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(SELECTED ARTICLES)

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CERTAIN PROBLEMS OF THE HYDRODYNAMICS OF RELAXING MEDIA

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This article examines certain general properties of the equations of motion of thermodynamically nonequilibrium media and the transition to "equilibrium" hydrodynamics in the limiting case of small relaxation times. Certain problems of the propagation of small disturbances in relaxing media are examined in a linear approximation.

In any motion of a liquid or gas, excluding certain special cases such as the uniform translational motion of a system as a whole, disturbances arise in the statistical equilibrium state of the medium. If the gradients of the hydrodynamic values are not too large, the equilibrium distribution function is only negligibly disturbed; however, from the molecular and kinetic viewpoint these slight disturbances are the basis of all dissipative processes in moving gases. Introducing terms with coefficients of viscosity, thermal conductivity, diffusion, etc., into ordinary hydrodynamic equations is the means used for calculating these small disturbances of statistical equilibrium; in these calculations the thermodynamic parameters of the medium are taken equal to their values in a state of complete thermodynamic equilibrium (local equilibrium).

However, in a number of cases this type calculation of statistical nonequilibrium is insufficient. The time τ for establishment of local statistical equilibrium varies within broad limits depending on the nature of the process. It is of the same order of time as the free path used for establishing Maxwell velocity distributions in a gas and is somewhat greater for the case of rotational motion of molecules. Such rapid processes are outside the range of applicability of hydrodynamic equations. Spatial regions with substantiate disturbance of statistical equilibrium of this type are described in hydrodynamics by mathematical discontinuity surfaces. However, for the case of molecular oscillations the relaxation time is of the order of thousands and ten thousands of free-path length times. The exchange of kinetic energy of translational motion between particles having great mass — the electrons and ions — is greatly hampered. Finally, considerable time is required to establish chemical or ionization equilibrium.

If the characteristic macroscopic reaction time is comparable in order of magnitude to the relaxation time τ for any of the processes occurring in the medium, the thermodynamic parameters can differ substantially from their equilibrium values, and the system of hydrodynamic equations should be supplemented with a kinetic equation describing the relaxation process. These conditions are encountered in supersonic flows, for example, in the zone directly behind the front of a shock wave which markedly disturbs the state of thermodynamic equilibrium [1]. The other well-known example of motion in which explicit calculation of the relaxation processes is necessary, is the propagation of ultra high-frequency sound. The studies of I. G. Kirkwood and L. I. F. Broer [2, 3] are devoted to obtaining a closed system of hydrodynamic and kinetic equations for different processes.

We will consider below certain general properties of the system of "relaxation hydrodynamic" equations, as well as the possibility of passing to the limit to the usual "equilibrium" hydrodynamics, hydrodynamics at small values of τ.

The latter is of interest because certain important properties of "relaxation hydrodynamic" equations when τ approaches zero are not transformed into properties of the equilibrium hydrodynamics equations. For example, the characteristics of equations of unsteady one-dimensional motion lacking viscosity and thermal conductivity described by L. I. F. Broer [3] do not depend on the value of time τ and do not coincide with the characteristics of equilibrium hydrodynamics.

As will be shown below, this difficulty is removed if we assume that the equations of equilibrium hydrodynamics are an approximate description of only the special class of solutions of relaxation hydrodynamic equations. These solutions are characterized by the fact that local nonequilibrium at every given moment is fully defined by the fields of hydrodynamic values regardless of the previous history of the medium. Moreover, kinetic equations describing the change in time of the parameters describing nonequilibrium of the medium drop out of relaxation hydrodynamics.

We shall also investigate the propagation of small disturbances in relaxing media. We shall show that in the region of the order of the "relaxation length" l, they will be propagated along the characteristics of the equations of relaxation hydrodynamics; therefore near obstacles weak discontinuities arise whose direction does not coincide with the usually observed Mach lines and which exponentially vanish at distances large in comparison with l. At large distances the propagation of the disturbances is determined by the equilibrium velocity of sound. However, unlike equilibrium hydrodynamics this disturbance is made up of disturbances arriving from different points; in this case as propagation proceeds, the shape of the disturbances changes so that the "fine details" gradually disappear.

1. Equations of relaxation hydrodynamics. We shall assume that the nonequilibrium states which occur are states of incomplete statistical equilibrium which can be described thermodynamically. As the characteristic function determining the thermodynamic properties of the medium, we choose the internal energy of a unit of mass, $\varepsilon(\rho, s, \xi)$, which can be considered as a function of density ρ, specific entropy s, and the additional parameter ξ (or several such parameters) [4, 5], which describes the deviation from complete thermodynamic equilibrium. The physical sense of the parameters ξ can be quite different. This can be for example, the temperature of the internal degrees of freedom or the concentration of the reacting component. In describing the chemical reaction it is especially convenient to introduce the parameter ξ by means of the equation [2]:

$$dm_i = \sum_k \nu_{ik} d\xi_k$$

Here dm_i is change of mass of the component i resulting from reaction k, and ν_{ik} is the corresponding stoichiometric coefficient.

The actual system, as a rule, is characterized by several such parameters ξ, because several reactions may simultaneously take place in the system, and moreover both the temperature of the different components and the temperature of their internal (for example, fluctuating) degrees of freedom can differ. We shall limit ourselves to a consideration of the simplest one-parameter relaxation process, for which, however, we can investigate the characteristic features of relaxation hydrodynamics.

In the state of equilibrium, ξ takes the value $\xi(\rho, s)$ defined by the equation

$$\varepsilon_\xi (\rho, s, \xi_0) = 0, \quad \varepsilon_\xi \equiv \left[\frac{\partial \varepsilon}{\partial \xi}\right]_{\rho, s} \tag{1.1}$$

The equations of continuity, momentum, and energy have their usual form

$$\begin{gathered} \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0 \\ \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\operatorname{grad} p + \nabla\cdot\Pi \qquad \left(p = \rho^2\left[\frac{\partial \varepsilon}{\partial \rho}\right]_{s,\xi}\right) \\ \frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho\varepsilon\right) = -\operatorname{div}\left\{\rho\mathbf{v}\left(\frac{v^2}{2} + \omega\right) - (\Pi\cdot\mathbf{v}) + \mathbf{q}\right\} \end{gathered} \tag{1.2}$$

Here $\mathbf{v}$ is velocity of the medium, p is pressure, ω is specific enthalpy, $\mathbf{q}$ is the density vector of the thermal flow, Π is the tensor with components π_{ik} definable from the relation

$$P_{ik} = -p\delta_{ik} + \pi_{ik} \tag{1.3}$$

where P_{ik} is the stress tensor. It further follows that Π need not coincide with the tensor of viscous stresses. The following form of the energy equation also follows from (1.2):

$$\frac{d\varepsilon}{dt} = p\frac{d}{dt}\frac{1}{\rho} - \frac{1}{\rho}\frac{\partial q_i}{\partial x_i} + \frac{\pi_{ik}}{\rho}\frac{\partial v_i}{\partial x_k} \qquad \left(\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) \tag{1.4}$$

From here and from the general thermodynamic expression

$$de = -p\,d\left(\frac{1}{\rho}\right) + T\,ds + e_\xi\,d\xi \qquad \left(T = \left(\frac{\partial e}{\partial s}\right)_{\xi,\rho}\right) \tag{1.5}$$

where T is temperature of the medium, we obtain

$$\frac{ds}{dt} = -\frac{1}{\rho T}\frac{\partial q_i}{\partial x_i} + \frac{1}{\rho T}\pi_{ik}\frac{\partial v_i}{\partial x_k} - \frac{e_\xi}{T}\frac{d\xi}{dt} \tag{1.6}$$

In the general case, the parameter ξ at a given point of the medium can vary both due to scalar flow r and to the vector flow **j**. In the future the physical sense of the parameter will not be specified; therefore the physical sense of flows r and **j** may not be involved.

In the particular case, when ξ defines the composition of the gas dissociating according to the equation $2A \rightleftarrows A_2$, flow r will be the dissociation and **j** will be the diffusion flow.

When defining r and **j** appropriately the equation for the rate of change of ξ can always be written as

$$\rho\frac{\partial\xi}{\partial t} = r - \operatorname{div} \mathbf{j} \tag{1.7}$$

Substituting (1.7) into (1.6), we obtain the equation of the entropy balance

$$\rho\frac{ds}{dt} + \frac{\partial}{\partial x_i}\left(\frac{q_i}{T} - \frac{e_\xi j_i}{T}\right) = \sigma \tag{1.8}$$

$$T\sigma = -q_i\frac{1}{T}\frac{\partial T}{\partial x_i} + \pi_{ik}\frac{\partial v_i}{\partial x_k} - r e_\xi - j_i T\frac{\partial}{\partial x_i}\frac{e_\xi}{T} \tag{1.9}$$

We easily see that

$$\frac{d(\rho s)}{dt} = -\operatorname{div}(\mathbf{j}_s + \rho s \mathbf{v}) + \sigma, \qquad \mathbf{j}_s = \frac{\mathbf{q} - e_\xi \mathbf{j}}{T} \tag{1.10}$$

The vector $\mathbf{j}_s$ is the density vector of the entropy flux, or to be more exact, that part of the density of the entropy flux which is associated with macroscopic motion of the medium; the total density of the entropy flux is $\mathbf{J}_s + \rho s \mathbf{v}$.

The right-hand side of (1.8) represents formation of entropy σ, i.e., the rate of change in the entropy of a unit volume due to the irreversible processes occuring in it: the thermal conductivity, internal friction, and processes describable by flows r and $\mathbf{J}$ (in the particular case, the processes can be chemical reaction and diffusion).

Here flows I and forces X in the thermodynamics sense of irreversible processes will be q_i, π_{ik}, r, J_i, and correspondingly

$$-\frac{1}{T}\frac{\partial T}{\partial x_i}, \quad +\frac{\partial v_i}{\partial x_k}, \quad -\varepsilon_\xi, \quad -T\frac{\partial}{\partial x_i}\frac{\varepsilon_\xi}{T}$$

satisfying the relation

$$\sigma = \frac{1}{T}\Sigma\, IX \tag{1.11}$$

By assuming a linear relation of flows to forces, for the isotropic medium we can write

$$\begin{gathered} q_i = -A\frac{1}{T}\frac{\partial T}{\partial x_i} - BT\frac{\partial}{\partial x_i}\frac{\varepsilon_\xi}{T} \\ \pi_{ik} = \eta\left\{\left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right) - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right\} + \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l} - M\delta_{ik}\varepsilon_\xi \\ r = -K'\varepsilon_\xi + L\frac{\partial v_l}{\partial x_l}, \qquad J_i = -C\frac{1}{T}\frac{\partial T}{\partial x_i} - DT\frac{\partial}{\partial x_i}\frac{\varepsilon_\xi}{T} \end{gathered} \tag{1.12}$$

Here A, B, C, D, K', L, M are kinetic coefficients which, generally speaking, are functions of the state of the medium, η and ζ are the shear and second coefficients of viscosity.

The requirements of Curie's theorem were taken into account. For example, the tensor force $\partial v_i/\partial x_k$ and scalar force ε_ξ cannot enter linearly into the expressions for $\mathbf{q}$ and $\mathbf{J}$, since in both cases a vector kinetic coefficient would be required, which is precluded for the case of an isotropic medium. For the same reasons the vector forces do not enter into the expressions for π_{ik} and r.

By virtue of the principle of symmetry of kinetic coefficients

$$B = C, \quad L = -M \tag{1.13}$$

As is apparent, in the expression for π_{ik} there is a term that is proportional to ε_ξ and not connected directly with the deformation of the velocity field. Thus, the total tensor of stresses in the relaxing medium is composed of three parts: 1) "thermodynamic pressure" (with the minus sign) $\eta - \delta_{ik} = -\rho' \varepsilon_\rho \delta_{ik}$, 2) viscous stresses, and 3) the term $M\delta_{ik}\varepsilon_\xi$. In accordance with (1.13) the term $L\partial v_l/\partial x_l$ appears in the expression for scalar flow r.

