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ESTIMATES OF THE BISPECTRUM OF STATIONARY RANDOM PROCESSES

by

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Estimates of the Bispectrum of Stationary Random Processes*

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1. Introduction.

A random process, $\{X_t(\omega)\}$, is an indexed set of random variables. Here t is an element of the index set T. The random variables and the index set can have quite general forms; however, usually the random variables are taken to be either real-valued or vector-valued and the index set is taken to be either the real line or the positive and negative integers. A time series, $\{x_t\}$, is a chronologically arranged set of observations from a random process where t is thought of as time.

Modern time series analysis has been said to encompass three fields: (i) the statistical theory of harmonic analysis, correlation analysis, and regression analysis, (ii) statistical communication and control theory, and (iii) the probabilistic (and Hilbert space) theory of random processes possessing finite second moments. The present paper is concerned with a particular area of harmonic analysis, that of the estimation (from a time series) of the bispectrum of a random process, where the bispectrum can be thought of as the Fourier transform of the third-order moment function (or sequence) of the process.

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Almost all work done heretofore on estimation and practical applications in the area of harmonic analysis has been concerned with the <u>spectrum</u> of the random process, the Fourier transform of the second-order moments. There is a vast literature in this area. For surveys of the theory see Bartlett [1], Hannan [7], Blackman and Tukey [2], Grenander and Rosenblatt [6], Jenkins [10], Parzen [14], [15], Rosenblatt [16], [17], and Tukey [19]. The applications of this theory are numerous and occur in a number of quite diverse phenomena -- signal detection, meteorological events, economics, genetics, aerodynamics, etc.

Recently interest has arisen in applications of higher order spectra and in particular in the bispectrum[1]. While existence problems in higher order spectral theory have been previously discussed[2], statistical problems have not. The present interest is primarily due to a desire to study various nonlinear effects in random phenomena. The spectrum (or second-order theory) provides insufficient information about such effects[3] while the bispectrum does furnish a first glimpse. Such was the case in a recent study by Hasselman, Munk, and MacDonald [8] where the bispectrum is used in

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[1] See, for example, Tukey [19] p. 312 and Hasselman, Munk, and MacDonald [8].

[2] See, for example, Blanc-Lapierre and Fortet [3].

[3] See discussion beginning on p. viii of Blackman and Tukey [2].

connection with oceanographic problems, among which, as the authors state, a number of interesting phenomena such as surf beats, wave breaking, and the energy transfer between wave components can be explained only by the nonlinearity of the wave motion.

The text of the present paper begins with a discussion of the bispectrum itself, some of its properties, and some assumptions on it and on the process. Intuitive reasons for choosing an estimate of the form discussed here are then given along with some convenient expressions for this estimate. Section 4 describes a further assumption on the process characteristic of this paper. This assumption involves the concept of cumulant functions[1] of the process. Sections 5, 6, 7 and 8 are concerned with the statistical properties of various estimates; that is with the asymptotic bias, second-order moments, and distribution of three estimates : (i) the third-order moment estimate, (ii) the weighted bispectral density (the bispectral distribution function) estimate, and finally (iii) the estimate of the bispectral density itself.

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[1] See, for example. Stratonovich [18], Chapter 1.

2. Preliminary definitions and assumptions. Symmetries of the bispectrum.

Let $\{X_t\}$, $EX_t = 0$, $EX_t^6 < \infty$, be a real-valued, sixth-order weakly stationary random process; so that for all t,

$$\begin{aligned} EX_tX_{t+v} &= m_2(t,t+v) = r(v), \\ EX_tX_{t+v_1}X_{t+v_2} &= m_3(t,t+v_1,t+v_2) = r_3(v_1,v_2), \\ &\vdots \\ EX_tX_{t+v_1}\cdots X_{t+v_5} &= m_6(t,t+v_1,\ldots,t+v_5) = r_6(v_1,\ldots,v_5). \end{aligned} \tag{2.1}$$

Throughout the paper, the index set, T, will be taken to be either the real line (in which case the process is called a continuous parameter process) or the positive and negative integers (in which case the process is called a discrete parameter process).

Both $r(v)$ and $r_3(v_1,v_2)$ are taken to be in L_1 (or ℓ_1) and, in the continuous parameter case, they are assumed to be continuous. It is well known from spectral theory that $r(v)$ has a Fourier-Stieltjes representation in terms of the spectral distribution function, $F(\lambda)$. Taking $F(\lambda)$ to be absolutely continuous with a continuous density, $f(\lambda)$, this representation becomes[1]

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[1] A "c" after a formula number denotes the formula is written for the continuous parameter case, similarly a "d" denotes the discrete case.

$$r(\nu) = \int_{-\infty}^{\infty} e^{i\nu\lambda} f(\lambda)d\lambda \qquad (2.2c)$$

$$r(\nu) = \int_{-\pi}^{\pi} e^{i\nu\lambda} f(\lambda)d\lambda. \qquad (2.2d)$$

Further

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\nu\lambda} r(\nu)d\nu \qquad (2.3c)$$

$$f(\lambda) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-i\nu\lambda} r(\nu). \qquad (2.3d)$$

Similarly define the bispectral density function as

$$g(\lambda_1,\lambda_2) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} e^{-i\nu_1\lambda_1 - i\nu_2\lambda_2} r_3(\nu_1,\nu_2)d\nu_1 d\nu_2 \qquad (2.4c)$$

$$g(\lambda_1,\lambda_2) = \frac{1}{(2\pi)^2} \sum_{\nu_1,\nu_2=-\infty}^{\infty}\sum e^{-i\nu_1\lambda_1 - i\nu_2\lambda_2} r_3(\nu_1,\nu_2) \qquad (2.4d)$$

and, assuming $g(\lambda_1,\lambda_2) \in L_1(R_2)$ (or ℓ_1),

$$r_3(\nu_1,\nu_2) = \iint_{-\infty}^{\infty} e^{i\nu_1\lambda_1 + i\nu_2\lambda_2} g(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \qquad (2.5c)$$

$$r_3(\nu_1,\nu_2) = \iint_{-\pi}^{\pi} e^{i\nu_1\lambda_1 + i\nu_2\lambda_2} g(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2. \qquad (2.5d)$$

Since the process is real, the following symmetries occur in the third-order functions:

$$r_3(\nu_1,\nu_2)=r_3(\nu_2,\nu_1)=r_3(-\nu_1,\nu_2-\nu_1)=r_3(-\nu_2,\nu_1-\nu_2)$$

$$= r_3(\nu_2-\nu_1,-\nu_1) = r_3(\nu_1-\nu_2,-\nu_2), \tag{2.6}$$

$$g(\lambda_1,\lambda_2)= g(\lambda_2,\lambda_1) = g(\lambda_1,-\lambda_1-\lambda_2) = g(\lambda_2,-\lambda_1-\lambda_2)$$

$$= g(-\lambda_1-\lambda_2,\lambda_1) = g(-\lambda_1-\lambda_2,\lambda_2) = \overline{g(-\lambda_1,-\lambda_2)} \text{ etc.} \tag{2.7}$$

The symmetries (2.6) imply that $r_3(\nu_1,\nu_2)$ is completely specified over the entire plane by its values in any one of the six sectors, ⓐ through ⓕ, shown in Figure I. These sectors include their boundaries so that, for example, sector ⓐ is

$$0\leq\nu_1<\infty, \quad 0\leq\nu_2\leq\nu_1. \tag{2.8}$$

Similarly $g(\lambda_1,\lambda_2)$ is, by (2.7), completely specified by its values in any one of the twelve sectors (including boundaries) shown in Figure II. Further, in the discrete parameter case, the periodicities of $g(\lambda_1,\lambda_2)$ imply that the sectors need only be of finite extent and natural regions to choose are those shown in Figure III where, for example, region (1) is given by

$$0 \leq \mu_1 \leq \pi, \quad 0 \leq \mu_2 \leq \mu_1, \quad \mu_2 \leq 2\pi - 2\mu_1. \tag{2.9}$$

It is also worth noting that the symmetries indicate that the integrals (sum) in equations (2.4) and (2.5) can be taken over one (or more) of the sectors instead of over the entire plane. For example, (2.4c) becomes[1]

$$g(\lambda_1,\lambda_2) = \frac{1}{(2\pi)^2} \int_0^\infty d\nu_1 \int_0^{\nu_1} d\nu_2 \; r_3(\nu_1,\nu_2) \cdot$$
$$(e^{-i\nu_1\lambda_1 - i\nu_2\lambda_2} + e^{-i\nu_2\lambda_1 - i\nu_1\lambda_2} + e^{-i\nu_1\lambda_2 + i\nu_2(\lambda_1+\lambda_2)}$$
$$+ e^{+i\nu_1(\lambda_1+\lambda_2) - i\nu_2\lambda_2} + e^{-i\nu_1\lambda_1 + i\nu_2(\lambda_1+\lambda_2)} \tag{2.4'c}$$
$$+ e^{+i\nu_1(\lambda_1+\lambda_2) - i\nu_2\lambda_1})$$

or

$$g(\lambda_1,\lambda_2) = \frac{1}{(2\pi)^2} \iint_0^\infty d\nu_1 d\nu_2 \; r_3(\nu_1,\nu_2) \cdot$$
$$(e^{-i\nu_1\lambda_1 - i\nu_2\lambda_2} + e^{-i\nu_1\lambda_1 + i\nu_2(\lambda_1+\lambda_2)} \tag{2.4''c}$$
$$+ e^{+i\nu_1(\lambda_1+\lambda_2) - i\nu_2\lambda_2}).$$

[1] See Tukey [19] for a concise notation.

3. Estimation.

Given observations, x_t, for $0 \le t \le N$, natural estimates for the bispectral density, $g(\lambda_1,\lambda_2)$, are those based on a function, $g_N(\lambda_1,\lambda_2)$, which is analogous to the periodogram[1] of second-order theory. The usual estimate for $r_3(v_1,v_2)$ is[2]

$$\rho_N(v_1,v_2) = \frac{1}{N} \int_{D_N(v_1,v_2)} x_t x_{t+v_1} x_{t+v_2} dt \qquad (3.1)$$

where the interval D_N restricts x_t, x_{t+v_1}, and x_{t+v_2} to the domain in which they are defined:

$$D_N(v_1,v_2) = \begin{cases} \Phi & |v_1| \text{ or } |v_2| \text{ or } |v_1-v_2| > N \\ [-\min[0,v_1,v_2], N-\max[0,v_1,v_2]] & \text{otherwise.} \end{cases}$$

It is intuitively plausible that if $r_3(v_1,v_2)$ is replaced by $\rho_N(v_1,v_2)$ in (2.4), the resulting function is an estimate of $g(\lambda_1,\lambda_2)$, thus

$$g_N(\lambda_1,\lambda_2) = \frac{1}{(2\pi)^2} \iint_{-N}^{N} e^{-iv_1\lambda_1 - iv_2\lambda_2} \rho_N(v_1,v_2)\, dv_1 dv_2. \qquad (3.2)$$

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[1] See, for example, Rosenblatt [17] for a discussion of periodogram theory.

[2] In Sections 3 through 7 all equations will be written for the continuous parameter case and the "c" will be omitted from the equation numbers. Except for Theorem 5 and definitions, the alterations for the discrete case are obvious.

There are two customary requirements made of such estimates:

(i) the estimate should be asymptotically unbiased, i.e. the expected value of the estimate should tend to the actual value of the quantity being estimated as the sample size, N, tends to infinity, and

(ii) the variance of the estimate should go to zero as $N \longrightarrow \infty$.

Theorem 1 states that under certain conditions on the process, $\{X_t\}$, $\rho_N(\nu_1,\nu_2)$ has both properties. However, $g_N(\lambda_1,\lambda_2)$, like the periodogram, has only property (i) and not property (ii)[1]. Prompted by the results of second-order theory, one corrects this difficulty by considering weighted estimates of the form

$$g_N^*(W) = \iint_{-\infty}^{\infty} W(\mu_1,\mu_2)\, g_N(\mu_1,\mu_2)d\mu_1 d\mu_2 \tag{3.3}$$

where $W(\mu_1,\mu_2)$ is a "bispectral averaging function" to be defined later. By Theorems 2 and 3, below, it is seen that under certain conditions on W and the process, $g_N^*(W)$ is an asymptotically unbiased estimate of

$$g(W) = \iint_{-\infty}^{\infty} W(\mu_1,\mu_2) g(\mu_1,\mu_2)d\mu_1 d\mu_2 \tag{3.4}$$

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[1]See Appendix for a discussion of the variance of the estimate (3.2).

and that $\sigma^2(G_N^*(W)) \to 0$ as $N \to \infty$. Therefore $g_N^*(W)$ suffices as an estimate of (3.4) but it obviously is not an asymptotically unbiased estimate of $g(\lambda_1,\lambda_2)$.

To get an estimate of the bispectral density itself with both properties (i) and (ii), a sequence of weight functions, $\{W_N(\mu_1,\mu_2)\}$, can be used so that

$$g_N^*(\lambda_1,\lambda_2) = \iint_{-\infty}^{\infty} W_N(\mu_1-\lambda_1,\mu_2-\lambda_2)g_N(\mu_1,\mu_2)d\mu_1 d\mu_2. \qquad (3.5)$$

The use of the weight functions tends to give $g_N^*(\lambda_1,\lambda_2)$ property (ii). As N increases, the weight functions narrow the region averaged over (that is they tend to behave more and more like a two-variate δ-function with the "spike" at the origin); this tends to give the estimate property (i). A proper rate of concentration by the weight functions must be chosen so that (under certain conditions on the process and on $\{W_N\}$) both properties can be obtained simultaneously (see Theorems 4 and 5).

Before continuing, it should be noted that when written in terms of the observables, x(t), (3.2) can be put into several forms. One form which appears quite convenient is

$$g_N(\lambda_1,\lambda_2)=\frac{1}{(2\pi)^2 N}\int_0^N e^{it(\lambda_1+\lambda_2)}x(t)dt \int_0^N e^{-iv_1\lambda_1}x(v_1)dv_1$$

$$\cdot\int_0^N e^{-iv_2\lambda_2}x(v_2)dv_2. \qquad (3.6)$$

Also note that $g_N(\lambda_1,\lambda_2)$ is subject to modifications corresponding to those of $g(\lambda_1,\lambda_2)$ illustrated by (2.4') and (2.4'').

4. Hypotheses characteristic of this paper.

Fourth and sixth-order moments arise during the investigation of the covariance properties of the estimates. It is necessary to make some kind of assumption on these moments. A very advantageous assumption involves the use of cumulant (semi-invariant) functions[1]. These functions are defined in a manner completely analogous to the way ordinary cumulants are defined for a single random variable.

Let a random vector $\eta = (\eta_1, \cdots, \eta_k)$ be given. Its characteristic function is

$$\varphi_\eta(\alpha_1, \cdots, \alpha_k) = Ee^{i\alpha_1\eta_1 + \cdots + i\alpha_k\eta_k}. \tag{4.1}$$

Assume $E|\eta_j|^n < \infty$, then the mixed moments

$$m^{(\nu_1, \cdots, \nu_k)} = E\eta_1^{\nu_1} \cdots \eta_k^{\nu_k} \tag{4.2}$$

exist for all $\nu_1, \cdots, \nu_k$ such that $\nu_j \geq 0$ and $\nu_1 + \cdots + \nu_k \leq n$.

Consequently φ_η has the Taylor expansion,

$$\varphi_\eta(\alpha_1, \cdots, \alpha_k) = \sum_{\nu_1 + \cdots + \nu_k \leq n} \frac{i^{\nu_1 + \cdots + \nu_k}}{\nu_1! \cdots \nu_k!} m^{(\nu_1, \cdots, \nu_k)} \alpha_1^{\nu_1} \cdots \alpha_k^{\nu_k}$$

$$+ o((|\alpha_1| + \cdots + |\alpha_k|)^n) \tag{4.3}$$

[1] See Stratonovich [18] Chapter 1 and Leonov and Shiryaev [11] for a discussion of cumulant functions.

where the sum is over all non-negative $v_1,\dots,v_k$ whose sum does not exceed n. Furthermore, log φ_η has a Taylor expansion exactly as in (4.3) except with $m^{(v_1,\dots,v_k)}$ replaced by the coefficient $s^{(v_1,\dots,v_k)}$. The quantities $s^{(v_1,\dots,v_k)}$ are called the cumulants of the vector η. Also $m^{(v_1,\dots,v_k)}$ can be expressed as a polynomial in the $s^{(\gamma_1,\dots,\gamma_k)}$, $0 \le \gamma_j \le v_j$, $j = 1,\dots,k$ (and vice versa)[1]. For a process with zero mean, the two expressions needed in the following are

$$m_4(v_1,\dots,v_4)=s_4(v_1,\dots,v_4)+\{s_2(v_1,v_2)s_2(v_3,v_4)\}_3 \qquad (4.4)$$

and

$$m_6(v_1,\dots,v_6)=s_6(v_1,\dots,v_6)+\{s_3(v_1,v_2,v_3)s_3(v_4,v_5,v_6)\}_{10}$$
$$+\ \{s_2(v_1,v_2)s_4(v_3,\dots,v_6)\}_{15}+\{s_2(v_1,v_2)s_2(v_3,v_4)s_2(v_5,v_6)\}_{15} \qquad (4.5)$$

where (for $\eta = (X_{v_1},\dots,X_{v_6})$) for example

$$s_2(v_1,v_2) = s^{(1,1,0,0,0,0)},$$

$$s_2(v_1,v_3) = s^{(1,0,1,0,0,0)},$$

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1 See Leonov and Shiryaev [11] p. 320

$$s_3(v_1,v_2,v_3) = s^{(1,1,1,0,0,0)},$$

$$s_4(v_1,v_2,v_5,v_6) = s^{(1,1,0,0,1,1)},$$

$$s_6(v_1,\cdots,v_6) = s^{(1,1,1,1,1,1)}, \text{ etc.},$$

and where the notation $\{\cdot\}_j$ denotes the sum of all j <u>different</u> terms obtained by interchanging the arguments of the terms in brackets (the order of the arguments of the s_j being immaterial)[1]. Thus (4.4) is

$$m_4(v_1,\cdots,v_4) = s_4(v_1,\cdots,v_4) + s_2(v_1,v_2)\, s_2(v_3,v_4)$$

$$+ s_2(v_1,v_3)s_2(v_2,v_4) + s_2(v_1,v_4)\, s_2(v_2,v_3).$$

Note that in the case of zero mean,

$$s_1(v) = 0,$$

$$s_2(v_1,v_2) = m_2(v_1,v_2), \tag{4.6}$$

$$s_3(v_1,v_2,v_3) = m_3(v_1,v_2,v_3).$$

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[1]See Tables I, II and III below.

