_n$$

Equations (3.6 - 3.9) are a set of n simultaneous linear algebraic equations for the n unknowns $-p_{2,m+1}$, $w_{m+1,1}$, $w_{m+1,2}$ ..., $w_{m+1,n-1}$. We may write them in the form

$$\underset{\sim}{A}_m \underset{\sim}{w}_{m+1} = \underset{\sim}{C}_m$$

where $\underset{\sim}{A}_m$ is an n x n matrix, while $\underset{\sim}{w}_{m+1}$ and $\underset{\sim}{C}_m$ are column matrices given by

$$\underset{\sim}{w}_{m+1} = \{-p_{2,m+1}, w_{m+1,1}; w_{m+1,2}, \ldots, w_{m+1,n+1}\}$$

$$C_m = \{C_{m,1}, C_{m,2}, \ldots, C_{m,n-1}, 2\gamma_n\}$$

The matrix $\underset{\sim}{A}_m$ can be written down easily from the left hand side of the equations (3.7, 3.6, 3.8 and 3.9). The solution of these equations is then obtained by usual methods of matrix inversion and is given by

$$\underset{\sim}{w}_{m+1} = \underset{\sim}{A}_m^{-1} \underset{\sim}{C}_m .$$

Iteration then gives the values of w at $x = x_2$. The same process can then be applied to find the solution at $x = x_2 + \ell$ and so on.

4. Results

The velocity profiles and the pressure distributions were calculated for seven different values of λ $(+1 \leq \lambda \leq .7)$ and are shown in figures 1-5. Near the entrance $x = 0$, the mesh size ℓ was taken very small (= 1/3200), since the variation of the velocity in this region is large. This was then gradually increased up to 1/400. Although the method is stable for any mesh ratio $\beta = \ell/h^2$, h was chosen such that $\beta = 1/4$ for the smallest ℓ. To ascertain whether the mesh size chosen was reasonable, the computations were repeated by taking doubled mesh size in x. The value of ϵ for convergence was chosen to be 10^{-6} and the number of iterations varied from 9 in the first step to 2 after a few steps, when the initial approximate solution at $x = x_2$ was chosen to be the same as at $x = x_1$. The computations were carried on up to a point where the velocity distribution at two neighboring points remained the same up to four decimal places.

The asymptotic solution of (2.5) satisfying (2.7) and (2.8) for large values of x is given by

$$u_\infty = 2[(1-r^2)\ln\lambda - (1-\lambda^2)\ln r]/[(1+\lambda^2)\ln\lambda + (1-\lambda^2)]$$

The velocity profiles obtained were compared with $u_f = \frac{n-1}{n} u_\infty$. The reason for introducing the factor $\frac{n-1}{n}$ was to satisfy the equation of continuity, since the velocity profile at $x = 0$ was taken to be unity at the mesh points and zero on the boundaries, but not uniform across the whole cross-section $x = 0$. The entrance length is then defined as the length at which the maximum velocity attains a particular percentage of the maximum velocity u_f. The entrance lengths for various percentage values are given in table 1.

The main results can be summarized as below.

1. The entrance length decreases with increase of λ. At first the decrease is rapid. This agrees qualitatively with the results of Chang and Atabek, however their numerical values are lower. A constant value ($\simeq .02$) for the entrance

length as has been taken by Sugino therefore seems to be a very rough approximation.

2. The pressure drop downstream of the flow depends upon λ as shown in figures (4-5). In order to compare the results with those given by Sugino we have also plotted pressure against $x_A = \frac{x'\nu}{(R_2-R_1)^2 u_o} = \frac{x}{(1-\lambda)^2}$, in figure (5). It is then confirmed that the pressure drop against x_A also depends on λ, although the variation is small near the entrance region.

3. To find the entrance length, only the maximum of the velocity was compared with its asymptotic value u_f. However, it was found that not only the maximum velocity reaches its final value but also the velocity at other mesh points also reach their asymptotic value. The final velocity profiles are in excellent agreement with the asymptotic velocity profiles.

