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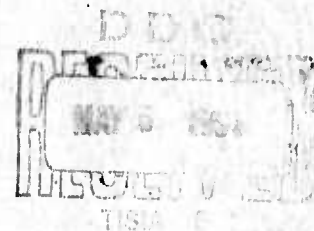
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PLANE PHASED ARRAYS:
DEPENDENCE OF GAIN ON
STEERING, THINNING AND
ELEMENT DIRECTIVITY

Worthie Doyle



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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

In this Memorandum the dependence of plane array gain upon thinning, steering and element directivity is discussed and a systematic way to avoid grating lobes in highly thinned arrays is introduced. The ideas are illustrated by some detailed examples. The Memorandum should interest people designing thinned, electronically steered, plane arrays for modern radars.

SUMMARY

Two related topics in the design of plane phased arrays are discussed. One is the dependence of gain upon thinning, steering and element directivity; the other is a systematic procedure for locating the elements of highly thinned, plane arrays.

A few general observations on the effects of thinning and steering can be made from the expression for the gain of an array of elements having an arbitrary directivity factor. The specializations to isotropic elements and to infinitesimal dipoles are then made and these observations are verified for some actual arrays.

A systematic way to randomize element locations in a thinned array while still approximating some basic excitation taper is also suggested and illustrated. The thinning scheme is based on an equal area approximation to a continuous excitation distribution, with the equal areas being laid out in the form of an expanding spiral. This procedure insures against accidental element alignment in any particular direction. Its effectiveness is shown by applying it to a 100-element array and plotting several sections of the resulting directivity pattern.

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This Memorandum is a direct result of some well-formed questions from John Mallett, whose conversation and suggestions provided steady assistance. It has also benefitted from a helpful critical reading by Larry Brennan.

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I. INTRODUCTION

Two related topics in the design of plane phased arrays are discussed. One is the dependence of gain upon thinning, steering and element directivity; the other is a systematic procedure for locating the elements of highly thinned, plane arrays.

A few general observations on the effects of thinning and steering can be made from the expression for the gain of an array of elements having an arbitrary directivity factor. The specializations to isotropic elements and to infinitesimal dipoles are then made and these observations are verified for some actual arrays.

A systematic way to randomize element locations in a thinned array while still approximating some basic excitation taper is also suggested and illustrated. The thinning scheme is based on an equal area approximation to a continuous excitation distribution, with the equal areas being laid out in the form of an expanding spiral. This procedure insures against accidental element alignment in any particular direction.

The dependence of gain on element spacing has been described by Milazzo and D'Angelo⁽¹⁾ for the case of a uniformly spaced, broadside linear array. Bickmore⁽²⁾ considers a square current sheet, linearly polarized parallel to one side of the square. For steering around the axis of polarization he derives the dependence of 3 db beamwidth on steering and shows that the cosine approximation for effective aperture holds to within a beamwidth or so of endfire. Von Aulock's⁽³⁾ expository article contains similar material on the beamwidths for

electronically steered rectangular arrays with uniform spacing. Although results like those in Ref. (1) are easily demonstrated for plane arrays also, the facts are apparently not universally appreciated. One difficulty is the number of parameters whose effects are to be disentangled and separately accounted for. In this Memorandum the effects will be discussed in connection with the detailed example which occupies about half the pages. Although some qualitative statements can be made from the general expression for gain, the actual fluctuations in gain as a thinned array is steered will depend strongly on the array configuration itself. For instance, a square array with uniform spacing of a wavelength or so would have very wide gain fluctuations, while the example to be considered has relatively mild gain fluctuations with steering.

The example also furnishes a convenient way to introduce a new approach to the problem of spurious main lobe reduction in highly thinned plane arrays. The approach of King, Packard and Thomas⁽⁴⁾ is essentially intelligent cut-and-try. Baklanov, Pokrovski, and Surdutovich⁽⁵⁾ apply a Newton-Raphson iterative procedure to solve for the locations of the N elements of a symmetrical linear array whose first $[N/2]$ side lobes have a pre-assigned level. Andreassen⁽⁶⁾ has also used an iterative procedure to deduce element distributions for which the nearest grating lobes or other large lobes are held down to a level appropriate to the number of elements. All three papers are concerned with linear arrays. For plane arrays or arbitrary volumetric distributions there have been some statistical analyses of large random sets of radiators. Cover's Memorandum⁽⁷⁾

is typical, though not directly concerned with spurious lobe reduction. Maher and Cheng⁽⁸⁾ consider the statistical effects of removing a relatively small number of elements from a linear array. The only reported explicit synthesis of a thinned, plane array seems to be that of Willey.⁽⁹⁾ The method used in this Memorandum is very close to Willey's, which is based on the idea of approximating a continuous aperture distribution with a discrete array of equal radiators by partitioning the continuous distribution into areas of equal weight and assigning a radiator to each such area.⁽¹⁰⁾

It differs from Willey's in providing a nearly automatic assignment of element locations without the need to make individual choices by hand from a small number of possibilities.

II. GAIN OF PLANE PHASED ARRAYS

Let N radiators be located at points, P_k , in the equatorial plane, $\theta = \pi/2$, of a spherical coordinate system (see Fig. 1). The locations will be specified in either rectangular coordinates, (x_k, y_k) , or polar coordinates, (r_k, ϕ_k) , as convenient. Let the

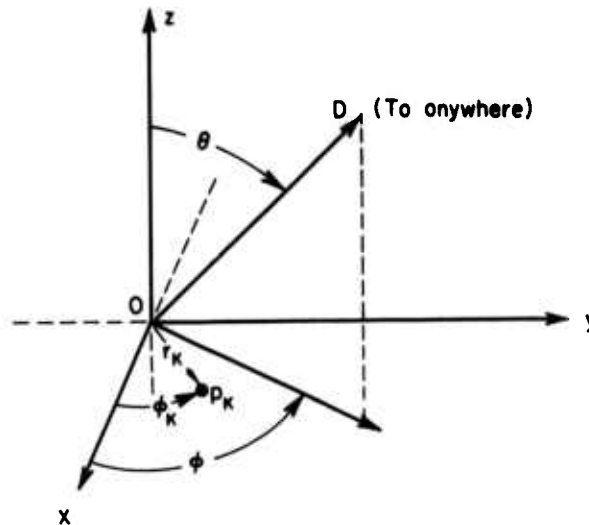


Fig. 1 — Coordinates

amplitude and phase of the excitation of the k^{th} element be a_k and b_k , so that the complex excitation is given by $\alpha_k = a_k \exp(ib_k)$.

Consideration will be restricted to cases describable by a scalar or one-dimensional analysis. These include acoustic radiators and linearly polarized electromagnetic radiators, where the far field contains only a single component. In particular, parallel linearly polarized current elements or parallel dipoles are covered.

The distant field in an arbitrary direction is then proportional to

$$V(\theta, \varphi) = E(\theta, \varphi) \sum_{k=1}^N a_k e^{ib_k + 2\pi i(r_k/\lambda) \sin \theta \cos(\varphi - \varphi_k)} \quad (1)$$

$E(\theta, \varphi)$ is the element directivity factor, assumed to be the same for all elements. The phase reference has been taken to be the origin, 0, of the coordinate system and the factor $(r_k/\lambda) \sin \theta \cos(\varphi - \varphi_k)$ is simply the number of wavelengths in the projection of OP_k onto OD. A pencil beam with a maximum in the direction (θ_0, φ_0) will be formed if the excitation phases, b_k , are so chosen as to annihilate each exponent in the direction (θ_0, φ_0) . This choice is

$$b_k = -2\pi(r_k/\lambda) \sin \theta_0 \cos(\varphi_0 - \varphi_k) \quad (2)$$

The gain in any direction, (θ, φ) , is the ratio of intensity at (θ, φ) to the average intensity over all directions:

$$G(\theta, \varphi) = \frac{|V(\theta, \varphi)|^2}{\frac{1}{4\pi} \int_{\Omega} |V(\theta, \varphi)|^2 d\omega}$$

where $\int_{\Omega} \dots d\omega$ indicates integration over the sphere. Gain, G , without qualification, will mean $G(\theta_0, \varphi_0)$, the gain in the direction of the beam maximum. This gain is given by

$$\begin{aligned}
 |E(\theta_0, \varphi_0)|^2 \left| \sum_{k=1}^N a_k \right|^2 / G &= \int_0^\pi |\sin \theta| d\theta \int_{-\pi}^\pi |V(\theta, \varphi)|^2 d\varphi / 4\pi \\
 &= \operatorname{Re} \sum_{j=1}^{N-1} \sum_{k=j+1}^N 2a_j a_k e^{i(b_k - b_j)} I_{jk} + \sum_{k=1}^N a_k^2 I_{kk} \quad (3)
 \end{aligned}$$

where

$$I_{jk} = \int_0^\pi |\sin \theta| \frac{d\theta}{4\pi} \int_{-\pi}^\pi e^{\frac{2\pi i \sin \theta}{\lambda} [r_k \cos(\varphi - \varphi_k) - r_j \cos(\varphi - \varphi_j)]} |E(\theta, \varphi)|^2 d\varphi \quad (4)$$

depends only upon the element locations and the element directivity factor. Notice that

$$I_{kk} = \int_0^\pi |\sin \theta| \frac{d\theta}{4\pi} \int_{-\pi}^\pi |E(\theta, \varphi)|^2 d\varphi \quad (5)$$

is merely the average element factor.