Due to stationarity, we write

$$s_2(t,t+v) = \xi_2(v),$$
$$\vdots \tag{4.7}$$
$$s_6(t,t+v_1,\cdots,t+v_5) = \xi_6(v_1,\cdots,v_5).$$

Then the basic assumption on the process used in the following is that $\xi_4(v_1,v_2,v_3) \in L_1(R_3)$ and $\xi_6(v_1,\cdots,v_5) \in L_1(R_5)$. This is a large class of processes which includes, for example, normal processes (trivially) and linear processes,

$$X_t = \sum_{j=-\infty}^{\infty} a_j \zeta_{t-j} \tag{4.8}$$

where $\{a_j\}$ is an ℓ_2 sequence of constants and $\{\zeta_j\}$, $E\zeta_j=0, E\zeta_j^6 < \infty$, is a sequence of independent identically distributed random variables[1]. More significantly, it includes all k-step dependent processes[2] (linear or nonlinear) which have sixth-order moments. The assumption that $\xi_4 \in L_1(R_3)$ was used by

[1]For the process (4.8),

$$\xi_4(v_1,v_2,v_3) = \hat{s}_4 \sum_{j=-\infty}^{\infty} a_j a_{j+v_1} a_{j+v_2} a_{j+v_3},$$

$$\xi_6(v_1,\cdots,v_5) = \hat{s}_6 \sum_{j=-\infty}^{\infty} a_j a_{j+v_1} \cdots a_{j+v_5}$$

where $\hat{s}_4$ and $\hat{s}_6$ are the fourth and sixth cumulants of the random variable ζ. See Bartlett [1] p. 147.

[2]See Hoeffding and Robbins [9] .

Magness [12] and Parzen [13] in works on spectral theory.

If one wishes the results written in the "frequency domain", a further assumption on ξ_4 and ξ_6 is employed solely for convenience in writing these results. This is that they have Fourier transforms, $h_4(\mu_1,\mu_2,\mu_3)$ and $h_6(\mu_1,\cdots,\mu_5)$, so that

$$\xi_4(v_1,v_2,v_3)=\iiint_{-\infty}^{\infty} e^{iv_1\mu_1+iv_2\mu_2+iv_3\mu_3} h_4(\mu_1,\mu_2,\mu_3)d\mu_1 d\mu_2 d\mu_3 \qquad (4.9)$$

and similarly for ξ_6 and h_6. Moreover, all the functions f,g,h_4 and h_6 are presumed to have the property that (for p and j integers, $0 \le p < j$),

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d\mu_1\cdots d\mu_j\, e^{iv_1\mu_1+\cdots+iv_j\mu_j} e^{iy[\mu_{j-p}+\cdots+\mu_j]} P(\mu_1,\cdots,\mu_j)$$

$$=2\pi \int_{-\infty}^{\infty} d\mu_1\cdots d\mu_{j-1}\, e^{iv_1\mu_1+\cdots+iv_{j-1}\mu_{j-1}+i(-\mu_{j-p}-\cdots-\mu_{j-1})v_j}$$

$$P(\mu_1,\cdots,\mu_{j-1},-\mu_{j-p}-\cdots-\mu_{j-1}) \qquad (4.10)$$

where $P(\mu_1,\cdots,\mu_j)$ is $h_6(\mu_1,\cdots,\mu_5)$ or $g(\mu_1,\mu_2)g(\mu_3,\mu_4)$ or $f(\mu_1)h_4(\mu_2,\mu_3,\mu_4)$ or $f(\mu_1)\, f(\mu_2)\, f(\mu_3)$ and the expression $(-\mu_{j-p}-\cdots-\mu_{j-1})$ on the right hand side of (4.10) is zero when $p = 0$.

5. Third-order moment estimation.

It was stated in Section 3 that, under certain restrictions, $\rho_N(v_1,v_2)$ is an asymptotically unbiased estimate of $r_3(v_1,v_2)$ and that $\sigma^2(\rho_N(v_1,v_2)) \to 0$ as $N \to \infty$. The specific result is

Theorem 1: (a) $\{X_t\}$, $EX_t=0$, is a real, 6th-order weakly stationary process,

(b) $r(v)$, $r_3(v_1,v_2)$, $\xi_4(v_1,v_2,v_3)$, $\xi_6(v_1,\cdots,v_5)$, $g(\lambda_1,\lambda_2) \in L_1$,

(c) $\gamma(N)$ is a real function that is $o(N)$

$$\Longrightarrow \quad (1) \quad \lim_{N\to\infty} \gamma(N)|E\rho_N(v_1,v_2)-r_3(v_1,v_2)| = 0, \tag{5.1}$$

$$(2) \quad \lim_{N\to\infty} N \operatorname{cov}[\rho_N(v_1,v_2),\ \rho_N(v_3,v_4)]$$

$$= \int_{-\infty}^{\infty} dy\ [\ r_6(v_1,v_2,y,y+v_3,y+v_4) - r_3(v_1,v_2)\ r_3(v_3,v_4)]$$

$$\begin{aligned} = \int_{-\infty}^{\infty} dy\ [&\xi_6(v_1,v_2,y,y+v_3,y+v_4) \\ &+ \{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10-1} \\ &+ \{m_2(0,v_1)\ s_4(v_2,y,y+v_3,y+v_4)\}_{15} \\ &+ \{m_2(0,v_1)m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}] \end{aligned} \tag{5.2}$$

where

$$\{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10-1} =$$

$$= \{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10} - r_3(v_1,v_2)r_3(v_3,v_4).$$

[Note: Equation (5.2) can be written in the "frequency domain" (using (4.10))as is done with the result in Theorem 3.]

Proof: Assertion (1) is immediate and (2) is almost as immediate but will be proved in order to introduce some notation.

$$NE[(\rho_N(v_1,v_2)-E\rho_N(v_1,v_2))(\rho_N(v_3,v_4)-E\rho_N(v_3,v_4))] =$$

$$= \frac{1}{N}\int_{D_N(v_1,v_2)} dt \int_{D_N(v_3,v_4)} d\tau[m_6(t,t+v_1,t+v_2,\tau,\tau+v_3,\tau+v_4)$$

$$-r_3(v_1,v_2)r_3(v_3,v_4)]. \qquad (5.3)$$

Let $y = \tau - t$, $t = t$ to get

$$\int_{-N}^{N} dy \frac{C_N(v_1,v_2,v_3,v_4,y)}{N} [r_6(v_1,v_2,y,y+v_3,y+v_4)$$

$$-r_3(v_1,v_2)\, r_3(v_3,v_4)] \qquad (5.4)$$

where C_N is defined as follows (see Figure IV):

1. construct the set $D_N(v_1,v_2) \times D_N(v_3,v_4) = \hat{D}_N$,
2. take the intersection of this set with the line $\tau-t=y$ and call that segment $\hat{C}_N$,
3. then $C_N(v_1,v_2,v_3,v_4,y)$ is equal to the length of the projection of $\hat{C}_N$ onto either axis.

Note that C_N can be written out analytically in terms of its arguments but the expression is cumbersome and not as enlightening as Figure IV. Also, $0 \leq \frac{C_N}{N} \leq 1$ and $\frac{C_N}{N} \longrightarrow 1$ as $N \longrightarrow \infty$

uniformly on any fixed finite set of C_N's arguments. Using (4.5), (5.4) becomes

$$\int_{-N}^{N} dy \frac{C_N}{N} [\xi_6(v_1,v_2,y,y+v_3,y+v_4)$$
$$+\{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10-1}$$
$$+\{m_2(0,v_1)s_4(v_2,y,y+v_3,y+v_4)\}_{15}$$
$$+\{m_2(0,v_1)m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}]. \qquad (5.5)$$

Tables I, II, and III display the individual terms (later called minor terms) which make up (5.5). Looking at each of these

minor term number	$s_3(\cdot,\cdot,\cdot)s_3(\cdot,\cdot,\cdot)$	
1	$r_3(v_1,v_2)$	$r_3(v_3,v_4)$
2	$r_3(v_1,y)$	$r_3(y+v_3-v_2,y+v_4-v_2)$
3	$r_3(v_2,y)$	$r_3(y+v_3-v_1,y+v_4-v_1)$
4	$r_3(v_1,y+v_3)$	$r_3(y-v_2,y+v_4-v_2)$
5	$r_3(v_1,y+v_4)$	$r_3(y-v_2,y+v_3-v_2)$
6	$r_3(v_2,y+v_3)$	$r_3(y-v_1,y+v_4-v_1)$
7	$r_3(v_2,y+v_4)$	$r_3(y-v_1,y+v_3-v_1)$
8	$r_3(y,y+v_3)$	$r_3(v_2-v_1,y+v_4-v_1)$
9	$r_3(y,y+v_4)$	$r_3(v_2-v_1,y+v_3-v_1)$
10	$r_3(y+v_3,y+v_4)$	$r_3(v_2-v_1,y-v_1)$

TABLE I

The ten terms which sum to give the expression

$$\{s_3(0,v_1,v_2)s_3(y,y+v_3,y+v_4)\}_{10}.$$

minor term number	$m_2(\cdot,\cdot)s_4(\cdot,\cdot,\cdot,\cdot)$	
1	$r(v_1)$	$\xi_4(v_2-y,v_3,v_4)$
2	$r(v_2)$	$\xi_4(v_1-y,v_3,v_4)$
3	$r(v_3)$	$\xi_4(v_1,v_2,v_4+y)$
4	$r(v_4)$	$\xi_4(v_1,v_2,v_3+y)$
5	$r(y)$	$\xi_4(v_2-v_1,y+v_3-v_1,y+v_4-v_1)$
6	$r(v_2-v_1)$	$\xi_4(-y,v_3,v_4)$
7	$r(v_4-v_3)$	$\xi_4(v_1,v_2,y)$
8	$r(y-v_1)$	$\xi_4(v_2,y+v_3,y+v_4)$
9	$r(y-v_2)$	$\xi_4(v_1,y+v_3,y+v_4)$
10	$r(y+v_3)$	$\xi_4(v_1-y,v_2-y,v_4)$
11	$r(y+v_4)$	$\xi_4(v_1-y,v_2-y,v_3)$
12	$r(y+v_3-v_1)$	$\xi_4(v_2,y,y+v_4)$
13	$r(y+v_3-v_2)$	$\xi_4(v_1,y,y+v_4)$
14	$r(y+v_4-v_1)$	$\xi_4(v_2,y,y+v_3)$
15	$r(y+v_4-v_2)$	$\xi_4(v_1,y,y+v_3)$

TABLE II

The fifteen terms which sum to give the expression

$$\{s_2(0_1 v_1)s_4(v_2,y,y+v_3,y+v_4)\}_{15}.$$

minor term number	$s_2(\cdot,\cdot)$	$s_2(\cdot,\cdot)$	$s_2(\cdot,\cdot)$
1	$r(v_1)$	$r(v_2-y)$	$r(v_3-v_4)$
2	$r(v_2)$	$r(v_1-y)$	$r(v_3-v_4)$
3	$r(v_1)$	$r(y+v_3-v_2)$	$r(v_4)$
4	$r(v_2)$	$r(y+v_3-v_1)$	$r(v_4)$
5	$r(v_1)$	$r(y+v_4-v_2)$	$r(v_3)$
6	$r(v_2)$	$r(y+v_4-v_1)$	$r(v_3)$
7	$r(y)$	$r(v_2-v_1)$	$r(v_4-v_3)$
8	$r(y)$	$r(y+v_3-v_1)$	$r(y+v_4-v_2)$
9	$r(y)$	$r(y+v_3-v_2)$	$r(y+v_4-v_1)$
10	$r(y+v_3)$	$r(v_2-v_1)$	$r(v_4)$
11	$r(y+v_4)$	$r(v_2-v_1)$	$r(v_3)$
12	$r(y+v_3)$	$r(y-v_1)$	$r(y+v_4-v_2)$
13	$r(y+v_3)$	$r(y-v_2)$	$r(y+v_4-v_1)$
14	$r(y+v_4)$	$r(y-v_1)$	$r(y+v_3-v_2)$
15	$r(y+v_4)$	$r(y-v_2)$	$r(y+v_3-v_1)$

TABLE III

The fifteen terms which sum to give the expression

$$\{s_2(0,v_1)\ s_2(v_2,y)\ s_2(y+v_3,y+v_4)\}_{15}.$$

40 terms separately, it is obvious that the expression in square brackets is absolutely integrable with respect to y on $(-\infty, \infty)$. Thus writing $\int_{-N}^{N} dy$ as $\int_{-\infty}^{\infty} dy\, \chi_N(y)$ where χ_N is the characteristic function of the interval $[-N, N]$, (5.5) can be looked upon as the infinite integral of a sequence of functions, $\chi_N(y) \frac{C_N}{N} [\ldots]$, which (by the properties of C_N) tend to zero pointwise and are dominated by the fixed, $L_1(-\infty, \infty)$, function given by the absolute value of the expression in square brackets. Consequently, Lebesgue's dominated convergence theorem gives the result.

Q. E. D.

6. Estimation of the weighted bispectral density.

Next consider estimates of the form (3.3),

$$g_N^*(W) = \iint_{-\infty}^{\infty} W(\mu_1,\mu_2)\, g_N(\mu_1,\mu_2)\, d\mu_1\, d\mu_2. \tag{3.3}$$

The following conditions will be placed on W.

Definition 1: A bispectral averaging function, $W(\mu_1,\mu_2)$, of order $\alpha > 0$ is

(i) real-valued,

(ii) $\in L_1 \cap L_2$, and

(iii) its Fourier transform (which is bounded and L_2),

$$w(\nu_1,\nu_2) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} e^{-i\nu_1\mu_1 - i\nu_2\mu_2} W(\mu_1,\mu_2)\, d\mu_1 d\mu_2, \tag{6.1}$$

is such that $|w(\nu_1,\nu_2)| = O(|\nu_1|+|\nu_2|)^{-\alpha}$.

Condition (iii) could be weakened. Note that (iii) implies there are constants K_1 and $b > 0$ such that for all $|\nu_1|$ and/or $|\nu_2| \geq b$,

$$|w(\nu_1,\nu_2)| \leq K_1(|\nu_1|+|\nu_2|)^{-\alpha}. \tag{6.2}$$

Theorem 2: (a) $\{X_t\}$ is 3rd-order weakly stationary, real,

(b) $r_2(\nu_1,\nu_2)$, $g(\lambda_1,\lambda_2) \in L_1(R_2)$,

(c) $W(\mu_1,\mu_2)$ is a bispectral averaging function of order $s+\varepsilon$, $\varepsilon > 0$, $0 < s < 1$,

$$\Rightarrow \lim_{N\to\infty} N^s |Fg_N^*(W) - g(W)| = 0.$$

<u>Proof</u>: By (3.2), (3.3), and (6.1)

$$E\, g_N^*(W) = E \iint_{-\infty}^{\infty} W(\mu_1,\mu_2) g_N(\mu_1,\mu_2)\, d\mu_1 d\mu_2$$

$$= E \iint_{-\infty}^{\infty} W(\mu_1,\mu_2)\Big[\frac{1}{(2\pi)^2} \iint_{-N}^{N} e^{-i\mu_1 v_1 - i\mu_2 v_2} \rho_N(v_1,v_2)\, dv_1 dv_2\Big] d\mu_1 d\mu_2$$

$$= E \iint_{-N}^{N} w(v_1,v_2)\rho_N(v_1,v_2)\, dv_1 dv_2.$$

Therefore

$$N^s |\, E g_N^*(W) - g(W)| = N^s |\, E \iint_{-N}^{N} w(v_1,v_2)[\rho_N(v_1,v_2) - r_3(v_1,v_2)]\, dv_1 dv_2$$

$$- \iint_{|v_1| \text{ or } |v_2| > N} w(v_1,v_2) r_3(v_1,v_2)\, dv_1 dv_2 \,|. \qquad (6.3)$$

The second integral on the left hand side of (6.3) tends to zero in absolute value by the bounded property of $w(v_1,v_2)$ and the L_1 property of $r_3(v_1,v_2)$. The first integral is

$$\leq N^s \Big| \iint_{-N}^{N} w(v_1,v_2)\Big[1 - \frac{c_N(v_1,v_2)}{N} - 1\Big] r_3(v_1,v_2)\, dv_1 dv_2 \Big| \qquad (6.4)$$

where

$$c_N(v_1,v_2) = \min[\, \hat{c}_N(v_1,v_2),\, N\,],$$

$$\hat{c}_N(v_1,v_2) = \max[\,|v_1|, |v_2|, |v_1 - v_2|\,].$$

Since $c_N \leq |v_1| + |v_2|$ and by (6.2), the integrand of (6.4), $\frac{c_N(v_1,v_2)}{N^{1-s}} w(v_1,v_2) r_3(v_1,v_2)$, is bounded in absolute value on

$(-\infty,\infty)$ by $K_2|r_3(v_1,v_2)|$ for some K_2. This integrand converges pointwise to zero so that again by Lebesgue's dominated convergence theorem the result is proved.

Q. E. D.

Property (i) of Section 3, the asymptotic unbiasedness, is therefore shown. Now turn to property (ii) of Section 3.

