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INVARIANT IMBEDDING AND A RESOLVENT OF THE PHOTON-DIFFUSION EQUATION

Richard Bellman, Robert Kalaba and Sueo Ueno

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PREFACE

In this Memorandum the authors present further mathematical results from their study of radiative transfer, using invariant-imbedding techniques. This subject has important implications for meteorology, astrophysics, and nuclear-blast detection.

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SUMMARY

In the study of radiative transfer, the classical Boltzmann—type approach focuses attention on the statistical behavior of the particles within the medium. There are, however, analytical and computational advantages in the invariant—imbedding approach, which leads to the determination of the diffusely reflected and transmitted intensities in slab geometry. In the present paper it is shown that the invariant—imbedding technique is also extremely useful for the study of the internal intensity of radiation in a finite, homogeneous, flat layer. In other words, the technique provides us directly with a complete solution of scattering problems of light in a finite plane—parallel atmosphere.

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INVARIANT IMBEDDING AND A RESOLVENT OF THE PHOTON-DIFFUSION EQUATION

1. INTRODUCTION

While it is known that the invariant-imbedding technique is powerful for the study of the diffuse reflection and transmission problems of radiation in slab geometry, the usefulness for inquiring into the outward and inward intensities of radiation at any level in the medium is not so well understood (see Bellman and Kalaba [1]; Wing [12]).

Recently, Sobolev ([7], [8]) has shown that when the Milne type of a linear integral equation with an arbitrary source distribution is considered in the case of isotropic scattering, the solution is obtained by reducing the determination of the resolvent in terms of two optical depth arguments of the integral equation to that of the function $\Phi(\tau)$ depending only on a single argument τ. The partial differential equation of the resolvent was found by using the physical and probabilistic manipulation. The invariant-imbedding technique was applied to the solution of an integral equation for the emission probability distribution in a semi-infinite, homogeneous, flat layer, and then the resolvent of the photon-diffusion equation was expressed in terms of the fundamental function $\Phi(\tau)$ in connection with the integro-differential equation of the emergence

probability of a photon (see Ueno [9]). Furthermore, the invariant-imbedding approach was used to determine the radiation field within the high-temperature stagnation region ahead of the blunt-nosed model in a hypersonic, rarefied gas stream.

In this paper we shall show how the invariant-imbedding technique can be used to find the source function, in the radiation field with arbitrary distribution of emitting sources, by the agency of the resolvent of the photon-diffusion equation. Thus, by evaluating the requisite intensity inside the homogeneous medium according to the standard procedure, a complete solution of the transfer equation of radiation is obtained. Finally, the relationship between the scattering function and the resolvent is given by means of the emergence probability of photon.

It is of interest to mention that the principles of invariance can also be applied to find the intensity at any level τ as follows (see Chandrasekhar [4]; Horak [5]):

$$I(\tau,+\mu) = I^*_{\tau_1-\tau}(\tau,+\mu) + \frac{1}{2\mu}\int_0^1 S(\tau_1-\tau;\mu,\mu')I(\tau,-\mu')d\mu',$$

$$I(\tau,-\mu) = I^*_{\tau}(\tau,-\mu) + \frac{1}{2\mu}\int_0^1 S(\tau;\mu,\mu')I(\tau,+\mu')d\mu',$$

where $I^*_{\tau_1-\tau}(\tau,+\mu)$ $(0 < \mu \leq 1)$ is the intensity which would be emitted in the outward direction at any level if there were no layer between 0 and τ, and $I^*_{\tau}(\tau,-\mu)$

$(0 < \mu \leq 1)$ is that in the inward direction at any level if there were no layer from τ to τ_1. Provided that $I^*_{\tau_1-\tau}(\tau,+\mu)$, $I^*_{\tau}(\tau,-\mu)$, $S(\tau;\mu,\mu_0)$, and $S(\tau_1-\tau;\mu,\mu_0)$ are known, the procedure of successive approximation leads to the requisite internal intensities. However, the approximate values of $I(\tau,\pm\mu)$ for starting the procedure should be evaluated by solving approximately the transfer equation. In the one-dimensional case it is shown that the reflected and transmitted fluxes can be considered as fundamental functions from which all other functions can be derived (cf. Bellman, Kalaba, and Wing [2]). Then the present approach may be called a direct procedure for obtaining the internal flux, whereas the above procedure is an indirect one.