4. The computations were carried out for seven different values of λ on a CDC1604 and took about one hour.

5. Final Comments

The method due to Hartree does not necessarily put any restriction on the use of higher order difference formulae. However in its present form we could only use the first order and central differences for the integration in the radial direction. Thus in effect the method is equivalent to implicit finite difference method of the type first suggested by Bodoia and Ostele [8] for problems of similar nature. However the motivation for using a particular finite difference scheme seems to be much clearer here; further the method is known to be stable. Attempts are being made to modify Hartree's method so as to enable one to use higher order formulae for integration in the radial direction and still adjust it in such a way that the whole computation can be carried out on a high-speed computer.

Acknowledgements

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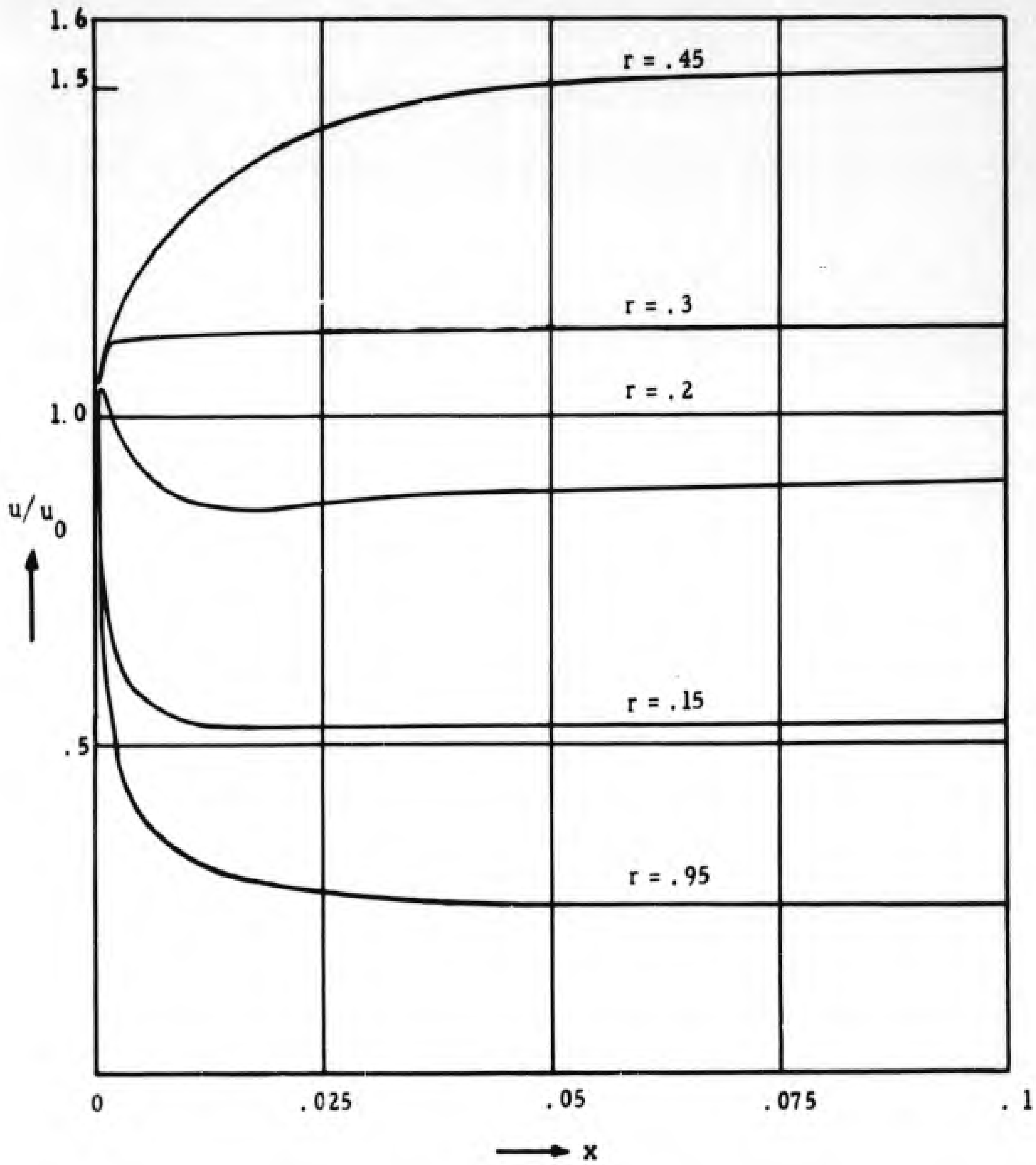
TABLE I