<u>Theorem 3</u>: (a) $\{X_t\}$ is 6th-order weakly stationary, real,

(b) $r(v)$, $r_3(v_1,v_2)$, $\xi_4(v_1,v_2,v_3)$, $\xi_6(v_1,\dots,v_5)$, $g(v_1,v_2)\in L_1$,

(c) $W_j(\mu_1,\mu_2)$, $j=1,2$, are bispectral averaging functions of order $1+\varepsilon_j$, $\varepsilon_j>0$,

$\Rightarrow$ (1) $\lim_{N\to\infty} N \operatorname{cov}[g_N^*(W_1), g_N^*(W_2)] =$

$$= \int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} dv_1\cdots dv_4 dy \ \ w_1(v_1,v_2)\overline{w_2(v_3,v_4)} \cdot$$

$$\cdot [\xi_6(v_1,v_2,y,y+v_3,y+v_4)+\{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10-1}$$

$$+\{m_2(0,v_1)s_4(v_2,y,y+v_3,y+v_4)\}_{15}$$

$$+\{m_2(0,v_1)m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}], \tag{6.5}$$

(2) Equation (6.5) is written in the "time domain"; if relations like (4.10) hold, the result in the "frequency domain" is

$$\frac{1}{2\pi} \lim_{N \to \infty} N \operatorname{cov}[g_N^*(W_1), g_N^*(W_2)]$$

$$= \int_{-\infty}^{\infty} W_1(\mu_1,\mu_2)W_2(\mu_3,\mu_4)h_6(\mu_1,\mu_2,\mu_3+\mu_4,-\mu_3,-\mu_4)d\mu_1 \ldots d\mu_4$$

$$+\int_{-\infty}^{\infty} [\tfrac{1}{2}\hat{W}_1(\mu_1,\mu_2)\hat{W}_2(-\mu_3,\mu_2+\mu_3) + \hat{W}_1(\mu_1,\mu_2)\hat{W}_2(-\mu_2,-\mu_3)$$

$$+ \tfrac{1}{2}\hat{W}_1(\mu_1,-\mu_1-\mu_2)\hat{W}_2(-\mu_2,-\mu_3) + \tfrac{1}{4}\hat{W}_1(\mu_1,-\mu_2-\mu_1)\hat{W}_2(-\mu_3,\mu_2+\mu_3)]$$

$$g(\mu_1,\mu_2)g(\mu_2,\mu_3)d\mu_1 d\mu_2 d\mu_3$$

$$+\int_{-\infty}^{\infty} [\tfrac{1}{2}\hat{W}_1(\mu_1,0)\hat{W}_2(-\mu_2,-\mu_3) + \tfrac{1}{2}\hat{W}_1(\mu_2,\mu_3)\hat{W}_2(\mu_1,0)$$

$$+ \tfrac{1}{4}\hat{W}_1(\mu_1,-\mu_1)\hat{W}_2(-\mu_2,-\mu_3) + \tfrac{1}{4}\hat{W}_1(\mu_2,\mu_3)\hat{W}_2(\mu_1,-\mu_1)]$$

$$f(\mu_1)h_4(0,\mu_2,\mu_3)d\mu_1 d\mu_2 d\mu_3$$

$$+\int_{-\infty}^{\infty} [\tfrac{1}{2}\hat{W}_1(\mu_2,\mu_1-\mu_2)\hat{W}_2(-\mu_1,-\mu_3) + \hat{W}_1(\mu_1,\mu_3)\hat{W}_2(\mu_1,-\mu_2)$$

$$+ \tfrac{1}{2}\hat{W}_1(\mu_1,\mu_3)\hat{W}_2(-\mu_2,-\mu_1+\mu_2) + \tfrac{1}{4}\hat{W}_1(\mu_2,\mu_1-\mu_2)\hat{W}_2(\mu_3+\mu_1,-\mu_3)]$$

$$f(\mu_1)h_4(\mu_2,\mu_3,\mu_1-\mu_2)d\mu_1 d\mu_2 d\mu_3$$

$$+\int_{-\infty}^{\infty} [\hat{W}_1(0,\mu_1)\hat{W}_2(0,\mu_2) + \tfrac{1}{2}\hat{W}_1(0,\mu_1)\hat{W}_2(\mu_2,-\mu_2)$$

$$+ \tfrac{1}{2}\hat{W}_1(-\mu_2,\mu_2)\hat{W}_2(\mu_1,0) + \tfrac{1}{4}\hat{W}_1(\mu_1,-\mu_1)\hat{W}_2(\mu_2,-\mu_2)]$$

$$f(\mu_1)f(\mu_2)f(0)d\mu_1 d\mu_2$$

$$+\int_{-\infty}^{\infty} [\hat{W}_1(\mu_2,-\mu_1-\mu_2)\hat{W}_2(\mu_1,-\mu_1-\mu_2) + \tfrac{1}{2}\hat{W}_1(\mu_1,\mu_2)\hat{W}_2(\mu_1,\mu_2)]$$

$$f(\mu_1)f(\mu_2)f(\mu_1+\mu_2)d\mu_1 d\mu_2 \qquad (6.6)$$

where $\hat{W}(\mu_1,\mu_2) = W(\mu_1,\mu_2) + W(\mu_2,\mu_1)$.

Proof: Recalling the proof of Theorem 1,

$$N\, E[(g_N^*(W_1) - Eg_N^*(W_1))\overline{(g_N^*(W_2) - Eg_N^*(W_2))}] =$$

$$= NE[\iint_{-N}^{N} w_1(v_1,v_2)(\rho_N(v_1,v_2) - E\rho_N(v_1,v_2))dv_1dv_2$$

$$\cdot \iint_{-N}^{N} \overline{w_2(v_3,v_4)(\rho_N(v_3,v_4) - E\rho_N(v_3,v_4))dv_3dv_4}]$$

$$= \iiiint\int_{-N}^{N} w_1(v_1,v_2)\, \overline{w_2(v_3,v_4)}\, \frac{C_N(v_1,\cdots,v_4,y)}{N}$$

$$[\xi_6(v_1,v_2,y,y+v_3,y+v_4) + \{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10-1}$$

$$+\{m_2(0,v_1)s_4(v_2,y,y+v_3,y+v_4)\}_{15} +$$

$$+ \{m_2(0,v_1)m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}]dv_1\cdots dv_4dy. \qquad (6.7)$$

There remains to show that the integrand with the $\frac{C_N}{N}$ term replaced by 1 is absolutely integrable on $(-\infty,\infty)$. To do this, consider the right hand side of (6.7) as made up of four "major terms" or 40 "minor terms", where the first major term and the first minor term are identical and the remaining minor terms are enumerated in accordance with Tables I, II, and III.

Each minor term is treated separately. Minor term 1 is obviously summable since w_1 and w_2 are bounded and $\xi_6 \in L_1(R_5)$. The remaining minor terms are grouped according to the major terms within which they are contained.

Lemma 1: (a) hypotheses of Theorem 3

$\Longrightarrow$ (1) $\iiiint\int_{-\infty}^{\infty} |\{m_3(0, v_1, v_2)m_3(y,y+v_3,y+v_4)\}_{10-1}$

$$w_1(v_1,v_2)\,\overline{w_2(v_3,v_4)}\;|\,dv_1\cdots dv_4dy < \infty, \quad (6.8)$$

(2) $\iiiint\int_{-\infty}^{\infty} |\,\{m_2(0,v_1)s_4(v_2,y,y+v_3,y+v_4)\}_{15}$

$$w_1(v_1, v_2)\overline{w_2(v_3,v_4)}\;|\,dv_1\cdots dv_4dy < \infty, \quad (6.9)$$

(3) $\iiiint\int_{-\infty}^{\infty} |\,\{m_2(0,v_1)\,m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}$

$$w_1(v_1,v_2)\overline{w_2(v_3,v_4)}\,|\;dv_1\cdots dv_4dy < \infty. \quad (6.10)$$

Proof: Let $\hat{w}$ be a bound on both $|w_1(v_1,v_2)|$ and $|w_2(v_3,v_4)|$, $\hat{r}$ be such that $\iint_{-\infty}^{\infty}|r_3(v_1, v_2)|\,dv_1dv_2 \leq \hat{r}$, and $\hat{\xi}_4$ be such that $\iiint_{-\infty}^{\infty}|\xi_4(v_1,v_2,v_3)|\,dv_1dv_2dv_3 \leq \hat{\xi}_4$. The first minor term involved in result (1) is then (see Table I)

$$\iiiint\int_{-\infty}^{\infty} r_3(v_1,y)r_3(y+v_3-v_2,y+v_4-v_2)$$

$$w_1(v_1,v_2)w_2(v_3,v_4)\,dv_1\cdots dv_4dy. \quad (6.11)$$

Let $\hat{v}_3 = y + v_3 - v_2$, $\hat{v}_4 = y + v_4 - v_2$, then the absolute value of (6.11) is (using (6.2))

$$\leq \hat{w} \iint dv_1 dy |r_3(v_1,y)| \int dv_2 |w_1(v_1,v_2)|$$

$$\iint d\hat{v}_3 d\hat{v}_4 |r_3(\hat{v}_3,\hat{v}_4)|$$

$$\leq \hat{w}\,\hat{r} \iint dv_1 dy |r_3(v_1,y)| \left[\int_{-b}^{b} dv_2 \hat{w} + \int_{|v_2|>b} dv_2 \frac{K_1}{|v_2|^{1+\varepsilon_1}}\right]$$

$$\leq \hat{w}\,\hat{r}^2 \left[2b\hat{w} + 2K_1 \frac{b^{-\varepsilon_1}}{\varepsilon_1}\right] < \infty.$$

Simple modifications of the above argument show that the eight remaining minor terms in major term 2 are also bounded. Therefore turn to result (2) and major term 3. Here the proof is just as easy. For example, the least obvious situation is represented by minor term 12:

$$\left| \iiiint\!\!\int_{-\infty}^{\infty} r(y+v_3-v_1)\, \xi_4(v_2,y,y+v_4)\right.$$

$$\left. w_1(v_1,v_2)\, \overline{w_2(v_3,v_4)}\, dv_1 \cdots dv_4 dy \right|$$

$$\leq \hat{w} \int dy \iint dv_2 dv_4 |\xi_4(v_2,y,v_4)| \int dv_1 |w_1(v_1,v_2)| \int dv_3 |r(v_3)|$$

$$\leq \hat{w}\,\hat{\xi}_4\,\hat{r} \left[2b\hat{w} + 2K_1 \frac{b^{-\varepsilon_1}}{\varepsilon_1}\right] < \infty.$$

Finally look at result (3) concerning major term 4. It is easily seen that by proper changes of variables, all 15 minor terms can be gotten into the form

$$\left| \int_{-\infty}^{\infty} dy\, r(y) \int_{-\infty}^{\infty} dv_1 r(v_1) \int_{-\infty}^{\infty} d v_3 r(v_3) \iint_{-\infty}^{\infty} d v_2 dv_4 \right.$$

$$\left. w_1(k_1(v_1,v_2,v_3,v_4,y),v_2)\, \overline{w_2(k_2(v_1,v_2,v_3,v_4,y),v_4)} \right| \quad (6.12)$$

where k_1 and k_2 are some linear functions of five variables. Using the boundedness properties and (6.2), (6.12) is

$$\leq \hat{r}^3 \left[2b_1\hat{w}+2K_1 \frac{b_1^{-\varepsilon_1}}{\varepsilon_1}\right] \left[2b_2\hat{w}+2K_2 \frac{b_2^{-\varepsilon_2}}{\varepsilon_2}\right] < \infty.$$

q. e. d.

Returning to the proof of Theorem 3, the previous statements and Lemma 1 show that there is an L_1 upper bound for all N on the integrand. Therefore the properties of $\frac{C_N}{N}$ and Lebesgue's dominated convergence theorem give (6.5).

To get (6.6) simply take each of the 40 terms of (6.5), write the cumulant functions (or moments) in terms of their transforms and apply (4.10).

Q. E. D.

7. Estimation of the bispectral density.

The most interesting properties are exhibited by the third form of estimate introduced in Section 3,

$$g_N^*(\lambda_1,\lambda_2)= \iint_{-\infty}^{\infty} W_N(\mu_1-\lambda_1,\mu_2-\lambda_2)\ g_N(\mu_1,\mu_2)d\mu_1 d\mu_2. \tag{3.5}$$

Properties (i) and (ii) of Section 3 will be discussed not for the general form (3.5) but for a subclass of estimates described by first defining a bispectral estimating kernel.

Definition 2: A bispectral estimating kernel, $w(v_1,v_2)$, of order $\alpha > 0$ is

(i) real-valued, $\in L_2$,

(ii) $|w(v_1,v_2)| \leq M_1 < \infty$,

(iii) $w(v_1,v_2) = w(v_2,v_1) = w(-v_1,v_2-v_1)$ (same symmetries as those of $r_3(v_1,v_2)$),

(iv) for every $\varepsilon > 0$ there exists an M_2 such that uniformly in v_2,

$$\int_{|v_1|>M_2} |w(v_1,v_2)|dv_1 \leq \varepsilon,$$

(v) and

$$w^{(\alpha)} = \sup_{\substack{\text{all possible}\\ \text{paths}}} \lim_{v_1,v_2 \to 0} \frac{|1-w(v_1,v_2)|}{(|v_1|+|v_2|)^{\alpha}}$$

is finite.

Letting $\{B_N\}$ be a sequence of positive constants tending to zero as $N \to \infty$, the estimates then discussed are of the form

$$g_N^*(\lambda_1,\lambda_2)=\frac{1}{(2\pi)^2}\iint_{-N}^{N} e^{-i\lambda_1 v_1 - i\lambda_2 v_2} w(B_N v_1, B_N v_2)\rho_N(v_1,v_2)dv_1 dv_2. \tag{7.1}$$

By condition (i) above, $w(v_1,v_2)$ is a Fourier transform in the L_2 sense,

$$w(v_1,v_2) = \iint_{-\infty}^{\infty} e^{-iv_1\lambda_1 - iv_2\lambda_2} W(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2. \tag{7.2}$$

Using this, (7.1) can be rewritten as

$$g_N^*(\lambda_1,\lambda_2)=\iint_{-\infty}^{\infty} \frac{1}{B_N^2} W\left(\frac{\mu_1-\lambda_1}{B_N}, \frac{\mu_2-\lambda_2}{B_N}\right) g_N(\mu_1,\mu_2)d\mu_1 d\mu_2 \tag{7.3}$$

which is of the form (3.5). Parzen [13] , uses an analogous estimate for the spectral density. The rate at which $B_N \to 0$ governs the rate of concentration of the weight functions.

For convenience of notation, define the generalized q^{th} bispectral derivative, $g^{(q)}(\mu_1,\mu_2)$, by

$$g^{(q)}(\mu_1,\mu_2)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} e^{-i\mu_1 v_1 - i\mu_2 v_2}(|v_1|+|v_2|)^q r_3(v_1,v_2)dv_1 dv_2. \tag{7.4}$$

Then the bias of (7.3) satisfies the following:

<u>Theorem 4</u>: (a) $\{X_t\}$ is 3rd-order weakly stationary, real,

(b) $r_3(v_1,v_2)$, $g(\lambda_1,\lambda_2) \in L_1(R_2)$,

(c) $\iint_{-\infty}^{\infty} (|v_1| + |v_2|)^q \, r_3(v_1,v_2) \, dv_1 dv_2 < \infty, \quad q>0,$

(d) $w(v_1,v_2)$ is a bispectral estimating kernel of order $\alpha \geq q$,

(e) B_N chosen such that as $N \to \infty$

(i) $B_N \to 0$,

(ii) $B_N N \to \infty$ if $q \leq 1$

$B_N^q N \to \infty$ if $q > 1$

$$\Rightarrow \lim_{N \to \infty} B_N^{-q} |Eg_N^*(\mu_1,\mu_2) - g(\mu_1,\mu_2)|$$

$$\begin{aligned} &= 0 && \text{if } \alpha > q \\ &= |w^{(q)} g^{(q)}(\mu_1,\mu_2)| && \text{if } \alpha = q \end{aligned} \tag{7.5}$$

where in the case $\alpha = q$, $w(v_1,v_2)$ is assumed to have the property that for almost all (v_1,v_2),

$$\lim_{N \to \infty} \frac{1-w(B_N v_1, B_N v_2)}{(|B_N v_1|+|B_N v_2|)^q} = w^{(q)}. \tag{7.6}$$

Proof: Break up the bias expression into three terms,

$$B_N^{-q}(Eg_N^*(\mu_1,\mu_2) - g(\mu_1,\mu_2)) =$$

$$\begin{aligned} &= \frac{B_N^{-q}}{(2\pi)^2} \Big[\iint_{-N}^{N} e^{-i\mu_1 v_1 - i\mu_2 v_2} (w(B_N v_1, B_N v_2)-1) r_3(v_1,v_2) dv_1 dv_2 \\ &- \iint_{-N}^{N} e^{-i\mu_1 v_1 - i\mu_2 v_2} \frac{\min[\max(|v_1|,|v_2|,|v_1-v_2|),N]}{N} \\ &\qquad \cdot w(B_N v_1, B_N v_2) r_3(v_1,v_2) dv_1 dv_2 \\ &- \iint_{|v_1| \text{or} |v_2| > N} e^{-i\mu_1 v_1 - i\mu_2 v_2} r_3(v_1,v_2) dv_1 dv_2 \Big]. \end{aligned} \tag{7.7}$$

The third term is

$$\le \frac{N^q}{(NB_N)^q} \iint_{|v_1| \text{ or } |v_2| > N} |r_3(v_1,v_2)|\, dv_1 dv_2$$

$$\le \frac{1}{(NB_N)^q} \iint_{|v_1| \text{ or } |v_2| > N} (|v_1|+|v_2|)^q |r_3(v_1,v_2)|\, dv_1 dv_2$$

$$\to 0$$

by (c) and (e). The second term of (7.7) is

$$\le \frac{M_1}{B_N^q N} \iint_{-N}^{N} (|v_1|+|v_2|)|r_3(v_1,v_2)|\, dv_1 dv_2 . \tag{7.8}$$

First suppose $q \le 1$, then (7.8) is

$$\le \frac{M_1}{(NB_N)^q} \iint_{-N}^{N} \frac{(|v_1|+|v_2|)^{1-q}}{N^{1-q}} (|v_1|+|v_2|)^q |r_3(v_1,v_2)|\, dv_1 dv_2$$

$$\to 0$$

by (c) and (e). Next suppose $q > 1$, then (7.8) is

$$\le \frac{M_1}{NB_N{}^q} \iint_{-N}^{N} (|v_1|+|v_2|)|r_3(v_1,v_2)|\, dv_1 dv_2$$

$$\to 0$$

again by (c) and (e). Therefore

$$\lim_{N \to \infty} B_N^{-q} |Fg_N^*(\mu_1,\mu_2) - g(\mu_1,\mu_2)| =$$

$$= \frac{1}{(2\pi)^2} \lim_{N\to\infty} |B_N^{-q} \iint_{-N}^{N} e^{-i\mu_1 v_1 - i\mu_2 v_2} [w(B_N v_1, B_N v_2) - 1]$$

$$r_3(v_1, v_2) dv_1 dv_2 |$$

$$= \frac{1}{(2\pi)^2} \lim_{N\to\infty} \left| \iint_{-N}^{N} e^{-i\mu_1 v_1 - i\mu_2 v_2} (|v_1| + |v_2|)^q \, r_3(v_1, v_2) \right.$$

$$\left. \frac{w(B_N v_1, B_N v_2) - 1}{(|B_N v_1| + |B_N v_2|)^\alpha} (|B_N v_1| + |B_N v_2|)^{\alpha - q} \, dv_1 dv_2 \right| . \quad (7.9)$$

Since $w(v_1, v_2)$ is a bispectral estimating kernel of order α, there exists a constant K_3 such that the integrand of (7.9) is bounded by

$$K_3(|v_1| + |v_2|)^q |r_3(v_1, v_2)|$$

which is integrable. The result follows immediately from Lebesgue's theorem.

Q. E. D.

Secondly, the variance of (7.1) also behaves itself under the assumptions listed in the next theorem. In the proof of this theorem the following lemmas are needed. Lemma 2 is motivated by the Riemann-Lebesgue lemma.