The powerfulness of the invariant-imbedding technique for obtaining a complete solution of the transfer problems of radiation in a finite flat layer lies in the fact that the back-scattering process does not play a role in the determination of the radiation field in slab geometry (see Uesugi [11]).

Later, the present approach may be applied to the determination of the perturbation velocity distribution in the linearized version of the Boltzmann transport equation, based on the Krook kinetic model in the classical Couette flow problem.

2. MILNE'S FIRST INTEGRAL EQUATION

Consider the diffuse reflection and transmission of parallel rays by a homogeneous flat layer of finite optical thickness $\tau_1 - \tau > 0$, scattering radiation isotropically.

Let a parallel beam of radiation, of net flux πF per unit area normal to itself, be falling on the surface $\tau = \tau_0$ of the medium in the direction $-\mu_0$ $(0 < \mu_0 \leq 1)$, where μ_0 denotes the cosine of the inclination to the inward normal at $\tau = \tau_0$. Following the notation of Chandrasekhar [4], the equation of transfer appropriate to the present case is written in the form

$$(2.1) \qquad \mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\omega \int_{-1}^{+1} I(\tau,\mu')d\mu' - \frac{\omega}{4} Fe^{-(\tau-\tau_0)/\mu_0},$$

where ω is the albedo for single scattering, i.e., the probability of survival of a photon.

Consider a photon-diffusion process in terms of the emission probability $p(\mu,\tau;\tau_0,\tau_1)$ that a photon absorbed at the level τ $(\tau_0 \leq \tau \leq \tau_1)$ will reappear in the direction $+\mu$ $(0 < \mu \leq 1)$ in the radiation emerging from the surface $\tau = \tau_0$. For convenience, we suppress the parameters τ_0 and τ_1 in $p(\mu,\tau;\tau_0,\tau_1)$.

Taking into account the random character of the photon-diffusion process, an integral equation governing $p(\mu,\tau)$ is expressed in the form (see [9])

$$(2.2) \qquad [1 - \omega\bar{\Lambda}]_\tau\{p(\mu,t)\} = \omega e^{-(\tau-\tau_0)/\mu},$$

where 1 on the left-hand side is an identity operator, and the truncated Hopf operator $\bar{\Lambda}$ is given by

$$(2.3) \qquad \bar{\Lambda}_\tau\{f(t)\} = \frac{1}{2}\int_{\tau_0}^{\tau_1} f(t)E_1(|t-\tau|)dt.$$

In Eq. (2.3), E_1 is the first exponential integral for the positive real argument

$$(2.4) \qquad E_1(\tau) = \int_0^1 e^{-\tau/\mu}\,\frac{d\mu}{\mu}.$$

With the aid of the combined operations method (see Busbridge [3]), it is readily shown that Eq. (2.2) is equal to the auxiliary equation, i.e., the Milne first integral equation for the source function $\mathfrak{J}(\tau,\mu)$.

Furthermore, consider the scattering of light in an emitting atmosphere of finite optical thickness $\tau_1 - \tau_0$ whose distribution of emitting sources is given by $B(\tau)$. The equation of transfer appropriate to this case takes the form

$$(2.5) \qquad \mu\,\frac{dI^*(\tau,\mu)}{d\tau} = I^*(\tau,\mu) - \mathfrak{J}(\tau),$$

where the source function $\mathfrak{J}(\tau)$ is

$$(2.6) \qquad \mathfrak{J}(\tau) = \frac{\omega}{2}\int_{-1}^{+1} I^*(\tau,\mu')d\mu' + B(\tau).$$

Let $I^*(\tau,+\mu)$, $0 < \mu \leq 1$, be the intensity of radiation at level τ directed towards the surface $\tau = \tau_0$, and let $I^*(\tau,-\mu)$, $0 < \mu \leq 1$, be that directed towards the surface $\tau = \tau_1$.

