

UNCLASSIFIED

AD NUMBER

AD437762

LIMITATION CHANGES

TO:

Approved for public release; distribution is unlimited.

FROM:

Distribution authorized to U.S. Gov't. agencies and their contractors;  
Administrative/Operational Use; MAR 1964. Other requests shall be referred to Defense Advanced Research Projects Agency, ASBD-TIO, 675 North Randolph Street, Arlington, VA 22203-2114.

AUTHORITY

rand ltr, via darpa dtd 31 mar 1966

THIS PAGE IS UNCLASSIFIED

**UNCLASSIFIED**

**AD 4 3 7 7 6 2**

**DEFENSE DOCUMENTATION CENTER**

**FOR**

**SCIENTIFIC AND TECHNICAL INFORMATION**

**CAMERON STATION, ALEXANDRIA, VIRGINIA**



**UNCLASSIFIED**

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

437762

ARPA ORDER NO. 189-61

MEMORANDUM  
RM-3960-ARPA  
MARCH 1964

4 3 7 7 6 2

CATALOGED BY DDC  
AS-AD No. \_\_\_\_\_

INVARIANT IMBEDDING AND SCATTERING  
OF LIGHT IN A ONE-DIMENSIONAL MEDIUM  
WITH A MOVING BOUNDARY

Richard Bellman, Robert Kalaba and Sueo Ueno

DDC  
MAY 6 1964  
RECEIVED  
TISIA B

PREPARED FOR:  
ADVANCED RESEARCH PROJECTS AGENCY

The RAND Corporation  
SANTA MONICA • CALIFORNIA

NO. OTS

MEMORANDUM  
RM-3960-ARPA  
MARCH 1964

INVARIANT IMBEDDING AND SCATTERING  
OF LIGHT IN A ONE-DIMENSIONAL MEDIUM  
WITH A MOVING BOUNDARY

Richard Bellman, Robert Kalaba and Sueo Ueno

This research is supported by the Advanced Research Projects Agency under Contract No. SD-79. Any views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of ARPA.

DDC AVAILABILITY NOTICE

Qualified requesters may obtain copies of this report from the Defense Documentation Center (DDC).

---

*The* **RAND** *Corporation*

1700 MAIN ST • SANTA MONICA • CALIFORNIA • 90406

PREFACE

Part of the RAND research program consists of basic supporting studies in mathematics. The present paper continues the work of the authors of invariant-embedding techniques for solving problems in radiative transport theory.

## SUMMARY

In the present paper it is shown how the invariant-embedding technique is used for the derivation of integral equations governing the reflection and the transmission coefficients of radiation in a one-dimensional medium with a moving boundary. In Part A, when formulating the source function, account is taken by the relaxation of photon emission under the assumption that during an elementary act of scattering a photon remains in a state of absorption for a certain duration of temporal capture. In Part B, the Schuster problem for a moving one-dimensional medium is exactly treated. In other words, when formulating the transfer equations, the influence of the Doppler effect caused by the macroscopic differential motions is considered as a principal agency of the change of frequency on scattering.

CONTENTS

PREFACE . . . . .	iii
SUMMARY . . . . .	v
Section	
1. INTRODUCTION. . . . .	1
2. THE EQUATION OF TRANSFER . . . . .	3
3. THE REFLECTED INTENSITY . . . . .	5
4. THE TRANSMITTED INTENSITY . . . . .	8
5. THE EQUATION OF TRANSFER . . . . .	11
6. THE REFLECTION COEFFICIENT . . . . .	14
7. THE TRANSMISSION COEFFICIENT . . . . .	16
REFERENCES . . . . .	19

## INVARIANT IMBEDDING AND SCATTERING OF LIGHT IN A ONE-DIMENSIONAL MEDIUM WITH A MOVING BOUNDARY

### 1. INTRODUCTION

In recent years, for analytical and numerical advantages, the principle-of-invariance method originally due to Ambarzumian [1] and Chandrasekhar [6] has been developed by several authors (Bellman, Kalaba, and Wing [2] and [3], Wing [11]). The invariant-imbedding technique thus proposed was successfully applied not only to various kinds of stationary transport problems for photons and neutrons, but also to time-dependent transfer problems. Many previous attempts at solving nonsteady transfer problems will be found in the list of references of the preceding papers (see also Bellman, Kalaba, and Ueno [4] and [5], Kaplan [8], and Wing [10]).