Lemma 2: (a) hypotheses of Theorem 5,

$$\Rightarrow \quad (1) \quad \iint_{-M}^{M} dv_1 dv_2 \, e^{-iv_1\mu_1 - iv_2(\mu_2/B_N)} \, w(B_N v_1, v_2) r(v_1)$$

$$\to 0 \quad \text{as } N \to \infty \quad \text{unless } \mu_2 = 0,$$

$$(2)\quad \int\int\int\int\int_{-M}^{M} dv_1 \ldots dv_4 dy\, e^{-iv_1(\mu_1/B_N)-iv_2\mu_2}\, e^{iv_3\mu_3+iv_4(\mu_4/B_N)}$$

$$w(v_1, B_N v_2+v_4) w(B_N v_3+v_1, v_4) r(y) r(y+v_3) r(y-v_2)$$

$$\to 0 \quad \text{unless} \quad \mu_1 = \mu_4 = 0.$$

<u>Proof</u>: First look at result (1). For $\mu_2 \neq 0$

$$\int\int_{-M}^{M} dv_1 dv_2\, e^{-iv_1\mu_1-iv_2(\mu_2/B_N)}\, w(B_N v_1, v_2) r(v_1)$$

$$= -\int\int_{-M}^{M} dv_1 dv_2\, e^{-iv_1\mu_1-i(\mu_2/B_N)(v_2+(\pi B_N/\mu_2))}\, w(B_N v_1, v_2) r(v_1).$$

Twice the integral in question is

$$\int_{-M}^{M} dv_1 e^{-iv_1\mu_1}\, r(v_1) \left[\int_{-M}^{M} dv_2 e^{-i(\mu_2/B_N)v_2}[w(B_N v_1, v_2)\right.$$

$$\left. - w(B_N v_1, v_2 - \frac{\pi B_N}{\mu_2})] + O(B_N)\right].$$

And this in absolute value is

$$\leq \int\int_{-M}^{M} dv_1 dv_2 \left| w(B_N v_1, v_2) - w(B_N v_1, v_2 - \frac{\pi B_N}{\mu_2}) \right| + O(B_N).$$

By (e) of Theorem 5, the integrand converges to zero a.e. Thus Lebesgue's dominated convergence theorem proves the result.

Result (2) is proved similarly. Twice the absolute value of the integral in question is for $\mu_1 \neq 0$

$$\leq \int\!\!\int\!\!\int\!\!\int\!\!\int_{-M}^{M} |w(v_1, B_N v_2 + v_4) w(B_N v_3 + v_1, v_4) -$$

$$w(v_1 - \frac{\pi B_N}{\mu_1}, B_N v_2 + v_4) w(B_N v_3 + v_1 - \frac{\pi B_N}{\mu_1}, v_4)|$$

$$|r(y) r(y+v_3) r(y-v_2)| dv_1 \ldots dv_4 dy + O(B_N).$$

This again tends to zero by the properties of w. A similar argument shows the result for $\mu_4 \neq 0$.

q.e.d

Lemma 3: (a) $\{\gamma(M,N)\}$ is a sequence of constants,

(b) $\gamma(M) = \lim_{N \to \infty} \gamma(M,N)$ exists for all M,

(c) $\gamma = \lim_{M \to \infty} \gamma(M)$ exists,

(d) for any $\varepsilon > 0$ there exists an $M_0(\varepsilon)$ such that for all $N > M > M_0$, $|\gamma(M,N) - \gamma(N,N)| < \varepsilon$,

$$\Rightarrow \lim_{N \to \infty} \gamma(N,N) = \gamma.$$

Proof: Take any $\varepsilon > 0$. First fix an M such that by (c) and (d)

$$|\gamma(M) - \gamma| < \varepsilon$$

$$|\gamma(M,N) - \gamma(N,N)| < \varepsilon.$$

Next choose $N_0 > M$ such that by (b)

$$|\gamma(M,N) - \gamma(M)| < \varepsilon \quad \text{for all } N > N_0.$$

Then for all $N > N_0(\varepsilon)$

$$|\gamma(N,N) - \gamma| \leq |\gamma(N,N) - \gamma(M,N)|$$

$$+ |\gamma(M,N) - \gamma(M)| + |\gamma(M) - \gamma| \leq 3\varepsilon$$

for arbitrary ε.

q.e.d.

<u>Theorem 5</u>: (a) $\{X_t\}$, 6th-order weakly stationary, real,

(b) $r(v)$, $r_3(v_1,v_2)$, $\xi_4(v_1,v_2,v_3)$, $\xi_6(v_1,\ldots,v_5)$, $g(\lambda_1,\lambda_2) \in L_1$,

(c) $w(v_1,v_2)$ is a bispectral estimating kernel,

(d) B_N is a sequence of positive constants such that

(i) $B_N \to 0$ as $N \to \infty$

(ii) $B_N^2 N \to \infty$ as $N \to \infty$,

(e) $w(v_1,v_2)$ is continuous a. e. and for $a < \infty$ $w(B_N a, v_2) \to w(0,v_2)$ for almost all v_2,

(f) for brevity in writing results, take (μ_1,μ_2) and (μ_3,μ_4) in their first sections of definition as shown in Figure II, i. e. $0 \leq \mu_1,\mu_3 < \infty$, $0 \leq \mu_2 \leq \mu_1$, $0 \leq \mu_4 \leq \mu_3$,

$$\Longrightarrow \lim_{N\to\infty} NB_N^2 \operatorname{cov}\left[g_N^*(\mu_1,\mu_2),\ g_N^*(\mu_3,\mu_4)\right]$$

$$= \frac{1}{2\pi}[f(\mu_1)f(\mu_2)f(\mu_1+\mu_2)f(\mu_3)f(\mu_4)f(\mu_3+\mu_4)]^{1/2}\cdot$$

$$\{w_1\ \delta(\mu_2)\delta(\mu_4)[1+2\delta(\mu_1)][1+2\delta(\mu_3)] + \tag{7.10}$$

$$w_2\delta(\mu_1-\mu_3)\delta(\mu_2-\mu_4)[1+\delta(\mu_1-\mu_4)+4\delta(\mu_1)\delta(\mu_2)]\},$$

where

$$w_1 = [\int_{-\infty}^{\infty} w(0,v)dv]^2,$$

$$w_2 = \iint_{-\infty}^{\infty} w^2(v_1,v_2)dv_1dv_2,$$

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Proof: Recalling Theorems 1 and 3, write

$$NB_N^2 \operatorname{cov}[\, g_N^*(\mu_1,\mu_2),\ g_N^*(\mu_3,\mu_4)\,]$$

$$= \frac{B_N^2}{(2\pi)^4} \int\!\!\int\!\!\int\!\!\int_{-N}^{N} dv_1 \ldots dv_4\ e^{-i\mu_1 v_1 - i\mu_2 v_2 + i\mu_3 v_3 + i\mu_4 v_4}$$

$$w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4) \int_{-N}^{N} dy\ \frac{C_N(v_1,\ldots,v_4,y)}{N} \cdot$$

$$[\xi_6(v_1,v_2,y,y+v_3,y+v_4) + \{m_3(0,v_1,v_2) m_3(v,y+v_3,y+v_4)\}_{10-1}$$

$$+\{m_2(0,v_1) s_4(v_2,y,y+v_3,y+v_4)\}_{15}$$

$$+\{m_2(0,v_1) m_2(v_2,y) m_2(y+v_3,y+v_4)\}_{15}]. \qquad (7.11)$$

The boundedness of w and $\frac{C_N}{N}$ and the $L_1(R_5)$ property of ξ_6 imply the first major term is $O(B_N^2)$. Further,

Lemma 4: (a) hypotheses of Theorem 5

$$\Longrightarrow \quad (1) \quad B_N^2 \left| \int\!\!\int\!\!\int\!\!\int\!\!\int_{-N}^{N} dv_1 \ldots dv_4 dy\ w(B_N v_1, B_N v_2)\ w(B_N v_3, B_N v_4) \right.$$

$$\left. \{m_3(0,v_1,v_2) m_3(y,y+v_3,y+v_4)\}_{10-1} \right| = O(B_N), \qquad (7.12)$$

$$(2) \quad B_N^2 \left| \int\!\!\int\!\!\int\!\!\int\!\!\int_{-N}^{N} dv_1 \ldots dv_4 dy\ w(B_N v_1, B_N v_2)\ w(B_N v_3, B_N v_4) \right.$$

$$\left. \{m_2(0,v_1)\ s_4(v_2,y,y+v_3,y+v_4)\}_{15} \right| = O(B_N). \qquad (7.13)$$

Proof: First consider result (1). The first minor term is

$$\leq B_N^2 \int\!\!\int\!\!\int\!\!\int\!\!\int_{-N}^{N} dv_1 \ldots dv_4 dy\ |w(B_N v_1, B_N v_2)\ w(B_N v_3, B_N v_4)|$$

$$|r_3(v_1,y)\ r_3(y+v_3-v_2, y+v_4-v_2)|. \qquad (7.14)$$

Let $\hat{v}_3 = v_3+y-v_2$, $\hat{v}_4 = v_4+y-v_2$, then (7.14) is

$$\leq B_N^2 M_1 \int_{-N}^{N} dv_1 \int_{-N}^{N} dv_2 |w(B_N v_1, B_N v_2)| \int_{-N}^{N} dy |r_3(v_1, y)|$$

$$\int_{-N+y-v_2}^{N+y-v_2} d\hat{v}_3 \int_{-N+y-v_2}^{N+y-v_2} d\hat{v}_4 |r_3(\hat{v}_3, \hat{v}_4)|$$

$$\leq B_N M_1 \hat{r}_3^2 \int_{-NB_N}^{NB_N} |w(B_N v_1, v_2)| dv_2 = O(B_N)$$

by the properties of a bispectral estimating kernel. The remaining 8 minor terms are treated by quite similar arguments.

Result (2) follows by like reasoning. For example, minor term 15 is

$$\leq B_N^2 \iiiint\!\!\int_{-N}^{N} dv_1 \ldots dv_4 dy |w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4)|$$

$$| r(y+v_4-v_2) \xi_4(v_1, y, y+v_3) |. \tag{7.15}$$

Let $\hat{v}_4 = v_4+y-v_2$, $\hat{v}_3 = v_3+y$; then (7.15) is

$$\leq B_N^2 M_1 \int_{-N}^{N} dy \int_{-N}^{N} dv_1 \int_{-N}^{N} dv_2 |w(B_N v_1, B_N v_2)| \int_{-N+y}^{N+y} d\hat{v}_3 \int_{-N+y-v_2}^{N+y-v_2} d\hat{v}_4$$

$$|r(\hat{v}_4) \xi_4(v_1, y, \hat{v}_3)|$$

$$\leq B_N M_1 \hat{r} \hat{\xi} \int_{-NB_N}^{NB_N} |w(B_N v_1, v_2)| dv_2 = O(B_N).$$

The proofs for the 14 other minor terms are just as simple.

q. e. d.

By Lemma 4,

$$\lim_{N\to\infty} NB_N^2 \operatorname{cov}[g_N^*(\mu_1,\mu_2), g_N^*(\mu_3,\mu_4)]$$

$$= \lim_{N\to\infty} \frac{B_N^2}{(2\pi)^4} \iiiint\!\!\int_{-N}^{N} dv_1 \ldots dv_4 dy\, e^{-i\mu_1 v_1 - i\mu_2 v_2 + i\mu_3 v_3 + i\mu_4 v_4}$$

$$w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4) \frac{C_N(v_1,\ldots,v_4,y)}{N}$$

$$\{m_2(0,v_1) m_2(v_2,y) m_2(y+v_3, y+v_4)\}_{15}.$$

Look at each of the 15 minor terms, listed by Table III, separately.

Minor term 1:

$$\lim_{N\to\infty} \frac{B_N^2}{(2\pi)^4} \iiiint\!\!\int_{-N}^{N} dv_1 \ldots dv_4 dy\, e^{-i\mu_1 v_1 - i\mu_2 v_2 + i\mu_3 v_3 + i\mu_4 v_4}$$

$$w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4) \frac{C_N(v_1,\ldots,v_4,y)}{N}$$

$$r(v_1) r(v_2 - y) r(v_3 - v_4). \qquad (7.16)$$

Let $\hat{y} = y - v_2$, $\hat{v}_4 = v_4 - v_3$, $\hat{v}_2 = B_N v_2$, $\hat{v}_3 = B_N v_3$; then (7.16) becomes

$$\lim_{N\to\infty} \frac{1}{(2\pi)^4} \int_{-N}^{N} dv_1 \int_{-NB_N}^{NB_N} d\hat{v}_2 \int_{-NB_N}^{NB_N} d\hat{v}_3 \int_{-N-\frac{\hat{v}_3}{B_N}}^{N-\frac{\hat{v}_3}{B_N}} d\hat{v}_4 \int_{-N-\frac{\hat{v}_2}{B_N}}^{N-\frac{\hat{v}_2}{B_N}} d\hat{y}$$

$$e^{-iv_1\mu_1 - i\hat{v}_2 \frac{\mu_2}{B_N}}\, e^{i\hat{v}_3 \frac{\mu_3+\mu_4}{B_N} + i\hat{v}_4\mu_4}\, w(B_N v_1, \hat{v}_2) w(-\hat{v}_3, B_N \hat{v}_4).$$

$$\frac{C_N(v_1,\frac{\hat{v}_2}{B_N},\frac{\hat{v}_3}{B_N},\hat{v}_4 + \frac{\hat{v}_3}{B_N},\hat{y} + \frac{v_2}{B_N})}{N} \; r(v_1)r(\hat{y})r(\hat{v}_4). \qquad (7.17)$$

An inspection of Figure IV reveals that for fixed $v_1,\ldots,v_4$ and y the sides of the rectangle $\hat{D}_N$ grow by ΔN when N increases by ΔN. Likewise the sides of $\hat{D}_N$ decrease in length by at most $\Delta v_1+\Delta v_2$ and $\Delta v_3+\Delta v_4$ if v_1 increases by Δv_1, v_2 by Δv_2, v_3 by Δv_3 and v_4 by Δv_4. Finally if y increases by Δy with $\hat{D}_N$ fixed, C_N decreases at most by Δy. All this implies that

$$\frac{C_N(v_1,\frac{v_2}{B_N},\frac{v_3}{B_N},v_4 + \frac{v_3}{B_N},y + \frac{v_2}{B_N})}{N} \rightarrow 1 \text{ as } N \rightarrow\infty \qquad (7.18)$$

since it at worst behaves like (for A and B some constants)

$$\frac{A + N - \frac{B}{B_N}}{N}$$

which still tends to one as $N \rightarrow \infty$.

Denote by $\square_M$, the j-dimensional hypercube centered at the origin with sides of length 2M parallel to the j axes. The dimension, j, will be obvious from the context. Also let $\square_M'$ denote the complement of $\square_M$ in R_j.

<u>Lemma 5</u>: (a) hypotheses of Theorem 5

$\Rightarrow$ for any $\varepsilon > 0$, there is an $M_2(\varepsilon)$ independent of N such that for all $M > M_2$ and for all N,

$$\int\!\!\int\!\!\int\!\!\int\!\!\int_{\square_M'} |w(B_N v_1,v_2)w(-v_3,B_N v_4)r(v_1)r(y)r(v_4)|dv_1\ldots dv_4 dy < \varepsilon .$$

__Proof:__ The integral in question is

$$\leq [\int_{|v_1|>M}\iiiint_{-\infty}^{\infty} + \int_{|v_2|>M}\iiiint_{-\infty}^{\infty} + \int_{|v_3|>M}\iiiint_{-\infty}^{\infty} + \int_{|v_4|>M}\iiiint_{-\infty}^{\infty} + \int_{|y|>M}\iiiint_{-\infty}^{\infty}]$$

$$|w(B_N v_1, v_2)w(-v_3, B_N v_4)r(v_1)r(y)r(v_4)|\,dv_1\ldots dv_4 dy. \qquad (7.19)$$

Each of the five integrals, by the L_1 property of $r(v)$ and property (iv) of a bispectral estimating kernel, can be made arbitrarily small uniformly in N by choosing M large enough.

q. e. d.

Let

$$\gamma_1(M,N) = \frac{1}{(2\pi)^4} \iiiint\int_{-M}^{M} dv_1\ldots dv_4\, dy\; e^{-iv_1\mu_1 - iv_2\frac{\mu_2}{B_N}}$$

$$e^{iv_3\frac{\mu_3+\mu_4}{B_N} + iv_4\mu_4}\; w(B_N v_1, v_2)w(-v_3, B_N v_4)r(v_1)r(y)r(v_4),$$

$$\gamma_1(M) = \lim_{N\to\infty} \gamma_1(M,N).$$

Lemma 5 and (7.18) indicate that the limit (7.17) is the same as $\lim_{N\to\infty} \gamma_1(N,N)$. Lemma 2 says that $\gamma_1(M) = 0$ unless $\mu_2=\mu_3+\mu_4=0$. Using the boundedness and continuity properties of w and Lebesgue's convergence theorem,

$$\gamma_1(M) = \frac{1}{(2\pi)^4} \iiiint\int_{-M}^{M} dv_1\ldots dv_4 dy\; e^{-iv_1\mu_1+iv_4\mu_4}$$

$$w(0,v_2)w(-v_3,0)r(v_1)r(y)r(v_4)\delta(\mu_2)\delta(\mu_3+\mu_4).$$

By Lemma 3,

$$\lim_{N\to\infty} \gamma_1(N,N) = \frac{1}{2\pi} w_1 f(0) f(\mu_1) f(\mu_3) \delta(\mu_2) \delta(\mu_3+\mu_4).$$

Minor term 2: Interchanging the roles of v_1 and v_2 appropriately in the above, the limit is

$$\frac{1}{2\pi} w_1 f(0) f(\mu_2) f(\mu_3) \delta(\mu_1) \delta(\mu_3+\mu_4).$$

Minor term 3: Letting $\hat{y} = y+v_3-v_2$, $\hat{v}_2 = B_N v_2$, $\hat{v}_3 = B_N v_3$, this term is seen to be (omitting the "hats")

$$\frac{1}{(2\pi)^4} \lim_{N\to\infty} \iint_{-N}^{N} dv_1 dv_4 \iint_{-NB_N}^{NB_N} dv_2 dv_3\, e^{-iv_1\mu_1 - iv_2\frac{\mu_2}{B_N} + iv_3\frac{\mu_3}{B_N} + iv_4\mu_4}$$

$$w(B_N v_1, v_2)\, w(v_3, B_N v_4) \int_{N+\frac{v_3-v_2}{B_N}}^{N+\frac{v_3-v_2}{B_N}} dy\, \frac{C_N(v_1, \frac{v_2}{B_N}, \frac{v_3}{B_N}, v_4, y-\frac{v_3-v_2}{B_N})}{N}$$

$$r(v_1) r(y) r(v_4).$$

As in the term 1 argument, $\frac{C_N}{N}$ still tends to one, and Lemmas 2, 3, and 5 say that the above equals

$$\frac{1}{(2\pi)^3} f(0) \lim_{M\to\infty} \iiiint_{-M}^{M} e^{-iv_1\mu_1 + iv_4\mu_4} w(0,v_2) w(v_3,0)$$

$$r(v_1) r(y) r(v_4)\quad \delta(\mu_2)\quad \delta(\mu_3)$$

$$= \frac{1}{2\pi} w_1\; f(0) f(\mu_1) f(\mu_4) \delta(\mu_2) \delta(\mu_3).$$

Minor terms 4, 5, 6: Interchange v_1 and v_2 and/or v_3 and v_4 correctly in minor term 3 to get, in the limit, respectively

$$\frac{1}{2\pi} w_1 \, f(0) \, f(\mu_2) \, f(\mu_4) \delta(\mu_1) \delta(\mu_3),$$

$$\frac{1}{2\pi} w_1 \, f(0) \, f(\mu_1) \, f(\mu_3) \delta(\mu_2) \delta(\mu_4), \text{ and}$$

$$\frac{1}{2\pi} w_1 f(0) \, f(\mu_2) \, f(\mu_3) \, \delta(\mu_1) \, \delta(\mu_4).$$