Then, as the formal solution of Eq. (2.5) we have

$$(2.7) \qquad I^*(\tau,+\mu) = \int_{\tau}^{\tau_1} \mathcal{J}(t)e^{-(t-\tau)/\mu}\,\frac{dt}{\mu},$$

$$(2.8) \qquad I^*(\tau,-\mu) = \int_{\tau_0}^{\tau} \mathcal{J}(t)e^{-(\tau-t)/\mu}\,\frac{dt}{\mu}.$$

Equations (2.7) and (2.8) show that once $\mathcal{J}(t)$ has been found, the inward and outward intensities at each level can be determined.

In what follows, it is shown that a resolvent of the photon-diffusion equation permits us to evaluate the source function $\mathcal{J}(t)$.

The Milne first integral equation for $\mathcal{J}(t)$ is given (see [3]) by

$$(2.9) \qquad [1 - \omega\Lambda]_{\tau}\{\mathcal{J}(t)\} = B(\tau).$$

With the aid of the resolvent $K(\tau,t;\tau_0,\tau_1)$, Eq. (2.9) takes the form

$$(2.10) \qquad \mathcal{J}(\tau) = B(\tau) + \int_{\tau_0}^{\tau_1} K(\tau,t;\tau_0,\tau_1)B(t)dt,$$

where the resolvent $K(\tau,t;\tau_0,\tau_1)d\tau dt$ is the probability that a photon emitted in the optical depth interval

$(t, t + dt)$ will be re-emitted between the optical depths τ and $\tau + d\tau$ after one or more scattering processes. Then, given the resolvent K-function, Eq. (2.10) enables us to evaluate the source function (see Fig. 1).

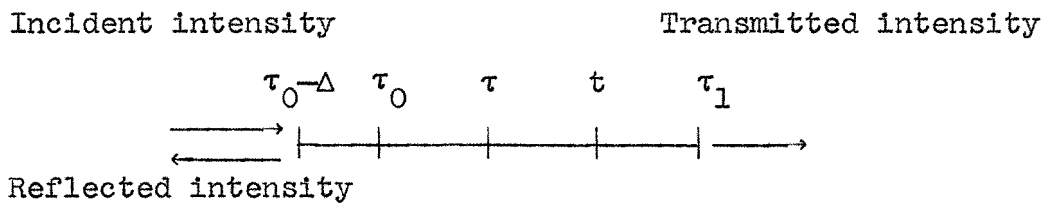

Fig. 1 Reflection and transmission of radiation in rod geometry

3. RESOLVENT OF THE PHOTON-DIFFUSION EQUATION

Let $\Phi(x;\tau_0,\tau_1)$ be the probability that a photon absorbed at $\tau = x$ $(\tau_0 \leq x \leq \tau_1)$ will reappear in the radiation emerging from the surface $\tau = \tau_0$ after successive scattering processes.

On replacing τ in Eq. (2.2) by x, multiplying it by $1/(2\mu)$ and integrating with respect to μ over $(0,1)$, we have

$$\Phi(x;\tau_0,\tau_1) = g(x-\tau_0) + \int_{\tau_0}^{\tau_1} \Phi(x';\tau_0,\tau_1)k(x',x)dx', \tag{3.1}$$

where $\tau_0 \leq x \leq \tau_1$;

$$\Phi(x;\tau_0,\tau_1) = \frac{1}{2}\int_0^1 p(\mu,x)\frac{d\mu}{\mu}, \tag{3.2}$$

$$g(x-\tau_0) = \frac{\omega}{2} E_1(x-\tau_0), \quad k(x',x) = \frac{\omega}{2} E_1(|x'-x|).$$

Alternatively,

$$(3.3)\qquad \Phi(x;\tau_0,\tau_1) = g(x-\tau_0) + \int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)g(x'-\tau_0)dx',$$

where $K(x,x';\tau_0,\tau_1)$ is the resolvent.

It is manifest that in Eqs. (3.1) and (3.3), $k(x,x')$ and $K(x,x';\tau_0,\tau_1)$ together are symmetric with respect to x and x' (see Mikhlin [6]).

Adding a layer of infinitesimal optical thickness Δ to the surface $\tau = \tau_0$ of the medium, we now consider the invariant-imbedding procedure.