When the element factor, $E(\theta, \varphi)$, has the twofold symmetry $|E(\theta, \varphi)| = |E(\theta, -\varphi)| = |E(\theta, \varphi + \pi)|$, as is the case for the examples to be considered, I_{jk} is real and Eq. (3) becomes

$$\begin{aligned}
 \frac{|E(\theta_0, \varphi_0)|^2 \left| \sum_{k=1}^N a_k \right|^2}{G} &= 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^N a_j a_k \cos(b_k - b_j) I_{jk} \\
 &\quad + \sum_{k=1}^N a_k^2 I_{kk} \quad (6)
 \end{aligned}$$

SWAMPING OF ELEMENT DIRECTIVITY WHEN THERE IS NO THINNING

The gain of a phased array will change as the beam is steered about. Various factors contribute to this change. In this section evidence is given that the element factor has little effect on the variation of gain with steering whenever the array is producing a sharp beam and is not thinned. In Section IV these assertions are verified by comparing the variations of gain with steering for two arrays, one of isotropic elements and the other of infinitesimal dipoles.

The quantity $|V(\theta, \varphi)|^2$ of Eq. (3) is the product of the squared magnitudes of the array factor

$$A(\theta, \varphi) = \sum_{k=1}^N \alpha_k e^{2\pi i (r_k/\lambda) \sin \theta \cos(\varphi - \varphi_k)}$$

and of the element factor, $E(\theta, \varphi)$. In terms of A and E, Eq. (3) may be rewritten as

$$|E(\theta_0, \varphi_0) A(\theta_0, \varphi_0)|^2 / G = \int_0^\pi |\sin \theta| \frac{d\theta}{4\pi} \int_{-\pi}^\pi |A(\theta, \varphi)|^2 |E(\theta, \varphi)|^2 d\varphi \quad (7)$$

Suppose the element directivity is a slowly varying and not very peaked function of position. For example, if the elements are parallel infinitesimal dipoles the element factor is $|E(\theta, \varphi)|^2 = \sin^2 \alpha$, where α is the angle between the direction (θ, φ) and the axis of a dipole. This pattern, in three dimensions, looks like a smooth, holeless doughnut. When the pencil beam is very sharp and does not come near a null of the slowly varying element factor and when most of the volume inside the array factor is the volume directly under

the peak, then a close approximation to the integral in Eq. (7) is obtained by setting $E(\theta, \varphi) = E(\theta_0, \varphi_0)$, its value at the peak of $A(\theta, \varphi)$, whereupon $E(\theta_0, \varphi_0)$ cancels out of Eq. (7).

Even if the total sidelobe volume of A is not negligible there is another effect that reduces the dependence of gain on the element factor: the contribution to the integral from outside the peak of A stays roughly constant as the beam is steered around. Denote this contribution by $E_1^2 A_1^2 (1 - B)$, where $E_1^2 A_1^2$ is an average value outside the peak of A and where B is the fraction of space occupied by the peak of A. Then Eq. (7) may be written as

$$\frac{E_0^2 A_0^2}{G} = E_1^2 A_1^2 (1 - B) + B E_0^2 A_0^2$$

or

$$\frac{1}{G} = B + (1 - B) \frac{E_1^2 A_1^2}{E_0^2 A_0^2}$$

The variation of G is produced mainly by the variation of E_0 . To the extent that $E_1^2 A_1^2 / E_0^2 A_0^2 < B$ the variation in the second term will be dampened by the first term, B. The two terms will be about equal when half the energy is radiated into the sidelobes, but even in this case the constant B will roughly halve the variation of G with E_0 .

EFFECT OF THINNING ON GAIN VARIATION WITH STEERING

A closely spaced array approximates a continuously excited area. For both these cases gain is proportional to $\cos \theta_0$ for angles, θ_0 ,

not too far from the perpendicular. For widely spaced arrays, however, this $\cos \theta_0$ factor will disappear. To see why this should be so, consider the two parts of Eq. (6). I_{kk} is a constant and hence the second term is independent of spacing or steering. In the first term, when spacing is large the angles b_k will contain many cycles so that the oscillating factors, $\cos(b_k - b_j)$, are equally likely to be positive or negative. There will be some fluctuation, but the average value of the first term will be zero. Thus for wide spacing, gain is approximately

$$G = \frac{|E(\theta_0, \varphi_0)|^2 \left(\sum_{k=1}^N a_k \right)^2}{I_{kk} \sum_{k=1}^N a_k^2}$$

This clearly does not vary as $\cos \theta_0$. Since I_{kk} , from Eq. (5), is the average of $|E(\theta, \varphi)|^2$, it follows that

$$\frac{|E(\theta_0, \varphi_0)|^2}{I_{kk}} = G_e(\theta_0, \varphi_0)$$

the gain of the element factor in the direction (θ_0, φ_0) . Thus for wide spacing the gain

$$G = G_e(\theta_0, \varphi_0) \frac{\left(\sum_{k=1}^N a_k \right)^2}{\sum_{k=1}^N a_k^2}$$

is simply a constant times the element gain in the direction of steering.

The argument of the preceding section said that element gain should be swamped. This section says gain follows the element factor. However, there is no contradiction, since the case of extreme thinning is precisely the case for which $E_1^2 A_1^2 / E_0^2 A_0^2$ is greater than B. All these remarks are checked for the particular examples computed in Section IV. In summary, when spacing is close, the element factor is unimportant and gain varies with steering as $\cos \theta_0$ out to a couple beamwidths away from the equatorial plane; when spacing is great, the variation of gain with steering follows the element gain.

GAIN IF ELEMENT FACTOR IS UNIFORM OVER A HEMISPHERE

Let

$$E(\theta, \varphi) = \begin{cases} 1 & \text{if } 0 \leq \theta \leq \pi/2 \\ 0 & \text{if } \pi/2 < \theta \leq \pi \end{cases}$$

Such a radiator will be referred to as "isotropic" for short, even though its effect is confined to a half-space.

From Eq. (4)

$$\begin{aligned} I_{jk} &= \int_0^{\pi/2} \sin \theta \frac{d\theta}{4\pi} \int_{-\pi}^{\pi} e^{\frac{2\pi i \sin \theta}{\lambda} [r_k \cos(\varphi - \varphi_k) - r_j \cos(\varphi - \varphi_j)]} d\varphi \\ &= \int_0^{\pi/2} \sin \theta \frac{d\theta}{4\pi} \int_{-\pi}^{\pi} e^{\frac{2\pi i \sin \theta}{\lambda} d_{jk} \cos(\varphi - \psi_{jk})} d\varphi \\ &= \int_0^{\pi/2} \sin \theta J_0(2\pi(d_{jk}/\lambda) \sin \theta) d\theta/2 \end{aligned} \quad (8)$$

$$I_{jk} = \frac{\sin(2\pi d_{jk}/\lambda)}{4\pi d_{jk}/\lambda} \quad (8)$$

(cont'd)

where

$$d_{jk}^2 = r_k^2 + r_j^2 - 2r_j r_k \cos(\varphi_k - \varphi_j) = (x_k - x_j)^2 + (y_k - y_j)^2$$

is the squared distance between the j -th and k -th elements.

When $j = k$

$$I_{kk} = \frac{1}{2} \quad (9)$$

From Eq. (3) the gain is given by

$$\frac{\left(\sum_{k=1}^N a_k \right)^2}{G} = \sum_{j=1}^{N-1} \sum_{k=j+1}^N a_j a_k \cos(b_k - b_j) \frac{\sin(2\pi d_{jk}/\lambda)}{2\pi d_{jk}/\lambda} + \frac{1}{2} \sum_{k=1}^N a_k^2 \quad (10)$$

An important special case is $a_1 = \dots = a_N = 1$; then

$$\frac{N}{G} = \frac{1}{2} + \frac{1}{N} \sum_{j=1}^{N-1} \sum_{k=j+1}^N \cos(b_k - b_j) \frac{\sin(2\pi d_{jk}/\lambda)}{2\pi d_{jk}/\lambda} \quad (11)$$

Some examples for this case appear in Section IV. For wide spacings the terms of the double sum go down because of the divisor d_{jk} . They also tend to cancel each other in a random way because of the large and irregular values of $(b_k - b_j)$. Hence for wide spacings $G \approx 2N$. The factor 2 results from choosing elements that radiate "isotropically" into a hemisphere.