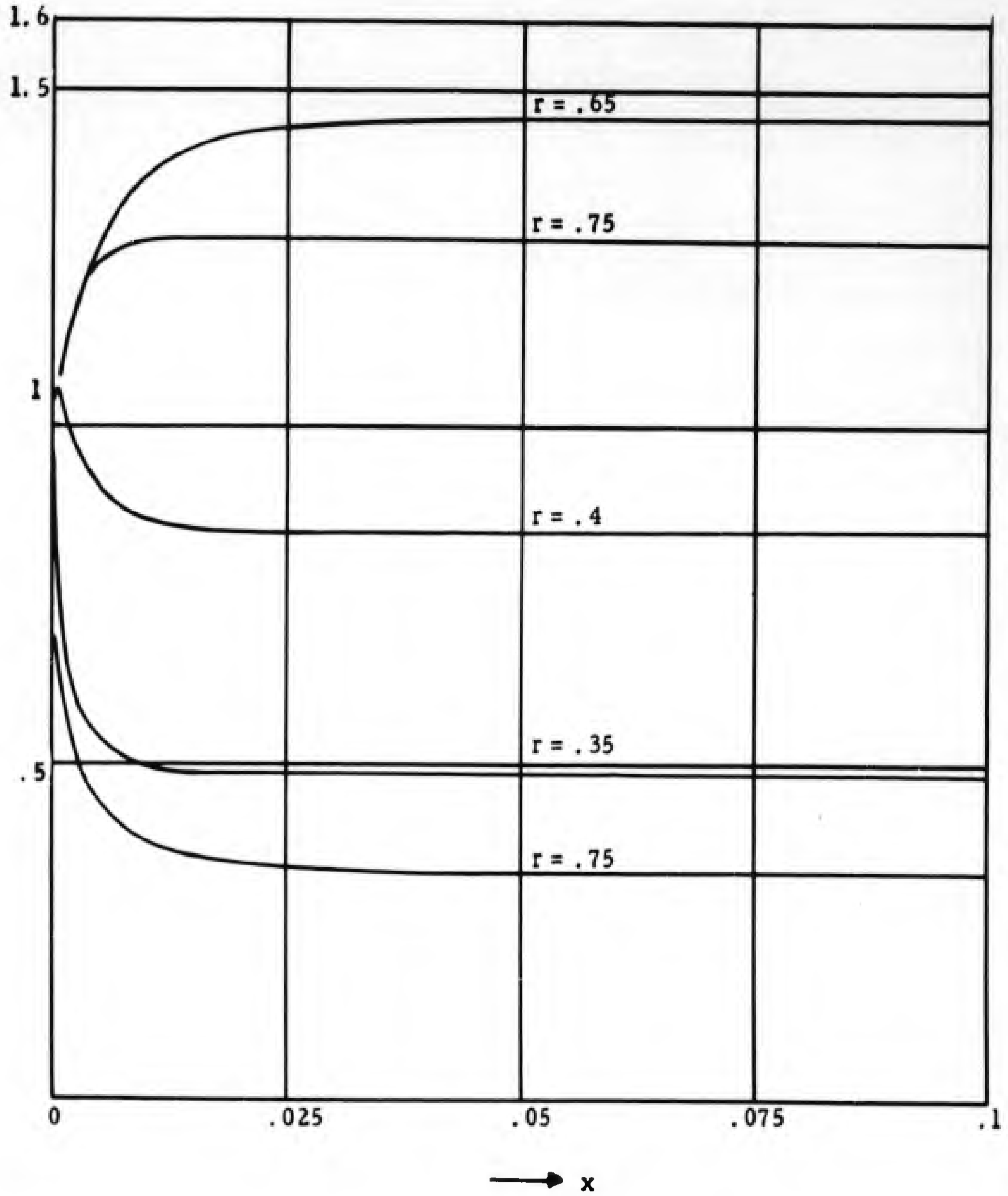
percentage	x $\lambda = 0.1$	x $\lambda = 0.3$	x $\lambda = 0.5$	x $\lambda = 0.7$
95%	.025	.012	.0056	.0018
99%	.053	.024	.011	.0037
99.9%	.097	.043	.019	.0064
99.99%	.137	.063	.028	.0091