Let

$$(3.4)\qquad \varphi(x) = g(x-\tau_0+\Delta) + \int_{\tau_0-\Delta}^{\tau_1} \varphi(x')k(x',x)dx'.$$

On the other hand, it is well known that the resolvent fulfills an integral equation given [6] by

$$(3.5)\qquad K(x,y;\tau_0,\tau_1) = k(x,y) + \int_{\tau_0}^{\tau_1} K(x,y';\tau_0,\tau_1)k(y',y)dy'.$$

Then, making use of Eq. (3.5), Eq. (3.4) reduces to

$$\begin{aligned}(3.6)\qquad \varphi(x) = {} & g(x+\Delta-\tau_0) + \varphi(\tau_0-\theta\Delta)k(\tau_0-\theta\Delta,x)\Delta \\ & + \int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)g(x'+\Delta-\tau_0)dx' \\ & + \varphi(\tau_0-\theta_0\Delta)\Delta\int_{\tau_0}^{\tau_1} K(x,y;\tau_0,\tau_1)k(\tau_0-\theta_0\Delta,y)dy \\ & + \mathcal{O}(\Delta),\end{aligned}$$

where $0 < \theta$ (or θ_0) < 1. The details of the computation will be referred to in the Appendix.

On the other hand, recalling Eq. (3.3), we get

$$(3.7)\qquad \begin{aligned}\varphi(x) &= g(x-\tau_0+\Delta) + \int_{\tau_0-\Delta}^{\tau_1} K(x,x';\tau_0-\Delta,\tau_1)g(x'-\tau_0+\Delta)dx' \\ &= g(x+\Delta-\tau_0) + \int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)g(x'-\tau_0+\Delta)dx' \\ &\quad - \Delta\int_{\tau_0}^{\tau_1} \frac{\partial K}{\partial\tau_0}(x,x';\tau_0,\tau_1)g(x'-\tau_0+\Delta)dx' \\ &\quad + K(x,\tau_0-\theta\Delta;\tau_0-\Delta,\tau_1)g(\Delta(1-\theta))\Delta + \mathcal{O}(\Delta),\end{aligned}$$

where $0 < \theta < 1$.

With Eq. (3.3), the combination of Eqs. (3.6) and (3.7) provides us with

$$(3.8)\qquad \begin{aligned}&\varphi(\tau_0-\theta\Delta)k(\tau_0-\theta\tau,x) \\ &+ \varphi(\tau_0-\theta_0\Delta)\int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)k(\tau_0-\theta_0\tau,x')dx' \\ &= K(x,\tau_0-\theta\Delta;\tau_0-\Delta,\tau_1) \\ &\cdot \left[\varphi(\tau_0-\theta\Delta) - \int_{\tau_0-\Delta}^{\tau_1} K(\tau_0-\theta\Delta,x';\tau_0-\Delta,\tau_1)g(x'-\tau_0+\Delta)dx'\right] \\ &- \int_{\tau_0}^{\tau_1} \frac{\partial K}{\partial\tau_0}(x,x';\tau_0,\tau_1)g(x'-\tau_0+\Delta)dx' + \mathcal{O}(\Delta).\end{aligned}$$

Passing to the limit as $\Delta \to 0$, from Eq. (3.8) we have

$$(3.9) \qquad \frac{\partial K}{\partial \tau_0}(x,y;\tau_0,\tau_1) = -K(x,\tau_0;\tau_0,\tau_1)K(\tau_0,y;\tau_0,\tau_1).$$

Furthermore, in what follows, we shall derive another type of partial differential equation. In a manner similar to that given before, we shall add a layer of small optical thickness Δ to the surface $\tau = \tau_1$.

Let

$$(3.10) \qquad \psi(x) = g(x-\tau_0) + \int_{\tau_0}^{\tau_1+\Delta} \psi(x')k(x',x)dx'.$$

Then, allowing for Eq. (3.5), Eq. (3.10) reduces to

$$(3.11) \qquad \begin{aligned} \psi(x) = {} & g(x-\tau_0) + \psi(\tau_1+\Delta)k(\tau_1+\Delta,x)\Delta \\ & + \int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)g(x'-\tau_0)dx' + \psi(\tau_1+\theta_0\Delta)\Delta \\ & \cdot \int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)k(\tau_1+\theta_0\Delta,x')dx + \mathcal{O}(\Delta), \end{aligned}$$

where $0 < \theta$ (or θ_0) < 1.