GAIN FOR DIPOLES RADIATING INTO A HEMISPHERE

The infinitesimal dipoles are taken to lie parallel to the x-axis. The element factor is then $(1 - \cos^2 \varphi \sin^2 \theta)$ in the northern hemisphere, $0 \leq \theta \leq \pi/2$, and vanishes in the southern. For the calculation of I_{jk} see the Appendix. The result is

$$4I_{jk} = \frac{\sin z}{z} + \frac{1}{z^2} \left(\frac{\sin z}{z} - \cos z \right) + \cos 2w \left[\frac{3}{z^2} \left(\frac{\sin z}{z} - \cos z \right) - \frac{\sin z}{z} \right] \quad (12)$$

where

$$z = 2\pi d_{jk}/\lambda \quad (13)$$

and

$$\cos 2w = \frac{(x_k - x_j)^2 - (y_k - y_j)^2}{d_{jk}^2} \quad (14)$$

When $j = k$

$$I_{kk} = \frac{1}{3} \quad (15)$$

Gain for this case is given by Eq. (3), with I_{jk} and I_{kk} given by Eqs. (12) through (15) and with $|E(\theta_0, \varphi_0)|^2 = 1 - \sin^2 \theta_0 \cos^2 \varphi_0$. For the special case of equal excitation, $a_1 = a_2 = \dots = a_N = 1$, the gain is given by

$$\frac{1 - \sin^2 \theta_o \cos^2 \varphi_o}{G/N} = \frac{1}{3} + \frac{2}{N} \sum_{j=1}^{N-1} \sum_{k=j+1}^N I_{jk} \cos(b_k - b_j) \quad (16)$$

Some examples for this case also appear in Section IV, where the behavior of gain with thinning and steering is compared with the corresponding behavior for elements with uniform radiation over a hemisphere.

III. DISCRETE APPROXIMATION OF APERTURE EXCITATIONS BY SPIRALS

Consider a plane array intended to produce a steerable pencil beam. The beam shape and gain will depend on the distribution of elements over the area of the aperture and on the aperture size. The beamwidths will depend mainly on the aperture diameters, while the relative distribution of elements will determine the sidelobe levels and will also determine whether spurious main lobes arise. The general behavior of near sidelobes can be controlled by making the element distribution approximate a continuous excitation having a known, and acceptable, set of sidelobes. For example, a suitable continuous excitation can be divided into areas of equal total excitation and an element of a discrete array assigned to each such area.

The actual partitioning of the illumination into equal increments can be done in many ways. For example, a square aperture can be divided into a regular array of rectangles and an element assigned to each rectangle, if a uniform excitation is to be approximated. When such a regular array is used, spurious lobes arise as soon as the spacing exceeds one wavelength. By randomizing the spacings, such lobes can be decreased. The general objective is so to space the elements that the distribution of their projections on any line is always irregular. This immediately suggests basing the distribution on a set of uniform distributions on circles, since circles already are free from preferred directions. If an annular region is to be occupied by a fixed number of elements, it would presumably be divided evenly and an element assigned to each equal part, perhaps

being randomly displaced inside its assigned area. The rings could be rotated with respect to one another so that no strongly lined-up elements formed spokes.

From the above discussion it is only a short jump to the idea of dividing the continuous excitation into equal pieces by winding out from the center in a spiral strip. The spiral strip will then be divided into roughly square pieces, each to be represented by one element of the final array. If the array is to approximate a constant excitation, then the spiral strip will soon reach a nearly constant width. If the illumination is to be tapered, then the strip should gradually widen as the spiral expands. Similarly, the spacing of elements along the strip would be even if a constant illumination is to be approximated, but would gradually increase if there is to be a tapering at the edge.

There is no guarantee that such a spiral plan for positioning the elements will not produce spokes, or even nearly parallel lines of elements. However, it seems probable that such accidents would be rare.

DISC WITH CONSTANT EXCITATION

Consider the problem of placing a given number of elements inside a circle so that the spacing between elements is roughly uniform but so that the elements do not tend to fall on straight lines. Suppose the elements are evenly spaced along the arc of an expanding spiral. After the spiral has grown a bit, the distance between successive elements on the spiral (chord or secant) will

soon be very nearly the same as the distance along the arc, hence choosing equal spacing along the arc guarantees roughly uniform spacing of elements in the θ direction. The direction of the discrepancy will be such as to produce a slight tapering down as the outside of the array is approached, which is a discrepancy in the right, or helpful, direction (since it means slightly lower sidelobes). To determine the spiral approximately, consider the relation between arc length, s , and radius, r . Let d be the constant distance between elements along the arc of the spiral; then the number of elements in length ds will be ds/d . Suppose the radius grows by dr during the space ds . If a constant illumination is to be approximated, then the number of elements in an annulus of width dr should be proportional to the area, $2\pi r dr$, of the ring. Hence the spiral is given by

$$ds = r dr/r_0 \quad (17)$$

where r_0 is a proportionality constant. Integrating Eq. (17) results in

$$r^2 = r_0^2 + 2r_0 s \quad (18)$$

From Eq. (18) and $ds^2 = r^2 d\theta^2 + dr^2$ it follows that

$$d\theta = \frac{\sqrt{2r_0 s} ds}{r_0^2 + 2r_0 s} \quad (19)$$

or

$$\theta = \sqrt{\frac{2s}{r_0}} - \tan^{-1} \sqrt{\frac{2s}{r_0}}$$

Equations (18) and (19) are parametric equations of the spiral. Spacing elements equally along the arc means that the k -th element corresponds to parameter $s_k = (k - 1)d$, for $k = 1, 2, \dots, N$, with the polar coordinates of the k -th element given by Eqs. (18) and (19). Let the outermost, or N -th, element lie at distance a from the center; thus a may be considered the radius of the roughly circular array. Then

$$a^2 = r_0^2 + 2r_0(N - 1)d \quad (20)$$

A choice for the starting radius, r_0 , has still to be made. The area represented by an individual element will be roughly square if the distance between successive turns of the spiral is equal to the distance, d , between elements along the spiral. This will make the radial and tangential components of the interelement distances about the same. An approximate solution for this requirement is possible if the spiral has several turns. Near the outside the term $\tan^{-1} \sqrt{2s/r_0}$ is approximately $\pi/2$ and can be ignored. From Eqs. (18) and (19), for two values of the parameter, t and s , corresponding to a change of 2π in θ

$$\frac{r^2 - r_0^2}{r_0^2} = \frac{2t}{r_0}, \quad \frac{(r - d)^2 - r_0^2}{r_0^2} = \frac{2s}{r_0}$$

and

$$\sqrt{\frac{2t}{r_0}} - \sqrt{\frac{2s}{r_0}} = 2\pi$$

hence

$$\sqrt{\frac{2t}{r_0}} - \sqrt{\frac{2s}{r_0}} \approx \frac{r}{r_0} - \frac{r-d}{r_0} \approx 2\pi$$

and

$$d \approx 2\pi r_0 \quad (21)$$

The elements are located where $s_k = (k-1)d$. Take the N-th element at the outer radius, a ; then

$$a^2 = r_0^2 + 2r_0(N-1)2\pi r_0$$

or

$$r_0 = \frac{a}{\sqrt{1 + 4\pi(N-1)}} \quad (22)$$

EXAMPLE--100 ELEMENTS OVER A DISC

If the outer radius is $a = 14.1$, then $r_0 = 14.1/\sqrt{1+4\pi(100-1)} \approx .400$ and $d = 2\pi r_0 = 2.51$. From Eqs. (18) and (19) there result

$$r_k = \sqrt{2.000k - 1.848} \quad (23)$$

and

$$\theta_k = \sqrt{12.57(k-1)} - \tan^{-1} \sqrt{12.57(k-1)} \quad (24)$$

The 100 locations for this example are plotted in Fig. 2. They look uniformly distributed and not lined up in any particular direction.

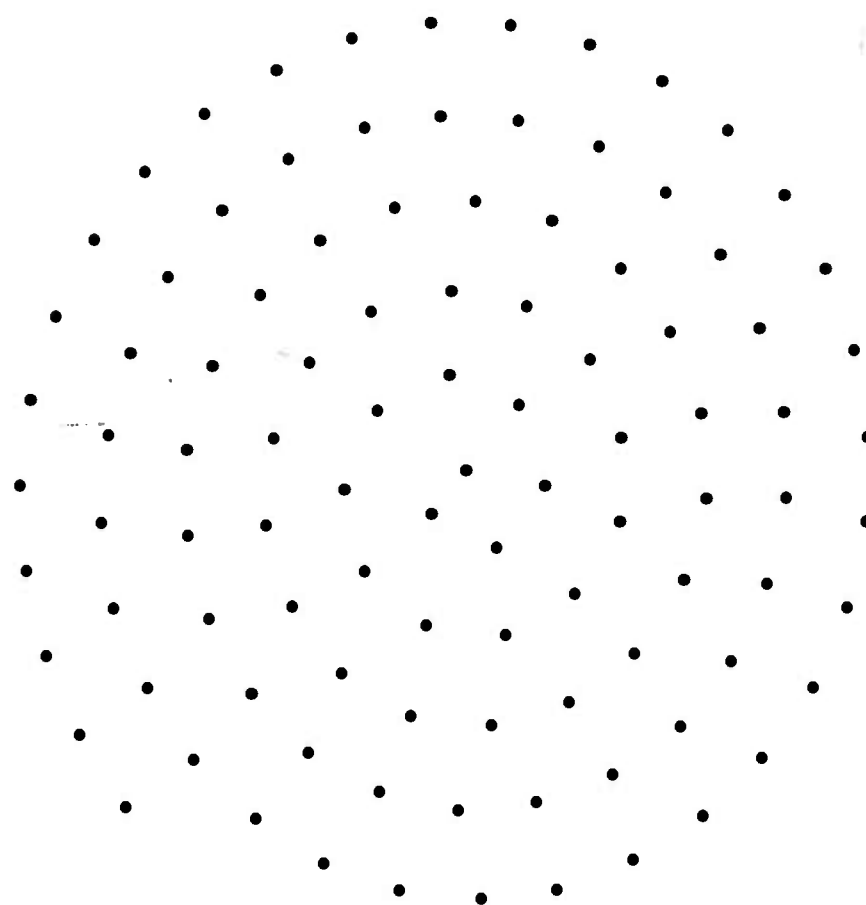


Fig. 2 — Spiral distribution approximating uniform excitation

It remains to discover the extent to which this arrangement is "random" and has succeeded in avoiding big sidelobes when the average interelement spacing is made noticeably greater than half a wavelength. The pattern produced by 100 elements arranged as in Eqs. (23) and (24) is described in Section IV, along with the other numerical results for this example.