FIGURE 1



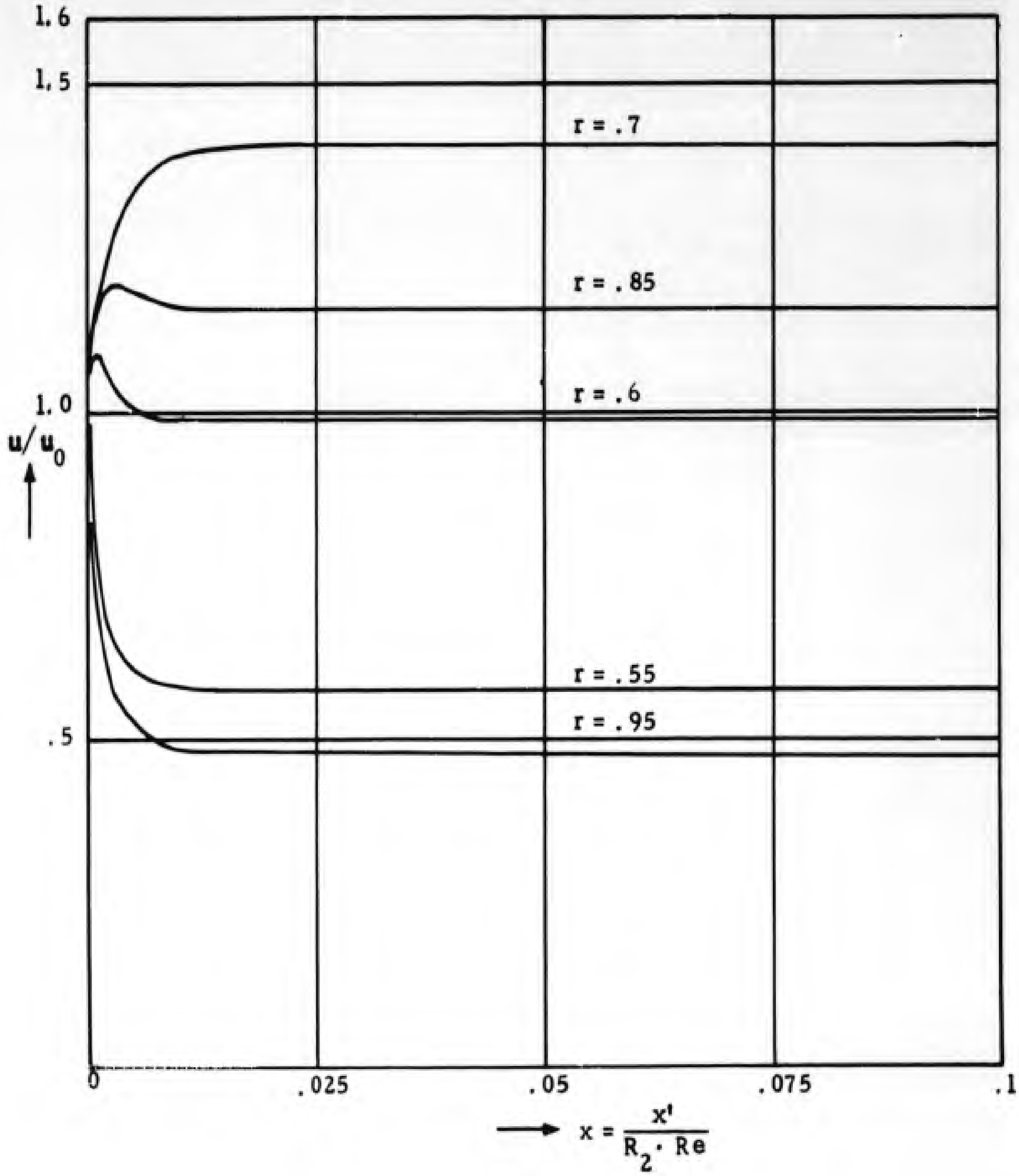
Velocity distribution $\lambda = 0.1$

FIGURE 2



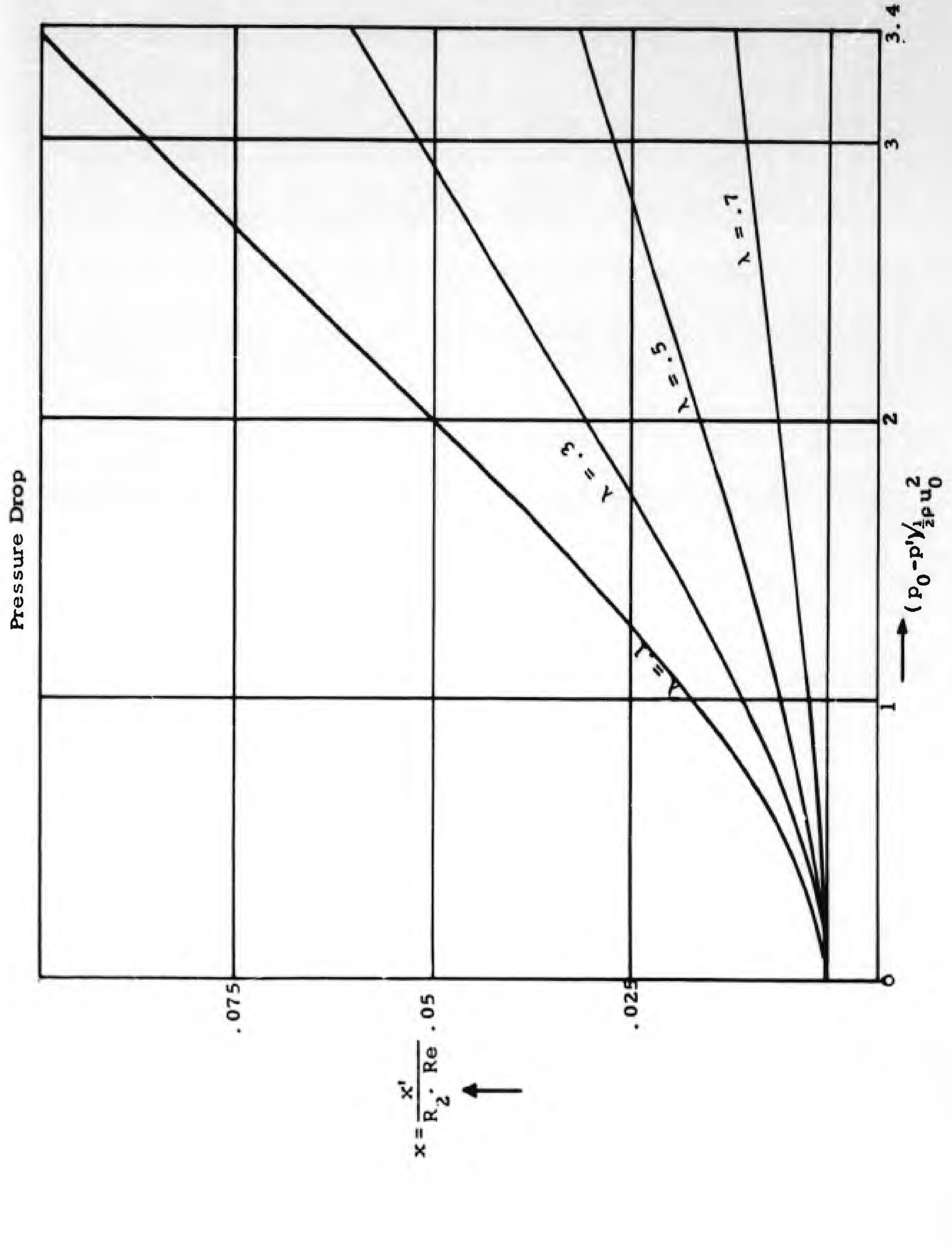
Velocity distribution $\lambda = .3$

FIGURE 3



Velocity distribution $\lambda = 0.5$

FIGURE 4


Pressure Drop
λ = .1
λ = .3
λ = .5
λ = .7
.025
.05
.075
0
1
2
3
3.4
$x = \frac{x'}{R_2 \cdot Re}$
$(p_0 - p')/\frac{1}{2}\rho u_0^2$