Recently, in the study of astrophysics, emphasis has been put on the elucidation of the photon diffusion process in a differentially moving atmosphere, whereas the exact treatment seems very difficult on account of the untractability of the transfer equation due to the Doppler shift in frequency. Such problems are encountered in the interpretation of spectral lines in novae, planetary nebulae, and the solar prominences and corona. This state of affairs is also found in the analysis of several physical experiments.

By reducing the transfer problem to a novel boundary-value problem for hyperbolic differential equations, Chandrasekhar [7] has given approximately the solutions illustrating the effects that may be expected in the rectangular contour of the scattering coefficient. Later, combining the Doppler effect due to the thermal motions of atoms and the macroscopic differential motions in finite one- and three-dimensional media and assuming an arbitrary contour of the scattering coefficient, Sobolev [9] has considered the approximate effect of the complete redistribution of radiation in frequency on the radiation pressure due to the Lyman- $\alpha$ .

In a preceding paper, the invariant-embedding technique was applied to the study of time-dependent scattering of light in a one-dimensional medium of a finite optical thickness. In this paper, extending the imbedding procedure to a one-dimensional moving medium, we shall consider a couple of nonsteady scattering processes as follows: In Part A the left-hand boundary of the medium moves as a prescribed function of time, and the source function is formulated under consideration of the relaxation of the photon emission. In Part B, we treat the Shuster problem for a differentially moving atmosphere such that, assuming that the only optical effect of large-scale motions is a change of frequency on scattering, the interaction of radiation fields with different frequencies is taken into account. In a later paper, extending this

procedure and allowing for the noncoherent scattering, we shall consider the formation of spectral lines in a differentially moving atmosphere in rod and slab geometries.

A. RELAXATION OF PHOTON EMISSION

2. THE EQUATION OF TRANSFER

Consider a one-dimensional medium of finite geometrical thickness  $z_1 - z_0 > 0$  with a moving boundary whose optical properties vary with its position and with time. Let a pencil of radiation of Dirac delta-function time-dependent intensity  $I_0(t) = F \delta(t - s)$  ( $t > s \geq 0$ ) be incident on the fixed boundary  $z_1$ , where  $F$  is a constant and  $\delta$  is the Dirac delta-function (see Fig. 1). At time  $t$ , the position of the left-hand boundary is  $z_0 = z_0(t)$ , whereas at  $t = 0$  it is equal to zero; further,  $|dz_0/dt|$  is less than the speed of light in the medium.

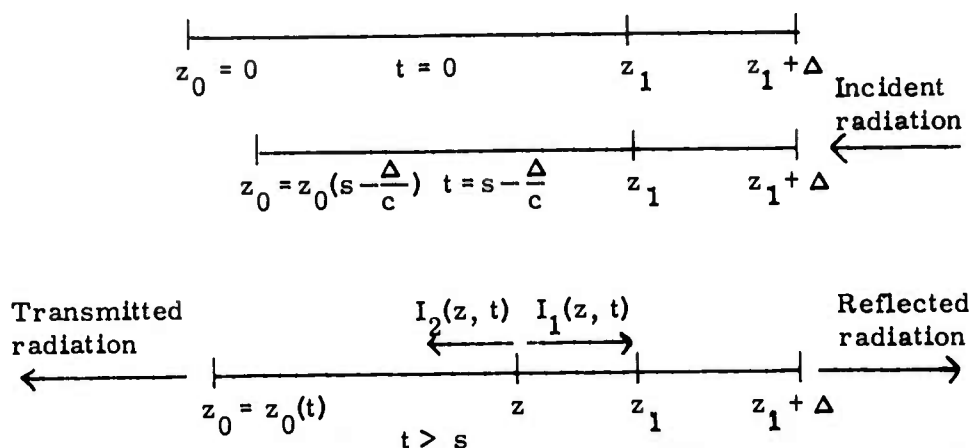


Fig. 1 The reflection and the transmission of radiation in a one-dimensional medium with a moving boundary

For the sake of simplicity, the scattering of light in either direction is assumed to be equally probable.

Let  $I_1(z, t)$  denote the specific intensity of radiation at depth  $z$ , at time  $t$ , directed toward the boundary  $z_1$ , and similarly let  $I_2(z, t)$  denote that directed toward the boundary  $z_0(t)$ .