IV. A DETAILED ILLUSTRATION

To verify the conclusions of Section II an example is worked out in detail. The variation of gain with steering and spreading (or thinning) is found for the spiral array of Fig. 2, using both isotropic elements and infinitesimal dipoles.* For no spreading, the gain varies with steering according to the well-known $\cos \theta_0$ approximation, while for great spreading the variation of array gain with steering follows the gain of the element factor.

THE THINNING PARAMETER, S

Up to now, thinning or spreading has been discussed in a rough quantitative way only. Before any calculation can be done, the notion of thinning needs to be made exact. For a regular array whose elements lie at the intersections in a square grid, the spacing parameter would naturally be defined to be the mesh dimension in, perhaps, half wavelengths. For the sort of irregular array presented by the spiral in this example, no such direct and simple definition is possible.

Any definition of the spacing parameter, S, will always be somewhat arbitrary. The one to be given here has two merits: it is fairly natural, and it reduces to the "right" number for regular, rectangular arrays. Let A be the area of an array which is occupied

*Because there was assumed to be no radiation into the southern hemisphere, the results may be 3 db higher than might be expected. No special virtue inheres in infinitesimal dipoles; they just happen to be a convenient example for which to verify the assertions made about the effect of the element factor on the variation of gain with thinning and steering.

by N elements. Then the spacing (or thinning, or spreading) parameter, S , is defined as

$$S = \frac{A}{N(\lambda/2)^2} \quad (25)$$

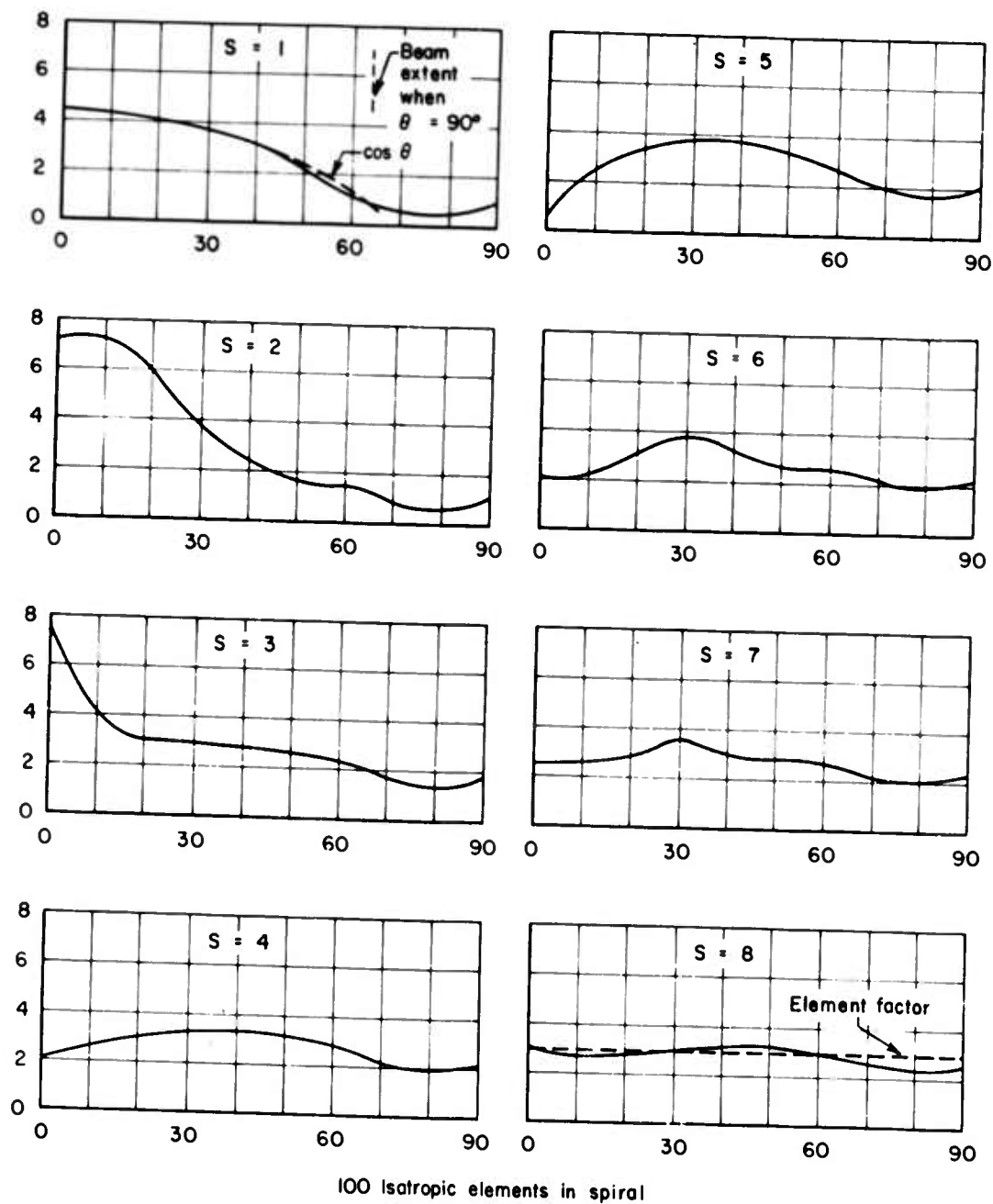
The units are square halfwaves per element. A regular array of N elements, spaced K halfwaves, has $S \approx K^2$, with $K^2 = S = 1$ the normal or unthinned case. For an array which has been literally thinned (by removing elements more or less uniformly, while leaving the same area occupied), S is the factor by which the number of elements in the array has been reduced from the normal case of halfwave spacing.

A question still to be settled is what area does the spiral of Fig. 2 actually "occupy." For the results to be exhibited, the area occupied is taken to be a circle whose diameter is half a wavelength greater than the diameter of the spiral. The diameter of the spiral is the maximum distance, $D_{\max}$, between a pair of elements on the spiral. Adding the extra half wavelength reflects a belief that the sphere of influence of a simple radiator like a dipole extends outward about a quarter wavelength. The spreading factor for the spiral array is then