On the other hand, we have

$$(3.12) \qquad \begin{aligned} \psi(x) = {} & g(x-\tau_0) + \int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)g(x'-\tau_0)dx' \\ & + \Delta \int_{\tau_0}^{\tau_1} \frac{\partial K}{\partial \tau_1}(x,y;\tau_0,\tau_1)g(y-\tau_0)dy \\ & + K(x,\tau_1+\theta\Delta;\tau_0,\tau_1+\Delta)g(\tau_1+\theta\Delta-\tau_0)\Delta + \mathcal{O}(\Delta), \end{aligned}$$

where $0 < \theta < 1$.

Making use of Eq. (3.3) and combining Eqs. (3.11) and (3.12), we get

$$(3.13)\qquad \psi(\tau_1+\theta\Delta)k(\tau_1+\theta\Delta,x)$$
$$+\ \psi(\tau_1+\theta_0\Delta)\Delta\int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)k(\tau_1+\theta_0\Delta,x')dx'$$
$$= K(x,\tau_1+\theta\Delta;\tau_0,\tau_1+\Delta)$$
$$\cdot\left[\psi(\tau_1+\theta\Delta) - \int_{\tau_0}^{\tau_1+\Delta} K(\tau_1+\theta\Delta,y;\tau_0,\tau_1+\Delta)g(y-\tau_0)dy\right]$$
$$+\ \int_{\tau_0}^{\tau_1} \frac{\partial K}{\partial\tau_1}(x,y;\tau_0,\tau_1)g(y-\tau_0)dy + \mathcal{O}(\Delta).$$

Using Eq. (3.1) and putting $\Delta \to 0$, from Eq. (3.13) we have

$$(3.14)\qquad \frac{\partial K}{\partial\tau_1}(x,y;\tau_0,\tau_1) = K(x,\tau_1;\tau_0,\tau_1)K(\tau_1,y;\tau_0,\tau_1).$$

Taking into account the optical homogeneity of the medium and putting $\tau_0 = 0$, Eq. (3.14) becomes

$$(3.15)\qquad \frac{\partial K}{\partial\tau_1}(x,y;\tau_1) = K(\tau_1-x,0;\tau_1)K(0,\tau_1-y;\tau_1),$$

which is equivalent to that yielded by Sobolev [8].

Because of the homogeneous medium, we get

$$(3.16)\qquad K(x+\Delta,y+\Delta;\tau_0,\tau_1) = K(x,y;\tau_0-\Delta,\tau_1-\Delta).$$

Then,

$$(3.17) \qquad \frac{\partial K}{\partial \tau_0} + \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} + \frac{\partial K}{\partial \tau_1} = 0.$$

On recalling Eqs. (3.9) and (3.14), from Eq. (3.17) we obtain

$$(3.18) \qquad \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} = K(x,\tau_0;\tau_0,\tau_1)K(\tau_0,y;\tau_0,\tau_1) - K(x,\tau_1;\tau_0,\tau_1)K(\tau_1,y;\tau_0,\tau_1).$$

When we allow for the homogeneity of the medium with respect to the optical depth, and write

$$(3.19) \qquad K(x,\tau_0;\tau_0,\tau_1) = K(x-\tau_0,0;\tau_1-\tau_0),$$

$$K(x,\tau_1;\tau_0,\tau_1) = K(\tau_1-x,0;\tau_1-\tau_0),$$

Eq. (3.18) becomes

$$(3.20) \qquad \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} = K(x-\tau_0,0;\tau_1-\tau_0)K(0,y-\tau_0;\tau_1-\tau_0) - K(\tau_1-x,0;\tau_1-\tau_0)K(0,\tau_1-y;\tau_1-\tau_0).$$