We note that there is no due clarity concerning the question of these terms in the literature. For example, De Groot's study [6] asserts that the velocity of the chemical reaction does not depend on the scalar combination of viscosity forces $\partial v_l/\partial x_l$, since hydrostatic pressure does not depend on the occurrence of a chemical reaction. It is apparent from Eqs. (1.12) and (1.13) derived above that within the scope of general phenomenological considerations these assertions will not be valid. In phenomonological theory, the question of the magnitude of the aforementioned effects remains open. They can be estimated in the molecular-kinetic theory.

The first two equations of (1.2), Eqs. (1.7) and (1.8) together with (1.12), (1.13) form a closed system of relaxation hydrodynamic equations. Here we take ρ, s, and ξ as independent variables defining the thermodynamic state of the medium. The remaining thermodynamic variables are determined from the characteristic function $\varepsilon(\rho, s, \xi)$, for example $T = (\partial\varepsilon/\partial s)\rho, \xi$.

When it is possible to disregard viscosity, thermal conductivity, and flow **j**, the system of equations takes the form

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0, \qquad \rho \frac{d\mathbf{v}}{dt} + \operatorname{grad} p = 0$$
$$\rho \frac{ds}{dt} = K' \frac{e_\xi^2}{T}, \qquad \rho \frac{d\xi}{dt} = -K' e_\xi \tag{1.14}$$

In these conditions L = M = 0, which follows, as is evident from the condition

$$\sum_{i,k} L_{ik} X_i X_k \geqslant 0 \tag{1.15}$$

From Eq. (1.14) a relation is established between coefficient K' and relaxation time τ of small deviations from local thermodynamic equilibrium. If in the expansion of the right-hand side of the last of Eqs. (1.14) in powers of $(\xi - \xi_0)$, we limit ourselves to linear terms and if we take into account that the changes in entropy based on (1.14) will be of the magnitude of the second order of smallness, we obtain

$$\tau = \frac{1}{K e_{\xi\xi}}, \qquad K = \frac{K'}{\rho} \tag{1.16}$$

2. Certain properties of the equations of relaxation hydrodynamics. Calculating the curl of the right- and left-hand sides of the second of Eqs. (1.2), we obtain

$$\frac{\partial \mathbf{R}}{\partial t} + \operatorname{curl}(\mathbf{R} \times \mathbf{v}) = \frac{1}{\rho^2} \nabla p \times \nabla \rho + \frac{1}{\rho} \operatorname{curl} \mathbf{P} - \frac{1}{\rho^2} \nabla \rho \times \mathbf{P}$$
$$\mathbf{R} = \operatorname{curl} \mathbf{v}, \qquad \mathbf{P} = \nabla \cdot \Pi \tag{2.1}$$

If we disregard viscosity forces and consider that hydrostatic pressure depends only on density, then (2.1) transforms into the familiar equation for an ideal fluid

$$\frac{\partial \mathbf{R}}{\partial t} + \operatorname{curl}(\mathbf{R} \times \mathbf{v}) = 0 \tag{2.2}$$

from which the maintenance of circulation follows. In the presence of the relaxation processes, $p = p(\rho, s, \xi)$, and thus circulation is not maintained even in the absence of viscous stresses ($\mathbf{P} = 0$).

Let us examine small deviations from the state of equilibrium. By retaining in (2.1) only the terms of the first order of smallness, we obtain

$$\frac{\partial \mathbf{R}'}{\partial t} = \frac{\eta}{\rho_0} \Delta \mathbf{R}', \qquad \mathbf{R}' = \operatorname{curl} \mathbf{v}',$$

where $\mathbf{v}'$ is the disturbances of velocity, ρ_0 and η are the undisturbed values of density and shear viscosity.

This equation does not contain sources and since the disturbances are small everywhere, the flow in this approximation is irrotational. By keeping in (2.1) the terms of the second order of smallness, we obtain

$$\frac{\partial \mathbf{R}''}{\partial t} = \frac{1}{\rho_0} \operatorname{curl} \mathbf{P}'' + \frac{p_\xi}{\rho_0} \nabla \rho' \times \nabla \xi' - \frac{1}{\rho_0^2} \nabla \rho' \times \mathbf{P}' \qquad (2.3)$$

In the last equation, the values of the first order of smallness are denoted by a single prime sign, while those of the second order of smallness have double prime sign. Since entropy changes are of the second order of smallness, we can assume that

$$\nabla p' = p_\rho \nabla \rho' + p_\xi \nabla \xi', \qquad p_\rho = \left(\frac{\partial p}{\partial \rho}\right)_{s,\xi}, \qquad p_\xi = \left(\frac{\partial p}{\partial \xi}\right)_{s,\rho} \qquad (2,4)$$

Due to the irrotational character of flow in the first approximation, we have

$$\Delta \mathbf{v}' = \nabla (\nabla \cdot \mathbf{v}') \qquad (2.5)$$

Therefore, by setting $M \approx 0$ in (1.12), we obtain

$$\mathbf{P}' = \eta \Delta \mathbf{v}' + \frac{1}{3} \eta \nabla (\nabla \cdot \mathbf{v}') + \zeta \nabla (\nabla \cdot \mathbf{v}') = \\ = \left(\frac{4}{3}\eta + \zeta\right) \nabla (\nabla \cdot \mathbf{v}') = -\left(\frac{4}{3}\eta + \zeta\right) \frac{1}{\rho_0} \nabla \frac{\partial \rho'}{\partial t} \qquad (2.6)$$

When calculating **P**'' we shall consider for simplicity the coefficients of viscosity to be constant and equal to their values in an unperturbed medium.

Then

$$\mathbf{P}'' = \eta\Delta\mathbf{v}'' + \left(\frac{\eta}{3} + \zeta\right)\nabla(\nabla\cdot\mathbf{v}'') \qquad (2.7)$$

Thus

$$\text{curl}\ \mathbf{P}'' = \eta\Delta\mathbf{R}'' \qquad (2.8)$$

Using (2.8) and (2.7), we write (2.3) as:

$$\frac{\partial\mathbf{R}''}{\partial t} - \frac{\eta}{\rho_0}\Delta\mathbf{R}'' = \frac{p_s}{\rho_0^2}\nabla\rho'\times\nabla s' + \frac{\frac{4}{3}\eta + \zeta}{\rho_0^3}\nabla\rho'\times\nabla\frac{\partial\rho'}{\partial t} \qquad (2.9)$$

It is known that in an ideal fluid, disturbances in the velocity curl are not displaced relative to the fluid (absence of transverse waves).

This is readily seen if we rewrite (2.2) as

$$\frac{\partial\mathbf{R}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{R} - (\mathbf{R}\cdot\nabla)\mathbf{v} + \mathbf{R}\,\text{div}\,\mathbf{v} = 0$$

For the case of a purely rotational field div **v** = 0 and $d\mathbf{R}'/dt =$ $= -(\mathbf{R}\ )\mathbf{v}$. It then follows that disturbances in the velocity curl R are "frozen" into the fluid. The presence of shear viscosity leads to a displacement of disturbances in the curl relative to the fluid; this process, however, is analogous to diffusion and not to the propagation of waves. The right-hand side of (2.9) defines the speed with which the eddy emerges at a given point due both to relaxation processes and viscosity. In this manner, both the relaxation processes and the second (dilatational) result in an eddy, however they cannot cause diffusion of the eddy through the fluid.

Let us consider the equations which determine the propagation of longitudinal waves in a relaxing medium. As we have already noted, in the absence of viscosity and thermal conductivity the equations of relaxation hydrodynamics can be written in the form of (1.14).

On examining the small deviations from the state of equilibrium and limiting ourselves to terms of the first order of smallness, we obtain

$$K e_\xi = K e_{\xi\xi}\xi' + K e_{\xi\rho}\rho', \qquad e_{\xi\xi} = \frac{\partial^2 e}{\partial \xi^2}, \qquad e_{\xi\rho} = \frac{\partial^2 e}{\partial \xi \partial \rho} \tag{2.10}$$

Taking into consideration (2.4) and (2.10), we write Eq. (1.14) as:

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial v_k}{\partial x_k} = 0, \qquad \rho_0 \frac{\partial v_i}{\partial t} + c_\infty^2 \frac{\partial \rho}{\partial x_i} + p_\xi \frac{\partial \xi}{\partial x_i} = 0$$
$$\frac{\partial \xi}{\partial t} = \frac{1}{\tau}\left(-\frac{e_{\xi\rho}}{e_{\xi\xi}}\rho - \xi\right) \qquad \left(\frac{1}{\tau} = K e_{\xi\xi},\ c_\infty^2 = p_\rho\right) \tag{2.11}$$

Here c_∞ is the propagation velocity of the sound oscillations having a frequency $\omega \gg 1/\tau$. The primes on the variables of the first order of smallness (v, ξ, ρ) are omitted.

The formal solution of the last equation of (2.11) is represented as

$$\xi = -\frac{e_{\xi\rho}}{e_{\xi\xi}}\Lambda^{-1}\rho \qquad \left(\Lambda = 1 + \tau\frac{\partial}{\partial t}\right) \tag{2.12}$$

Here Λ^{-1} is an operator inverse to the operator Λ. Using (2.12) we eliminate ξ from the second equation of (2.11) and then taking into consideration that operators Λ and Λ^{-1} are commutative with $\partial/\partial x_1$, we obtain

$$\rho_0 \frac{\partial v_i}{\partial t} + c_\infty^2 \frac{\partial \rho}{\partial x_i} - p_\xi \frac{e_{\xi\rho}}{e_{\xi\xi}}\Lambda^{-1}\frac{\partial \rho}{\partial x_i} = 0 \tag{2.13}$$

Differentiating the first of Eqs. (2.11) with respect to **x** and Eq. (2.13) with respect to **t**, we can eliminate ρ from (2.13).

$$A\left\{\frac{\partial^2 v_i}{\partial t^2} + c_\infty^2 \frac{\partial^2 v_k}{\partial x_i \partial x_k}\right\} + p_\xi \frac{e_{\xi\rho}}{e_{\xi\xi}} \frac{\partial^2 v_k}{\partial x_i \partial x_k} = 0 \qquad (2.14)$$

As noted earlier, in the approximation under consideration the motion can be considered irrotational, and therefore

$$\frac{\partial^2 v_k}{\partial x_i \partial x_k} = \Delta v_i \qquad (2.15)$$

The equilibrium velocity of sound c_0 is related to c_∞ by the relationship [4]:

$$c_0^2 = c_\infty^2 - p_\xi \frac{e_{\xi\rho}}{e_{\xi\xi}} \qquad (2.16)$$

Using (2.15) and (2.16), we rewrite (2.14) as:

$$\tau \frac{\partial}{\partial t}\left\{\frac{\partial^2 v_i}{\partial t^2} - c_\infty^2 \Delta v_i\right\} + \frac{\partial^2 v_i}{\partial t^2} - c_0^2 \Delta v_i = 0 \qquad (2.17)$$

Equation (2.17) describes the propagation of longitudinal disturbances in the relaxing medium. As $\tau \to 0$ and $\tau \to \infty$, Eq. (2.17) transforms into the ordinary wave equations of acoustics, in which the role of sound velocity is played by c_0 and c_∞ respectively. This equation was obtained and analyzed earlier [5, 7].

It is of interest to consider the characteristics and characteristic form of Eqs. (1.14). If the disturbances are not assumed to be small, then

$$\frac{\partial p}{\partial x_i} = c_\infty^2 \frac{\partial \rho}{\partial x_i} + p_s \frac{\partial s}{\partial x_i} + p_\xi \frac{\partial \xi}{\partial x_i}, \qquad p_s = \frac{\partial p}{\partial s} = \rho^2 \frac{\partial^2 \varepsilon}{\partial s \partial \rho} \qquad (2.18)$$

By using customary methods [8], we find that the equations of the characteristics for a one-dimensional flow have the following form

$$\frac{dx}{dt} = v \pm c_\infty, \qquad \frac{dx}{dt} = v \qquad (2.19)$$

Thus, through every point on the surface x, t pass three characteristics, one of which coincides with the trajectory of the particle, and the other two coinciding with the "trajectories" of high-frequency disturbances ($\omega \gg 1/\tau$) propagating in the moving medium. Introducing the operator

$$\frac{D_{\pm}}{Dt} \equiv \frac{\partial}{\partial t} + (v \pm c_\infty)\frac{\partial}{\partial x} \equiv \frac{d}{dt} \pm c_\infty \frac{\partial}{\partial x} \tag{2.20}$$

we write the relaxation hydrodynamic equation (1.14) in the characteristic form

$$\begin{gathered} c_\infty^2 \frac{D_{\pm}\rho}{Dt} \pm \rho c_\infty \frac{D_{\pm} v}{Dt} + p_s \frac{D_{\pm} s}{Dt} + p_\xi \frac{D_{\pm}\xi}{Dt} = \pm K e_\xi \left(\frac{e_\xi p_s}{T} - p_\xi\right) \\ \frac{ds}{dt} = \frac{K}{T} e_\xi^2, \qquad \frac{d\xi}{dt} = -K e_\xi \end{gathered} \tag{2.21}$$

3. Transfer to equilibrium hydrodynamics. Equations (2.19) do not contain τ. Therefore, the characteristics of relaxation hydrodynamics do not change as τ tends to zero and do not convert to the characteristics of equilibrium hydrodynamics definable by the velocity c_0. Thus, the change to equilibrium hydrodynamics describing movement of the medium for small values of τ, requires special consideration.