The equation of transfer appropriate to the present case is written in the form

$$(2.1) \quad \frac{\partial I_1}{\partial z} + \frac{1}{c} \frac{\partial I_1}{\partial t} = -\ell(z, t) I_1(z, t) + B(z, t),$$

$$(2.2) \quad -\frac{\partial I_2}{\partial z} + \frac{1}{c} \frac{\partial I_2}{\partial t} = -\ell(z, t) I_2(z, t) + B(z, t),$$

where  $c$  is the speed of light,  $\ell$  is the extinction coefficient, and  $B(z, t)$  is the source function.

In a manner similar to that used in a preceding paper [5], the source function takes the form

$$(2.3) \quad B(z, t) = \frac{1}{2} \sigma(z, t) \int_s^t \{I_1(z, t') + I_2(z, t')\} e^{-\frac{t-t'}{t_1}} \frac{dt'}{t_1}$$

where  $\sigma$  is the scattering coefficient and  $t_1$  is the duration of temporal capture. Equation (2.3) represents the relaxation of emission of absorbed energy.

Equations (2.1) and (2.2) should be solved subject to the boundary and initial conditions

$$(2.4) \quad I_1(z_0(t), t) = 0, \quad I_2(z_1, t) = F\delta(t - s),$$

$$(2.5) \quad I_1(z, s) = 0, \quad I_2(z, s) = 0 \quad (z_0 \leq z < z_1, 0 \leq s < t),$$

where  $I_1(z_1, t)$  and  $I_2(z_0, t)$  are respectively the reflected and the transmitted intensities.

### 3. THE REFLECTED INTENSITY

Let the coefficient of reflection be denoted by  $R(z_1; t, s)$ .

Then the reflected intensity is expressed in terms of the reflection coefficient  $R$  as follows:

$$(3.1) \quad I_1(z_1, t) = \int_s^t R(z_1; t, t') I_2(z_1, t') dt',$$

where  $I_2(z_1, t)$  is given by (2.4).

We shall inquire into an integral equation for  $R(z_1; t, s)$  with the aid of the invariant-embedding technique.

When we add a layer of infinitesimal thickness  $\Delta$  to the boundary  $z_1$ , the imbedding procedure of the scattering in the added layer yields

$$(3.2) \quad I_1(z_1 + \Delta, t + \frac{\Delta}{c}) = I_1^*(z_1, t) + \{-l(z_1, t) I_1(z_1, t) + B(z_1, t)\} \Delta + \mathcal{O}(\Delta),$$

where  $\mathcal{O}(\Delta)$  is of the order of magnitude of the infinitesimal  $\Delta^2$ .

Equation (3.2) shows that the reflected intensity at the boundary  $z_1 + \Delta$  at time  $t + \frac{\Delta}{c}$ , due to the incident intensity  $I_2(z_1 + \Delta, t')$ ,

is produced by the intensity at the boundary  $z_1$  at time  $t$ , due to a modified incident radiation source  $\hat{I}_2(z_1, t)$ , in addition to the intensity produced by the interaction of the radiation field in  $(z_1, z_1 + \Delta)$ .

The term  $I_1$  on the left-hand side of (3.2) is given by

$$(3.3) \quad I_1(z_1 + \Delta, t + \frac{\Delta}{c}) = \int_{s - \frac{\Delta}{c}}^{t + \frac{\Delta}{c}} R(z_1 + \Delta; t + \frac{\Delta}{c}, x) I_2(z_1 + \Delta, x) dx,$$

where  $I_2(z_1 + \Delta, t) = F \delta(t - s + \frac{\Delta}{c})$ .

On the other hand, recalling (2.2), we obtain the modified incident source given by

$$(3.4) \quad \hat{I}_2(z_1, t) = I_2(z_1 + \Delta, t - \frac{\Delta}{c}) - l(z_1, t) I_2(z_1, t) l + \frac{1}{2} \sigma(z_1, t) \Delta \int_s^t \{I_1(z_1, x) + I_2(z_1, x)\} e^{-\frac{(t-x)}{t_1}} \frac{dx}{t_1} + O(\Delta).$$