Minor term 7: By the same reasoning used on previous terms, the seventh becomes

$$\frac{w_1}{2\pi} f(0) \, f(\mu_1) \, f(\mu_3) \; \delta(\mu_1+\mu_2) \, \delta(\mu_3+\mu_4).$$

Minor term 8: A slight variation occurs with the eighth minor term. Put $\hat{v}_3=v_3-v_1$, $\hat{v}_2=v_2-v_4$, $\hat{v}_1=B_N v_1$, $\hat{v}_4=B_N v_4$ to get (again omitting the "hats")

$$\frac{1}{(2\pi)^4} \lim_{N\to\infty} \iint_{-NB_N}^{NB_N} dv_1 dv_4 \int_{-N-\frac{v_4}{B_N}}^{N-\frac{v_4}{B_N}} dv_2 \int_{-N-\frac{v_1}{B_N}}^{N-\frac{v_1}{B_N}} dv_3 e^{-iv_1\frac{\mu_1-\mu_3}{B_N}-iv_2\mu_2}$$

$$e^{iv_3\mu_3+iv_4\frac{\mu_4-\mu_2}{B_N}} \quad w(v_1, B_N v_2+v_4) w(B_N v_3+v_1, v_4)$$

$$\int_{-N}^{N} dy \, \frac{C_N(\frac{v_1}{B_N}, v_2+\frac{v_4}{B_N}, \; v_3+\frac{v_1}{B_N}, \; \frac{v_4}{B_N}, \; y)}{N} \quad r(y) r(y+v_3) r(y-v_2). \quad (7.20)$$

Again it is seen that $\frac{C_N}{N} \to 1$ as $N \to \infty$. Corresponding to Lemma 5 is

<u>Lemma 6</u>: (a) hypotheses of Theorem 5

$\Longrightarrow$ for any $\varepsilon>0$, there is an $M_3(\varepsilon)$ independent of N such that for all $M>M_3(\varepsilon)$ and for all $N>M$,

$$\iiiint\!\!\int_{\square_M'} dv_1 \ldots dv_4 dy \, |w(v_1, B_N v_2+v_4) w(B_N v_3+v_1, v_4)$$

$$r(y) r(y+v_3) r(y-v_2)| < \varepsilon \ .$$

<u>Proof</u>: The integral is

$$\leq \Big[\iiint_{\square_M'} dv_2 dv_3 dy \iint_{-\infty}^{\infty} dv_1 dv_4 + \iiint_{\square_M} dv_2 dv_3 dy \iint_{|v_1| \text{ or } |v_4| > M} dv_1 dv_4\Big]$$

$$|w(v_1, B_N v_2+v_4) w(B_N v_3+v_1, v_4) r(y) r(y+v_3) r(y-v_2)| \qquad (7.21)$$

Schwarz's inequality implies

$$\iint dv_1 dv_4 |w(v_1, B_N v_2+v_4) w(B_N v_3+v_1, v_4)|$$

$$\leq \Big[\iint dv_1 dv_4 \, w^2(v_1, B_N v_2+v_4) \iint dv_1 dv_4 w^2(B_N v_3+v_1, v_4)\Big]^{1/2}.$$

Using this, the first integral of (7.21) can obviously be made small since $w(v_1,v_2) \in L_2(R_2)$. The second integral, which is

$$\leq \iiint_{\square_M} dv_2 dv_3 dy |r(y)r(y+v_3)r(y-v_2)| \cdot$$

$$\left[\iint_{\substack{|v_1|>M \\ \text{or} \\ |v_4|>M-B_N M}} dv_1 dv_4\, w^2(v_1,v_4) \iint_{-\infty}^{\infty} dv_1 dv_4\, w^2(v_1,v_4)\right]^{1/2},$$

can also be made small since eventually B_N becomes $< a < 1$.

q. e. d.

Define

$$\gamma_8(M,N) = \frac{1}{(2\pi)^4} \iiiint\!\!\int_{-M}^{M} dv_1 \ldots dv_4 dy \; e^{-iv_1 \frac{\mu_1-\mu_3}{B_N} - iv_2\mu_2}$$

$$e^{iv_3\mu_3 + iv_4 \frac{\mu_4-\mu_2}{B_N}} \; w(v_1, B_N v_2 + v_4) w(B_N v_3 + v_1, v_4)$$

$$r(y)r(y+v_3)r(y-v_2),$$

$$\gamma_8(M) = \lim_{N \to \infty} \gamma_8(M,N).$$

Lemma 2 and Lebesgue's theorem imply

$$\gamma_8(M) = \frac{1}{(2\pi)^4} \iiiint\!\!\int_{-M}^{M} dv_1 \ldots dv_4 dy \; e^{-iv_2\mu_2 + iv_3\mu_3}$$

$$w^2(v_1, v_4) r(y) r(y+v_3) r(y-v_2)\, \delta(\mu_1-\mu_3)\delta(\mu_2-\mu_4).$$

And Lemma 3 gives minor term 8 as

$$\lim_{N\to\infty} \gamma_8(N,N) = \frac{1}{2\pi} w_2\, f(\mu_2)f(\mu_3)f(\mu_2+\mu_3)\,\delta(\mu_1-\mu_3)\delta(\mu_2-\mu_4).$$

Minor term 9: Following the proof of minor term 8 get

$$\frac{1}{2\pi} w_2\, f(\mu_1)f(\mu_2)f(\mu_1+\mu_2)\delta(\mu_1-\mu_4)\delta(\mu_2-\mu_3).$$

Minor terms 10 and 11: The reasoning for minor term 1 shows that the respective limits are

$$\frac{1}{2\pi} w_1 f(0)f(\mu_1)f(\mu_3)\,\delta(\mu_1+\mu_2)\delta(\mu_4), \text{ and}$$

$$\frac{1}{2\pi} w_1 f(0)f(\mu_1)f(\mu_4)\,\delta(\mu_1+\mu_2)\delta(\mu_3).$$

Minor term 12: Set $\hat{y} = y+v_3$, $\hat{v}_1=v_3+v_1$, $\hat{v}_4=v_4-v_3-v_2$, $\hat{v}_3=B_N v_3$, $\hat{v}_2 = B_N v_2$ and minor term 12 becomes (omitting the "hats")

$$\frac{1}{(2\pi)^4}\lim_{N\to\infty}\iint_{-NB_N}^{NB_N} dv_2\,dv_3\int_{-N+\frac{v_3}{B_N}}^{N+\frac{v_3}{B_N}} dv_1\int_{-N-\frac{v_3+v_2}{B_N}}^{N-\frac{v_3+v_2}{B_N}} dv_4\, e^{-iv_1\mu_1-iv_2\frac{\mu_2-\mu_4}{B_N}}$$

$$e^{iv_3\frac{\mu_1+\mu_2+\mu_3}{B_N}+iv_4\mu_4}\qquad w(B_N v_1-v_3,v_2)w(-v_3,B_N v_4+v_2)$$

$$\int_{-N+\frac{v_3}{B_N}}^{N+\frac{v_3}{B_N}} dy\,\frac{C_N\left(v_1-\frac{v_3}{B_N},\frac{v_2}{B_N},\frac{v_3}{B_N},v_4+\frac{v_3+v_2}{B_N},y-\frac{v_3}{B_N}\right)}{N}\,r(y)r(y-v_1)r(y+v_4). \tag{7.24}$$

Again $\frac{C_N}{N}\to 1$ as $N\to\infty$ and Lemma 6 permits (7.24) to be replaced by $\lim_{N\to\infty}\gamma_{12}(N,N)$, where

$$\Upsilon_{12}(M,N) = \frac{1}{(2\pi)^4} \iiiint\!\!\int_{-M}^{M} dv_1 \ldots dv_4 dy \; e^{-iv_1\mu_1 - iv_2 \frac{\mu_2-\mu_4}{B_N}}$$

$$e^{iv_3 \frac{\mu_1+\mu_2+\mu_3}{B_N} + iv_4\mu_4} \; w(B_N v_1 - v_3, v_2) w(-v_3, B_N v_4 + v_2)$$

$$r(y) r(y-v_1) r(y+v_4),$$

$$\Upsilon_{12}(M) = \lim_{N\to\infty} \Upsilon_{12}(M,N).$$

As before

$$\lim_{N\to\infty} \Upsilon_{12}(N,N) = \lim_{M\to\infty} \Upsilon_{12}(M)$$

$$= \frac{1}{2\pi} w_2 \, f(\mu_1) f(\mu_4) f(\mu_1+\mu_4) \, \delta(\mu_2-\mu_4) \delta(\mu_1+\mu_3+\mu_4).$$

Minor terms 13, 14, and 15: These terms are the same as term 12 except for certain interchanging of indices and therefore tend to

$$\frac{1}{2\pi} \; w_2 \; f(\mu_2) f(\mu_4) f(\mu_2+\mu_4) \; \delta(\mu_1-\mu_4) \; \delta(\mu_2+\mu_3+\mu_4),$$

$$\frac{1}{2\pi} \; w_2 \; f(\mu_1) f(\mu_3) f(\mu_1+\mu_3) \; \delta(\mu_2-\mu_3) \delta \; (\mu_1+\mu_3+\mu_4),$$

$$\frac{1}{2\pi} \; w_2 \; f(\mu_2) f(\mu_3) f(\mu_2+\mu_3) \delta \; (\mu_1-\mu_3) \delta \; (\mu_2+\mu_3+\mu_4).$$

Restricting μ_1, μ_2, μ_3, and μ_4 as in hypothesis (f), the result is obtained.

Q. E. D.

Certain generalizations of Theorem 5 could readily be obtained. For example, two weight functions, w_1 and w_2, could be used; different (but interelated) sequences, $\{B_N^{(1)}\}$ and $\{B_N^{(2)}\}$, could also be used; and as pointed out earlier certain restrictions on w and the process could be relaxed slightly. It is also interesting to look at real and imaginary parts; this is done to a certain extent in the next section.

8. Asymptotic distribution of the estimates.

The final property of the estimates to be discussed is the asymptotic distribution. The estimate of interest to look at here is $g_N^*(\lambda_1,\lambda_2)$ as given by (7.1). This estimate will, under certain conditions on the process, $\{X_t\}$, have a distribution which tends to a complex normal distribution. In this section, one such group of conditions is investigated. The process will be a discrete parameter process and will demonstrate how arguments are carried out in the discrete case.

Consider a real, strictly stationary process, $\{X_t\}$, of the following form. Let

$$\eta = (\ldots, \eta_{-1}, \eta_o, \eta_1, \ldots)$$

be a doubly infinite sequence of independent, identically distributed random variables. We could take the η's to be uniformly distributed on [0,1] or normally distributed. Let T be the shift operator on η, i.e.

$$T\eta = (\ldots, \eta_o, \eta_1, \eta_2, \ldots).$$

Take h to be a Borel measurable function of the doubly infinite vector and define

$$X_t = h(T^t \eta) \quad t = 0,\pm 1, \ldots . \tag{8.1}$$

Also let

$$X_{t,k} = E[X_t \mid \eta_{t-k},\ldots, \eta_{t+k}] \tag{8.2}$$

(i.e. $X_{t,k}$ is the projection of X_t onto the Borel field, $\mathcal{B}_{t-k}^{t+k}$, generated by $\eta_{t-k},\ldots,\eta_{t+k}$) and

$$r^{(\infty,k)}(v) = EX_t X_{t+v,k} \,,$$
$$r^{(k,k)}(v) = EX_{t,k} X_{t+v,k} \,, \qquad (8.3)$$
$$r_3^{(\infty,\infty,k)}(v_1,v_2) = EX_t X_{t+v_1} X_{t+v_2,k} \,,$$

and similarly define $r_3^{(\infty,k,k)}$, $r_3^{(k,k,k)}$, $r_4^{(\infty,\infty,\infty,k)}$, ..., $r_4^{(k,k,k,k)}$, $r_6^{(\infty,\infty,\infty,k,k,k)}$ and $r_6^{(k,k,k,k,k,k)}$. Corresponding to these there are as in (4.4), (4.5) and (4.7)

$$\xi_2^{(\infty,k)}(v),\ \xi_2^{(k,k)}(v),$$
$$\xi_3^{(\infty,\infty,k)}(v_1,v_2),\ \ldots,\ \xi_3^{(k,k,k)}(v_1,v_2),$$
$$\xi_4^{(\infty,\infty,\infty,k)}(v_1,v_2,v_3),\ldots,\ \xi_4^{(k,k,k,k)}(v_1,v_2,v_3), \qquad (8.4)$$
$$\xi_6^{(\infty,\infty,\infty,k,k,k)}(v_1,\ldots,v_5),\ \text{and}\ \xi_6^{(k,k,k,k,k,k)}(v_1,\ldots,v_5).$$

Next, the definition of the bispectral estimating kernel must be adapted to the discrete parameter case. Conditions (i) and (iv) of Definition 2 become (letting, for convenience, $A_N = \frac{1}{B_N}$ be an integer for all N)

(i') for any $\varepsilon > 0$, there is an $M_1(\varepsilon)$ such that for all $M > M_1$ and uniformly in $N > M$,

$$B_N^2 \sum_{|v_1|,|v_2| > MA_N} \sum w^2(B_N v_1, B_N v_2) < \varepsilon,$$

(iv') for any $\varepsilon > 0$, there is an $M_2(\varepsilon)$ such that for all $M > M_2$ and uniformly in $N > M$ and in v_1,

$$B_N \sum_{|v_2| > MA_N} |w(v_1, B_N v_2)| < \varepsilon.$$

Further, the continuity conditions assumed in Theorem 4 become that (condition (v) of Definition 2 is not needed in the following)

(v') for all fixed numbers a and b, any fixed $M > 0$, and for any $\varepsilon > 0$ there is an $N_o(\varepsilon, M, a, b)$ such that for all $N > N_o$,

$$B_N^2 \Big| \sum_{v_1, v_2 = -MA_N}^{MA_N} w(B_N v_1 + B_N a, B_N v_2) w(B_N v_1, B_N v_2 + B_N b) - \sum_{v_1, v_2 = -MA_N}^{MA_N} w^2(B_N v_1, B_N v_2) \Big| < \varepsilon$$

and

$$B_N \Big| \sum_{v_1 = -MA_N}^{MA_N} w(B_N v_1, B_N a) - \sum_{v_1 = -MA_N}^{MA_N} w(B_N v_1, 0) \Big| < \varepsilon .$$

This last condition is certainly satisfied if $w(v_1, v_2)$ is continuous.

<u>Theorem 6</u>: (a) $\{X_t\}$ is a process as defined above,

(b) $EX_t^{12} < \infty$,

(c) all cumulant functions,

$\xi_2, \ldots, \xi_2^{(j,j)}, \xi_3, \ldots, \xi_3^{(j,j,j)}$,

$\xi_4, \ldots, \xi_4^{(j,j,j,j)}, \xi_6, \ldots, \xi_6^{(j,j,j,j,j,j)}$

are $\in L_1$ uniformly in j ,

(d) the hypotheses of Theorem 4 (adapted to the discrete parameter case[1])

---

[1] Due to the periodicities of the discrete parameter case the sector of definition is sector (I) of Figure III (see eq.(2.9)).

$$\Rightarrow \quad \sqrt{N}\, B_N[g_N^*(\mu_1,\mu_2) - Eg_N^*(\mu_1,\mu_2)]$$

converges in distribution to a complex normal random variable, X+iY, where X and Y have zero mean, are jointly normal, are independent and have the following variances[1]:

$$\sigma^2(X) = \frac{W_1}{2\pi} f(\mu_1)f(\mu_2)f(\mu_1+\mu_2)[8\delta(\mu_1) + \delta(\mu_2)] + A + B,$$

$$\sigma^2(Y) = A - B$$

where

$$A = \frac{1}{2}\frac{W_2}{2\pi} f(\mu_1)f(\mu_2)f(\mu_1+\mu_2)\{[1 + \delta(\mu_1-\mu_2)][1 + \delta(\mu_1+2\mu_2-2\pi) + \delta(2\mu_1+\mu_2-2\pi)] + 4\delta(\mu_1)\},$$

$$B = \frac{1}{2}\frac{W_2}{2\pi} f(\mu_1)f(\mu_2)f(\mu_1+\mu_2)\{5\delta(\mu_1) + \delta(\mu_2)[1 + \delta(\mu_1-\pi)]\}.$$

Proof: Due to the length of the proof, a precursory sketch will be given. The random variable of interest, V_N, (see (8.5)) is first approximated in mean square by V_{NM} (see (8.8)) uniformly in N for M large. Next V_{NM} is similarly approximated for fixed M by $V_{NM}^{(k)}$ (defined later) for k large. A lemma is then quoted to show that

$$V_{NM}^{(k)} \xrightarrow{\text{dist.}} N(0,\sigma_{MkR}, \sigma_{MkI}, r_{Mk}) \quad \text{as } N \to \infty$$

---

1 Note that if (μ_1,μ_2) lies inside the region (I) and not on its boundaries the variances reduce to

$$\sigma^2(X) = \sigma^2(Y) = \frac{1}{2}\frac{W_2}{2\pi} f(\mu_1)\, f(\mu_2)f(\mu_1+\mu_2).$$

where $N(0,\sigma_R,\sigma_I,r)$ denotes a complex normal random variable, mean zero, real part variance σ_R, imaginary part variance σ_I, and covariance r between real and imaginary parts. It is also true that $\sigma_{MkR} \rightarrow \sigma_{MR}$, $\sigma_{MkI} \rightarrow \sigma_{MI}$, and $r_{Mk} \rightarrow r_M$ as $k \rightarrow \infty$ and that $\sigma_{MR} \rightarrow \sigma_R$, $\sigma_{MI} \rightarrow \sigma_I$, and $r_M \rightarrow r$ as $M \rightarrow \infty$,

Recall the weak convergence metric, L(F,G), for two distribution functions, F(x) and G(x)[1]:

$$L(F,G) = \text{infinium over } \delta \text{ such that for all } x,$$

$$F(x-\delta) - \delta \leq G(x) \leq F(x+\delta) + \delta .$$

In carrying out the proofs, one does not deal with the complex-valued random variables themselves but rather with linear combinations of their real and imaginary parts. Denote such linear combinations by affixing a "hat" to the notation so that, for example, $\hat{V}_N = \lambda_1 \operatorname{Re} V_N + \lambda_2 \operatorname{Im} V_N$. In addition let

$F_N(x)$ be the distribution function of $\hat{V}_N$,

$F_{NM}(x)$ be the distribution function of $\hat{V}_{NM}$,

$F_{NMk}(x)$ be the distribution function of $\hat{V}_{NM}^{(k)}$,

$G_{Mk}(x)$ be the distribution function of $\hat{N}(0,\sigma_{MkR},\sigma_{MkI}, r_{Mk})$,

$G_M(x)$ be the distribution function of $\hat{N}(0,\sigma_{MR},\sigma_{MI},r_M)$,

$G(x)$ be the distribution function of $\hat{N}(0,\sigma_R,\sigma_I,r)$.