We shall introduce the parameter $\mu = c\tau/L$, where c is the velocity of sound (c_0 or c_∞), L is the characteristic macroscopic length (the distance over which the hydrodynamic values markedly change), and, assuming $\mu \ll 1$, we shall consider the solution of system (1.14) of the form y (μ, r, t; μ), where y is some kind of magnitude of ρ, **v** etc. Thus we are considering flows in which spatial gradients of the n-th order are values of the n-th order of smallness.

Expanding $\underline{y}$ in a power series with respect to μ

$$y = y(\mu \mathbf{r}, t; \mu) = y^0(\mu \mathbf{r}, t) + \mu y'(\mu \mathbf{r}, t) + \dots \qquad (3.1)$$

we derive from (1.14) that

$$\frac{d\rho^0}{dt} = 0, \qquad \frac{d\mathbf{v}^0}{dt} = 0, \qquad \frac{d\xi^0}{dt} = -K e^0_{\xi} \qquad (3.2)$$

The last equation describes the relaxation process with characteristic time τ. Changes in density and velocity fields, etc., occur in the second approximation which is divergent with time. Therefore, the expansion (3.1) is advantageously used for describing movement only into a flow of sufficiently small time intervals. The characteristics of the thus obtained equations are determined by velocity c_{∞}. Consequently, by means of expansion (3.1) we cannot arrive at the equations of equilibrium hydrodynamics. The difficulties arising here can be due to the small-valued parameter (τ) being the coefficient for higher derivatives (see, for example, Eq. (2.17)).

To change to equations of equilibrium hydrodynamics, we shall consider a flow from the moment that the relaxation process in the zero approximation described by Eq. (3.2) is virtually complete and at each later moment, the deviations of parameter ξ from equilibrium values are determined by the fields ρ, s, $\mathbf{v}$. Under these conditions we can write the following expansion for

$$\xi = \xi_0(\rho, s) + \mu \xi^{(1)}(\mu \mathbf{r}; s, \rho, \mathbf{v}) + \mu^2 \xi^{(2)}(\mu \mathbf{r}; \rho, s, \mathbf{v}) + \dots \qquad (3.3)$$

where $\xi^{(n)}$ are functionals of the fields $\rho(\mathbf{r})$, $s(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$. Thus, the coefficients of expansion (3.3) depend on the time implicitly in terms of the fields ρ, s, $\mathbf{v}$.

Under such an assumption of the character of the change of ξ with time, it is evident that the following expansion for $\partial s/\partial t$ etc., is also valid.

$$\frac{\partial\rho}{\partial t} = \mu N^{(1)}(\mathbf{r}; \rho, s, \mathbf{v}) + \mu^2 N^{(2)}(\mathbf{r}; \rho, s, \mathbf{v}) + \dots \tag{3.4}$$

$$\frac{\partial\mathbf{v}}{\partial t} = \mu \mathbf{V}^{(1)}(\mathbf{r}; \rho, s, \mathbf{v}) + \mu^2 \mathbf{V}^{(2)}(\mathbf{r}; \rho, s, \mathbf{v}) + \dots \tag{3.5}$$

$$\frac{\partial s}{\partial t} = \mu S^{(1)}(\mathbf{r}; \rho, s, \mathbf{v}) + \mu^2 S^{(2)}(\mathbf{r}; \rho, s, \mathbf{v}) + \dots \tag{3.6}$$

where the coefficients in powers of μ are also functionals of fields $\rho(\mathbf{r})$, $s(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$. The explicit form of these coefficients can be obtained by equating (3.4) – (3.6) with the expressions for $\partial\rho/\partial t$ etc., obtainable from (1.14). For example, considering that for grad ρ and div $\mathbf{v}$ there are values of the order of μ, we find from (3.4) and (1.14)

$$\mu N^{(1)} = -(\mathbf{v}\cdot\mathrm{grad}\,\rho) - \rho\,\mathrm{div}\,\mathbf{v}, \qquad N^{(2)} = N^{(3)} = \dots = 0 \tag{3.7}$$

Taking into consideration the expansion for p

$$p = p_0(\rho, s) + \mu\left(\frac{\partial p}{\partial \xi}\right)_{\xi_0} \xi^{(1)} + \dots \tag{3.8}$$

where p_0 is equilibrium pressure, from (3.14) and (1.14) we find

$$\mu\mathbf{V}^{(1)} = -(\mathbf{v}, \nabla)\mathbf{v} - \frac{1}{\rho}\,\mathrm{grad}\,p_0, \qquad \mu\mathbf{V}^{(2)} = -\frac{1}{\rho}\,\mathrm{grad}\,(p_\xi \xi^{(1)}) \tag{3.9}$$

From (3.4) and (1.14) we find

$$\mu S^{(1)} = -(\mathbf{v}, \mathrm{grad}\,s) \tag{3.10}$$

Leaving terms of the first order with respect to μ in expansions (3.4)-(3.6) and taking into account (3.7), (3.10), we obtain the equations of motion of an ideal fluid with an equilibrium value of pressure.

Note that the change to equilibrium hydrodynamic equations by expansions of type (3.3), (3.4)-(3.6) are completely analogous to the methods worked out in nonlinear mechanics and statistical physics in connection with the appearance of secular terms in expansions of type (3.1).

To obtain equations of motion in the next approximation, we need to find $\xi^{(1)}$ entering into (3.9).

From (3.3)-(3.6) we find, correct to μ^2

$$\frac{\partial \xi^{(1)}}{\partial t} = \mu \frac{\partial \xi_0}{\partial \rho} N^{(1)} + \mu \frac{\partial \xi_0}{\partial s} S^{(1)} \tag{3.11}$$

Substituting (3.7) and (3.10) into (3.11), we obtain

$$\frac{\partial \xi^{(1)}}{\partial t} = -\mathbf{v} \cdot \operatorname{grad} \xi_0 - \frac{\partial \xi_0}{\partial \rho} \rho \operatorname{div} \mathbf{v} \tag{3.12}$$

On the other hand, with the same approximation

$$-K \varepsilon_\xi = -\mu \frac{1}{\tau} \xi^{(1)} \tag{3.13}$$

Substituting (3.12) and (3.13) into the last of Eqs. (2.21), we determine $\xi^{(1)}$

$$\frac{\mu}{\tau} \xi^{(1)} = \frac{\partial \xi_0}{\partial \rho} \rho \operatorname{div} \mathbf{v}$$

By substituting this value of $\xi^{(1)}$ into (3.9), we obtain the equation of motion as

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p_0 + \operatorname{grad} (\zeta \operatorname{div} \mathbf{v}) \tag{3.14}$$

where

$$\zeta = -\left(\frac{\partial p}{\partial \xi}\right)_{\xi=\xi_0} \tau \frac{\partial \xi_0}{\partial \rho} \rho \quad \text{when} \quad \zeta = \tau \rho \left(c_\infty^2 - c_0^2\right) \tag{3.15}$$

plays the role of the second coefficient of viscosity.

This conclusion assumes $\mu \ll 1$. The conclusion, however, is not restricted by the conditions of smallness of the change in the hydrodynamic values and is thus also valid outside the framework of accoustic approximation (compare the studies of V. Finkel'nburg [1] and V. Granovskiy [4]).

The propagation of small disturbances in relaxing media have a number of important features. Due to dispersion, a relaxing medium does not have a definite velocity of sound, although its properties can be characterized by two magnitudes having a dimensionality of velocity: c_∞ and c_0. The complete set of these two values, as will be shown below, fully define the law of propagation of small disturbances and thus, to a certain degree, replaces the velocity of sound in relaxation hydrodynamics.

In equilibrium hydrodynamics, small disturbances are propagated along the characteristics. In relaxing media this assumption does not hold by virtue of dispersion, and thus the characteristic directions take on a different meaning. This has already been seen from the fact that the characteristic directions of the equations of relaxation hydrodynamics for all values of relaxation time τ are determined by velocity c_∞ simultaneously with the propagation of small disturbances for small values of τ.

To clarify both the laws of propagation of small disturbances in relaxing media and the role of the characteristic directions, we shall consider the following problem [7].

4. Weak shock wave in a relaxing medium. Let a cylindrical tube be closed at one end by a piston which, when $t < 0$, is at rest at point $x = 0$, while at time $t = 0$ it is set into motion and subsequently proceeds with constant velocity u_0 in a positive direction of the x-axis. The velocity of the piston is small in comparison with the velocity of sound ($u_0 < c_0 < c_\infty$). Let us consider the propagation of disturbances arising from piston motion.

The equation describing propagation of a small disturbance of velocity v in one dimension has the following form (see (2.17))

$$\tau \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial t^2} - c_\infty^2 \frac{\partial^2 v}{\partial x^2}\right) + \frac{\partial^2 v}{\partial t^2} - c_0^2 \frac{\partial^2 v}{\partial x^2} = 0 \qquad (4.1)$$

The boundary and initial conditions in this problem can be stated in the following manner:

$$v(x, t) = 0 \text{ when } t < 0, \qquad v(0, t) = f(t) = \begin{cases} 0 & \text{when } t < 0 \\ u_0 & \text{when } t > 0 \end{cases} \qquad (4.2)$$

A solution is found only for the region $x > 0$. The discontinuous function $f(t)$ from (4.2) can be presented as a contour integral

$$f(t) = -\frac{u_0}{2\pi i}\int \frac{e^{-i\omega t}}{\omega} d\omega \qquad (4.3)$$

Then the solution of the problem is also represented by a contour integral

$$v(x, t) = -\frac{u_0}{2\pi i}\int \frac{e^{i(kx-\omega t)}}{\omega} d\omega, \qquad k = \omega \sqrt{\frac{1 - i\omega\tau}{c_0^2 - ic_\infty^2 \omega\tau}} \qquad (4.4)$$

Integration in (4.3) is carried out along the actual axis with a by-pass of the origin of the coordinates with respect to the upper half-plane.

In the lower half-plane, function $k(\omega)$ contains two branching points

$$\omega_1 = -i \frac{c_0^2}{c_\infty^2} \frac{1}{\tau}, \qquad \omega_2 = -\frac{i}{\tau}$$

The section for transferring to the other sheet of the Riemann surface extends along an imaginary axis between these points. Since the path of integration does not pass through the section, we can select for the entire path one definite branch of the function $K(\omega)$, namely that for which $k = +\omega/c_\infty$ when $\omega \rightarrow 0$.

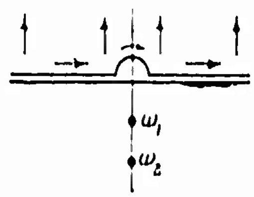

Fig. 1.

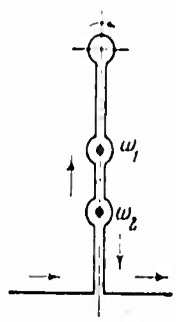

Fig. 2.

It is readily seen that (4.4) satisfies both differential equation (4.1) and boundary condition (4.2) if we represent f(t) in the form (4.3). To verify fulfillment of initial condition (4.2), we note that in the upper-part of the half-plane the integrand does not contain the characteristic features, then the contour of integration can be carried out to infinity along the positive direction of the imaginary axis (Fig. 1).