Thus, the intensity of radiation due to the modified source  $\hat{I}_2$  is

$$(3.5) \quad I_1^*(z_1, t) = \int_s^t R(z_1; t, x) \hat{I}_2(z_1, x) dx \\ = \int_s^t R(z_1; t, x) I_2(z_1 + \Delta, x - \frac{\Delta}{c}) dx - \Delta \int_s^t R(z_1; t, x) l(z_1, x) I_2(z_1, x) dx \\ + \frac{1}{2} \Delta \int_s^t R(z_1; t, x) \sigma(z_1, x) dx \int_s^x I_1(z_1, y) e^{-\frac{(x-y)}{t_1}} \frac{dy}{t_1}$$

$$+ \frac{1}{2} \Delta \int_s^t R(z_1; t, x) \sigma(z_1, x) dx \int_s^x I_2(z_1, y) e^{-\frac{(x-y)}{t_1}} \frac{dy}{t_2} + \mathcal{O}(\Delta).$$

On substituting (3.3) and (3.5) into (3.2) and putting  $I_2(z_1, t) = I_0(t)$ , we get

$$(3.6) \quad R(z_1 + \Delta; t + \frac{\Delta}{c}, s - \frac{\Delta}{c}) = R(z_1; t, s) - l(z_1, s) R(z_1; t, s) \Delta \\ + \frac{1}{2} \Delta \int_s^t R(z_1; t, x) \sigma(z_1, x) dx \int_s^x R(z_1; y, x) e^{-\frac{(x-y)}{t_1}} \frac{dy}{t_1} \\ + \frac{1}{2} \Delta \int_s^t R(z_1; t, x) \sigma(z_1, x) e^{-\frac{(x-s)}{t_1}} \frac{dx}{t_1} \\ + \Delta \left[ -l(z_1, t) R(z_1; t, s) + \frac{1}{2} \sigma(z_1, t) \int_s^t R(z_1; x, s) e^{-\frac{(t-x)}{t_1}} \frac{dx}{t_1} \right. \\ \left. + \frac{1}{2} \sigma(z_1, t) e^{-\frac{(t-s)}{t_1}} \right] + \mathcal{O}(\Delta).$$

Then, in the limit as  $\Delta \rightarrow 0$ , (3.6) becomes

$$(3.7) \quad \frac{\partial R}{\partial z_1} + \frac{1}{c} \frac{\partial R}{\partial t} - \frac{1}{c} \frac{\partial R}{\partial s} + (l(z_1, t) + l(z_1, s)) R \\ = \frac{1}{2} \sigma(z_1, t) e^{-\frac{(t-s)}{t_1}} + \frac{1}{2} \sigma(z_1, t) \int_s^t R(z_1; x, s) e^{-\frac{(t-x)}{t_1}} \frac{dx}{t_1} \\ + \frac{1}{2} \int_s^t R(z_1; t, x) \sigma(z_1, x) e^{-\frac{(x-s)}{t_1}} \frac{dx}{t_1} \\ + \frac{1}{2} \int_s^t R(z_1; t, x) \sigma(z_1, x) dx \int_s^x R(z_1; y, s) e^{-\frac{(x-y)}{t_1}} \frac{dy}{t_1}.$$

Equation (3.7) is the requisite integral equation for the reflection coefficient. The boundary and initial conditions imposed on  $R$  are given by

$$(3.8) \quad R(z_1; t, s) = 0 \quad \text{for} \quad z_0(s) = z_1,$$

$$(3.9) \quad R(z_1; t, s) = 0 \quad \text{for} \quad s \geq t \geq 0.$$

When the optical properties of the medium are steady and homogeneous with respect to time and depth, and we put

$$(3.10) \quad \tau = lz, \quad t_2 = \frac{1}{cl}, \quad \tilde{\omega} = \frac{\sigma}{l},$$

where  $\tau$  is the optical depth,  $t_2$  is the mean free time, and  $\tilde{\omega}$  is the albedo for single scattering, (3.7) reduces to

$$(3.11) \quad \frac{\partial R}{\partial \tau_1}(\tau_1; t-s) + t_2 \frac{\partial R}{\partial t} - t_2 \frac{\partial R}{\partial s} + 2R \\ = \frac{\tilde{\omega}}{2} \left[ \frac{e^{-(t-s)/t_1}}{t_1} + 2 \int_s^t R(\tau_1; t-x) e^{-(x-s)/t_1} \frac{dx}{t_1} \right. \\ \left. + \int_s^t R(\tau_1; t-x) dx \int_s^x R(\tau_1; y-s) e^{-(x-y)/t_1} \frac{dy}{t_1} \right].$$

#### 4. THE TRANSMITTED INTENSITY

Consider the case of illumination of the fixed boundary  $z_1$ . Let the coefficient of transmission be denoted by  $T(z_1; z_0(t), t; z_0(s), s)$ . Then the transmitted intensity at the boundary  $z_0$  at time  $t$  is expressed in the form

$$(4.1) \quad I_2(z_0(t), t) = \int_s^{t^*} T(z_1; z_0(t), t; z_0(x), x) I_2(z_1, x) dx,$$

where  $I_2(z_1, t)$  is given by (2.4) and  $t^* = t - (z_1 - z_0(t))/c$ .