$$S = \frac{\pi(D_{\max} + .5\lambda)^2}{N \lambda^2}$$

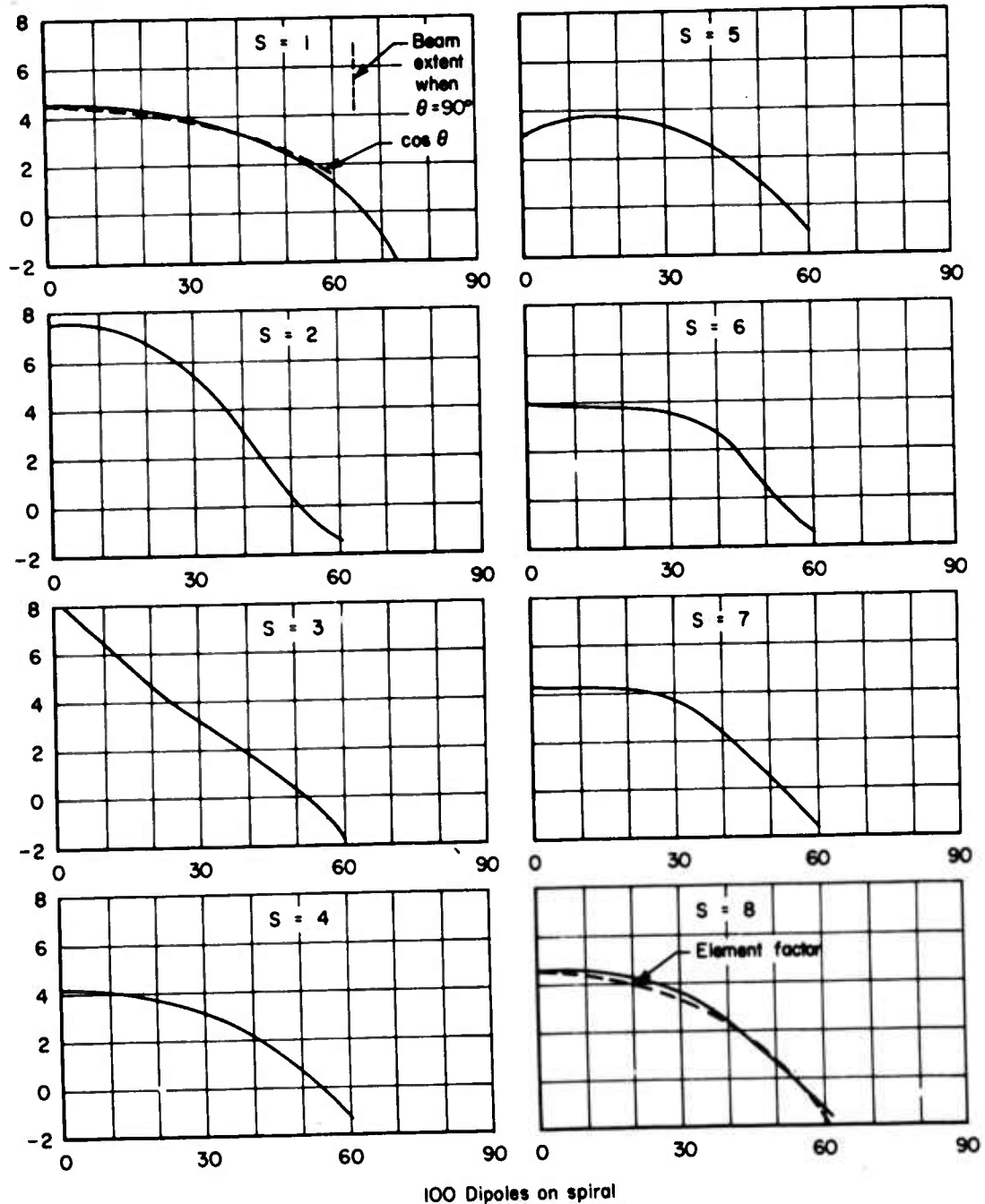
VARIATION OF GAIN WITH THINNING

Figures 3 and 4 are almost self-explanatory. For a beam steered from the north pole to the equator along longitude 0° the gain (normalized by N , the number of elements) is plotted vs θ for values of the thinning parameter, $S = 1, \dots, 8$. Figure 3 is for so-called



S = square halfwaves per element
 $\phi^\circ = 0$

Fig. 3 — G/N (db) versus θ°



S = square halfwaves per element
 $\phi = 0$

Fig. 4 — G/N (db) versus θ°

isotropic elements (uniform radiation over hemisphere), while Fig. 4 is for dipoles except that, again, radiation into the southern hemisphere was assumed zero.

To show how well the $\cos \theta$ law is followed for the case $S = 1$, the 3 db limit of the endfire beam has been sketched. Since the $\cos \theta$ rule would predict zero gain at $\theta = 90^\circ$, it obviously cannot hold over the whole range. The example shows that the rule is closely followed out to within a beamwidth (endfire) of 90° .

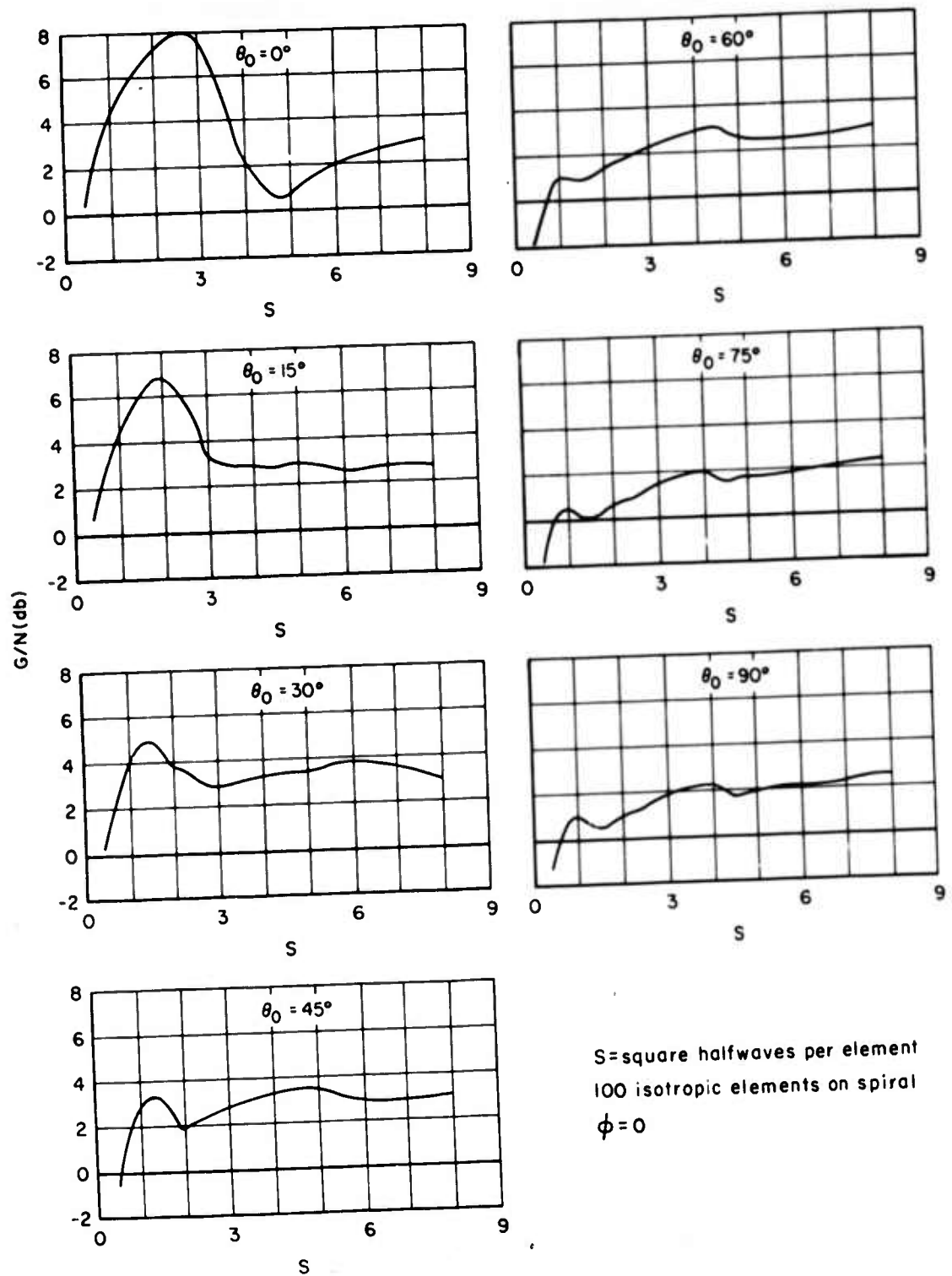
The approximation, $\cos \theta$, is shown for the unthinned case ($S = 1$), and the element factor is shown for the most thinned case ($S = 8$). For dipoles along OX, the element gain varies as $\cos^2 \theta$, which is the element factor plotted in Fig. 4 for $S = 8$.

Figure 5 contains the same data as Fig. 3, but plotted against spacing factor for constant steering angle, θ .

Figure 6 exhibits the variation of gain for a wide range of spacings when the beam is aimed broadside ($\theta = 0^\circ$). The solid curve is for isotropic elements while the circles are for infinitesimal dipoles. Note that at low S the gains are virtually identical, while as S increases the gains gradually come to differ by about 1.8 db, corresponding to the gain $(3/2)$ of a dipole.