In this case

$$kx - \omega t = \omega\left(\frac{k}{\omega}x - t\right) \approx \omega\left(\frac{x}{c_\infty} - t\right)$$

Thus, if $t < x/c_\infty$, the integrand, when integration is carried out to infinity, contains an exponentially damping coefficient, and, consequently, $v(x, t) = 0$ when $t < x/c_\infty$. Hence, in particular, the initial condition (4.2) follows for $x \geq 0$

Let us consider the behavior of the solution along the straight line $x - c_\infty t = 0$, which coincides with one of the characteristics of Eqs. (4.1). As shown above, on one side of this straight line, namely where $t < x/c_\infty$, the disturbance is absent.

Let us consider the nature of the disturbance when $t - x/c_\infty = t' > 0$. We deform the contour of integration into a semi-circle of large radius located in the upper half-plane with segments of the real axis adjacent to it. Such an integral, taken from a mirror image of this curve in the lower half-plane, equals zero, since on integrating to $-\infty$ along the imaginary axis the integrand, when $t' > 0$, decreases exponentially. Combining the integrals with respect to these curves, we represent v(x, t) as an integral over a circle of large radius with its center at the origin. In this circle the function $k(\omega)$ can be expanded into a series in powers of $(\omega\tau)^{-1}$

$$k = \frac{\omega}{c_\infty}\left\{1 - \frac{1}{2}\frac{\alpha^2}{i\omega\tau} + \left(\frac{1}{2} - \frac{3}{8}\alpha^2\right)\frac{\alpha^2}{\omega^2\tau^2} + \dots\right. \qquad \left(\alpha^2 = \frac{c_\infty^2 - c_0^2}{c_\infty^2} > 0\right) \tag{4.5}$$

Thus, if we limit ourselves to the first three terms of expansion (4.5), then

$$v(x, t) \approx -\frac{u_0}{2\pi i}\exp\left(-\frac{1}{2}\frac{\alpha^2}{c_\infty{}^2\tau}x\right)\oint\frac{d\omega}{\omega}\exp\left\{i\left(\frac{\lambda}{\omega} - \omega t'\right)\right\} \tag{4.6}$$

where

$$\lambda = \frac{\alpha^2}{2}\left(1 - \frac{3}{4}\alpha^2\right)\frac{x}{c_\infty\tau^2}, \qquad t' = t - \frac{x}{c_\infty} \tag{4.7}$$

By substituting variables ω $t'/\lambda = \lambda'$, we find

$$v(x, t) = -\frac{u_0}{2\pi i}\exp\left(-\frac{1}{2}\frac{\alpha^2}{c_\infty\tau}x\right)\oint\frac{d\lambda'}{\lambda'}\exp\left\{-i\sqrt{\lambda t'}\left(\lambda' - \frac{1}{\lambda'}\right)\right\} \tag{4.8}$$

If we use the familiar expression for Bessel's function with a whole number index

$$J_n(z) = \frac{1}{2\pi i}\oint\frac{d\lambda}{\lambda^{n+1}}\exp\left\{\frac{1}{2}z\left(\lambda - \frac{1}{\lambda}\right)\right\} \tag{4.9}$$

where the integration is carried out along any closed contour around $\lambda = 0$, and we rewrite (4.8) as

$$v(x, t) = u_0\exp\left(-\frac{1}{2}\frac{\alpha^2}{c_\infty\tau}x\right)J_0\left(-2i\sqrt{\lambda t'}\right) \tag{4.10}$$

To clarify the criteria of the applicability of the obtained expression, we retain terms of higher order in expansion (4.5).

As a result we obtain

$$v(x, t) = u_0 \exp\left(-\frac{1}{2}\frac{\alpha^3}{c_\infty \tau} x\right)\left\{J_0(-2i\sqrt{\lambda t'}) + \frac{A}{\lambda} t' J_2(-2i\sqrt{\lambda t'})\right\} \quad (4.11)$$

where

$$A = \frac{\alpha^3}{2}\left(1 - \frac{3}{2}\alpha^2 + \frac{5}{8}\alpha^4\right)\frac{x}{c_\infty \tau}$$

The second term in (4.11) can be neglected, if $t' \ll \lambda/A \approx \tau$. Therefore (4.10) really represents the solution in a narrow field (less than τ in width) along the characteristic in question. Since $J_0(2i\sqrt{\lambda t'}) \approx 1$ when $t' \ll \tau$, the results thus obtained can be written as:

$$v(x,t) = \begin{cases} 0 & \text{when } t' = t - x/c_\infty < 0 \\ u_0 e^{-\gamma x} & \text{when } t' > 0,\ t'/t \ll 1 \end{cases} \qquad \left(\gamma = \frac{1}{2}\frac{\alpha^3}{c_\infty \tau}\right) \quad (4.12)$$

Thus, along the characteristic in question, there arises a discontinuity, whose intensity decreases exponentially with increasing distance from the piston. At the distance of $x \gg c_\infty \tau$ the discontinuity on the characteristic virtually vanishes along with the disturbance propagating along the characteristic.

To obtain a complete representation of the flow when $x \gg c_\infty \tau$, we must investigate the region far from the characteristic (i.e., when $t' \gg \tau$). We shall shift the path of integration toward negative values of the imaginary axis (Fig. 2). When $t' > 0$, the integral with respect to the path displaced toward $-\infty$ vanishes, and the solution is determined only by the integrals around special points. The integrand contains the coefficient $e^{-i\omega t}$ which for $\omega = \omega_1$,

$\omega = \omega_2$ decreases exponentially. Thus, when $i\omega_1 t' \gg 1$, and when $i\omega_2 t' \gg 1$, i.e., when $t' \gg \tau$, the integral actually reduces to a remainder near the pole $\omega = 0$, therefore

$$v(x, t) = u_0 \quad \text{when} \quad t' \gg \tau \tag{4.13}$$

This also indicates that when $t' \gg \tau$, the disturbance is determined by small frequencies $\omega \ll 1/\tau$, as should be expected from physical considerations. To obtain the dependence of $\underline{v}$ on the coordinates and time, we shall consider the following expression

$$\frac{\partial v}{\partial t} = \frac{u_0}{2\pi} \int_{-\infty}^{+\infty} e^{i(kx-\omega t)}\, d\omega \tag{4.14}$$

Since the integrand in (4.14) does not contain singularities when $\omega = 0$, integration can then be carried out along the real axis. Expanding $k(\omega)$ into a series for $\omega\tau \ll 1$, we get

$$k \approx \frac{\omega}{c_0}\left(1 + \frac{i}{2}\mu_0^2\omega\tau\right), \qquad \mu_0^2 = \frac{c_\infty^2 - c_0^2}{c_0^2} > 0 \tag{4.15}$$

Substituting (4.15) into (4.14), and then integrating, we find

$$\frac{\partial v}{\partial t} = u_0 \sqrt{\frac{c_0}{2\pi\mu_0^2\tau x}} \exp\left\{-\frac{(c_0 t - x)^2}{{}^1\!/_2\mu_0^2 c_0 \tau x}\right\}$$

From which it follows that

$$v = \frac{1}{2} u_0 \Phi\left(\sqrt{\frac{1}{\mu_0^2 c_0 \tau x}}\,(c_0 t - x)\right) + \text{const} \tag{4.16}$$

where

$$\Phi(z) = \sqrt{\frac{2}{\pi}} \int_0^z e^{-{}^1\!/_2\eta^2}\, d\eta \tag{4.17}$$

Because of (4.13) the constant in (4.16) must be chosen so that $v \to u_0$ when $t - x'/c_0 \to \infty$, then

$$u(x, t) = \frac{1}{2} u_0 \left\{ \Phi\left(\frac{c_0 t - x}{\sqrt{\mu_0^2 c_0^2 \tau x}} \right) + 1 \right\} \qquad (4.18)$$

Thus when $x \gg c_\infty \tau$, the discontinuity with respect to the characteristic gives way to a continuous disturbance which arises along the straight line $x - c_0 t = 0$, and which does not coincide with the characteristic. The width of the region in which disturbance appears is of the order of

$$\delta = \sqrt{\mu_0^2 c_0 \tau x} \qquad (4.19)$$

and therefore its relative dimensions decrease with distance from the piston

$$\frac{\delta}{x} = \sqrt{\frac{\mu_0^2 c_0 \tau}{x}} \qquad (4.20)$$

Thus the small perturbation which arises at a certain point is propagated close to it ($x < c_\infty \tau/\alpha^2$) along the characteristic passing through this point; however, as distance increases, an ever greater part of the perturbation is displaced from the characteristic on which it arises, and finally when $x \gg c_\infty \tau/\alpha^2$ the propagation of the perturbation is determined by the velocity c_0. Since there are no perturbations having a propagation velocity greater than c_∞, the characteristic $x = c_\infty \tau$ separates the unperturbed region. However, due to rapid damping of the high-frequency components at large distances compared to $c_\infty \tau$, the boundary between the perturbed and unperturbed flow does not proceed along this characteristic.

According to the result obtained from analyzing stationary solutions in Waldmann's work [9] (see also Finkel'burg's, et al. work [1]), a weak shock wave in a relaxing medium does not have to contain discontinuities. Its width is determined in S. P. D'yakov's work [10]. The results obtained in this work illustrate a law, in accordance with which a weak discontinuity in a relaxing medium disappears. Note, however, as will be shown below, that this conclusion is not wholly applicable to plane flows. Near the barrier whose width is comparable with the relaxation length l, and even in the presence of relaxation processes, there may arise a steady discontinuity of arbitrarily small intensity which, however, vanishes exponentially with distance from the wall.

The unrestricted widening of the region in which the disturbance arises (see (4.19)) is associated with neglect of nonlinear effects. We know that small nonlinear effects accumulate and at sufficiently great distances lead to marked deviations from a linear solution. For instance, during propagation of a disturbance arising from harmonic oscillations, the leading edge of the wave front gradually becomes steeper. In the case under consideration the same effects ultimately limit the increase in width of the disturbance.

However, if the disturbance is sufficiently weak and limited in time, it may disappear due to absorption before a steady-state sets in. Actually, if at the moment $t = T$ the piston stops, there arises a disturbance which propagates according to laws described above, but at a distance $\Delta x \approx c_0 T$ from it (taking into consideration that $c_0 \approx c_\infty$). Due to this, when

$$\delta \equiv \sqrt{\mu_0{}^2 c_0 \tau x} \approx c_0 T$$

both disturbances are superposed on and weaken each other. Therefore, a disturbance of duration T disappears at distances significantly greater than $c_0 T^2/\mu_0 \tau$.

5. <u>Flow past a thin wedge</u>. We shall consider the shape of a plane, steady supersonic flow in a relaxing medium and, in particular, the problem of the flow past a thin wedge (having an apex angle of 2φ) of a supersonic flow at zero angle of attack [11, 12]. Let the x-axis coincide with the direction of movement of an unperturbed flow whose velocity we denote by U, and let us assume that $U_\infty > c_\infty > c_0$. The equations of relaxation hydrodynamics for the plane, steady flow are written as (see (1.14)):

$$v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + \rho \frac{\partial v_x}{\partial x} + \rho \frac{\partial v_y}{\partial y} = 0 \tag{5.1}$$

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} + \frac{\partial p}{\partial x} = 0, \qquad v_x \frac{\partial s}{\partial x} + v_y \frac{\partial s}{\partial y} = \frac{K\xi^2}{T}$$

$$\rho v_x \frac{\partial v_y}{\partial x} + \rho v_y \frac{\partial v_y}{\partial y} + \frac{\partial p}{\partial y} = 0, \qquad v_x \frac{\partial \xi}{\partial x} + v_y \frac{\partial \xi}{\partial y} = -K\xi$$

where v_x, v_y are the velocity components along the x- and y-axes. In these equations, due to thermal conductivity and diffusion, the anisotropic portion of the stress tensor and change in entropy can be neglected. The influence of these effects is evaluated below.

Since the perturbations caused by the wedge are negligible, Eq. (5.1) can be linearized by setting $v_x = U + u$, $v_y = v$, etc. Thus we obtain for <u>v</u>

$$l \frac{\partial}{\partial x}\left\{(M_\infty^2 - 1)\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2}\right\} + (M_0^2 - 1)\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0 \qquad \left(l = \frac{c_\infty^2}{c_0^2} U\tau\right) \tag{5.2}$$

Here $M_\infty = U/c_\infty$, $M_0 = U/c_0$ are the Mach numbers. Similar equations can be obtained for density and velocity disturbances along the x-axis.