For convenience, we shall write

$$(4.2) \quad T(z_1; z_0(t), t; z_0(s), s) = T(z_1; t, s).$$

In what follows, we shall seek an integral equation for the T-function. On adding a layer of infinitesimal thickness  $\Delta$  to the boundary  $z_1$ , we get (3.4). On applying the imbedding argument to the transmitted intensity, we obtain

$$(4.3) \quad I_2^0(z_0(t), t) = I_2^*(z_0(t), t).$$

On the other hand, we have

$$(4.4) \quad I_2^0(z_0(t), t) = \int_{s - \frac{\Delta}{c}}^{t^* - \frac{\Delta}{c}} T(z_1 + \Delta; t, x) I_2(z_1 + \Delta, x) dx,$$

and

$$(4.5) \quad I_2^*(z_0(t), t) = \int_s^{t^*} T(z_1; t, x) \hat{I}_2(z_1, x) dx.$$

Substitution of (3.4) into (4.5) yields

$$(4.6) \quad I_2^*(z_0(t), t)/F = T(z_1; t, s) - l(z_1, s) T(z_1; t, s)\Delta \\ + \frac{1}{2} \Delta \int_s^{t^*} T(z_1; t, x) \sigma(z_1, x) e^{-\frac{(x-s)}{t_1}} \frac{dx}{t_1} \\ + \frac{1}{2} \Delta \int_s^{t^*} T(z_1; t, x) \sigma(z_1, x) dx \int_s^x R(z_1; y, s) e^{-\frac{(x-y)}{t_1}} \frac{dy}{t_1} + O(\Delta).$$

On combining (2.4) and (4.3)-(4.5), we get

$$\begin{aligned}
 (4.7) \quad T(z_1 + \Delta; t, s - \frac{\Delta}{c}) &= T(z_1; t, s) - l(z_1, s) T(z_1; t, s) \Delta \\
 &+ \frac{1}{2} \Delta \int_s^{t^*} T(z_1; t, s) \sigma(z_1, x) e^{-\frac{(x-s)}{t_1}} \frac{dx}{t_1} \\
 &+ \frac{1}{2} \Delta \int_s^{t^*} T(z_1; t, x) \sigma(z_1, x) dx \int_s^x R(z_1; y, s) e^{-\frac{(x-y)}{t_1}} \frac{dy}{t_1} \\
 &+ \mathcal{O}(\Delta).
 \end{aligned}$$

Letting  $\Delta \rightarrow 0$  in (4.6) gives

$$\begin{aligned}
 (4.8) \quad \frac{\partial T}{\partial z_1} - \frac{1}{c} \frac{\partial T}{\partial s} + l(z_1, s) T \\
 = \frac{1}{2} \int_s^{t^*} T(z_1; t, x) \sigma(z_1, x) e^{-\frac{(x-s)}{t_1}} \frac{dx}{t_1} \\
 + \frac{1}{2} \int_s^{t^*} T(z_1; t, x) \sigma(z_1, x) dx \int_s^x R(z_1; y, s) e^{-\frac{(x-y)}{t_1}} \frac{dy}{t_1}.
 \end{aligned}$$

Equation (4.8) is the required integral equation governing the transmission coefficient, where  $T(z_1; t, s) = 0$  for

$$t - s \leq \frac{z_1 - z_0(t)}{c}.$$

Let the reduced and the diffuse components of the transmission coefficient be denoted by  $T^0(z_1; t, s)$  and  $T^*(z_1; t, s)$ , respectively, where

$$(4.9) \quad T^0(z_1; t, s) = \exp \left[ - \int_{z_0(t)}^z l(z, s + \frac{z_1 - z}{c}) dz \right] \delta \left( t - \frac{z_1 - z_0(t)}{c} - s \right).$$

Then the insertion of  $T = T^0 + T^*$  into (4.8) provides us with an integral equation for the  $T^*$ -function.