COMPARISON OF SPREADING AND THINNING

All the results plotted in Figs. 3 through 6 have been for a fixed number (100) of elements. They therefore show the effects of spreading a fixed number of elements over a greater and greater area,

Fig. 5 — G/N (db) versus S

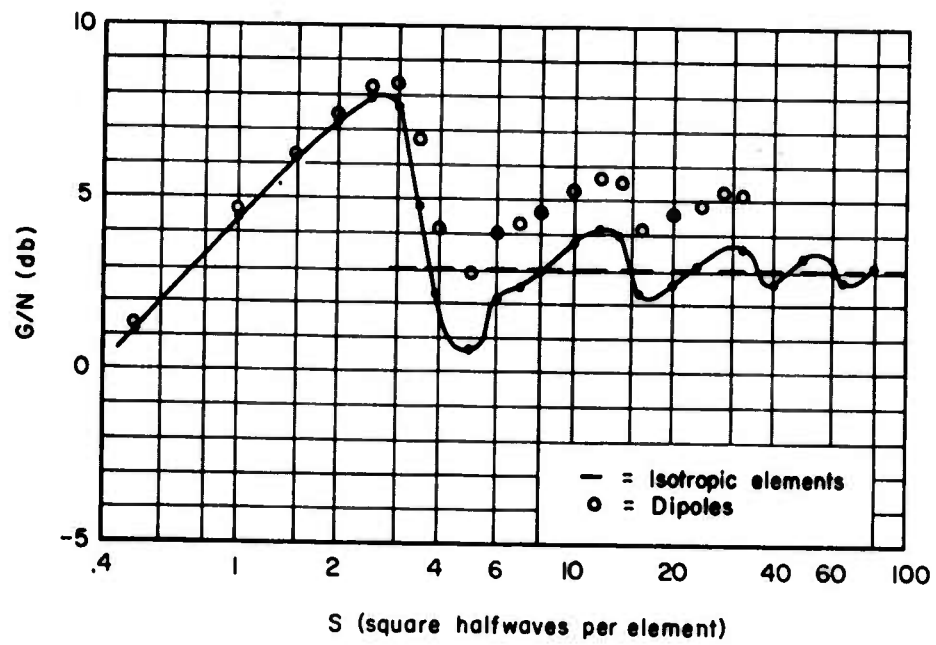


Fig. 6 — Gain versus spreading - 100 elements on spiral -
 $\theta_0 = \phi_0 = 0^\circ$

keeping the configuration similar to the original. What if the area were to be held fixed and a given distribution actually thinned by, say, halving and quartering the number of elements? The answer to this question for the spiral example appears in Fig. 7. The three curves on the left are repeated from Fig. 3. The three curves on the right also correspond to spacing factors of $S = 1, 2$ and 4 , but these factors were produced by reducing the number of elements rather than by spreading 100 elements over a greater area.

The cases considered fall in the most critical region of spacing--the transition region between no thinning and great thinning. Nevertheless, the variations of gain with steering for these two kinds of thinning (increasing the area vs decreasing the number of elements) are in remarkable accord.

FREEDOM OF ARRAY FACTOR FROM GRATING LOBES

The detailed array factor,

$$A(\theta, \varphi) = \sum_{k=1}^N \exp\{(2\pi i/\lambda) r_k [\sin \theta \cos(\varphi - \varphi_k) - \sin \theta_0 \cos(\varphi_0 - \varphi_k)]\}$$

for an array of equally excited elements may be written in either of the universal forms

$$A(U, V) = \sum_{k=1}^N \exp\{(2\pi i/\lambda) [(x_k/\sqrt{S})U + (y_k/\sqrt{S})V]\}$$

or

$$A(Z, \psi) = \sum_{k=1}^N \exp\{(2\pi i/\lambda) (r_k/\sqrt{S})Z \cos(\psi - \varphi_k)\}$$

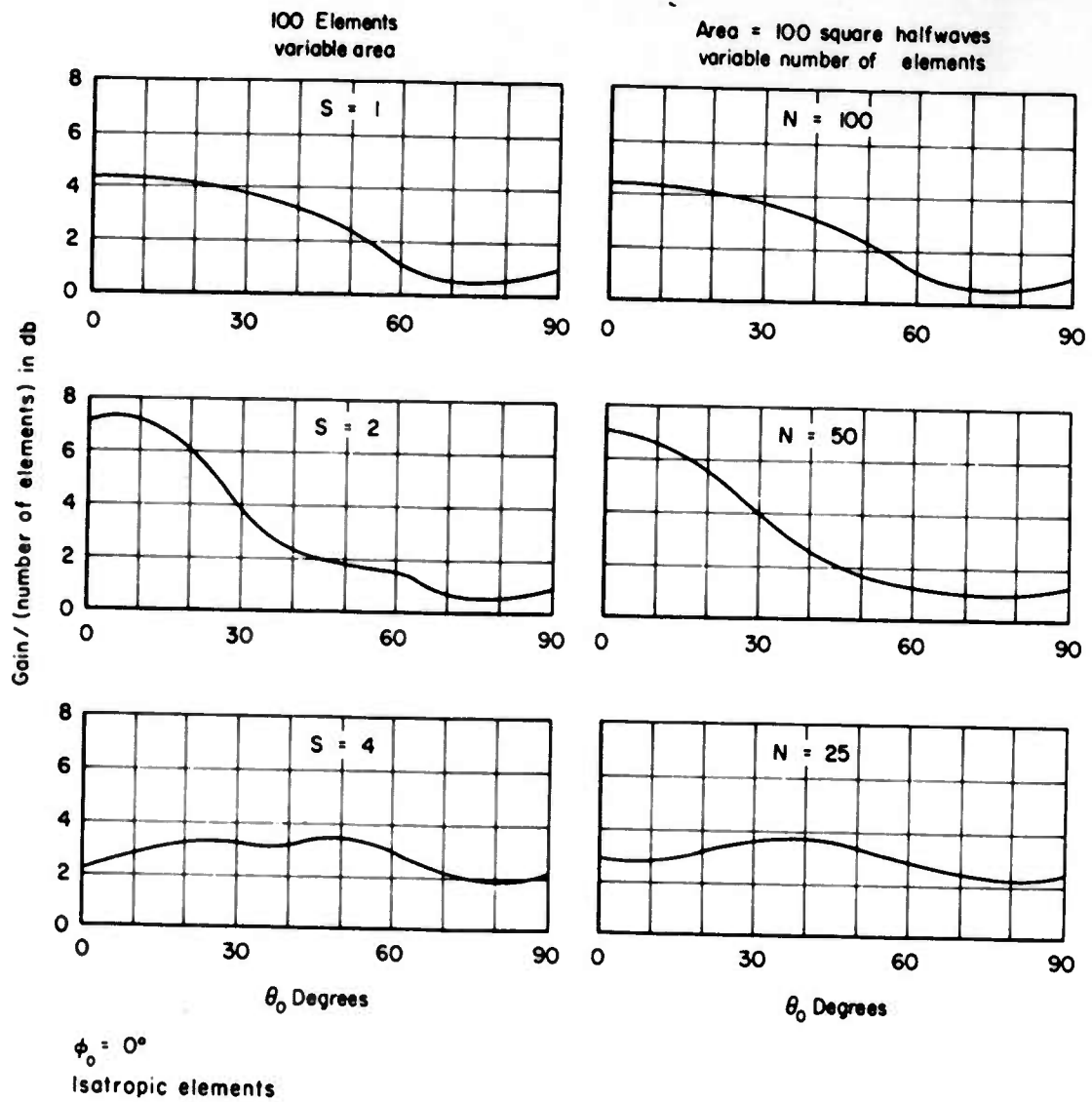


Fig. 7 — Variation of gain with steering and thinning

where

$$U = Z \cos \Psi = \sqrt{S}(\sin \theta \cos \varphi - \sin \theta_0 \cos \varphi_0)$$

$$V = Z \sin \Psi = \sqrt{S}(\sin \theta \sin \varphi - \sin \theta_0 \sin \varphi_0)$$

The scaling $x_k/\sqrt{S}$, $y_k/\sqrt{S}$, $r_k/\sqrt{S}$ produces a normalized set of element locations for which the new spacing is one square halfwave per element. A universal array factor may be computed in terms of the pair of variables (U,V) or the pair (Z,Ψ) . These pairs embody both steering and spreading variations. If $S = 1$ and the beam is aimed broadside ($\theta_0 = 0$), then $|Z| \leq 1$ and $-1 \leq U, V \leq 1$ cover all possible points in real space. If there is to be steering down to $\theta_0 = \pi/2$, then the ranges of U, V and Z must be doubled to account for all of real space and for steering as well. Finally, if there is to be thinning by the factor, S , then the ranges of U, V and Z are multiplied by a further $\sqrt{S}$. For example, if the array of the example is to be fully steered and if thinning by a factor $S = 16$ is to be used, then the ranges of U, V and Z are $0 \leq Z^2 = U^2 + V^2 \leq 64$ or $0 \leq Z \leq 8$. Several slices ($\Psi = 0^\circ, 30^\circ, 60^\circ, 90^\circ$) were plotted for the array factor of the example and are shown in Figs. 8 and 9. In addition a coarse survey of the array factor was made by computing it at a 101 by 201 grid of points in the rectangular interval $0 \leq U \leq 50$, $-50 \leq V \leq 50$. The highest value encountered was 10.7 db down from the main beam. Allowing a factor of 2 for steering, it appears that this 100-element configuration may be free of grating lobes even when average interelement spacing is 12 wavelengths, though the survey calculations were not dense enough to guarantee that some