---

1 See, for example, Gnedenko and Kolmogorov [5], p. 33.

Then by the previous statements, we first choose $M(\varepsilon)$ so that

(i) $L(F_N, F_{NM}) < \varepsilon$ uniformly in N, and

(ii) $L(G_M, G) < \varepsilon$.

Next choose $k(\varepsilon, M)$ so that

(iii) $L(F_{NM}, F_{NMk}) < \varepsilon$ uniformly in N, and

(iv) $L(G_M, G_{Mk}) < \varepsilon$.

Finally, choose $N_0(\varepsilon, M, k)$ such that for all $N > N_0$

(v) $L(F_{NMk}, G_{Mk}) < \varepsilon$.

In conclusion, by the properties of a metric, for any $\varepsilon > 0$, there exists an $N_0(\varepsilon)$ such that for all $N > N_0$

$$L(F_N, G) \le L(F_N, F_{NM}) + L(F_{NM}, G)$$

$$\le L(F_N, F_{NM}) + L(F_{NM}, F_{NMk}) + L(F_{NMk}, G_{Mk})$$

$$+ L(G_{Mk}, G_M) + L(G_M, G) \le 5\varepsilon ,$$

and the theorem follows.

To get to the detail, separate

$$V_N = \sqrt{N}\, B_N(g_N^*(\mu_1, \mu_2) - Eg_N^*(\mu_1, \mu_2)) \tag{8.5}$$

into its real and imaginary parts

$$\mathrm{Re} V_N = \frac{B_N}{(2\pi)^2 \sqrt{N}} \sum_{v_1, v_2 = -N}^{N} \cos(\mu_1 v_1 + \mu_2 v_2) w(B_N v_1, B_N v_2) \cdot \sum_{t \in D_N(v_1, v_2)} (X_t X_{t+v_1} X_{t+v_2} - r_3(v_1, v_2)), \tag{8.6}$$

$$\mathrm{Im}V_N = \frac{-B_N}{(2\pi)^2\sqrt{N}} \sum_{\nu_1,\nu_2=-N}^{N}\sum \sin(\mu_1\nu_1+\mu_2\nu_2)w(B_N\nu_1,B_N\nu_2)\cdot$$

$$\sum_{t\in D_N(\nu_1,\nu_2)} (X_tX_{t+\nu_1}X_{t+\nu_2} - r_3(\nu_1,\nu_2)). \tag{8.7}$$

Truncating the ν_1, ν_2 summation and extending the t summation, define

$$V_{NM} = \frac{B_N}{(2\pi)^2\sqrt{N}} \sum_{\nu_1,\nu_2=-MA_N}^{MA_N}\sum e^{-i\mu_1\nu_1-i\mu_2\nu_2}\, w(B_N\nu_1,B_N\nu_2)\cdot$$

$$\sum_{t=1}^{N} (X_tX_{t+\nu_1}X_{t+\nu_2} - r_3(\nu_1,\nu_2)). \tag{8.8}$$

<u>Lemma 7</u>: (a) hypotheses of Theorem 6

$\Longrightarrow$ for any $\varepsilon > 0$, there is an $M_o(\varepsilon)$ such that for all $M > M_o$, $N > M$;

$\sigma^2(V_N - V_{NM}) < \varepsilon$.

<u>Proof</u>: Let $\hat{V}_N$ be V_N with the domain of summation, $D_N(\nu_1,\nu_2)$, set equal to $[1,N]$ for all ν_1 and ν_2. Then

$$\sigma^2(V_N-V_{NM})^{1/2} \le \sigma^2(V_N-\hat{V}_N)^{1/2} + \sigma^2(\hat{V}_N - V_{NM})^{1/2}. \tag{8.9}$$

First bound

$$\sigma^2(V_N-\hat{V}_N) \le \frac{B_N^2}{N} \sum_{\nu_1,\ldots,\nu_4=-N}^{N}\sum\sum\sum |w(B_N\nu_1,B_N\nu_2)w(B_N\nu_3,B_N\nu_4)|$$

$$\left[\sum_{t=1}^{N}\ \sum_{\tau\in\tilde{D}_N(\nu_3,\nu_4)} + \sum_{t\in\tilde{D}_N(\nu_1,\nu_2)}\ \sum_{\tau=1}^{N}\right]\cdot$$

$$|m_6(t,t+\nu_1,t+\nu_2,\tau,\tau+\nu_3,\tau+\nu_4) - r_3(\nu_1,\nu_2)r_3(\nu_3,\nu_4)| \tag{8.10}$$

where

$$\tilde{D}_N(v_1,v_2) = [1,N] \ominus D_N(v_1,v_2)$$

is the symmetric set difference. Looking at Figure IV, (8.10) yields

$$\sigma^2(V_N-\hat{V}_N) \le B_N^2 \sum\sum\sum\sum_{v_1,\dots,v_4=-N}^{N} |w(B_N v_1, B_N v_2)$$

$$w(B_N v_3, B_N v_4)| [\frac{|v_3|+|v_4|}{N} \sum_{y=-N}^{N} + \frac{|v_1|+|v_2|}{N} \sum_{y=-N}^{N}] \cdot$$

$$|\xi_6(v_1,v_2,y,y+v_3,y+v_4) + \{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10-1} +$$

$$+\{m_2(0,v_1)s_4(v_2,y,y+v_3,y+v_4)\}_{15}+\{m_2(0,v_1)m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}|. \tag{8.11}$$

It is readily seen that this tends to zero by arguments similar to those used in the proof of Theorem 5 only here, instead of $\frac{C_N}{N} \to 1$, the expression $\frac{|v_1|+|v_2|}{N} \to 0$ occurs. Next

$$\sigma^2(\hat{V}_N-V_{NM}) \le B_N^2 \sum_{(v_1,\dots,v_4)\in \square'_{MA_N} \cap \square_N} \sum \sum \sum \sum_{y=-N}^{N}$$

$$|w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4) [\xi_6(v_1,v_2,y,y+v_3,y+v_4)$$

$$+ \{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10-1} +$$

$$+ \{m_2(0,v_1)s_4(v_2,y,y+v_3,y+v_4)\}_{15} +$$

$$+ \{m_2(0,v_1)m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}]|.$$

$$\leq B_N^2 \sum\sum\sum\sum_{\Box'_{MA_N} \cap \Box_N} \sum_{y=-N}^{N} |w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4)$$

$$\{m_2(0, v_1) m_2(v_2, y) m_2(y+v_3, y+v_4)\}_{15}| + O(B_N) \qquad (8.12)$$

by the discrete analog of Lemma 4. Again each minor term must be treated separately. The first minor term is, letting $\hat{y} = y - v_2$, $\hat{v}_4 = v_4 - v_3$,

$$\leq B_N^2 \sum_{(v_1, v_2, v_3, v_4) \in \Box'_{\frac{MA_N}{2}}} \sum\sum\sum \sum_{y=-\infty}^{\infty} |w(B_N v_1, B_N v_2) w(-B_N v_3, B_N v_4)$$

$$r(v_1) r(y) r(v_4)|, \qquad (8.13)$$

which can be made small by choosing M large as is seen by breaking (8.13) up into four sums similar to (7.19) and using the properties of w and r. Minor terms 2, 10, and 11 follow similarly. It is even simpler to bound terms 3, 4, 5 and 6. Minor term 7 is also bounded by (8.13). Minor term 8 is

$$\leq B_N^2 \sum_{(v_1, v_2, v_3, v_4) \in \Box'_{\frac{MA_N}{2}}} \sum\sum\sum \sum_{y=-N}^{N} |w(B_N v_1, B_N v_2 + B_N v_4)$$

$$w(B_N v_1 + B_N v_3, B_N v_4)\ r(y) r(y+v_3) r(y-v_2)|$$

$$\leq B_N^2 \sum_{y=-N}^{N} |r(y)| \{ \sum_{(v_2, v_3) \in \Box'_{\frac{MA_N}{2}}} \sum |r(y+v_3) r(y-v_2)| \cdot$$

$$\cdot\Big[\sum_{\nu_1,\nu_4=-\infty}^{\infty}\sum w^2(B_N\nu_1,B_N\nu_2+B_N\nu_4)\cdot$$

$$\sum_{\nu_1,\nu_4=-\infty}^{\infty}\sum w^2(B_N\nu_1+B_N\nu_3,B_N\nu_4)\Big]^{1/2}+$$

$$+\sum_{\nu_2,\nu_3=-\infty}^{\infty}\sum |r(y+\nu_3)r(y-\nu_2)|\cdot\Big[\sum_{(\nu_1,\nu_4)\in\Box'_{\frac{MA_N}{2}}}\sum w^2(B_N\nu_1,B_N\nu_2+B_N\nu_4)$$

$$\cdot\sum_{(\nu_1,\nu_4)\in\Box'_{\frac{MA_N}{2}}}\sum w^2(B_N\nu_1+B_N\nu_3,B_N\nu_4\Big]^{1/2}\Big\}$$

and again this can be made small uniformly in N. Minor term 9 has identical behavior. Minor terms 13, 14, 15 and 16 are only trivially more complicated.

q.e.d.

Lemma 7 indicates that the asymptotic distribution in question is the same as that of V_{NM}. The second modification is to replace the X_j in V_{NM} by $X_{j,k}$ to get $V_{NM}^{(k)}$. The next lemma shows that this can be done -- that $\sigma^2(V_{NM}-V_{NM}^{(k)})$ can be made smaller than any previously chosen $\varepsilon > 0$ uniformly in N for k sufficiently large (M being fixed).

<u>Lemma 8</u>: (a) hypotheses of Theorem 6

$\Rightarrow$ for every M, $\varepsilon > 0$, there is a $k_o(\varepsilon,M)$ independent of N and a constant K independent of both M and N such that for all $k > k_o(\varepsilon,M)$;

$$\sigma^2(V_{NM}-V_{NM}^{(k)}) \leq \varepsilon + KB_N\ .$$

Note, the conditions are sufficient to remove the KB_N term on the bound (see remark at end of proof).

Proof:

$$\sigma^2(V_{NM}-V_{NM}^{(k)}) \leq \frac{B_N^2}{N} \sum\sum\sum\sum_{v_1,\ldots,v_4=-MA_N}^{MA_N} \sum_{t=1}^{N} \sum_{\tau=1}^{N}$$

$$|w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4)|$$

$$|E[X_t X_{t+v_1} X_{t+v_2} - r_3(v_1,v_2)][X_\tau X_{\tau+v_3} X_{\tau+v_4} - r_3(v_3,v_4)]$$

$$-E[X_t X_{t+v_1} X_{t+v_2} - r_3(v_1,v_2)][X_{\tau,k} X_{\tau+v_3,k} X_{\tau+v_4,k} - r_3^{(k,k)}(v_3,v_4)]$$

$$-E[X_{t,k} X_{t+v_1,k} X_{t+v_2,k} - r_3^{(k,k)}(v_1,v_2)][X_\tau X_{\tau+v_3} X_{\tau+v_4} - r_3(v_3,v_4)]$$

$$+E[X_{t,k} X_{t+v_1,k} X_{t+v_2,k} - r_3^{(k,k)}(v_1,v_2)][X_{\tau,k} X_{\tau+v_3,k} X_{\tau+v_4,k} - r_3^{(k,k)}(v_3,v_4)]|. \tag{8.14}$$

As in the beginning of the proof of Theorem 5, and using the discrete version of Lemma 4, (8.14) is

$$\leq B_N^2 \sum\sum\sum\sum_{v_1,\ldots,v_4=-MA_N}^{MA_N} \sum_{y=-N}^{N} |\{m_2(0,v_1)m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}$$

$$+ \{m_2^{(k,k)}(0,v_1)m_2^{(k,k)}(v_2,y)m_2^{(k,k)}(y+v_3,y+v_4)\}_{15} -$$

$$- 2\{m_2^{(\cdot,\cdot)}(0,v_1)m_2^{(\cdot,\cdot)}(v_2,y)m_2^{(\cdot,\cdot)}(y+v_3,y+v_4)\}_{15}| + O(B_N) \tag{8.15}$$

where $(\cdot,\cdot)$ stands for one of either (k,k), (k,∞), (∞,k) or

(∞,∞) depending on the arguments of m_2. Looking at the first minor terms of (8.15),

$$B_N^2 \sum\sum\sum\sum_{-MA_N}^{MA_N} \sum_{y=-N}^{N} |w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4)|$$

$$|r(v_1)r(y-v_2)r(v_4-v_3)+r^{(k,k)}(v_1)r^{(k,k)}(y-v_2)r^{(k,k)}(v_4-v_3)$$

$$-r(v_1)r^{(\infty,k)}(y-v_2)r^{(k,k)}(v_4-v_3)-r^{(k,k)}(v_1)r^{(\infty,k)}(y-v_2)r(v_4-v_3)|$$

$$= B_N^2 \sum_{v_1,v_2,v_3=-MA_N}^{MA_N}\sum\sum \sum_{v_4=-MA_N-v_3}^{MA_N-v_3} \sum_{y=-N(1+\frac{v_2}{N})}^{N(1-\frac{v_2}{N})} |w(B_N v_1, B_N v_2)$$

$$w(-B_N v_3, B_N v_4)|\,|r(v_1)r(y)r(v_4) + r^{(k,k)}(v_1)r^{(k,k)}(y)r^{(k,k)}(v_4)$$

$$- r(v_1)r^{(\infty,k)}(y)r^{(k,k)}(v_4) - r^{(k,k)}(v_1)r^{(\infty,k)}(y)r(v_4)|$$

$$\leq \hat{w}^2 \sum_{v_1,v_4=-2MA_N}^{2MA_N}\sum \sum_{y=-N-MA_N}^{N+MA_N} |r(v_1)r(y)r(v_4) +$$

$$r^{(k,k)}(v_1)r^{(k,k)}(y)r^{(k,k)}(v_4) - r(v_1)r^{(\infty,k)}(y)r^{(k,k)}(v_4)$$

$$- r^{(k,k)}(v_1)r^{(\infty,k)}(y)r(v_4)| \qquad (8.16)$$

where

$$B_N \sum_{v_2=-\infty}^{\infty} |w(a, B_N v_2)| < \hat{w} \quad \text{for all } a.$$

This can obviously be approximated arbitrarily closely (uniformly in N) by a finite sum. Then having only a finite number of terms, the fact that $r^{(\infty,k)}(v) \to r(v)$ and $r^{(k,k)} \to r(v)$ gives the desired result.[1] A similar argument holds for all other terms.

Due to the fact that $EX_t^{12} < \infty$, all the moments defined in (8.3) have the property that they tend to the corresponding moments of the X_t process. Thus one would not need to eliminate the contributions of the first, second, and third major terms as was done in (8.15) but treat them the same as the fourth major term thereby making the $O(B_N)$ in Lemma 8 unnecessary. q.e.d.

To apply a central limit theorem, write

$$U_R^{(N)} = \operatorname{Re} V_{NM}^{(k)} = \frac{1}{\sqrt{N}} \sum_1^N Y_t^{(k,N,M)},$$

$$U_I^{(N)} = \operatorname{Im} V_{NM}^{(k)} = \frac{1}{\sqrt{N}} \sum_1^N Z_t^{(k,N,M)}$$

where $$Y_t^{(k,N,M)} = \frac{B_N}{(2\pi)^2} \sum_{v_1,v_2=-MA_N}^{MA_N} \cos(\mu_1 v_1+\mu_2 v_2)$$

$$w(B_N v_1, B_N v_2)[X_{t,k} X_{t+v_1,k} X_{t+v_2,k} - r_3^{(k,k,k)}(v_1,v_2)]$$

and $Z_t^{(k,M,N)}$ is as above except with a sine instead of cosine. For any two real parameters, λ_1 and λ_2, form

$$U_N(\lambda_1,\lambda_2) = \lambda_1 U_R^{(N)} + \lambda_2 U_I^{(N)} \tag{8.17}$$

with

$$U_t^{(k,N,M)} = \frac{1}{\sqrt{N}}(\lambda_1 Y_t^{(k,N,M)} + \lambda_2 Z_t^{(k,N,M)}).$$

Note that the $\{U_t^{(k,N,M)}\}$ sequence is a $2MA_N+2k$ step dependent

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[1] See, for example, Chapter VII of Doob [4] on martingales.

process.[1] This prompts one to use the following lemma from Rosenblatt [16] p. 262.

<u>Lemma 9</u>: (a) $\{V_t^{(N)}\}$ is a sequence of $d(N)$-dependent strictly stationary random variables,

(b) $d(N) \to \infty$ as $N \to \infty$,

(c) $\frac{d(N)}{N} \to 0$ as $N \to \infty$,

(d) $E|V_t^{(N)}|^{2+\delta} < \infty$ for some $\delta > 0$,

(e) $t(N)$ is an integer-valued function

(i) $t(N) \to 0$

(ii) $d(N) = o(t(N))$

(iii) $t(N) = o(N)$,

(f) for $\{r_\nu^{(N)}\}$ the covariance sequence of $\{V_t^{(N)}\}$,

$$\sum_{|\nu| \leq t(N)} |\nu| r_\nu^{(N)} = o\Big(\sum_{|\nu| \leq t(N)} r_\nu^{(N)} t(N)\Big) \quad \text{as } N \to \infty,$$

(g)
$$\frac{E\Big|\sum_{1}^{t(N)} V_t^{(N)}\Big|^{2+\delta}}{N^{\frac{\delta}{2}} t(N) \Big(\sum_{-d(N)}^{d(N)} r_\nu^{(N)}\Big)^{1+\frac{\delta}{2}}} \to 0 \quad \text{as } N \to \infty$$

$\Rightarrow \sum_{1}^{N} V_t^{(N)}$ is asymptotically normally distributed with mean zero and variance $2\pi N h_N(0)$, where $h_N(\lambda)$ is the spectral density of $\{V_t^{(N)}\}$.