In the steady and homogeneous case with respect to time and depth, by (3.10), (4.8) becomes

$$(4.10) \quad \frac{\partial T}{\partial T_1}(\tau_1; t-s) - t_2 \frac{\partial T}{\partial s} + T \\ = \frac{\omega}{2} \left[ \int_s^{t^*} T(\tau_1; t-x) e^{-\frac{(x-s)}{t_1}} \frac{dx}{t_1} \right. \\ \left. + \int_s^{t^*} T(\tau_1; t-x) dx \int_s^x R(\tau_1; y-s) e^{-\frac{(x-y)}{t_1}} \frac{dy}{t_1} \right],$$

where  $t^* = t - (\tau_1 - \tau_0)t_2$ .

## B. CHANGE OF FREQUENCY ON SCATTERING DUE TO DOPPLER EFFECT

### 5. THE EQUATION OF TRANSFER

Consider a one-dimensional, differentially expanding medium of finite geometrical thickness  $z_1 - z_0 > 0$  whose velocity  $v(z) > 0$  at  $z$  is a linear function of the depth,  $z_0 \leq z \leq z_1$ . Let  $\sigma(v, z)$  denote the monochromatic volume scattering coefficient at  $z$  for the frequency  $\nu$  as judged by a stationary observer. The observer may be considered to be at rest with respect to the position at  $z = 0$ . For simplicity, the quantity  $\sigma$  is assumed to be  $\rho(z) \sigma^*(\nu)$ , where  $\rho(z)$  is the mass density and  $\sigma^*(\nu)$  is the mass scattering coefficient. Furthermore, we assume the characteristic feature of  $\sigma^*$ , that the only optical effect of the large-scale motions  $v(z)$  is a change of frequency on scattering.

Let  $I_1(\nu, z)$  denote the specific intensity of radiation at  $z$  directed toward the boundary  $z_1$  and of frequency  $\nu$  as judged by the stationary observer, and further let  $I_2(\nu, z)$  denote the specific intensity directed toward the boundary  $z_0$  (see Fig. 2).

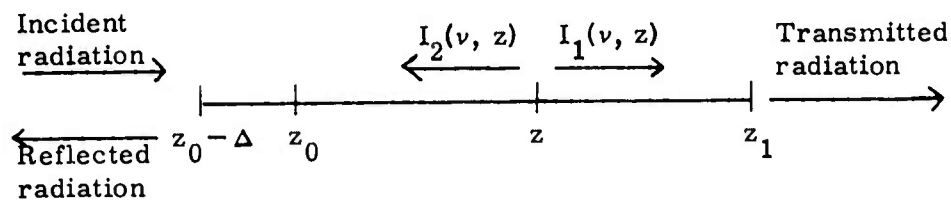


Fig. 2. The reflected and transmitted intensities of radiation by a differentially moving one-dimensional medium

Consider a pencil of radiation of constant intensity  $F$  in all frequencies to be incident on  $z_0$ . When we take into account the Doppler shift in frequency,  $-\frac{\nu}{c} \nu(z) (1 \pm 1)$ , caused by the large-scale motions, the equations of transfer appropriate to the present case take the form (cf. Chandrasekhar [7])

$$(5.1) \quad \frac{\partial I_1(\nu, z)}{\partial z} = -\frac{1}{2} \sigma(\nu(1 - \frac{\nu}{c}), z) [I_1(\nu, z) - I_2(\nu(1 - 2\frac{\nu}{c}), z)],$$

$$(5.2) \quad -\frac{\partial I_2(\nu, z)}{\partial z} = -\frac{1}{2} \sigma(\nu(1 + \frac{\nu}{c}), z) [I_2(\nu, z) - I_1(\nu(1 + 2\frac{\nu}{c}), z)],$$

where  $c$  is the speed of light.

In what follows, in a manner similar to that used by Chandrasekhar [7], we shall make some simplifications of the transfer equations. In the evaluation of the Doppler shift in frequency,  $\nu(1 - \frac{\nu}{c})$  is reasonably approximated by  $\nu - \nu_0 \frac{\nu}{c}$ , where  $\nu_0$  is the

frequency of the line center.