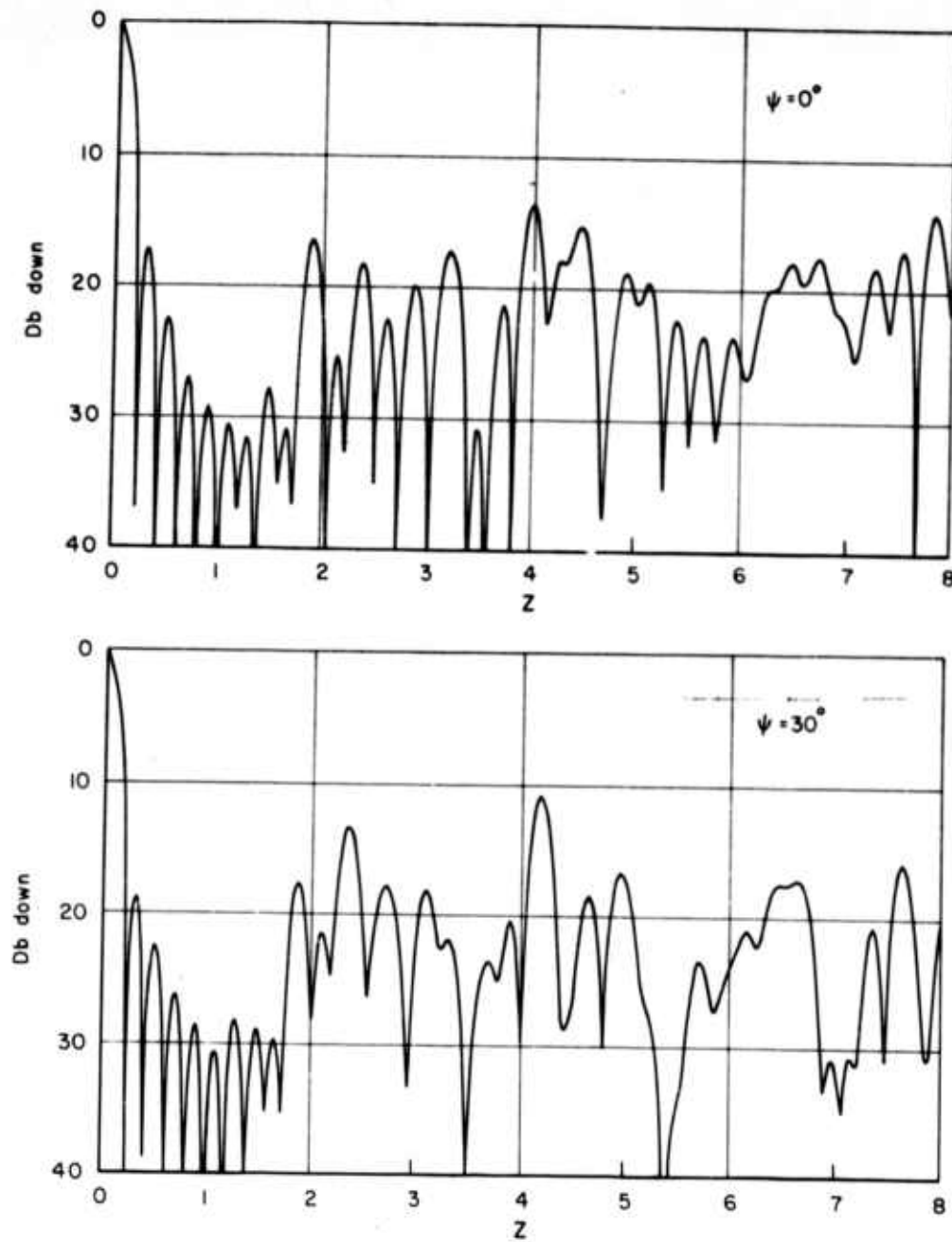


Fig. 8 — Slices of space factor of 100-element spiral

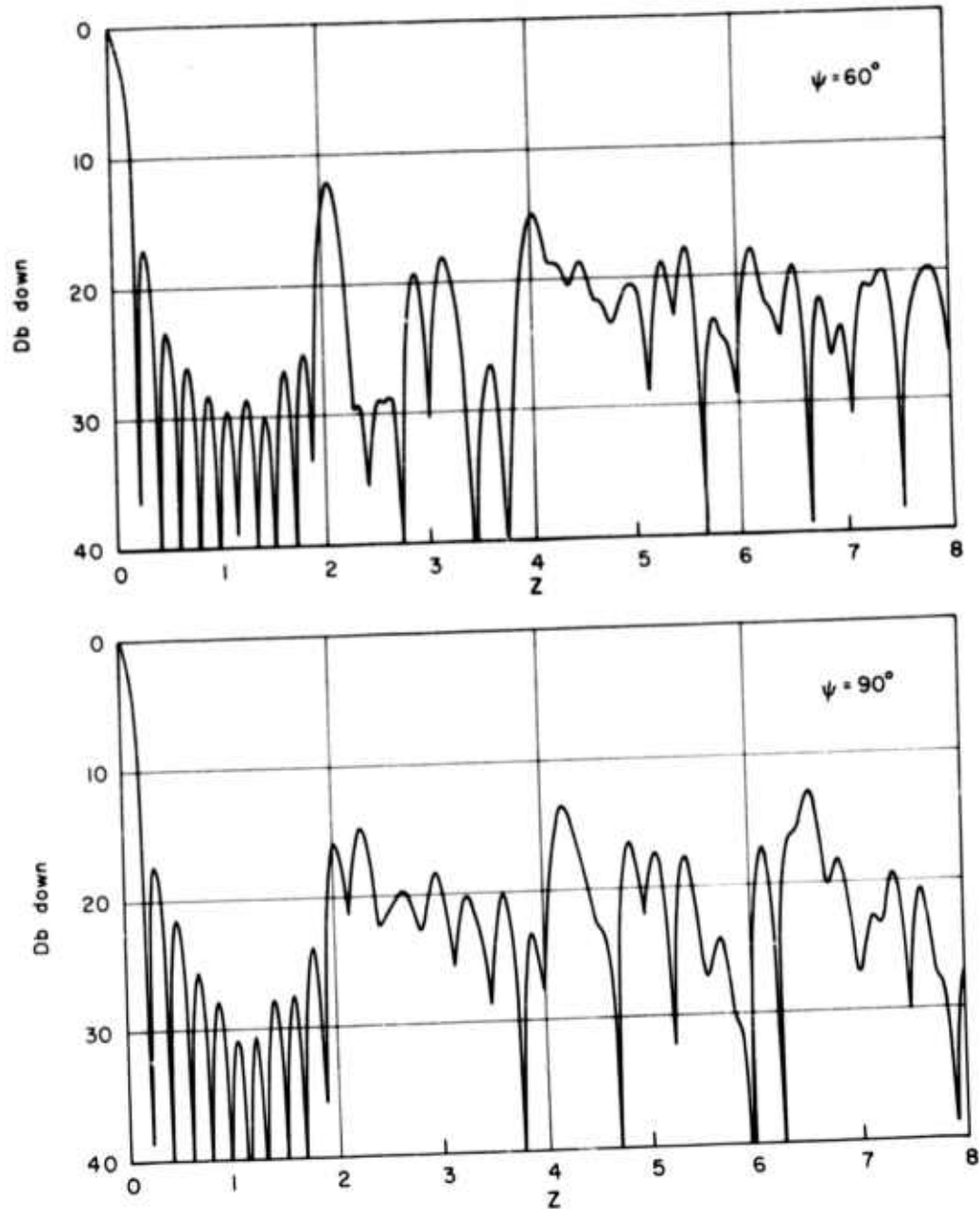


Fig. 9 — Slices of space factor of 100-element spiral

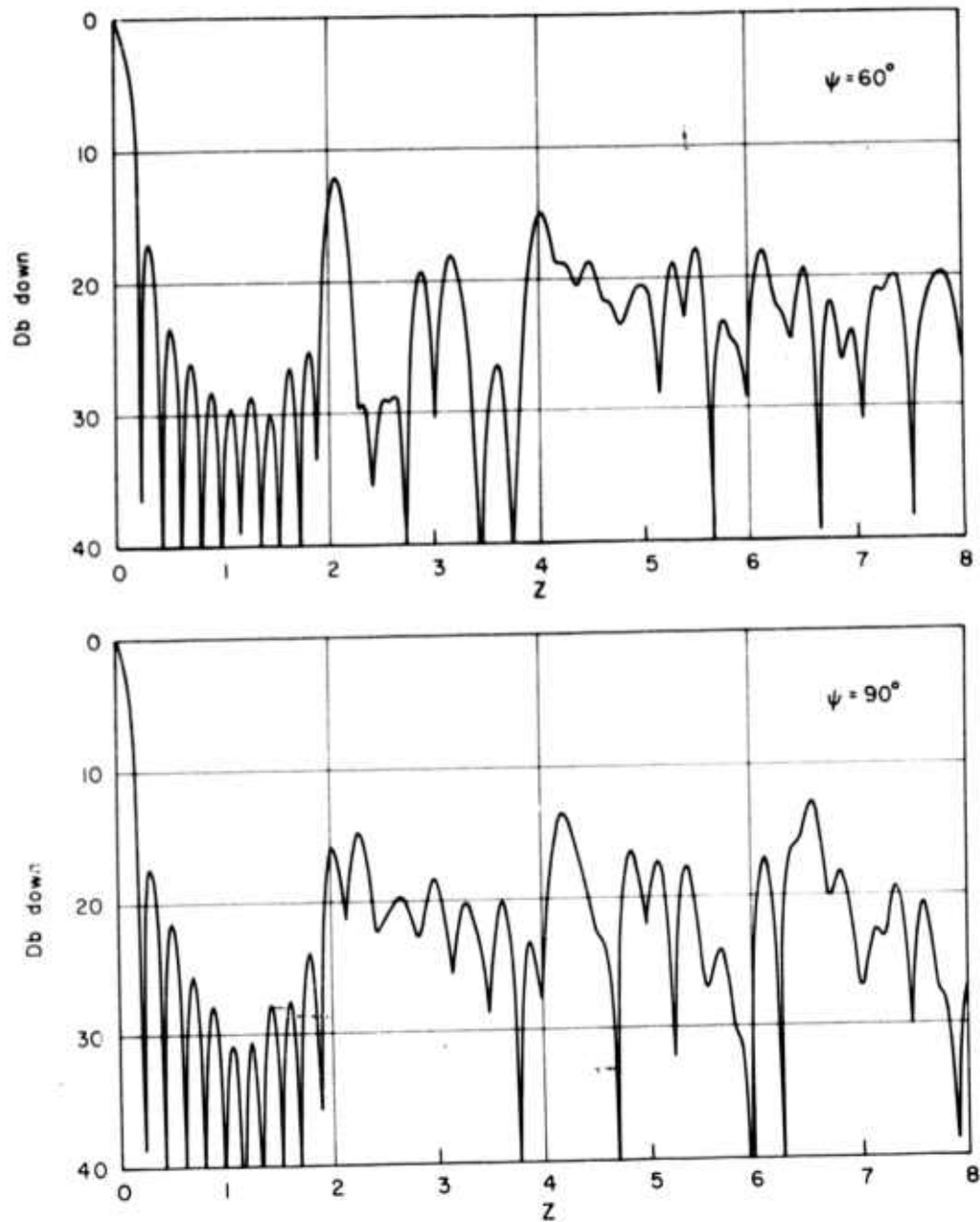


Fig. 9 — Slices of space factor of 100-element spiral

lobe has not been missed. The survey amounts to taking about one point for every three or four lobes, the samples being equally spaced while the lobe spacings are random.

The purpose of this discussion is not to suggest that the particular example presented is an outstanding array (after all, it has 17 db nearby sidelobes), but rather to illustrate the success of the spiral layout in avoiding grating lobes when spread or thinned to an average interelement space of 2λ , with full steering. As a comparison, notice that a 10 by 10 square array with full steering will produce a grating lobe at $\lambda/2$ spacing (at endfire it is bi-directional), and that its first sidelobe is 12 db down. The 17 db first sidelobe of the example is about what one expects from a uniformly excited circular area.

Appendix

 I_{jk} FOR INFINITESIMAL DIPOLES

The element factor for an infinitesimal dipole is $\sin^2 A$ (intensity), where A is the angle between the dipole axis and a vector to the distant point. If the dipoles are all parallel to the x-axis, then $\sin^2 A = 1 - \sin^2 \theta \cos^2 \varphi$ and

$$I_{jk} = \int_0^\pi |\sin \theta| d\theta \int_{-\pi}^\pi e^{\frac{2\pi i \sin \theta}{\lambda} [r_k \cos(\varphi - \varphi_k) - r_j \cos(\varphi - \varphi_j)]} (1 - \sin^2 \theta \cos^2 \varphi) d\varphi$$

The first step in evaluating I_{jk} is to rewrite the bracket in the exponent as

$$r_k \cos(\varphi - \varphi_k) - r_j \cos(\varphi - \varphi_j) = d_{jk} \cos(\varphi - \psi_{jk})$$

where

$$d_{jk}^2 = r_k^2 + r_j^2 - 2r_j r_k \cos(\varphi_k - \varphi_j) = (x_k - x_j)^2 + (y_k - y_j)^2$$

$$\cos \psi_{jk} = \frac{r_k \cos \varphi_k - r_j \cos \varphi_j}{d_{jk}} = \frac{x_k - x_j}{d_{jk}}$$

$$\sin \psi_{jk} = \frac{r_k \sin \varphi_k - r_j \sin \varphi_j}{d_{jk}} = \frac{y_k - y_j}{d_{jk}}$$

Shifting the φ - integral by an amount ψ_{jk} gives

$$I_{jk} = 2 \int_0^{\frac{\pi}{2}} \sin \theta d\theta \int_{-\pi}^{\pi} e^{\frac{2\pi i \sin \theta}{\lambda}} d_{jk}^{\cos \varphi} d\varphi$$

$$- 2 \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta \int_{-\pi}^{\pi} e^{\frac{2\pi i \sin \theta}{\lambda}} d_{jk}^{\cos \varphi} \cos^2(\varphi + \psi_{jk}) d\varphi$$

Now put $2\cos^2(\varphi + \psi_{jk}) = 1 + \cos 2\varphi \cdot \cos 2\psi_{jk} - \sin 2\varphi \sin 2\psi_{jk}$ and split one of the $\sin^3 \theta$ factors into $\sin \theta (1 - \cos^2 \theta)$. The result is