Equation (5.2) is of the hyperbolic type for $M_\infty > 1$ and of the elliptical type for $M_\infty < 1$. In the first case, three characteristics pass through each point, two of which have the form

$$x = \pm\sqrt{M_\infty^2 - 1}\,y + \text{const} \tag{5.3}$$

while the third is parallel to the x-axis. The direction of these characteristics does not depend on τ and does not coincide with the direction of the Mach lines of equilibrium hydrodynamics, $x = \pm\sqrt{M_0^2 - 1}\,y + \text{const}$. In order that the function $\exp(Bx - Dy)$ represent the partial solution (5.2), it is necessary that

$$\frac{B}{D} = \frac{i(M_\infty^2 - 1)B + (M_0^2 - 1)}{iB + 1} \tag{5.4}$$

We will consider the origin as coinciding with the apex of the wedge and we shall write the boundary conditions of the problem as

$$v(x, 0) \equiv f(x) = \begin{cases} 0 & \text{when } x < 0 \\ \varphi U & \text{when } x > 0 \end{cases} \tag{5.5}$$

Representing $f(x)$ as a contour integral, we will write the solution of the problem in the following form:

$$v(x, y) = \frac{\varphi U}{2\pi i}\int \frac{1}{\eta}\exp\{i\eta(x' - a(\eta)y')\}\,d\eta \tag{5.6}$$

where

$$a(\eta) = \sqrt{\frac{i(M_\infty^2 - 1)\eta + (M_0^2 - 1)}{i\eta + 1}}, \qquad x' = \frac{x}{l}, \qquad y' = \frac{y}{l} \tag{5.7}$$

In (5.6), integration is carried out along the real axis with a bypass of origin of the coordinates in the lower half-plane.

The function $a(\eta)$ in the plane of the complex variable is analogous according in its own properties to the function $k(\omega)$ introduced in Section 4; special points of this function are distributed in the upper half-plane $\eta_1 = i$, $\eta_2 = i(M_0^2 - 1)/(M_\infty^2 - 1)$. The path of integration in this case also passes through the cross section. When $\eta \to \infty$ and $\eta \to 0$, $a(\eta)$ takes on the real values

$$a(\infty) = \pm\sqrt{M_\infty^2 - 1}, \qquad a(0) = \pm\sqrt{M_0^2 - 1} \tag{5.8}$$

If we select that branch of (5.7) which as $\eta \to \infty$ approaches $+\sqrt{M - 1}$, then by transferring the contour of integration to $-\infty$ along the imaginary axis, we can show that the solution vanishes when

$$x - \sqrt{M_\infty^2 - 1}\,y < 0 \tag{5.9}$$

otherwise vanishing takes place when

$$x + \sqrt{M_\infty^2 - 1}\,y < 0 \tag{5.10}$$

Subsequently in the solution for $y > 0$ it is necessary to use that branch of the radical (5.7) which corresponds to the plus sign in (5.8). The second branch should yield perturbations propagating upstream and which should be discarded for physical considerations. When $y < 0$, we must choose, for the same considerations, a branch corresponding to the negative sign in (5.8).

Equation (5.9) is one of the characteristics passing through the apex of the wedge. Let us consider the behavior of the solution along this characteristic in a layer of width << 1. Deforming the contour of integration in the same manner used in the problem of the piston motion, we find

$$v(x, y) = \begin{cases} 0 & \text{when } x - \sqrt{M_\infty^2 - 1}\,y < 0 \\ \varphi U \exp\left[-\tfrac{1}{2}\beta^2 (y/l)\right] J_0(z') & \text{when } x - \sqrt{M_\infty^2 - 1}\,y > 0 \end{cases} \tag{5.11}$$

Here $J_o(z')$ is Bessel's function

$$\beta^2 = \frac{M_0^2 - M_\infty^2}{\sqrt{M_\infty^2 - 1}} > 0, \qquad z' = i\left(2\beta^2\left[1 + \frac{\beta^4}{\sqrt{M_\infty^2 - 1}}\,\frac{y}{l}\,\frac{x - \sqrt{M_\infty^2 - 1}\,y}{l}\right]\right)^{1/2}$$

The function $J_o(z') \approx 1$ in the vicinity of the characteristic (5.9).

Thus near the apex of the wedge along the characteristic passing through it, a discontinuity arises, which, however, exponentially decreases with distance. When $x \gg 1$, $y \gg l$ the perturbation along the characteristic virtually disappears. Here the main contribution is determined by the long-wave components of the perturbation. If in (5.6) we let the quantity $a(\eta)$ be approximately $\sqrt{M_0^2 - 1}$, then after integrating we obtain a result which follows from equilibrium hydrodynamics

$$v(x, t) = \begin{cases} 0 & \text{when } x - \sqrt{M_0^2 - 1}\, y < 0 \\ \varphi U & \text{when } x - \sqrt{M_0^2 - 1}\, y > 0 \end{cases} \tag{5.12}$$

However if in $a(\eta)$ we take into account the terms of a higher order with respect to η, we find that actually there will be no discontinuity along the line $x - \sqrt{M_0^2 - 1}y = 0$. Proceeding in the same manner as in the problem of the piston motion, we find

$$v(x, y) = \frac{\varphi U}{2} \left\{ \Phi \left(\frac{x - \sqrt{M_0^2 - 1}\, y}{\sqrt{\nu^2 l y}} \right) + 1 \right\} \qquad \begin{pmatrix} x \gg l \\ y \gg l \end{pmatrix} \tag{5.13}$$

Here Φx is the integral of probability (4.17)

$$\nu^2 = \frac{M_0^2 - M_\infty^2}{\sqrt{M_0^2 - 1}} > 0 \tag{5.14}$$

Equations (5.13) and (5.14) yield the shape of the Mach line produced by the effect of relaxation processes. The characteristic width of the line is of the order of $\sqrt{\nu^2 l y}$.

The flow under consideration can be integrated as a weak shock wave arising near the wedge in a supersonic flow. As noted earlier [9, 1] in a relaxing medium there can be no discontinuities of small magnitude propagating with a velocity less than c_∞ to the unperturbed medium. In the problem in question we are concerned with the more involved case where the value of perturbation (φU) is small, but the velocity of the wedge is greater than c_∞. In this case, as evident from the previously derived solution (5.11), in the vicinity of the wall and in the region of the order of $l \approx U\tau$ a steady discontinuity of arbitrarily small intensity can arise. Within the framework of relaxation hydrodynamics, we have to consider this discontinuity as a geometrical surface. Its width is determined by the time of the free-path length.

Another feature of these discontinuities which arise near the barrier is that the discontinuities are directed (for small intensities) along the characteristics of the relaxation hydrodynamic equations and thus do not coincide with ordinary Mach lines. They vanish when the valocity of the wedge becomes less than c_∞, but not less than c_0. This is because Eq. (5.2) becomes elliptical when $U < c_\infty$.

To elicit the possibility of experimental detection of these discontinuities, we must evaluate the said assumptions. Basically they reduce to neglect of the effects of viscosity and thermal conductivity. When the solution contains frequencies close to relaxation, we can neglect the dissipation effects associated with viscosity and thermal conductivity. Actually, the effect exerted by the latter is of the order of $l_0/L = \tau_0 c/L$ (l_0 is free-path length, τ_0 is free-path length time, $\underline{c}$ is average thermal velocity of the molecules, L is the characteristic dimension of the region in question), on the other hand, the influence of the relaxation effects is of the order of $l/L = U\tau/L >> l_0/L$, since $\tau >> \tau_0$.

With distance from the wall the intensity of the discontinuities under consideration decrease exponentially, and thus they can be detected experimentally only provided the "relaxation length",

$l = (c_\infty^2/c_0^2)\ U\ \tau$, is significant. In particular it is necessary that l be significantly greater than the thickness of the boundary layer. We readily see that under normal pressure, the thickness of the boundary layer generally exceeds the dimensions of the relaxation boundary. We know, however, that the relationship of the boundary layer Δ to the characteristic dimensions of the streamlined body L equals

$$\frac{\Delta}{L} \approx \frac{1}{\sqrt{R}} \qquad \left(R = \frac{UL}{\eta}\right)$$

where R is the Reynolds number. Since η is approximately $1/3\ c l_0$, $\Delta \approx \ l_0 L/3M$. Since the relationship $l/l_0 = N \gg 1$ is virtually independent of density, the relationship

$$\frac{\Delta}{l} = \sqrt{\frac{1}{3M^2}}\,\frac{1}{N}\sqrt{\frac{L}{l_0}}$$

decreases as the free-path length increases, i.e., the dimensions of the relaxation zone increase on rarefaction more rapidly than the thickness of the boundary layer. Here it is absolutely necessary that the free-path length remain small in comparison with L. Because l usually exceeds l_0 by several orders, $l_0/l = 1/N$ is a small value and thus the conditions are possible when

$$\frac{\Delta}{l} = \frac{1}{\sqrt{3M^{1/2}}}\,\frac{1}{N}\sqrt{\frac{L}{l_0}} \ll 1$$

despite the fact that $L/l_0 \gg 1$.

In the case of strong shock waves, the behavior of the shock wave in a relaxation zone near the wall should be specially investigated. From similarity considerations we can expect that in the vicinity of the wedge at distance $x \ll l$, the flow should be analogous to a gas flow having a frozen relaxation process. Thus there should arise a discontinuity whose direction is determined by the Mach number

$M_\infty = U/c_\infty$. At the same time, at distances $x >> l$ from the apex there should occur an ordinary shock wave (expanded owing to the effect of relaxation processes), whose angle of inclination is determined by the velocity of sound c_0. Since the leading front of this type wave has a discontinuity whose width is of the order of the free-path length [9, 1], we can also expect that in the case of a strong shock wave the leading front is curved as it approaches the apex of the wedge, gradually assuming a direction characteristic of a shock wave in a medium lacking internal degrees of freedom. The characteristic dimension within whose limits curving is noticeable should be of the order of l.

In the relaxation zone, the disturbances are propagated along directions differing from those which follow from equilibrium hydrodynamic considerations. At distances significantly exceeding the zone of relaxation, propagation is localized in a relatively narrow region along the characteristics of equilibrium hydrodynamics. The width of this region increases proportionally as the square root of distance, although it does not begin to exert an influence on the nonlinear effects. Thus, the perturbation issuing from two barriers at a distance L from each other and which merge into a single Mach line when $y > L^2/v^2 l$. In exactly the same manner the barrier having linear dimensions L yields a Mach line of finite length, which vanishes at a distance of the order of $L^2/v^2 l$ from the x-axis due to superposition of disturbances issuing from extreme points. We shall consider in greater detail the change in the nature of the disturbance as distance from the site of its origin increases.

6. Propagation of small disturbances in the relaxation zone. Suppose on the line of flow, $y = 0$, there arises an arbitrary disturb-

ance of some hydrodynamic magnitude (velocity, density, temperature)

$$F(x,0)=\psi(x) \tag{6.1}$$

If the disturbance is steady, it is described by the following equation:

$$l\frac{\partial}{\partial x}\left\{(M_\infty^2-1)\frac{\partial^2F}{\partial x^2}-\frac{\partial^2F}{\partial y^2}\right\}+(M_0^2-1)\frac{\partial^2F}{\partial x^2}-\frac{\partial^2F}{\partial y^2}=0 \tag{6.2}$$

which must be integrated with boundary condition (6.1). In the case of equilibrium hydrodynamics, the disturbance is governed by the wave equation

$$(M_0^2-1)\frac{\partial^2F}{\partial x^2}-\frac{\partial^2F}{\partial y^2}=0 \tag{6.3}$$

which may be obtained from (6.2), having set the relaxation length equal to zero. If we discard disturbances propagating upstream, the solution of (6.3) for $y > 0$ is then written as

$$F(x,y)=\psi\left(x-\sqrt{M_0^2-1}\,y\right) \tag{6.4}$$

Solution of (6.2) can be written as:

$$F(x,y)=\frac{1}{\sqrt{2\pi}}\int g(\eta)\exp\left\{i\eta\left(\frac{x}{l}-a(\eta)\frac{y}{l}\right)\right\}d\eta \tag{6.5}$$

where

$$g(\eta)=\frac{1}{\sqrt{2\pi}}\int\frac{\psi(x)}{l}e^{-i\eta\frac{x}{l}}dx \tag{6.6}$$

and function $a(\eta)$ was taken from (5.7). When $y \gg 1$, only the low-frequency components of the initial disturbance $\psi(x)$ contribute to (6.5).