Furthermore, in place of considering the equations of transfer for some fixed frequency  $\nu$ , we shall somewhat modify them in such a way that  $\nu$  in (5.1) and  $\nu$  in (5.2) are respectively replaced by  $\nu_1$  and  $\nu_2$ , where

$$(5.3) \quad \nu_1 = \nu + \nu_0 \frac{v}{c}, \quad \nu_2 = \nu - \nu_0 \frac{v}{c}.$$

Allowing for (5.3), (5.1) and (5.2) are rewritten as follows:

$$(5.4) \quad \frac{\partial I_2}{\partial z}(\nu_1, z) = -\frac{1}{2} \sigma(\nu, z) [I_1(\nu_1, z) - I_2(\nu_2, z)],$$

$$(5.5) \quad -\frac{\partial I_2}{\partial z}(\nu_2, z) = -\frac{1}{2} \sigma(\nu, z) [I_2(\nu_2, z) - I_1(\nu_1, z)].$$

Writing

$$(5.6) \quad I_i(\nu_i, z) = \Psi_i(\nu, z) \quad (i = 1, 2),$$

we have

$$(5.7) \quad \frac{\partial I_i}{\partial z}(\nu_i, z) = \frac{\partial \Psi_i}{\partial z} + Q \frac{\partial \Psi_i}{\partial \nu} \quad \text{for } i = \begin{matrix} 1 \\ 2 \end{matrix},$$

where  $Q$  is a constant given by

$$(5.8) \quad Q = \frac{\nu_0}{c} \frac{dv(z)}{dz}.$$

By (5.7), (5.4) and (5.5) become

$$(5.9) \quad \frac{\partial \Psi_1}{\partial z} - Q \frac{\partial \Psi_1}{\partial v} = -\frac{1}{2} \sigma(v, z) [\Psi_1 - \Psi_2],$$

$$(5.10) \quad \frac{\partial \Psi_2}{\partial z} + Q \frac{\partial \Psi_2}{\partial v} = -\frac{1}{2} \sigma(v, z) [\Psi_1 - \Psi_2].$$

Equations (5.9) and (5.10) should be solved subject to the boundary conditions

$$(5.11) \quad \Psi_1(v, z_0) = F, \quad \Psi_2(v, z_1) = 0,$$

where  $F$  is a constant for all frequencies.

The quantities  $I_2(v, z_0)$  and  $I_1(v, z_1)$  represent respectively the reflected and the transmitted intensities.

## 6. THE REFLECTION COEFFICIENT

Let the coefficient of reflection be denoted by  $R(z_0, z_1; v)$ . Then the reflected intensity  $\Psi_2(v, z_0)$  is expressed in terms of  $R$  as follows:

$$(6.1) \quad \Psi_2(v, z_0) = F R(z_0, z_1; v).$$

In a manner similar to that used in Sec. 3, we shall seek an integral equation for the  $R$ -function by means of the invariant-imbedding technique.

By addition of a layer of infinitesimal thickness  $\Delta$  to the boundary  $z_0$ , and use of (5.10), the imbedding procedure gives

$$(6.2) \quad \Psi_2(\nu - Q\Delta, z_0 - \Delta) = \Psi_2^*(\nu, z_0) + \frac{1}{2} \sigma(\nu, z_0) \Delta (\Psi_1(\nu, z_0) - \Psi_2(\nu, z_0)) + \mathcal{O}(\Delta).$$

By (5.9), the modified intensity at  $z$  in an added layer is provided by

$$(6.3) \quad \hat{\Psi}_1(\nu, z_0) = \Psi_1(\nu + Q\Delta, z_0 - \Delta) - \frac{1}{2} \sigma(\nu, z_0) \Delta [\Psi_1(\nu, z_0) - \Psi_2(\nu, z_0)] + \mathcal{O}(\Delta).$$

On making use of (6.3), we see that the reflected intensity of radiation due to the modified source  $\hat{\Psi}_1$  becomes

$$(6.4) \quad \Psi_2^*(\nu, z_0) = F [R(z_0, z_1; \nu) - \frac{1}{2} \sigma(\nu, z_0) R(z_0, z_1; \nu) \Delta + \frac{1}{2} \sigma(\nu, z_0) R^2(z_0, z_1; \nu) \Delta] + \mathcal{O}(\Delta).$$

The combination of (6.1), (6.2), and (6.4) provides us with

$$(6.5) \quad R(z_0 - \Delta, z_1; \nu - Q\Delta) = R(z_0, z_1; \nu) + \frac{1}{2} \sigma(\nu, z_0) \Delta - \sigma(\nu, z_0) R(z_0, z_1; \nu) \Delta + \frac{1}{2} \sigma(\nu, z_0) R^2(z_0, z_1; \nu) \Delta + \mathcal{O}(\Delta).$$

Then, letting  $\Delta \rightarrow 0$ , we have

$$(6.6) \quad \frac{\partial R}{\partial z_0} + Q \frac{\partial R}{\partial \nu} = -\frac{1}{2} \sigma(\nu, z_0) + \sigma(\nu, z_0) R - \frac{1}{2} \sigma(\nu, z_0) R^2.$$

Equation (6.6) is the required integral equation for the reflection coefficient  $R$ .