$$I_{jk} = 2 \int_0^{\frac{\pi}{2}} \sin \theta d\theta \int_{-\pi}^{\pi} e^{\frac{2\pi i \sin \theta}{\lambda}} d_{jk}^{\cos \varphi} d\varphi$$

$$- \int_0^{\frac{\pi}{2}} (\sin \theta - \sin \theta \cos^2 \theta) d\theta \int_{-\pi}^{\pi} e^{\frac{2\pi i \sin \theta}{\lambda}} d_{jk}^{\cos \varphi} d\varphi$$

$$- \cos 2\psi_{jk} \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta \int_{-\pi}^{\pi} e^{\frac{2\pi i \sin \theta}{\lambda}} d_{jk}^{\cos \varphi} \cos 2\varphi d\varphi$$

$$- \sin 2\psi_{jk} \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta \int_{-\pi}^{\pi} e^{\frac{2\pi i \sin \theta}{\lambda}} d_{jk}^{\cos \varphi} \sin 2\varphi d\varphi$$

The last integral vanishes because the integrand is an odd function of φ . The first two integrals simplify a bit, so that finally

$$\begin{aligned}
I_{jk} &= 2\pi \int_0^{\pi/2} \sin\theta J_0(2\pi d_{jk} \sin\theta/\lambda) d\theta \\
&+ 2\pi \int_0^{\pi/2} \sin\theta \cos^2\theta J_0(2\pi d_{jk} \sin\theta/\lambda) d\theta \\
&+ 2\pi \cos 2\psi_{jk} \int_0^{\pi/2} \sin^3\theta J_2(2\pi d_{jk} \sin\theta/\lambda) d\theta
\end{aligned}$$

Use has been made of the fact that

$$\frac{1}{\pi} \int_0^\pi e^{iz \cos\varphi} \cos n\varphi d\varphi = e^{i n \frac{\pi}{2}} J_n(z)$$

The three integrals in I_{jk} are special cases (see Ref. 11, p. 373) of

$$B(a, b, z) = \int_0^{\pi/2} J_a(z \sin\theta) \sin^{a+1}\theta \cos^{2b+1}\theta d\theta = \frac{2^b \Gamma(b+1)}{z^{b+1}} J_{a+b+1}(z)$$

Particular cases needed are

$$a = 0, \quad b = \frac{1}{2}$$

$$B(0, \frac{1}{2}, z) = \frac{\sqrt{2} \Gamma(3/2)}{z^{3/2}} J_{3/2}(z) = \frac{\sin z}{z^3} - \frac{\cos z}{z^2}$$

$$a = 0, \quad b = -\frac{1}{2}$$

$$B(0, -\frac{1}{2}, z) = \frac{\Gamma(1/2)}{\sqrt{2} z} J_{1/2}(z) = \frac{\sin z}{z}$$

and

$$a = 2, \quad b = -\frac{1}{2}$$

$$B(2, -\frac{1}{2}, z) = \frac{\Gamma(1/2)}{\sqrt{2} z} J_{5/2}(z) = \frac{3}{z^2} \left(\frac{\sin z}{z} - \cos z \right) - \frac{\sin z}{z}$$

Let $z_{jk} = 2\pi d_{jk}/\lambda$; then

$$\begin{aligned} \frac{I_{jk}}{2\pi} &= \frac{\sin z_{jk}}{z_{jk}} + \frac{\sin z_{jk}}{z_{jk}^3} - \frac{\cos z_{jk}}{z_{jk}^2} \\ &\quad + \cos 2\psi_{jk} \left[\frac{3}{z_{jk}^2} \left(\frac{\sin z_{jk}}{z_{jk}} - \cos z_{jk} \right) - \frac{\sin z_{jk}}{z_{jk}} \right] \end{aligned}$$

This expression is indeterminate for $j = k$ (and hence $z_{jk} = 0$); however, its limit as $z_{jk} \rightarrow 0$ is

$$I_{kk} = \frac{8\pi}{3}$$

This value is also obtained directly from the basic formula:

$$I_{kk} = 2 \int_0^{\pi/2} \sin \theta \, d\theta \int_{-\pi}^{\pi} (1 - \sin^2 \theta \cos^2 \varphi) d\varphi = \frac{8\pi}{3}$$

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