Expanding $a(\eta)$ into a series in η

$$a(\eta)=\sqrt{M_0^2-1}-i\eta\frac{\nu^2}{2}+\cdots$$

and limiting ourselves to the first two terms of the expansion, we find

$$F(x, y) \approx \frac{1}{2\pi} \iint \exp\left\{i\eta\left(\frac{x}{l} - \sqrt{M_0^2 - 1}\,\frac{y}{l} - \frac{\zeta}{l}\right)\right\} \times$$

$$\times \exp\left(-\eta^2 \frac{y}{l} \frac{\nu^2}{2}\right) \psi(\zeta) \frac{1}{l}\, d\eta\, d\zeta =$$

$$= \frac{1}{2\pi} \iint \exp\left\{i\eta \frac{\zeta}{l} - \eta^2 \frac{\nu^2}{2} \frac{y}{l}\right\} \frac{\psi(x - \sqrt{M_0^2 - 1}\,y - \zeta)}{l}\, d\eta\, d\zeta$$

where F is also determined in the same way as in (5.14)

Integrating with respect to η, we obtain

$$F(x, y) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{\nu^2 l y}} \exp\left(-\frac{\zeta^2}{2ly\nu^2}\right) \psi(x - \sqrt{M_0^2 - 1}\,y - \zeta)\, d\zeta \qquad (6.7)$$

Equation (6.7) more exactly describes the laws of propagation of small disturbances for $y \gg 1$ than does Eq. (6.4). Since in equilibrium hydrodynamics the disturbance at any point $P(x, y)$ is fully defined by the value of the disturbance at a definite point $P_0(x_0, 0)$ of the x-axis, whose coordinate is $x_0 = x - \sqrt{M - 1}\,y$. If we take into account the relaxation processes, the disturbance, as evident from (6.7), at any point is represented as the sum of disturbances transmitted by different points of the x-axis, however, the weight with which the contributions of the different points are introduced, decrease exponentially with distance from the point $P_0(x_0, 0)$; the width Δx of the region of the x-axis influencing the disturbance at point P is of the order of

$$\Delta x = \sqrt{2lyv^2}$$

It also follows from (6.7) that the initial disturbance $\psi(x)$ is transmitted with a distortion: the integration results in a gradual smoothing out of the initial disturbance and disappearance of fine details as distance from the x-axis increases.

REFERENCES

1. L. D. Landau and Ye. M. Lifshits. Mekhanika sploshnykh sred. M., GTTI, 1953

2. I. G. Kirkwood and W. W. Wood. Hydrodynamics of a Reacting and Relaxing Fluid. J. Appl. Phys., 1957, Vol. 28, No. 4, 395.

3. L. I. F. Broer. Characteristics of the Equations of Motion of a Reacting Gas. J. Fl. Mech., 1958, 4, 276.

4. L. I. Mandel'shtam and M. A. Leontovich. K teorii poglos-shcheniya zvuka v zhidkostyakh, ZhETF, 1937, Vol. 7, P. 438.

5. Ye. V. Stupochenko and I. P. Stakhanov. Ob uravneniyakh relaksatsionnoy gidrodinamiki. DAN SSSR, 1960, Vol. 134, No. 4, p. 872.

6. S. R. De Groot. Termodinamika neobratimykh protsessov. M., GTTI, 1956.

7. Ye. V. Stupochenko and I. P. Stakhanov. K teorii nestat-sionarnykh razryvov v relaksiruyushchikh sredakh. DAN SSSR, 1957, Vol. 117, No. 1, p. 65.

8. G. Curant and K. Friedricks. Sverxzvukovoye techeniye i udarnyye volny [Russian translation]. M., IL, 1950.

9. Ya. B. Zel'dovich. Teoriya udarnykh voln i vvedeniye v gazodinamiku. AN SSSR, 1946.

10. S. P. D'yakov. Udarnyye volny v relaksiruyushchey srede. ZhETF, 1954, Vol. 27, No. 6, p. 728.

11. I. P. Stakhanov and Ye. V. Stupochenko. O strukture liniy Makha v relaksiruyushchikh sredakh, DAN SSSR, 1960, Vol. 134, No. 5, p. 1044.

12. J. F. Clarke. The Linearized Flow of a Dissociating Gas. I. Fl. Mech. 1960, Vol. 7, No. 4, 577.

THE BREAKUP OF DROPS AND A FLUID STREAM BY AN AIR SHOCK WAVE

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While studying the disintegration of drops and a fluid stream by a shock wave, it was found [1-2] that disintegration results from the protracted effect of the dynamic pressure of gas flow behind a shock wave front, and that the relative velocity of gas flow and drop size considerably affects the nature of disintegration. Here the basic defining parameter is the relationship between the pressure forces of gas flows on the drop and the surface tension forces, equal to $\rho u^2 d/\sigma$. Here $\underline{u}$ is the relative velocity of gas flow, ρ is gas density, $\underline{d}$ is the parameter of the liquid drop, and σ is the coefficient of the surface tension. From evidence in the literature [3-5], disintegration does not occur at certain less-than-critical values of this parameter, and the liquid drops are stable to a gas flow of subcritical velocity. This is because the capillary forces which impede breakdown of the drop's spherical stability predominate over the dynamic pressure straining forces.

At relatively low supercritical flow velocities, the strains produced by dynamic pressure increase and the drop begins to break up. The mechanism of this breakup is as follows: the windward side of the drop is pressed inward and the drop becomes parachute-like. The thus formed liquid film, convex in the flow direction, bursts, forming numerous minute droplets. In certain cases, drops can be broken up by increasing oscillations of the drop.

If the velocity of the gas flow greatly exceeds critical velocity, the drop disintegrates by separation of the windward surface layer of the liquid from the drop, whereas its bulk is displaced and deformed negligibly. It was noted that this disintegration process is preceded by the emergence of minute capillary waves on the drop surface.

Whereas the mechanism of drop breakup at critical and low supercritical velocities has been sufficiently well studied and can be theoretically described [6-8], the circumstances of the occurrence of "supercritical" breakup at gas flow velocities comparable to sound velocities are insufficiently clear.

Described below are the results of an experimental study of supercritical breakup of droplets and streams of water by an air shock wave. We also studied the qualitative picture of breakup and have proposed certain empirically verified formulas for breakup time. A value was found for the critical relationship between the governing parameters at which the aforementioned supercritical mechanism of disintegration starts.

1. Description of experimental setup. Figure 1 shows photographs of the breakup process of water drops at different instants after passage of the shock wave (the numbers below the picture indicate in microseconds the time elapsed from the start of shock wave flow

past the drop), and Fig. 2 shows photographs of the breakup process of water streams by a gas flow behind the shock wave front: (0-1) is the time of the preliminary breakup stage, (1-2) is the time of the actual breakup. In the photographs, the velocity of gas flow is designated in m/sec (the number 6 in the last photograph corresponds to a velocity of 172 m/sec).

These photographs were obtained in a shock tube (Fig. 3) of constant cross section (110 x 110 mm), which ensured a greater than 2 msec duration of virtually constant pressure and gas flow velocity in the shock wave. This duration of gas flow of constant parameters was entirely sufficient for completing the entire breakup process. The gas flow velocity behind the shock wave front, which was varied in different experiments from 50 to 200 m/sec, was determined by the pressure in the high-pressure section, which was separated from the working part of the shock tube by a cellophane diaphragm. To obtain the required pressure difference, the diaphragm was shot out of an air gun. A diagram of the experimental device is shown in Fig. 3, where 1 is the working part of the shock tube, 2 the high-pressure section, 3 the water inlet nozzle, 4 the pressure-sensitive detector, 5 a double pulse oscillograph, 6 a delay line spark light source, 7 the aperture of the phototracer, and 8 the photorecorder magazine.

A part of the shock tube was made out of plexiglass in order to see the breakup process. The photographs were made by the open shutter method, using a spark impulse bias lighting triggered at regulatable time intervals from pressure detectors. Moreover, to obtain a continuous time exposure of the breakup process we used a camera which enabled us to evolve an image on a film moving at a speed of 100 m/sec.

Along with photographing the processes, shock-wave pressure diagrams were oscillographed for a more accurate determination of flow velocity and gas density.

The droplets and streamlets of water, whose diameter in different experiments varied from 0.8 to 3.5 mm, were created by using droppers inserted into the shock tube through the upper wall.

2. Experimental results. It is apparent from the pictures in Figs. 1 and 2 that from the moment of the passage of the shock wave past the drops or water streams, the entire disintegration process is distinctly divided into two stages.

During the first stage, the preliminary stage, the drops or water streams remain intact, however, disturbances of the surface capillary wave type occur on their surface. These disturbances increase with time and are transformed into stable crests, whose crowns are on the side surfaces of the drops and streams.

The formation of these crests, from which a gradual separation of water begins, concludes the first stage — preparation for breakup. Depending on size, the shape of deformed drop and the number of crests can differ. For example, while relatively large drops of water with diameters of 3-4 mm have two or three series of crests from each of which disintegration of the water occurs, drops with diameters up to 1 mm are surrounded by only a single crown of the crests.

The second stage of breakup (1-2 in Fig. 2), which lasts until total disappearance of the drop or stream is the stage of actual disintegration. During this stage, there forms a jet of atomized water consisting of almost imperceptible droplets and vapors. No special study of the degree of atomization was made, but individual photographs under the microscope show that the atomization jet contains liquid droplets with dimensions of 20-50 μ. During the second stage we observed nonuniformity in the atomization process. By the degree of opacity of the atomization jet, it is possible to distinguish between the initial ejection of water, evidently associated with final formation of crests, the increase of flow rate to maximum, and the final, rapid diminishing of flow rate by the end of atomization.

The times of the first and second stages depend essentially on velocity of gas flow and diameter of water streams; when flow velocity drops to 60-70 m/sec and stream diameter decreases to 1 mm, the change in the mechanism of atomization is noticeable. The breakup of the streams begins to resemble disintegration into relatively large pieces, and not finely dispersed atomization. An example of this type of disintegration is shown in the fourth photograph from the top in Fig. 2.

It must be pointed out that at the gas flow velocities being considered, both the times of the first and second stages of atomization are so small that there is insufficient time for the streams or drops to be substantially displaced and the internal streams do not have enough time to develop so intensely in them. Therefore, longitudinal instability of the drop, which generally influences the disintegration process, is less substantial. In addition, this explains the fact that the flow rate proceeds from the windward side while the leeward side of the drops and streams remain undisturbed for a long time.

3. Atomization time of drops and fluid streams. To calculate the time of the preparation stage of atomization during which surface waves and crests form on the surface of drops or the stream we use the equation of steady motion of a fluid

$$\frac{dv}{dt} + (v\,\mathrm{grad})\,v = -\frac{1}{\rho}\,\mathrm{grad}\,P + \nu\Delta v \qquad (3.1)$$

Considering water as an ideal liquid, in the equation we can disregard the term which takes into account viscous friction forces. Since the surface crests are small relative to the over-all dimensions of the drop or stream, we can also discard the second term of the equation. In this case, Eq. (3.1) takes the form

$$\frac{dv}{dt} = -\frac{\mathrm{grad}\,P}{\rho} \qquad (3.2)$$

Let us estimate the order of the left- and right-hand sides of Eq. (3.2). For the left-hand side we have

$$dv/dt \sim d/t_1^2 \qquad (3.3)$$

where the diameter of the drop or stream d is taken as the characteristic dimension, and where the time of the preparation stage of atomization t_1 is taken as the characteristic time. Since here we are considering a mechanism of atomization corresponding to large values of the parameter $\rho u^2 d/\sigma$, we can conclude that the chief force deforming the drop surface is that of the dynamic pressure of the oncoming gas flow. By taking this into account, Eq. (3.2) is presented as

$$\frac{\mathrm{grad}\,P}{\rho} \sim \frac{\rho u^2}{\rho_* d} \qquad (3.4)$$

where ρ_* is the density of water, ρ is the density of gas, and u is the relative velocity of the gas flow. Combining (3.2), (3.3) and (3.4), we finally obtain

$$t_1 = k_1 \frac{d}{u} \sqrt{\frac{\rho_*}{\rho}} \qquad (3.5)$$

An analogous dependence for the disintegration time of a liquid drop in a gas is carried out in Gordon's work [6], where the proportionality factor is taken to be 2. As follows from our measurements, this value for the factor is overestimated. This is understandable since the author in his calculations assumed the deformations of the drop determining the time required for disintegration to be comparable with the initial diameter, an assumption which in the case of "super-critical" atomization is false.