To apply this lemma to (8.17), put

$$V_t^{(N)} = U_t^{(k,N,M)},$$

$$d(N) = 2MA_N + 2k,$$

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[1] See Hoeffding and Robbins [9].

$$t(N) = MA_N^2 ,$$

$$\delta = 2.$$

Conditions (a), (b), (c), and (e) of Lemma 9 are certainly satisfied. Condition (d) is satisfied since there is a constant K so that

$$E|V_t^{(N)}|^4 \le \frac{B_N^4}{N^2} \sum_{v_1,\ldots,v_8=-MA_N}^{MA_N} \cdots \sum$$

$$|w(B_N v_1, B_N v_2) \ \ldots \ w(B_N v_7, B_N v_8)| \cdot KEX_t^{12}$$

$$\le \hat{w}^4 KEX_t^{12} \ \frac{2}{(NB_N^2)^2} < \infty .$$

Condition (f) involves

$$\frac{\sum_{|v| \le MA_N^2} |v| r_v^{(N)}}{\sum_{|v| \le MA_N^2} r_v^{(N)} MA_N^2} = \frac{N \sum_{|v| \le MA_N^2} \frac{|v|}{MA_N^2} r_v^{(N)}}{N \sum_{|v| \le MA_N^2} r_v^{(N)}} . \qquad (8.18)$$

But,

$$N \sum_{|v| \le MA_N^2} r_v^{(N)} = N \sum_{|v| \le MA_N^2} EV_0^{(N)} V_v^{(N)}$$

$$= \frac{B_N^2}{(2\pi)^4} \sum_{|v| \le MA_N^2} \sum_{|v_1|,|v_2| \le MA_N} \sum [\lambda_1 \cos(\mu_1 v_1 + \mu_2 v_2) +$$

$$\lambda_2 \sin(\mu_1 v_1 + \mu_2 v_2)] \ w(B_N v_1, B_N v_2)$$

$$\sum_{|v_3|,|v_4| \le MA_N} \sum [\lambda_1 \cos(\mu_1 v_3 + \mu_2 v_4) + \lambda_2 \sin(\mu_1 v_3 + \mu_2 v_4)]$$

$$w(B_N v_3, B_N v_4) [r_6(v_1, v_2, v, v+v_3, v+v_4) - r_3(v_1, v_2) r_3(v_3, v_4)] \quad (8.19)$$

and this, from earlier results, converges absolutely uniformly in N. Therefore provided $Nh_N(0) \neq 0$ and since $\frac{|v|}{MA_N^2}$ converges to zero pointwise, (8.18) tends to zero. Finally condition (g) leads to

$$\frac{E[\sum_{t=1}^{MA_N^2} V_t^{(N)}]^4}{N\, MA_N^2(\Sigma\, r_v^{(N)})^2} \quad . \tag{8.20}$$

The denominator $\sim \frac{1}{NB_N^2}$ by (8.19). Define

$$D_j = \sum_{t=(j-1)d(N)+1}^{j\, d(N)} V_t^{(N)} \qquad 1 \leq j \leq 2u_o,$$

$$2u_o = \text{largest even integer} \leq \frac{MA_N^2}{d(N)} \ ,$$

$$D_{2u_o+1} = \begin{cases} 0 & MA_N^2 = 2u_o d(N) \\ \displaystyle\sum_{2u_o\, d(N)+1}^{\min[MA_N^2,(2u_o+1)d(N)]} V_t^{(N)} & MA_N^2 > 2u_o d(N), \end{cases}$$

$$D_{2u_o+2} = \begin{cases} 0 & MA_N^2 \leq (2u_o+1)d(N) \\ \displaystyle\sum_{(2u_o+1)d(N)+1}^{MA_N^2} V_t^{(N)} & MA_N^2 > (2u_o+1)d(N), \end{cases}$$

then

$$\sum_{t=1}^{MA_N^2} V_t^{(N)} = \sum_{j=1}^{u_o+1} D_{2j-1} + \sum_{j=1}^{u_o+1} D_{2j} \ .$$

By Minkowski's inequality, the fourth root of the numerator of (8.20) is

$$\leq [E(\sum_{j=1}^{u_o+1} D_{2j-1})^4]^{1/4} + [E(\sum_{j=1}^{u_o+1} D_{2j})^4]^{1/4} . \tag{8.21}$$

Note that the D_j's in the first term are independent of one another and have zero mean so that

$$E(\sum_{j=1}^{u_o+1} D_{2j-1})^4 = \sum_{j=1}^{u_o+1} E\, D_{2j-1}^4 + 3 \sum_{j_1,j_2=1}^{u_o+1}\sum ED_{2j_1-1}^2\, ED_{2j_2-1}^2$$

$$- 3 \sum_{j=1}^{u_o+1} ED_{2j-1}^2\, ED_{2j-1}^2 . \tag{8.22}$$

Further

$$E(\sum_{1}^{d(N)} V_t)^4 \leq \frac{B_N^4}{N^2} \sum_{t_1,\ldots,t_4=1}^{d(N)} \cdots \sum \sum_{v_1,\ldots,v_8=-MA_N}^{MA_N} \cdots \sum M_1^4$$

$$|E\{[X_{t_1,k}X_{t_1+v_1,k}X_{t_1+v_2,k} - r_3^{(k,k)}(v_1,v_2)] \cdots$$

$$[X_{t_4,k}X_{t_4+v_7,k}X_{t_4+v_8,k} - r_3^{(k,k)}(v_3,v_4)]\}| . \tag{8.23}$$

To get a bound on this recall that the $X_{j,k}$ are k-step dependent and have mean zero. This implies that each $X_{j,k}$ must be within k-steps of at least one other $X_{j,k}$ otherwise the expected value of the product is zero. Hence the sum in (8.23) has many zero terms. The fact that $EX_t^{12} < \infty$ says that all non-zero terms are bounded. Thus we can get a bound by enumerating the non-zero terms. Look first at the terms with indices such that all X_j's are "tied together", that is there is no way of dividing the X_j's into 2 groups so that one group of random variables is independent of the other. It is easily seen that there can be only $O(d(N)k^{11})$ number of such terms. One could proceed down the line and look at those groups of indices such that the X_j's divide into 2 and only 2 independent groups of 2 in the first group and 10 in the second or of 3 in the first and 9 in the second, etc. then go on to the various combinations of 3 groups, 4 groups, 5 groups, and lastly 6 groups. It is seen that the highest order number of non-zero terms occurs when there are 6 independent groups (implies two X_j's in each group). Here there are $O(d^6(N)k^6)$ such non-zero terms. This means that the first term of (8.22) has $O(u_o A_N^6)$ non-zero terms all of which are $O(\frac{B_N^4}{N^2})$, so that the contribution of it to the numerator of (8.20) is $O(\frac{A_N^3}{N^2})$ and comparing this with the denominator obtain $O(\frac{A_N}{N}) \to 0$.

The next term in (8.22) is

$$3(\sum_{j=1}^{u_o} ED_{2j-1}^2)^2 = O(u_o\ d(N) \sum_{-d(N)}^{d(N)} r_v^{(N)})^2.$$

Comparing this with the denominator get

$$O(\frac{A_N^4(\sum r_v^{(N)})^2}{NA_N^2(\sum r_v^{(N)})^2}) = O(\frac{1}{NB_N^2}) \to 0.$$

The third term of (8.22) is contained in the case just discussed. Thus Lemma 9 applies.

The proof of the theorem is complete except for the evaluation of the variances and covariances of the real and imaginary parts. Lemma 9 states that

$$\text{Re } V_{NM}^{(k)} + i \text{ Im } V_{NM}^{(k)} \xrightarrow{\text{dist.}} X_M^{(k)} + iY_M^{(k)} \quad \text{as} \quad N \to \infty$$

where $X_M^{(k)}$ and $Y_M^{(k)}$ are jointly normal with mean zero and

$$E(X_M^{(k)})^2 = \sigma_{kMR}^2 ,$$

$$E(Y_M^{(k)})^2 = \sigma_{kMI}^2 ,$$

$$E(X_M^{(k)} Y_M^{(k)}) = r_{kM} .$$

As $k \to \infty$, $\sigma_{kMR}^2 \to \sigma_{MR}^2$, $\sigma_{kMI}^2 \to \sigma_{MI}^2$, and $r_{kM} \to r_M$. A lemma is needed to evaluate σ_{MR}^2, σ_{MI}^2, and r_M.

Lemma 10: (a) $h(x)$, $h(x,y)$ are bounded and their respective sets, D_1 and D_2, of points of discontinuity have measure zero,

(b) A_N and B_N as in Theorem 6,

(c) μ, μ_1, μ_2 are real constants

$\Rightarrow$ (1) $$B_N \sum_{\nu=-MA_N}^{MA_N} \sin \nu\mu h(B_N\nu) \to 0 \quad \text{as} \quad N \to \infty ,$$

(2) $$B_N \sum_{\nu=-MA_N}^{MA_N} \cos \nu\mu\, h(B_N\nu) \to 0 \quad \text{as} \quad N \to \infty$$

provided $\mu = 0, \pm 2\pi, \ldots,$

(3) $B_N^2 \sum_{\nu_1,\nu_2=-MA_N}^{MA_N} \sin \nu_1 \mu_1 \sin \nu_2\mu_2 \, h(B_N\nu_1,B_N\nu_2) \to 0$

as $N \to \infty$,

(4) $B_N^2 \sum_{\nu_1,\nu_2=-MA_N}^{MA_N} \sin \nu_1\mu_1 \cos \nu_2\mu_2 \, h(B_N\nu_1,B_N\nu_2) \to 0$

as $N \to \infty$,

(5) $B_N^2 \sum_{\nu_1,\nu_2=-MA_N}^{MA_N} \cos \nu_1\mu_1 \cos \nu_2\mu_2 \, h(B_N\nu_1,B_N\nu_2) \to 0$

as $N \to \infty$ unless $\mu_1 = 0, \pm 2\pi, \ldots$ and $\mu_2 = 0, \pm 2\pi, \ldots$

Proof: For (1), note that if $\mu = 0, \pm 2\pi, \ldots$, the result is trivially true, so assume $\mu \neq 0, \pm 2\pi, \ldots$. Let $h_q(x)$ be a step function with q "steps". It follows immediately that the lemma holds for $h = h_q$; for example, let $q = 2$ and $h_2(x)$ be as shown in Figure V. Then

$$B_N \sum_{-MA_N}^{MA_N} \sin \nu\mu \, h_2(B_N\nu) =$$

$$= B_N c_1 \sum_{\nu=-MA_N}^{\nu \leq aA_N} \sin \nu\mu + B_N c_2 \sum_{\nu > aA_N}^{MA_N} \sin \nu\mu$$

$$= c_1 B_N \left[\frac{2 \cos \frac{1}{2}\mu - \cos(MA_N + \frac{1}{2})\mu - \cos(aA_N + \frac{1}{2})\mu}{2 \sin \frac{1}{2}\mu}\right]$$

$$+ c_2 B_N \left[\frac{\cos(aA_N + \frac{1}{2})\mu - \cos(MA_N + \frac{1}{2})\mu}{2 \sin \frac{1}{2}\mu}\right]$$

$$= O(B_N).$$

The generalization of this argument to any $h_q(x)$ is obvious. Now for any $\varepsilon > 0$, there exists an $h_q(x)$ such that

$$\int_{-M}^{M} |h(x) - h_q(x)|\, dx < \varepsilon . \tag{8.24}$$

Also

$$|B_N \sum_{-MA_N}^{MA_N} \sin \nu\mu \, (h(B_N\nu) - h_q(B_N\nu))|$$

$$\leq B_N \sum_{-MA_N}^{MA_N} |h(B_N\nu) - h_q(B_N\nu)| .$$

For N large enough this approaches to within ε

$$\int_{-M}^{M} |h(x) - h_q(x)|dx$$

which is small by (8.24). Thus by choosing N large enough so that in addition

$$|B_N \sum_{-MA_N}^{MA_N} \sin \nu\mu \, h_q(B_N\nu)| < \varepsilon ,$$

the lemma is proved.

The arguments for parts (2), (3), (4) and (5) are analogous. In the two-dimensional cases $h_q(x,y)$ is taken to be a function which is constant on each of q disjoint rectangles whose union is $[-M,M] \times [-M,M]$.

q.e.d.

The three quantities σ^2_{MR}, σ^2_{MI} and r_M must be evaluated separately for each of the fifteen minor terms of major term 4. Due to the number of calculations, only minor terms 1 and 8 will be calculated. The other terms follow by similar proofs. Henceforth, consider the δ-function as periodic in 2π.

(i) Calculation of σ^2_{MR}

Minor term 1:

$$\frac{B_N^2}{(2\pi)^4} \sum_{v_1,\dots,v_4 = -MA_N}^{MA_N} \sum_{y=-N}^{N} \frac{N-|y|}{N} \cos(v_1\mu_1 + v_2\mu_2)$$

$$\cos(v_3\mu_1 + v_4\mu_2) w(B_N v_1, B_N v_2) w(B_N v_3, B_N v_4)$$

$$r(v_1) r(y-v_2) r(v_4-v_3) .$$

By earlier results, this behaves like

$$\frac{f(0)}{(2\pi)^3} B_N^2 \sum\sum\sum\sum_{v_1,\dots,v_4=-MA_N}^{MA_N} \cos(v_1\mu_1+v_2\mu_2)\cos(v_3(\mu_1+\mu_2) + \mu_2 v_4)$$

$$w(0, B_N v_2)\ w(-B_N v_3, 0)\ r(v_1) r(v_4)$$

using the modified continuity conditions, (v'). Using trigonometric identities this is

$$\frac{f(0)}{(2\pi)^3} \sum\sum\sum\sum_{-MA_N}^{MA_N} [\cos v_1\mu_1 \cos v_2\mu_2 - \sin v_1\mu_1 \sin v_2\mu_2]$$

$$[\cos v_3(\mu_1+\mu_2)\cos v_4\mu_2 - \sin v_3(\mu_1+\mu_2)\sin v_4\mu_4]$$

$$w(0, B_N v_2) w(-B_N v_3, 0) r(v_1) r(v_4) \tag{8.25}$$

$$\to \frac{f(0)}{(2\pi)^3}\left[\int_{-M}^{M} w(0, v)dv\ \right]^2 \sum\sum_{-\infty}^{\infty} \cos v_1\mu_1 \cos v_2\mu_2$$

$$r(v_1) r(v_2)\ \delta(\mu_1+\mu_2)\ \delta(\mu_2)$$

$$= \frac{W_1}{2\pi} f(0) f(\mu_1) f(\mu_2)\ \delta(\mu_1+\mu_2)\ \delta(\mu_2)$$

by Lemma 10.

Minor term 8:

$$\frac{1}{(2\pi)^4} \sum\sum\sum\sum_{v_1,\ldots,v_4=-MA_N}^{MA_N} \sum_{y=-N}^{N} \cos(v_1\mu_1 + v_2\mu_2)\cos(v_3\mu_1+v_4\mu_2)$$

$$w(B_Nv_1,B_Nv_2)w(B_Nv_3,B_Nv_4)\ r(y)r(y+v_3-v_1)r(y+v_4-v_2)$$

$$\sim \frac{1}{(2\pi)^4} \sum\sum\sum\sum_{-MA_N}^{MA_N} \sum_{y=-N}^{N} \cos(v_1\mu_1+v_2\mu_2)\cos((v_1+v_3)\mu_1 + (v_2+v_4)\mu_2)$$

$$w(B_Nv_1,B_Nv_2)w(B_Nv_1+B_Nv_3,B_Nv_2+B_Nv_4)\ r(y)r(y+v_3)r(y+v_4)$$

$$= \frac{B_N^2}{(2\pi)^4} \sum_{-N}^{N} r(y) \sum\sum\sum\sum_{-MA_N}^{MA_N} \tfrac{1}{2}\{\cos(v_3\mu_1 + v_4\mu_2) +$$

$$\cos(2v_1\mu_1+2v_2\mu_2)\cos(v_3\mu_1+v_4\mu_2) - \sin(2v_1\mu_1+2v_2\mu_2)\cdot$$

$$\sin(v_3\mu_1+ v_4\mu_2)\}\ w(B_Nv_1,B_Nv_2)w(B_Nv_1+B_Nv_3,B_Nv_2+B_Nv_4)$$

$$r(y+v_3)r(y+v_4). \tag{8.26}$$

Again applying Lemma 10 this

$$\to \frac{1}{(2\pi)^4} \int_{-M}^{M} w^2(v_1,v_2)dv_1dv_2 \sum_{-N}^{N} \sum\sum_{-MA_N}^{MA_N} \tfrac{1}{2}\cos(v_3\mu_1 + v_4\mu_2)$$

$$[1 + \delta(2\mu_1)\delta(2\mu_2)]\ r(y)r(y+v_3)r(y+v_4)$$

$$\to \frac{1}{2}\frac{1}{2\pi} \int_{-M}^{M} w^2(v_1,v_2)dv_1dv_2\ f(\mu_1)f(\mu_2)f(\mu_1+\mu_2)$$

$$(1 + \delta(2\mu_1)\delta(2\mu_2)).$$

(ii) <u>Calculation of σ^2_{MI}</u>

<u>Minor term 1</u>: In place of (8.25) get

$$\frac{f(0)}{(2\pi)^3} \sum\sum\sum\sum_{-MA_N}^{MA_N} [\sin \nu_1\mu_1 \cos \nu_2\mu_2 + \cos \nu_1\mu_1 \sin \nu_2\mu_2]$$

$$[\sin \nu_3(\mu_1+\mu_2)\cos \nu_4\mu_2 + \cos \nu_3(\mu_1+\mu_2)\sin \nu_4\mu_2]$$

$$w(0,B_N\nu_2)\ w(-B_N\nu_3,0)\, r(\nu_1) r(\nu_4)$$

$$\rightarrow \frac{f(0)}{(2\pi)^3} [\int_{-M}^{M} w(0,\nu)d\nu\ ]^2 \sum\sum_{-\infty}^{\infty} \sin \nu_1\mu_1 \sin \nu_4\mu_2$$

$$r(\nu_1)r(\nu_4)\ \delta(\mu_1+\mu_2)\ \delta(\mu_2)$$

$$= 0$$

since $f(\lambda)$ is real.