## 7. THE TRANSMISSION COEFFICIENT

Consider the transmitted intensity of radiation emergent from the boundary  $z_1$  for the illumination of the boundary  $z_0$ .

Let the coefficient of transmission be denoted by  $T(z_0, z_1; \nu)$ . The transmitted intensity at  $z_1$  is expressed in the form

$$(7.1) \quad \Psi_1(\nu, z_1) = F T(z_0, z_1; \nu).$$

On adding a layer of infinitesimal thickness  $\Delta$  to the boundary  $z_0$ , we get (6.3). By imbedding the photon diffusion process in the added  $\Delta$ -layer, we obtain

$$(7.2) \quad \Psi_1^0(\nu, z_1) = \Psi_1^*(\nu, z_1).$$

On the other hand, we have

$$(7.3) \quad \Psi_1^0(\nu, z_1) = F T(z_0 - \Delta, z_1; \nu + Q\Delta),$$

and

$$(7.4) \quad \Psi_1^*(\nu, z_1) = T(z_0, z_1; \nu) \hat{\Psi}_1(\nu, z_0).$$

Then, combining (6.3) and (7.3)-(7.4), we obtain (7.2) in the form

$$(7.5) \quad T(z_0 - \Delta, z_1; \nu + Q\Delta) = T(z_0, z_1; \nu) \left[ 1 - \frac{1}{2} \sigma(\nu, z_0) \Delta \{1 - R(z_0, z_1; \nu)\} \right] + \mathcal{O}(\Delta).$$

In the limit as  $\Delta \rightarrow 0$ , we have

$$(7.6) \quad \frac{\partial T}{\partial z_0} - Q \frac{\partial T}{\partial \nu} = \frac{1}{2} \sigma(\nu, z_0) T - \frac{1}{2} \sigma(\nu, z_0) T R.$$

Equation (7.6) is the requisite integral equation governing the transmission coefficient  $T$ .

REFERENCES

1. Ambarzumian, V. A., "Diffuse Reflection of Light by a Foggy Medium," Doklady Akad. Nauk SSSR., Vol. 38, 1943, pp. 229-232.
2. Bellman, R., R. E. Kalaba, and G. M. Wing, "Invariant Imbedding and Mathematical Physics—I: Particle Processes," J. Math. Phys., Vol. 1, 1960, pp. 280-308.
3. Bellman, R., R. E. Kalaba, and G. M. Wing, "Invariant Imbedding and Neutron Transport in a Rod of Changing Length," Proc. Nat. Acad. Sci. USA, Vol. 46, 1960, pp. 128-130.
4. Bellman, R., R. E. Kalaba, and S. Ueno, "Invariant Imbedding and Time-Dependent Diffuse Reflection by a Finite Inhomogeneous Atmosphere," ICARUS, Vol. 1, 1962, pp. 191-199.
5. Bellman, R., R. E. Kalaba, and S. Ueno, Invariant Imbedding and Scattering of Light in a One-Dimensional Medium, The RAND Corporation, RM-3603-ARPA, 1963.
6. Chandrasekhar, S., Radiative Transfer, Oxford University Press, London, 1950.
7. Chandrasekhar, S., "The Formation of Absorption Lines in a Moving Atmosphere," Rev. Mod. Phys., Vol. 17, 1945, pp. 138-156.
8. Kaplan, S. A., "On the Theory of Light Scattering in a Non-Steady State Medium," Astron. J. USSR., Vol. 39, 1962, pp. 702-709.
9. Sobolev, V. V., "The Diffusion of  $L_{\alpha}$ -Radiation in Nebulae and Stellar Envelopes," Astron. J. USSR., Vol. 34, 1957, pp. 694-705.
10. Wing, G. M., "Analysis of a Problem of Neutron Transport in a Changing Medium," J. Math. Mech., Vol. 11, 1962, pp. 21-34.
11. Wing, G. M., An Introduction to Transport Theory, John Wiley and Sons, New York, 1962.

**UNCLASSIFIED**

**UNCLASSIFIED**