As shown empirically, during the time of actual atomization, the liquid film peels from the windward side of the drop or stream. Here it is appropriate to consider the actual atomization process as a process of the spreading of water over the circumfluent gas flow. This spreading is observed when the fluid is pierced by a liquid "hammerhead." Therefore, to determine the actual atomization time we can use the expression for the depth of the opening being perforated by a cumulative stream [9]

$$L = l\sqrt{\frac{\rho_*}{\rho}} \qquad (3.6)$$

Here l is length of stream, ρ_* is density of the stream matter, and ρ is density of the matter comprising the barrier. Identifying the depth of the opening with the length of the drop in the gas flow, we have

$$L \sim t_2 u \qquad (3.7)$$

Hence

$$t_2 = k_2 \frac{d}{u}\sqrt{\frac{\rho_*}{\rho}} \qquad (3.8)$$

Here the diameter d of the drop or stream is taken as the characteristic dimension of the drop or stream.

4. Comparison with the experiment. For verifying the obtained calculated relationships, we carried out a series of experiments to

determine the time of the preliminary stage and the actual atomization time by measuring the diameters of the streams and the gas flow velocities within a wide range.

On photographs similar to those shown in Fig. 2, the following measurements were carried out: the preliminary stage time, t_1, was determined, counting it from the instant the shock wave (0) flows past the stream up to the instant of the appearance of the first ejection of water (1); the time of actual atomization, t_2, was determined from the appearance of the first ejection of water (1) until a maximum flow rate of water or until breakup of the stream is achieved (2). In each experiment, we recorded the initial diameter of the stream and the velocity of the shock wave front, thus making it possible to calculate the proportionality factors k_1 and k_2 in Formulas (3.5) and (3.8).

The results of measuring k_1 and k_2 in relation to parameter $\rho u^2 d/\sigma$ are shown in Figs. 4 and 5. It turns out that at values of this parameter of 10^3 and greater, which correspond to flow velocities higher than 60-70 m/sec and to diameters of the water streams greater than 1 mm, the values of k_1 and k_2 vary negligibly, thus indicating the correct choice of force factors used in evaluating the atomization time. With decreasing velocity of the gas flow and with diameters less than the indicated limits, the values of k_1 and k start to increase sharply, because of the increasing effect of the forces exerted by surface tension during deformation and breakup of water streams. It is evident here that the proposed mechanism of atomization becomes invalid.

Experiments on spontaneous measurements of atomization time of sperical drops were not conducted, although the dimensional relationships (3.5) and (3.8) also hold for them with a slight difference of k_1 and k_2 on the decreasing side.

GRAPHIC NOT REPRODUCIBLE

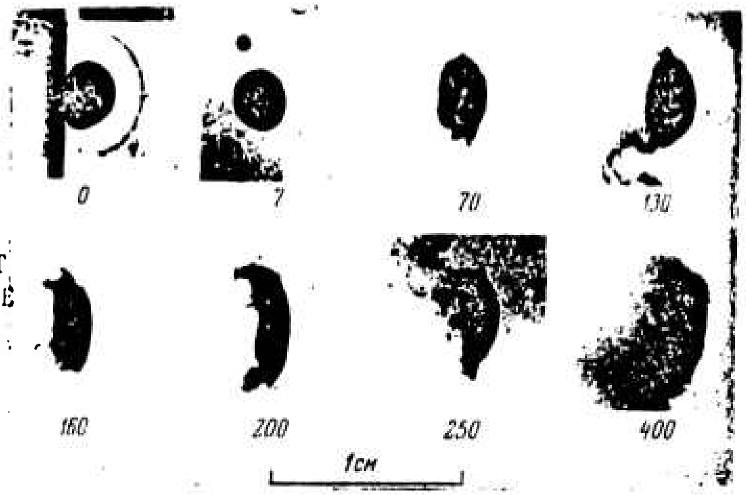

Fig. 1.

GRAPHIC NOT REPRODUCIBLE

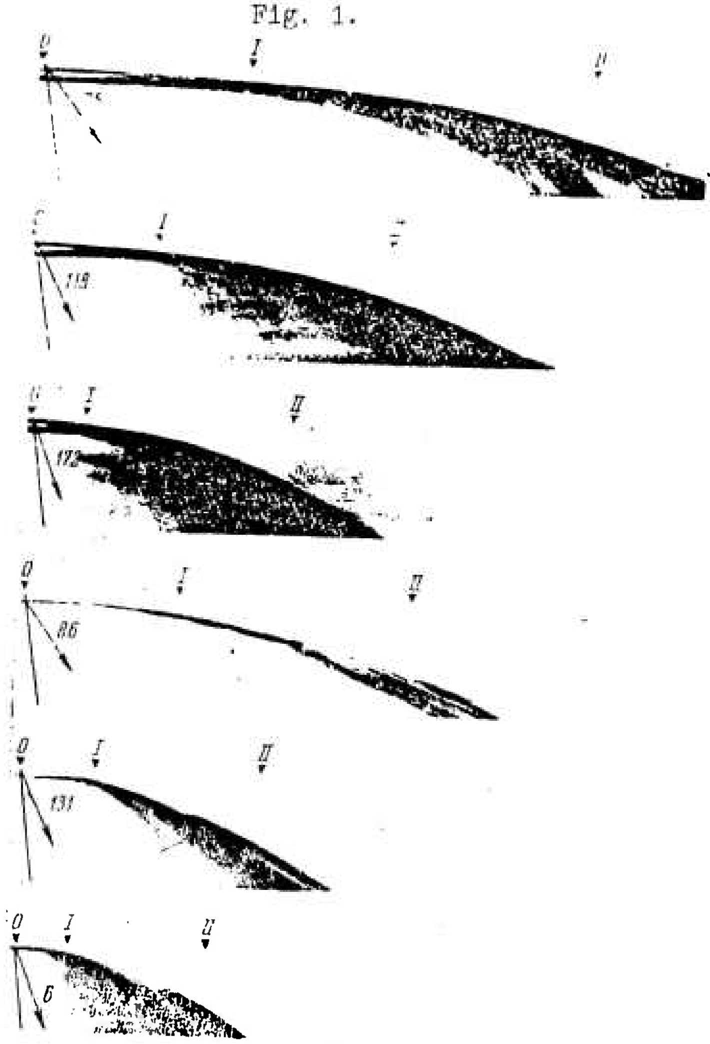

Fig. 2.

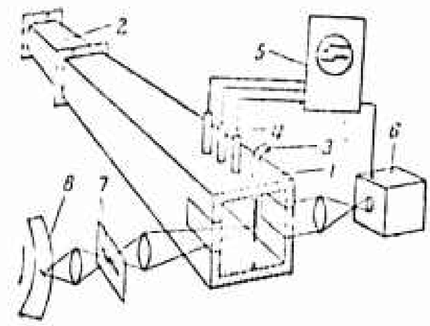

Fig. 3.

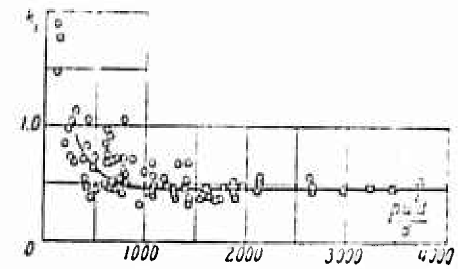

Fig. 4.

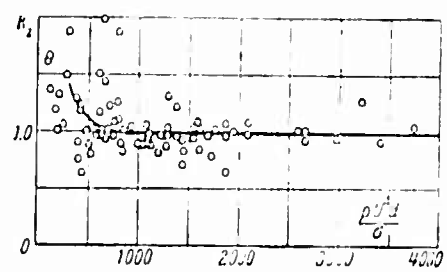

Fig. 5.

REFERENCES

1. W. R. Lane. Shatter of drops in stream of air. Ind Eng. Chem. 43, 6, 1951.

2. R. J. Prime. Raxrusheniye vodyanykh kapel' i bryzg udarnoy volnoy. IL., Voprosy raketnoy tekhniki, 45, No. 3, 1958.

3. M. S. Volynskiy. Izucheniye drobleniya kapel' v gazovom potoke. DAN SSSR, Vol. 68, No. 2., 1949.

4. S. V. Bukhman. Eksperimental'noy issledovaniye raspada kapel'. Vestn. AN Kazakh. SSR, No. 11, 1954.

5. R. H. Magarvey, and B. W. Taylor. Free Fall Breakup of Large Drops. Journ. of Appl. Phys., Vol. 27, No. 10, 1956.

6. Gordon. Mechanism and Speed of Breakup of Drops. Journ. of Appl. Phys. Vol. 30, No. 11, 1959.

7. K. N. Dodd. On the disintegration of drops in an air stream. Journ. of Fluid Mech. Vol. 9, 1960.

8. V. A. Borodin, Yu. F. Dityakin, and V. I. Yagodkin. O droblenii sfericheskoy kapli v gazovom potoke. PMTF, No. 1, 1962.

9. M. A. Lavrent'yev, and B. V. Shabat. Metody teorii funktsiy kompleksnogo peremennogo, GIFML, 1958.

THE STRUCTURE OF A FLAME FRONT

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We found in the first two approximations (from the square of the Mach number, M^2) solutions of hydrodynamic equations with respect to chemical reactions. We noted that in the stationary case the pressure in front of the flame front, which varies proportionally as M^2, increases when $P > 1$ or decreases when $P < 1$ (P is the Prandtl number). For the nonstationary case, qualitative distributions of temperature, pressure, density, and velocity are given with respect to the flow.

In the presence of diffusion processes and chemical reactions, to the familiar hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0 \qquad (1)$$

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma'_{ik}}{\partial x_k} \qquad (2)$$

$$\frac{\partial}{\partial t}\left(\rho e + \frac{\rho v^2}{2}\right) = -\operatorname{div}\left[\rho \mathbf{v}\left(w + \frac{v^2}{2}\right) - (\sigma' \mathbf{v}) + \mathbf{q}\right] \qquad (3)$$

(the notations here agree with those of L. D. Landau [1]) we must add K equations of continuity for each chemical component K_i ($i = 1, \ldots, K$) participating in R reactions, R_s ($s = 1, \ldots, R$). The following form of these equations is used below:

$$\rho \frac{dc_i}{dt} = -\operatorname{div} \mathbf{i}_i - \sum_s m_i \nu_{is} \varphi_s \qquad (4)$$

where c_i is a portion of the mass, i_i is diffusion flow, m_i is the molecular weight [g/mole] of the component K_i, and then φ_s is the velocity of R_s [mole/cm³ sec] and ν_{is} is stoichiometric coefficient of K_i in R_s.