Thus, for $x > y$ we get

$$(3.21) \qquad K(x,y;\tau_0,\tau_1) = K(x-y,0;\tau_1-\tau_0) + \int_{\tau_0}^{y} \Big[K(t+x-y-\tau_0,0;\tau_1-\tau_0)K(0,t-\tau_0;\tau_1-\tau_0) - K(\tau_1-t-x+y,0;\tau_1-\tau_0)K(0,\tau_1-t;\tau_1-\tau_0)\Big]dt.$$

Then, the function $K(x,y;\tau_0,\tau_1)$ of the two arguments, x and y, is expressed in terms of the function $K(x-\tau_0,0;\tau_1-\tau_0)$ of a single argument x. When $\tau_0 = 0$, Eqs. (3.20) and (3.21) reduce to those given by Sobolev [8], with the aid of the probabilistic approach.

From Eq. (3.5) we have

$$(3.22) \qquad K(x-\tau_0,0;\tau_1-\tau_0) = g(x-\tau_0) + \int_{\tau_0}^{\tau_1} K(t-\tau_0,0;\tau_1-\tau_0)k(t,x)dt.$$

Thus, given $g(x-\tau_0)$, the one-parametric resolvent $K(x-\tau_0,0;\tau_1-\tau_0)$ can be determined from Eq. (3.22).

In a manner similar to that given by the authors in a preceding paper [9], we shall verify an identity $K(x-\tau_0,0;\tau_1-\tau_0) = \Phi(x;\tau_0,\tau_1)$. Following the notation of Chandrasekhar [4], $X(\mu;\tau_0,\tau_1)$ is given by

$$(3.23) \qquad X(\mu;\tau_0,\tau_1) = 1 + \frac{1}{2}\int_0^1 S(\tau_0,\tau_1;\mu,\mu')\,\frac{d\mu'}{\mu'},$$

where the scattering function S is

$$(3.24) \qquad S(\tau_0,\tau_1;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} p(\mu_0;t)e^{-(t-\tau_0)/\mu}dt.$$

Putting $\tau = \tau_0$ in Eq. (2.2) and recalling Eq. (2.10), we have

$$(3.25) \qquad p(\mu,\tau_0) = \omega + \omega\int_{\tau_0}^{\tau_1} K(\tau_0,t;\tau_0,\tau_1)e^{-(t-\tau_0)/\mu}dt.$$

Writing

$$(3.26) \qquad p(\mu,\tau_0) = \omega X(\mu;\tau_0,\tau_1),$$

from Eq. (2.25) we get

$$(3.27) \qquad X(\mu;\tau_0,\tau_1) = 1 + \int_0^{\tau_1-\tau_0} K(\tau_0,t+\tau_0;\tau_0,\tau_1)e^{-t/\mu}dt.$$

On the other hand, Eq. (3.23) becomes

$$(3.28) \qquad X(\mu;\tau_0,\tau_1) = 1 + \int_0^{\tau_1-\tau_0} \Phi(t+\tau_0;\tau_0,\tau_1)e^{-t/\mu}dt.$$

Recalling Eq. (3.19), the comparison of Eqs. (3.27) and (3.28) yields

$$(3.29) \qquad K(x-\tau_0,0;\tau_1-\tau_0) = \Phi(x;\tau_0,\tau_1).$$

Then, Eq. (3.21) for $x > y$ becomes

$$(3.30) \qquad K(x,y;\tau_0,\tau_1) = \Phi(x-y;\tau_0,\tau_1) + \int_{\tau_0}^{y} \Big[\Phi(t+x-y-\tau_0;\tau_0,\tau_1)\Phi(t-\tau_0;\tau_0,\tau_1) - \Phi(\tau_1-t-x+y;\tau_0,\tau_1)\Phi(\tau_1-t;\tau_0,\tau_1)\Big]dt.$$

Recently, expanding the function $\Phi(x;0,\tau_1)$ by a power series of ω

$$(3.31) \qquad \Phi(x;0,\tau_1) = \sum_{i=1} \omega^i\Phi_i(x,\tau_1),$$

Sobolev and Minin [8] computed numerically $\Phi_i(x,\tau_1)$ for

several values of τ_1. They mention that the approximation of the above result for the computation of the resolvent is of practical interest only when just a few terms of the expansion formula are sufficient for the accuracy required.