<u>Minor term 8</u>: In place of (8.26) the imaginary parts give

$$\frac{B_N^2}{(2\pi)^4} \sum_{-N}^{N} r(y) \sum\sum\sum\sum_{-MA_N}^{MA_N} \tfrac{1}{2}\{ \cos(\nu_3\mu_1 + \nu_4\mu_2)$$

$$- \cos(2\nu_1\mu_1+2\nu_2\mu_2)\ \cos(\nu_3\mu_1+\nu_4\mu_2) + \sin(2\nu_1\mu_1+2\nu_2\mu_2) \cdot$$

$$\sin(\nu_3\mu_1 + \nu_4\mu_2) \}$$

$$\rightarrow \frac{1}{(2\pi)^4}[\int_{-M}^{M} w^2(\nu_1,\nu_2)\ d\nu_1 d\nu_2] \sum\sum\sum_{-\infty}^{\infty} r(y)r(y+\nu_3)r(y+\nu_4)$$

$$\cos(\nu_3\mu_1 + \nu_4\mu_2)\ [1 - \delta(2\mu_1)\delta(2\mu_2)]$$

$$\rightarrow \frac{1}{2\pi}[\int_{-M}^{M} w^2(\nu_1,\nu_2)d\nu_1 d\nu_2] f(\mu_1)f(\mu_2)f(\mu_1+\mu_2)[1-\delta(2\mu_1)\delta(2\mu_2)].$$

(iii) Calculation of r_M

Minor term 1: Equation (8.25) now becomes

$$\frac{f(0)}{(2\pi)^3} \Sigma\,\Sigma \sum_{-MA_N}^{MA_N} \Sigma \,[\cos v_1\mu_1 \cos v_2\mu_2 - \sin v_1\mu_1 \sin v_2\mu_2]$$

$$[\sin v_3(\mu_1+\mu_2)\cos v_4\mu_2 + \cos v_3(\mu_1+\mu_2)\sin v_4\mu_2]$$

$$w(0,B_N v_2)\ w(-B_N v_3,0)\ r(v_1)r(v_4)$$

$$\rightarrow \frac{f(0)}{(2\pi)^3} [\int_0^M w(0,v)dv]^2 \sum_{-\infty}^{\infty}\Sigma \cos v_1\mu_1 \sin v_2\mu_2$$

$$r(v_1)r(v_2)\ \delta(\mu_2)\delta(\mu_1+\mu_2)$$

$$= 0.$$

Minor term 8: Here (8.26) becomes

$$\frac{B_N^2}{(2\pi)^4} \sum_{-N}^{N} r(y)\,\Sigma\,\Sigma \sum_{-MA_N}^{MA_N} \Sigma\ \tfrac{1}{2}\{\sin(v_3\mu_1 + v_4\mu_2)$$

$$+ \cos(2v_1\mu_1+2v_2\mu_2)\ \sin(v_3\mu_1+ v_4\mu_2) + \sin(2v_1\mu_1 + 2v_2\mu_2)$$

$$\sin(v_3\mu_1 + v_4\mu_2)\}\ w(B_N v_1,B_N v_2)\ w(B_N v_1 + B_N v_3, B_N v_2 + B_N v_4)$$

$$r(y+v_3)\ r(y+v_4)$$

$$\rightarrow 0 .$$

Q.E.D.

<u>Appendix</u>: <u>The asymptotic variance of</u> $g_N(\lambda_1,\lambda_2)$.

<u>Theorem</u>: (a) $\{\xi_t\}$ is a pure white noise process (discrete parameter) with moments $E\xi_t = 0$, $E\xi_t^2 = \alpha$, $E\xi_t^3 = \beta$, $E\xi_t^4 = \gamma$ and $E\xi_t^6 = \theta$ all finite,

(b) $X_t = \sum_{-\infty}^{\infty} a_j \xi_{t-j}$, $t = 0, \pm 1, \pm 2, \ldots$, (linear process) where $\{a_j\}$ is a sequence of constants with $a_j > 0$ and $\{a_j\} \in \ell_1$

$$\Longrightarrow \quad \sigma^2(g_N(0,0)) \sim N.$$

<u>Proof</u>:

$$E[g_N(\lambda_1,\lambda_2)\overline{g_N(\lambda_1,\lambda_2)}] - |Eg_N(\lambda_1,\lambda_2)|^2$$

$$= \frac{1}{(2\pi)^4} \sum\sum\sum\sum_{-N}^{N} e^{-iv_1\lambda_1 - iv_2\lambda_2 + iv_3\lambda_1 + iv_4\lambda_2}$$

$$\sum_{y=-N}^{N} \frac{C_N}{N^2}[\xi_6(v_1,v_2,y,y+v_3,y+v_4)$$

$$+ \{m_3(0,v_1,v_2)m_3(y,y+v_3,y+v_4)\}_{10-1}$$

$$+ \{m_2(0,v_1)\varepsilon_4(v_2,y,y+v_3,y+v_4)\}_{15}$$

$$+ \{m_2(0,v_1)m_2(v_2,y)m_2(y+v_3,y+v_4)\}_{15}]. \qquad \text{(A.1)}$$

Now it is not difficult to see that[1]

$$\xi_2(v) = \alpha \sum_{j=-\infty}^{\infty} a_j a_{j+v} ,$$

$$\xi_3(v_1,v_2) = \beta \sum_{j=-\infty}^{\infty} a_j a_{j+v_1} a_{j+v_2} , \qquad \text{(A.2)}$$

---

[1] See a partial verification in Bartlett [1] p. 147.

$$\xi_4(v_1,v_2,v_3) = \hat{\xi}_4 \sum_{j=-\infty}^{\infty} a_j a_{j+v_1} a_{j+v_2} a_{j+v_3},$$

$$\xi_6(v_1,\ldots,v_5) = \hat{\xi}_6 \sum_{j=-\infty}^{\infty} a_j a_{j+v_1} \cdots a_{j+v_5}, \qquad \text{(A.2) Cont'd.}$$

where $\hat{\xi}_4$ and $\hat{\xi}_6$ are respectively the fourth and sixth cumulants of ξ_0.

The ξ_6 term of (A.1) is then

$$\leq \frac{|\hat{\xi}_6|}{N} \sum_{v_1,\ldots,v_4=-N}^{N} \sum\sum\sum \sum_{y=-N}^{N} \sum_{j=-\infty}^{\infty} |a_j a_{j+v_1} a_{j+v_2} a_{j+y} a_{j+y+v_3} a_{j+y+v_4}|$$

$$= O(\tfrac{1}{N})$$

since $\{a_j\} \in \ell_1$. The second major term is

$$\leq \frac{1}{N} \sum\sum\sum\sum_{-N}^{N}\sum |\{m_3(0,v_1,v_2) m_3(y,y+v_3,y+v_4)\}_{10-1}|$$

$$= O(1)$$

looking at Table II and using (A.2) and hypothesis (b). Similarly the fourth major term is O(1). Hence

$$\sigma^2(g_N(0,0)) = \frac{1}{(2\pi)^4} \sum\sum\sum\sum_{-N}^{N}\sum \frac{C_N}{N^2}$$

$$\{m_2(0,v_1) m_2(v_2,y) m_2(y+v_3,y+v_4)\} + C(1). \qquad \text{(A.3)}$$

Due to the fact that the a_j are positive (A.3) must be $\sim N$.

Q.E.D.

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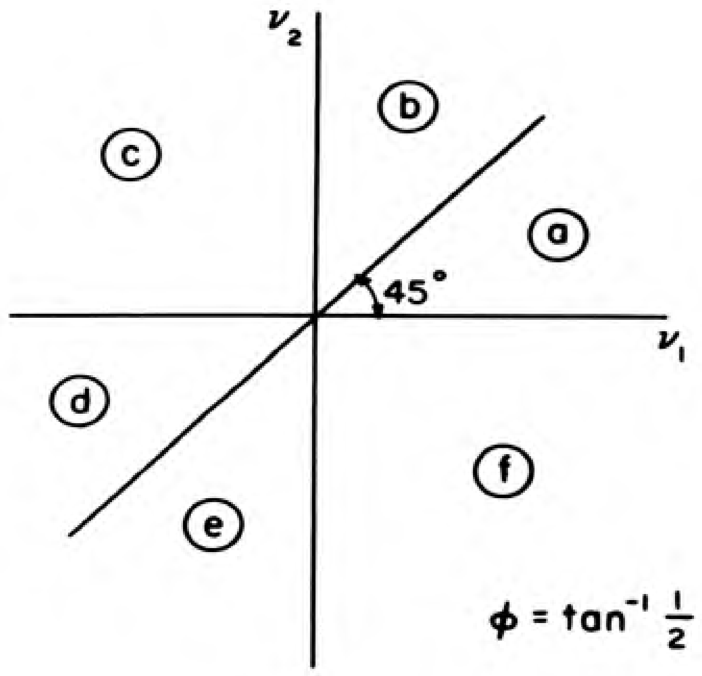

ν_2
ν_1
a
b
c
d
e
f
45°
$\phi = \tan^{-1} \frac{1}{2}$

FIG. I

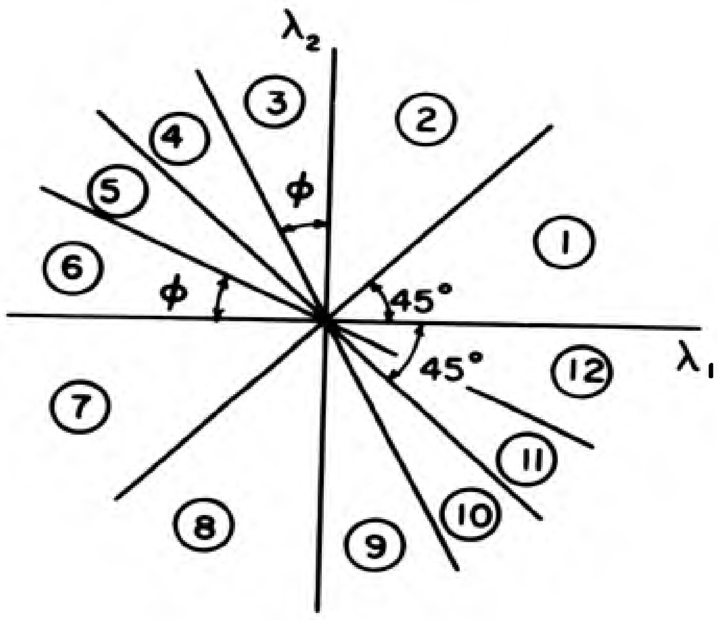
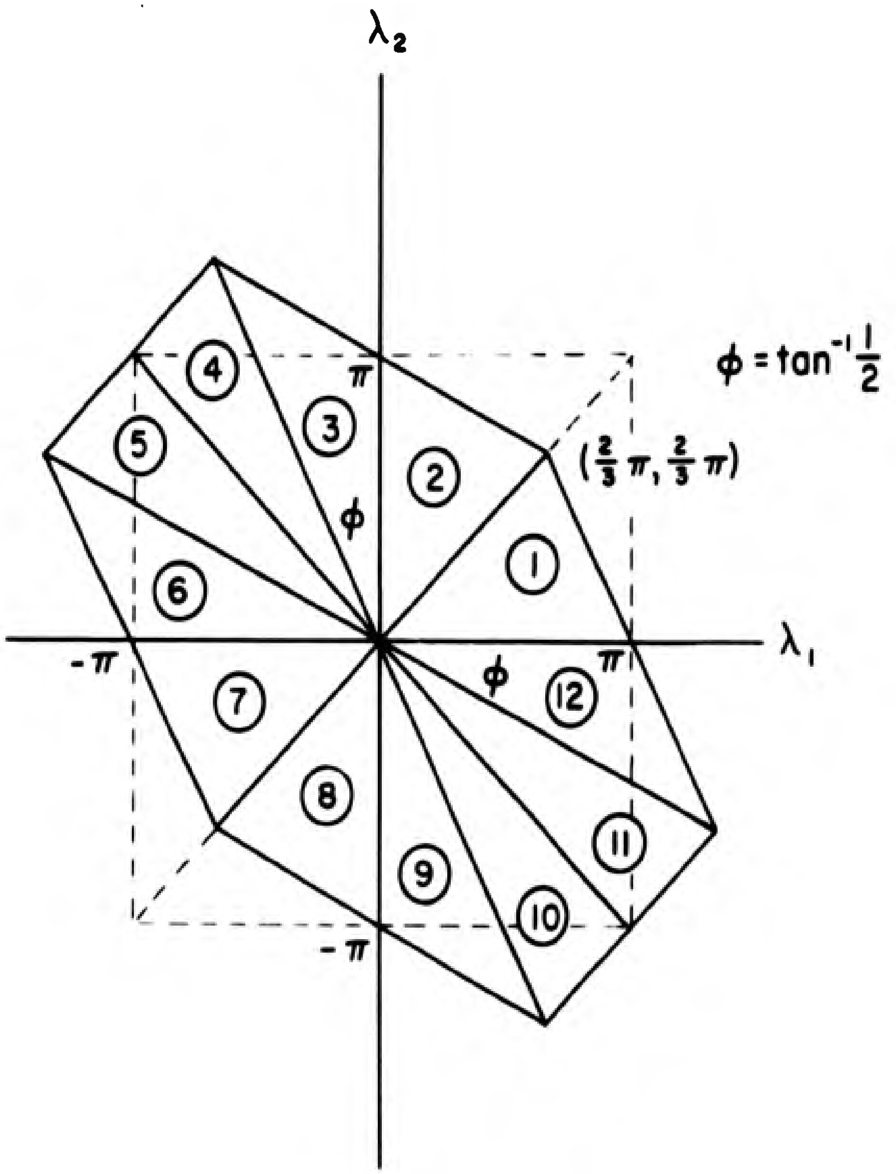

λ_2
λ_1
1
2
3
4
5
6
7
8
9
10
11
12
ϕ
ϕ
45°
45°

FIG. II



λ_2
λ_1
$\phi = \tan^{-1}\frac{1}{2}$
$(\frac{2}{3}\pi, \frac{2}{3}\pi)$
π
$-\pi$
π
$-\pi$
ϕ
ϕ
1
2
3
4
5
6
7
8
9
10
11
12

FIG. III

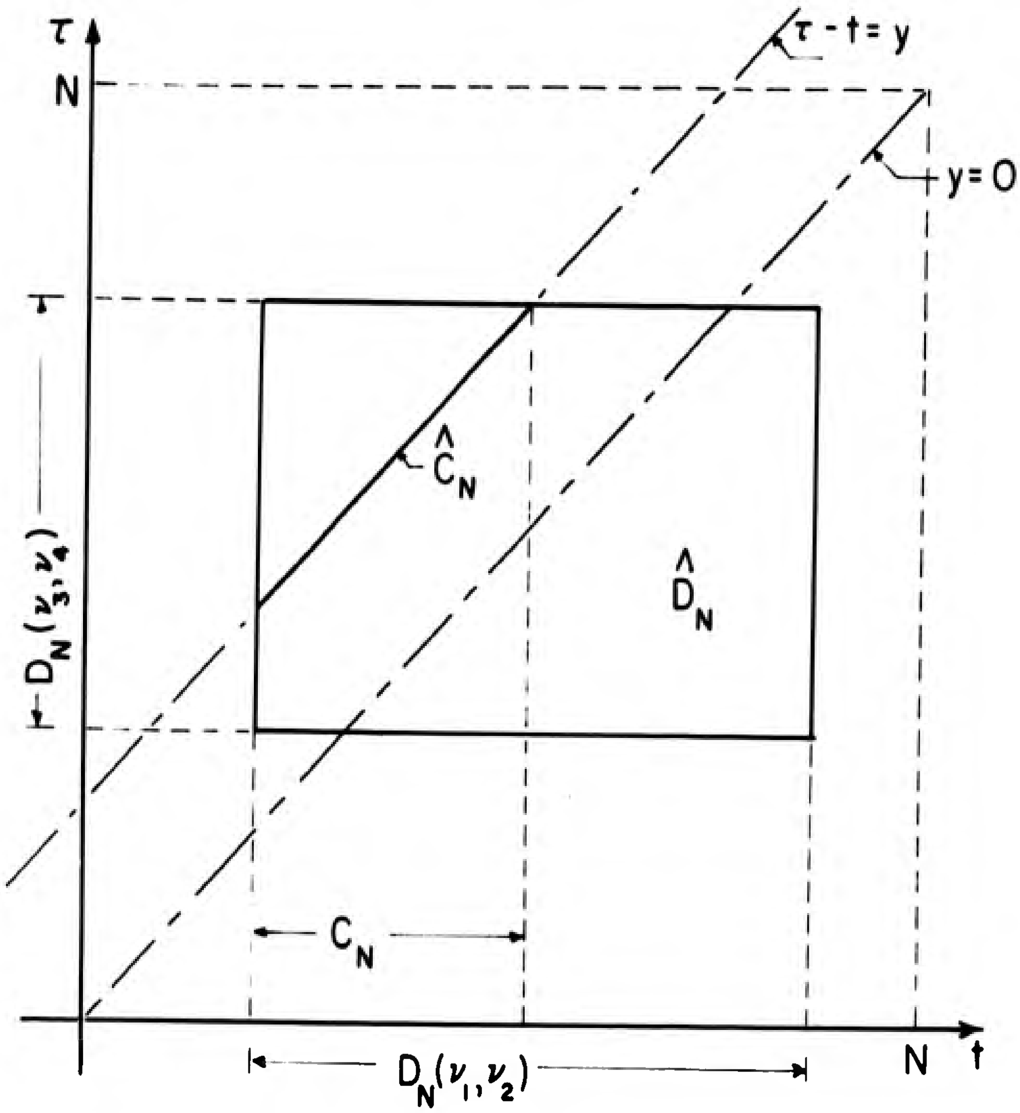


FIG. IV

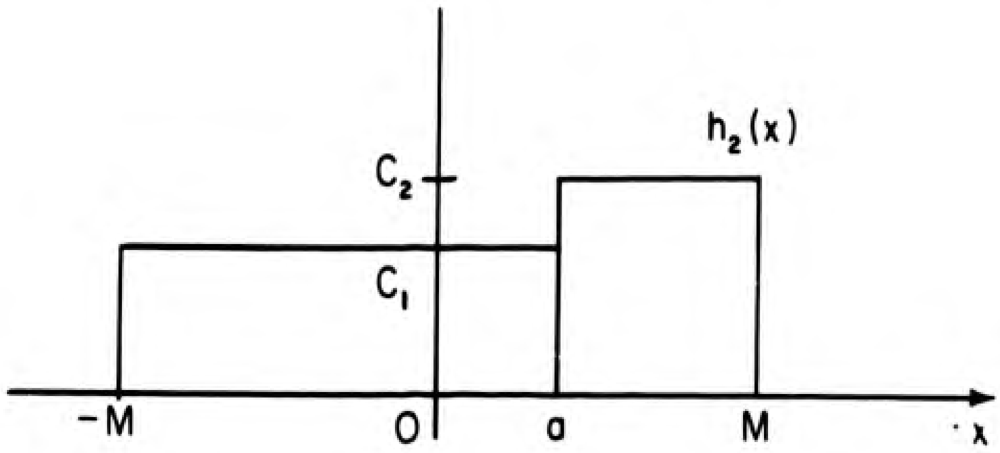


FIG. V