For one-dimensional steady flows, a single integration of the extracted equations yield (in Landau's notations [1])

$$\rho v = j, \quad C_{1i} j c_i + i_i + \sum_s m_i \nu_{is} \int_{-\infty}^{x} \varphi_s dx = C_{2i}, \qquad (p + \rho v^2) - \left(\frac{4}{3}\eta + \zeta\right)\frac{dv}{dx} = C_3 \qquad (5)$$

$$j\left(w + \frac{v^2}{2}\right) - \left(\frac{4}{3}\eta + \zeta\right) v \frac{dv}{dx} + q = C_4 \qquad (6)$$

where C with the subscripts are constants of integrations. We shall introduce for consideration the expression for thermal flow [1]

$$q = -\varkappa \frac{dT}{dx} + \sum_i w_i i_i$$

and the dimensionless variables .

$$\chi = j \int \frac{c_p}{\varkappa} dx, \quad \theta = \frac{T}{T_{1,2}}, \quad \pi = \frac{p}{p_{1,2}}, \quad \lambda = \frac{V}{V_{1,2}} = \frac{\rho_{1,2}}{\rho}$$

Considering the thermal capacities constant, we represent (5) and (6) in the dimensionless form

$$(\pi - 1) + \gamma \varepsilon \left[(\lambda - 1) - P \frac{d(\lambda - 1)}{d\chi}\right] = 0 \qquad (7)$$

$$\left[(\theta - 1) - \frac{d(\theta - 1)}{d\chi}\right] + \frac{\gamma - 1}{2} \varepsilon \left[(\lambda^2 - 1) - P \frac{d(\lambda^2 - 1)}{d\chi}\right] - \frac{1}{\gamma c_v T_{1,2}} \sum_s Q_{p_s} \int_{\mp\infty}^{x} \varphi_s dx = 0$$

$$\left(\gamma = \frac{c_p}{c_v}, \; Q_{p_s} = \sum_i \nu_{is} m_i w_i, \; \sqrt{\varepsilon} = M_{1,2} = \frac{v}{c_{1,2}}, \; P = \frac{(4/3\eta + \zeta) c_p}{\varkappa}\right) \qquad (8)$$

(the subscripts 1,2 refer to the states for $\lambda = \mp \infty$, respectively). Here Q_{p_s} is the molar thermal effect of R_s when p = const, $M_{1,2}$ is the Mach number, and P is the Prandtl number).*

We take the equation of state in the form

$$\pi\lambda\mu = 0 \qquad \left(\mu = \frac{m}{m_{1,2}}\right)$$

For slow combustion the parameter ε is small, and therefore we will seek the solution of the extracted system in the form

$$f = 1 + \sum_{k=0}^{\infty} \varepsilon^k f^{(k)}$$

For simplicity we shall consider P = const.

It is easily confirmed that only the identical solutions $\theta = \pi = \lambda = 1$ correspond to the state behind the flame front ($\lambda > 0$).

For the condition in front of the flame front ($\lambda < 0$), in a zero approximation we have

$$\pi^{(0)} = 0, \quad \theta^{(0)} = \lambda^{(0)} = C_1 e^{\chi} > 0. \tag{9}$$

where C_1 is a constant definable from the condition of joining solutions. The expression for $\theta^{(0)}$ is the familiar Michelson solution [1]. In the following approximation

$$\pi^{(1)} = \gamma(P-1)\,\theta^{(0)} \qquad \lambda^{(1)} = \theta^{(1)} - \pi^{(1)}(1+\theta^{(0)})$$
$$\theta^{(1)} = C_2 e^{\chi} - \frac{\gamma - 1}{2}\left[(2P-1)\,\theta^{(0)} + 2(P-1)\,\chi\right]\theta^{(0)} \tag{10}$$

where C_2 is a constant analogous to C_1, and, finally, in the second approximation

$$\pi^{(2)} = -\frac{\gamma(\gamma-1)}{2}\left[(2P-1)\,\theta^{(0)} + 2(P-1)\right]\theta^{(0)} \tag{11}$$

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* The Prandtl number used here, if we abstract it from the second viscosity is 4/3 greater than the customary expression $P = \eta c_p/\varkappa$.

To change the pressure in front of the flame front, we get

$$\frac{p - p_1}{p_1} = \gamma M_1^2 (P - 1) \frac{T - T_1}{T_1} + O(M_1^4) \tag{12}$$

It is essential that when P > 1, the pressure in front of the flame front increases on approach to the front and conversely, when P < 1, the pressure decreases. Note, that an increase in the pressure in front of the flame front when P > 1 can be a source of disturbance.

For the state in the burning zone, we have in the zero approximation

$$\pi^{(0)} = 0, \quad \lambda^{(0)} = \theta^{(0)} - \frac{d\theta^{(0)}}{d\xi} + \frac{1}{\mu c_p T_{1,2}} \sum_s Q_{p_s} \int_{-\infty}^{\xi} \varphi_s \, d\xi \tag{13}$$

and in the first approximation

$$\pi^{(1)} = \frac{\gamma}{c_p T_{1,2}} \sum_s Q_p \int \varphi_s \, d\xi \tag{14}$$

These results are obtained provided P = 1, and μ = 1. For electrochemical reactions it is evident that $\pi^{(1)} < 0$.

Thus, if in front of the flame front we have $\delta_{p1}/_{p1} > 0$ or $\delta_{p1}/_{p1} < 0$ for P > 1 and P < 1, respectively, and is proportional to M_1^2, then in the region of the front $\delta_{p1}/_{p1} < 0$ and is proportional to M_1^2 (q/c_1^2) (q is the thermal effect of the reaction [cal/g]); here $M_1^2 << 1$ and $q/c_1^2 \sim 10$, if, for example, $Q_p \sim 10^4$ cal/mole, $\mu \sim 50$ g/mole, and $c_1 \sim 300$ m/sec.

Let us now consider the nonstationary system of equations outside of the flame front. It is more convenient to use in place of Eq. (3) the equivalent form:

$$\rho \frac{dw}{dt} - \frac{dp}{dt} = \tau'_{ik} \frac{\partial r_i}{\partial r_k} - \operatorname{div} q$$

Considering as before the thermal capacity of the medium to be constant, let us examine the last equation in a zero approximation (with respect to M^2)

$$\rho c_p\left(\frac{\partial}{\partial t}+(\mathbf{v},\nabla)\right)T = \operatorname{div}(\varkappa\nabla T)$$

Combining this equation with the equation of continuity (1), and by using the equation of state, where p = const, we get

$$\delta\mathbf{v} = \mathbf{v}-\mathbf{v}_{1,2} = (\varkappa / c_p)\nabla\Gamma \tag{15}$$

(In the cylindrical and spherical cases $v_{1,2}$ = 0).

In the same approximation the equation for determining ρ for centrally symmetrical flows has the form

$$\frac{\partial\rho}{\partial t}+r^{-n}\frac{\partial}{\partial r}\left[r^n\rho\left(v_{1,2}+\frac{\varkappa}{c_p}\frac{\partial}{\partial r}\frac{1}{\rho}\right)\right]=0 \tag{16}$$

where n = 0, 1, 2, respectively, in the plane, cylindrical, and spherical case. Thus, the calculation of these flows in the zero approximation reduces to a solution of one nonlinear parabolic equation, for which it is still necessary to formulate specific initial and boundary conditions.

We turn now to the calculation for changing the pressure, limiting ourselves to the one-dimensional case. The combination of Eqs. (1) and(2) yields the expression

$$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}\left[p+\rho v^2-\left(\frac{4}{3}\eta+\zeta\right)\frac{\partial v}{\partial x}\right]$$

Hence

$$p=\left(\frac{4}{3}\eta+\zeta\right)\frac{\partial v}{\partial x}-\rho v^2+\iint\frac{\partial^2\rho}{\partial t^2}\,dx\,dx+\text{const}$$

In the dimensionless variables already used, and also in the new ones

$$\tau=v_{1,2}\,/\int\frac{c_p}{\varkappa}\,dt\,,\qquad \nu=\frac{\rho}{\rho_{1,2}}\,,\qquad \sigma=\frac{v}{v_{1,2}}$$

the last equation has the form

$$\frac{\pi - 1}{\gamma p} = p\frac{\partial\sigma}{\partial\chi} - \nu\sigma^2 + 1 + \iint \frac{\partial^2\nu}{\partial\tau^2}\,d\chi\,d\chi \tag{17}$$

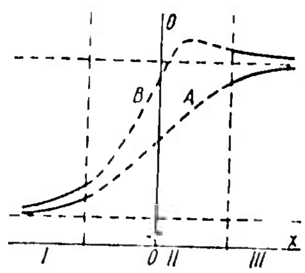

Fig. 1.

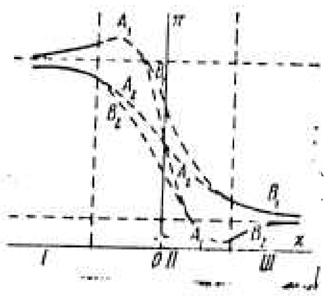

Fig. 2.

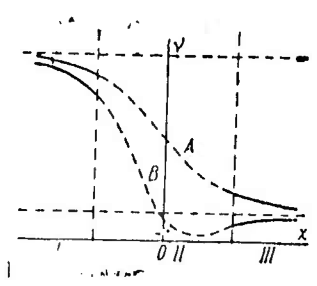

Fig. 3.

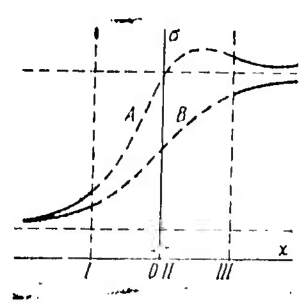

Fig. 4.

Instead of (15) and (16) we will have, respectively

$$\sigma^{(0)} = \frac{\partial\theta^{(0)}}{\partial\chi}, \qquad \frac{\partial v^{(0)}}{\partial\tau} + \frac{\partial v^{(0)}}{\partial\chi} = \frac{\partial^2}{\partial\chi^2}\ln(1 + v^{(0)}).$$

After appropriate substitutions in (17), we obtain

$$\pi^{(1)} = \gamma(P-1)\frac{\partial^2\theta^{(0)}}{\partial\chi^2} = \gamma(P-1)\frac{\partial\sigma^{(0)}}{\partial\chi} \tag{18}$$

It can be shown that for centrally symmetric flows the corresponding equation for p in the functions ρ and v has the form

$$\frac{\partial p}{\partial r} = \left(\frac{4\eta}{3} + \zeta - \frac{\varkappa}{c_p}\right)\frac{\partial}{\partial r}\left[\frac{1}{r^n}\frac{\partial}{\partial r}(r^n v)\right] - n\frac{\rho v^2}{r}$$

A comparison of (18) and (16) with consideration of the equations defining $\theta^{(0)}$ in both cases, leads us to the conclusion that when the initial temperature distribution coincided with stationary steady solution in the case of a subsequent increase in temperature, unsteadiness increases, with respect to the modulus, the change in pressure in front of the flame front (because $\partial\theta^{(0)}/\partial\tau > 0$). The sign of pressure change is usually determined by the sign of the inequality $P \gtrless 1$. Unlike the stationary case, the states behind the flame front here can be described not only by identity solutions.

If the temperature distribution is a monotonically increasing function of the coordinate, its curvature behind the flame front is negative and the sign of the change in pressure behind the front is the inverse of that ahead of the flame front. From considerations of continuity, the pressure distribution when $P > 1$ should in this case have a maximum at the leading edge of the flame front and a minimum at the trailing edge; for $P < 1$ the pressure distribution will be a monotonically decreasing function of the coordinate (in any case exterior to the flame front). The density distribution as evident from the equation of state, when p = const, will be monotonically decreasing function of the coordinate (in any case outside the flame front). From (18) we see that here the velocity in a zero approximation should pass through the maximum at the flame front.

If the temperature distribution has a maximum at the flame front, its curvature outside the front at a sufficient distance from it are positive, and the changes in the sign of the pressure in front of and behind the front are identical. When $P > 1$, the pressure distribution should have a maximum at the leading edge of the flame front, and for $P < 1$ it should have a minimum at the trailing edge. The density distribution should have a minimum at the flame front, and the velocity distribution will be a monotonically increasing function of the coordinate (in any case outside the flame front).

To illustrate the obtained results, Figs. 1-4 show qualitative graphs of the dimensionless characteristics of the flow (the temperature θ, pressure π, density ν, velocity σ) as a function of the dimensionless coordinate χ. The numerals I, II, and III on the graphs correspond to the regions up to the flame front, the flame front, and behind the flame front, respectively. Letters A and B refer to the two possible temperature distributions: the montonically increasing distribution (A) and the distribution with the maximum at the flame front (B). The numbers 1,2 refer to the cases $P > 1$ and $P < 1$.

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REFERENCES

1. L. D. Landau and Ye. M. Lifshits. Mekhanika sploshnykh sred. Gostekhizdat, 1953.

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