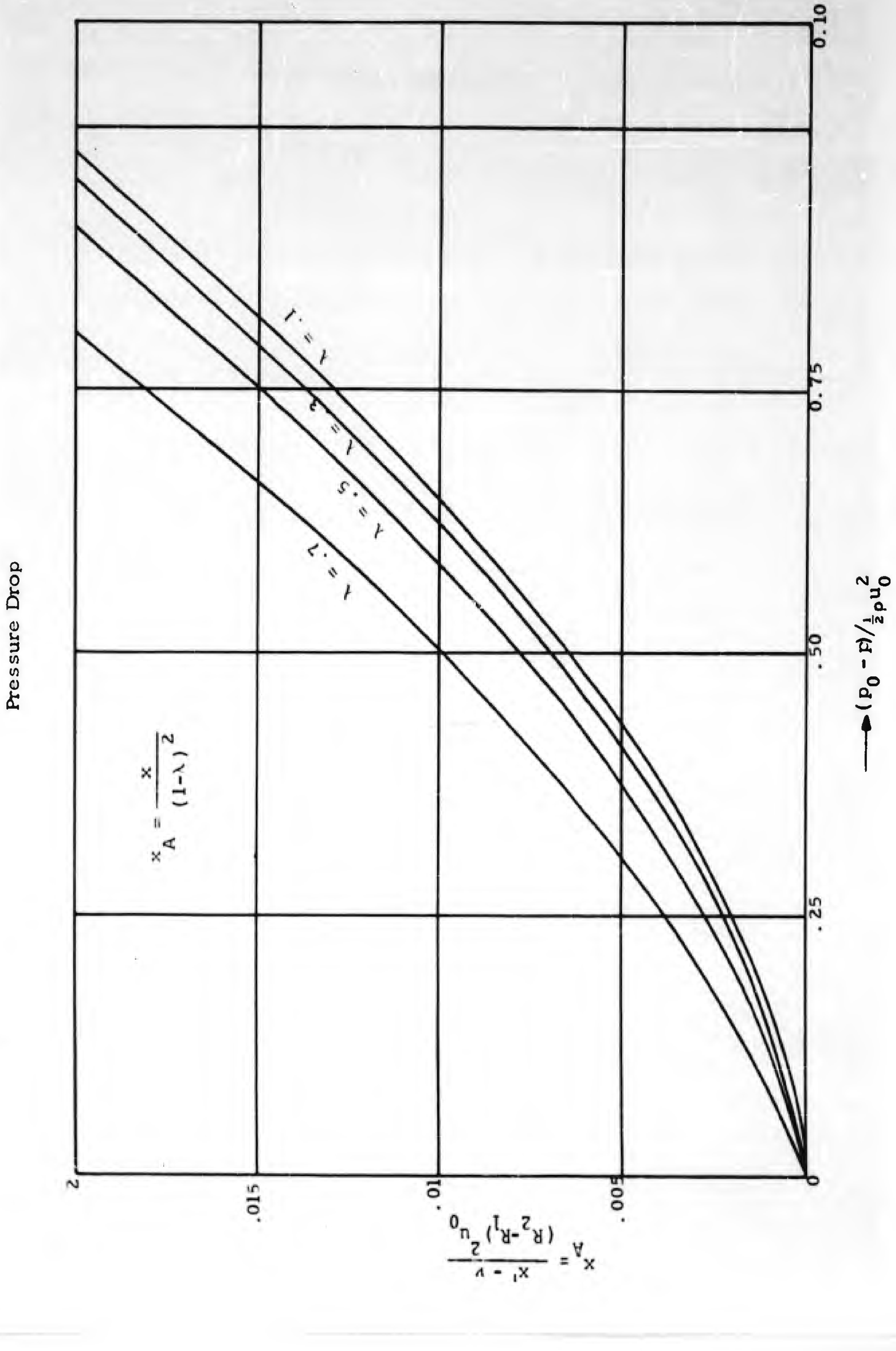
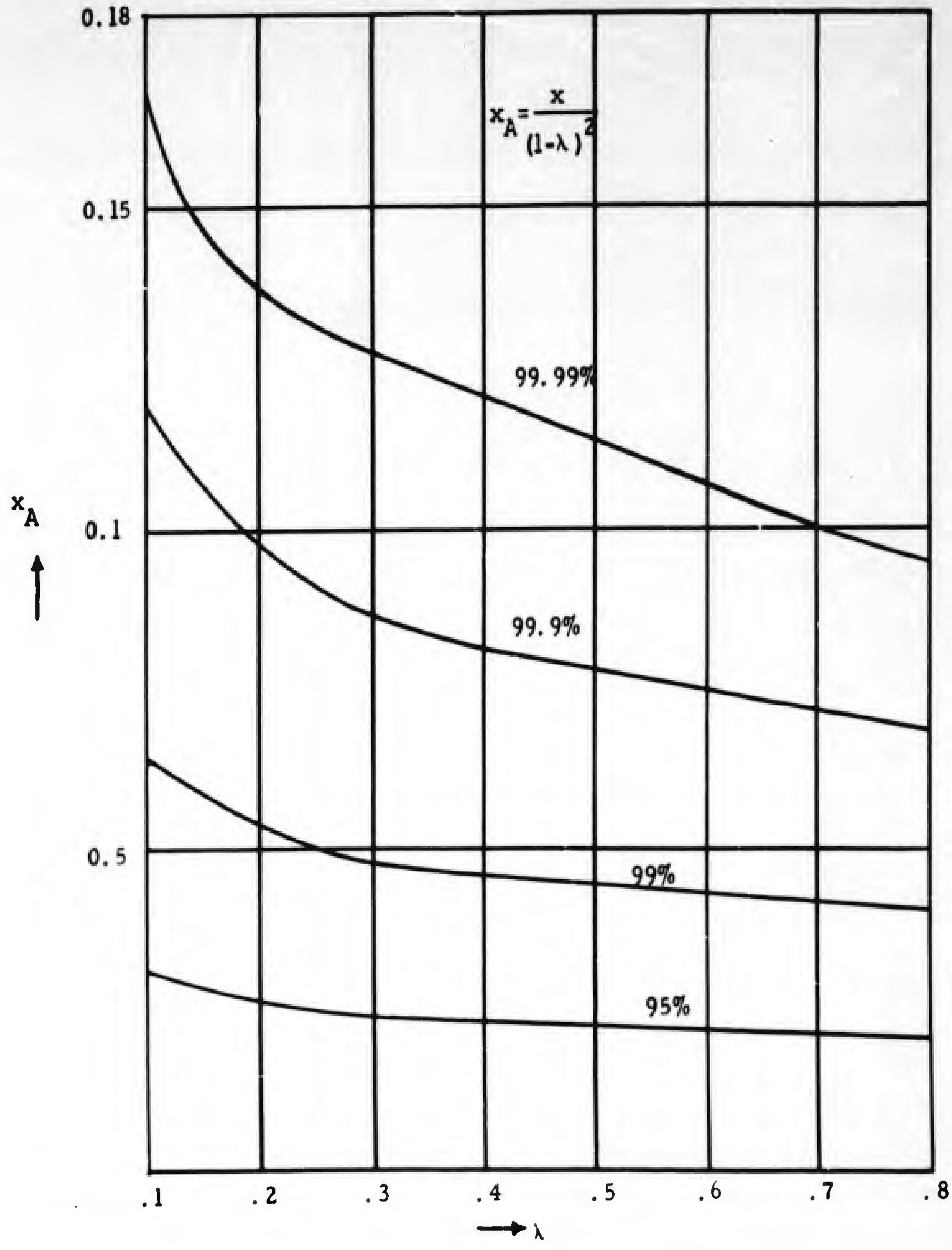

Pressure Drop
$x_A = \frac{x}{(1-\lambda)^2}$
λ = .1
λ = .3
λ = .5
λ = .7
$(p_0 - p)/\frac{1}{2}\rho u_0^2$
.25
.50
0.75
0
.005
.010
.015
$x_A = \frac{x' - \nu}{(R_2 - R_1)^2 u_0}$

FIGURE 6



Entrance Length

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