APPENDIX

Allowing for Eq. (3.5), recalling the symmetrical property of $K(x,x';\tau_0,\tau_1)$ and $k(x,x')$ with respect to x and x', and inverting the order of integration, we have

$$\text{(A.1)} \quad \int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)dx' \int_{\tau_0}^{\tau_1} \varphi(y)k(y,x')dy$$
$$= \int_{\tau_0}^{\tau_1} k(x,x')dx' \int_{\tau_0}^{\tau_1} \varphi(y)k(y,x')dy$$
$$+ \int_{\tau_0}^{\tau_1} dx' \int_{\tau_0}^{\tau_1} K(x,y;\tau_0,\tau_1)k(y,x')dy$$
$$\cdot \int_{\tau_0}^{\tau_1} \varphi(y')k(y',x')dy',$$

and

$$\text{(A.2)} \quad \int_{\tau_0}^{\tau_1} \left[g(x'+\Delta-\tau_0) + \varphi(\tau_0-\theta_0\Delta)k(\tau_0-\theta_0\Delta,x')\Delta\right]dx'$$
$$\cdot \int_{\tau_0}^{\tau_1} K(x,y;\tau_0,\tau_1)k(y,x')dy$$
$$= \int_{\tau_0}^{\tau_1} \left[\varphi(x') - \int_{\tau_0}^{\tau_1} \varphi(x'')k(x'',x')dx''\right]dx'$$
$$\cdot \int_{\tau_0}^{\tau_1} K(x,y;\tau_0,\tau_1)k(y,x')dy$$
$$= \int_{\tau_0}^{\tau_1} K(x,y;\tau_0,\tau_1)dy \int_{\tau_0}^{\tau_1} \varphi(x')k(x',y)dx' -$$

$$-\int_{\tau_0}^{\tau_1} dx' \int_{\tau_0}^{\tau_1} \varphi(x'')k(x'',x')dx'' \int_{\tau_0}^{\tau_1} K(x,y;\tau_0,\tau_1)k(y,x')dy$$
$$= \int_{\tau_0}^{\tau_1} k(x,x')dx' \int_{\tau_0}^{\tau_1} \varphi(y)k(y,x')dy.$$

Making use of Eqs. (A.1) and (A.2), Eq. (3.4) becomes

$$\text{(A.3)} \quad \varphi(x) = g(x+\Delta-\tau_0) + \varphi(\tau_0-\theta\Delta)k(\tau_0-\theta\Delta,x)\Delta$$
$$+ \int_{\tau_0}^{\tau_1} \varphi(x')k(x',x)dx' + \mathcal{O}(\Delta)$$
$$= g(x+\Delta-\tau_0) + \varphi(\tau_0-\theta\Delta)k(\tau_0-\theta\Delta,x)\Delta$$
$$+ \int_{\tau_0}^{\tau_1} \Big[g(x'+\Delta-\tau_0) + \varphi(\tau_0-\theta_0\Delta)k(\tau_0-\theta_0\Delta,x')\Delta$$
$$+ \int_{\tau_0}^{\tau_1} \varphi(y)k(y,x')dy\Big]$$
$$\cdot \Big[K(x,x';\tau_0,\tau_1) - \int_{\tau_0}^{\tau_1} K(x,y;\tau_0,\tau_1)k(y,x')dy\Big]dx'$$
$$+ \mathcal{O}(\Delta)$$
$$= g(x+\Delta-\tau_0) + \varphi(\tau_0-\theta\Delta)k(\tau_0-\theta\Delta,x)\Delta$$
$$+ \int_{\tau_0}^{\tau_1} K(x,x';\tau_0,\tau_1)g(x'+\Delta-\tau_0)dx'$$
$$+ \varphi(\tau_0-\theta_0\Delta)\Delta \int_{\tau_0}^{\tau_1} K(x,y;\tau_0,\tau_1)k(\tau_0-\theta_0\Delta,y)dy$$
$$+ \mathcal{O}(\Delta).$$

The above result coincides with Eq. (3.6).

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