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THE SCATTERING OF ELECTROMAGNETIC WAVES FROM PLASMA CYLINDERS: PART II

Phyllis Greifinger

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THE SCATTERING OF ELECTROMAGNETIC WAVES FROM PLASMA CYLINDERS: PART II

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PREFACE

This Memorandum is one result of a continuing investigation of the radar returns from missile wakes. In a companion Memorandum, RM-3573-ARPA, The Scattering of Electromagnetic Waves from Plasma Cylinders: Part I, a general theory was developed for the short-wavelength limit of the electromagnetic radiation by radially varying electron-density distributions. This Memorandum considers the techniques for calculating the long-wavelength limit of the scattering.

The work is part of RAND's study of midcourse phenomenology, conducted under the Advanced Research Projects Agency's Defender program.

SUMMARY

In this Memorandum we consider the long-wavelength limit of the scattering of electromagnetic radiation in the transverse magnetic mode by infinite plasma cylinders with radially varying electron-density distributions. Techniques for calculating the scattering cross section are described for both increasing and decreasing electron-density distributions. Resonance phenomena peculiar to increasing electron-density distributions are discussed in some detail with specific numerical examples.

Finally, we present a résumé of methods one can use to calculate the scattering cross section for radiation of arbitrary wavelength for monotonically decreasing electron-density distributions. Examples are given for the Gaussian electron distribution.

ACKNOWLEDGMENT

The author is deeply grateful to Jeannine McGann for programming all the machine calculations and for preparing all the figures.

CONTENTS

SYMBOLS

B = amplitude factor in wave function $\psi_{oK}(y)$

C = Euler's constant

C_n = coefficient of ℓny in asymptotic form of $Y_n(y)$

c = velocity of light

D_n = constant term in asymptotic form of $Y_n(y)$

E = electric field

e = electronic charge

${}_1F_1(a; b; Z)$ = confluent hypergeometric function

$|f(\varphi)|^2$ = differential-scattering cross section

$|f_B(\varphi)|^2$ = differential-scattering cross section in Born approximation

$|f_G(\varphi)|^2$ = geometrical-optics-scattering cross section

$g(y)$ = electron-density distribution; defined by

$$g(y) = \frac{n_e - n_{e\infty}}{n_{eo} - n_{e\infty}}$$

$J_m(Ky)$ = Bessel function regular at origin

K = $k\rho_o$

k = $\dfrac{\sqrt{\omega^2 - \omega^2_{p_\infty}}}{c}$

m = integer

m_e = electron mass

$N_m(Ky)$ = Neumann function

n_e = electron-number density

$S(\varphi)$ = angular distribution of scattered radiation

$W_K(y)$ = wave function for $m = 0$ and $\beta = 0$

$Y_n(y)$ = coefficient of β^n in power-series expansion of $\psi_{oo}(y)$

y = ρ/ρ_o

α = $\dfrac{\omega^2_{p_o} - \omega^2_{p_\infty}}{\omega^2 - \omega^2_{p_\infty}}$

β = $K^2\alpha$

γ_m = $\left[\dfrac{d\psi_m}{dy}\dfrac{1}{\psi_m}\right]_{y=1}$

$\dfrac{1}{\gamma}$ = constant term in $W_o(y)$

ϵ_m = 1 when $m = 0$; = 2 when $m \neq 0$

η_m = phase shift of m^{th} partial wave

ρ = perpendicular distance from cylinder axis

ρ_o = characteristic radius of scattering region

φ = scattering angle

$\psi_m(y)$ = m^{th} partial wave function

ω = frequency of electromagnetic wave

ω_p = plasma frequency

ω_{p_o} = plasma frequency on cylinder axis ($\rho = 0$)

ω_{p_∞} = plasma frequency as $\rho \to \infty$

I. INTRODUCTION

In a previous Memorandum,* hereafter referred to as Part I, we developed a general theory for the scattering of electromagnetic radiation by radially varying electron-density distributions under the conditions that the incident wave was a plane wave traveling perpendicular to the cylinder axis and polarized parallel to the cylinder axis, the electron-density distribution was monotonic, and collisions were negligible. Techniques for calculating the cross section in the short-wavelength limit were described and applied to a few specific cases.

In this Memorandum we will consider the long-wavelength limit of the scattering. The conditions of the scattering and, therefore, the formulation of the problem are the same as in Part I. However, the techniques appropriate for calculating the scattering cross section differ in the long- and short-wavelength limits.

In our consideration of the long-wavelength limit, we will place special emphasis on the scattering by monotonically increasing electron-density distributions (analogous to the scattering by attractive potentials in particle-scattering theory), since distributions of this type have not been previously considered in the literature except in the small perturbation limit. Such distributions can give rise to anomalies in the scattering not encountered with decreasing electron-density distributions. These anomalies are associated with partial-wave resonances, familiar in the

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*Phyllis Greifinger, The Scattering of Electromagnetic Waves From Plasma Cylinders: Part I, The RAND Corporation, RM-3573-ARPA, August 1963.

analogous situation of the scattering of low-energy particles from spherically symmetric attractive potentials.

Despite the emphasis placed on the increasing electron-density distributions, the methods described for calculating the cross section in the long-wavelength limit will be equally applicable to increasing and decreasing electron-density distributions.

II. REVIEW OF PARTIAL-WAVE EXPANSION

The formulation of the problem, the expansion of the wave into partial waves, and the expression of the scattering cross section in terms of the phase shifts of the partial waves were presented in some detail in Part I. We will present a brief review here, with a few changes in notation which are introduced for convenience in considering the long-wavelength limit of the scattering.

Dropping the time-dependent factor $e^{-i\omega t}$, the wave equation in cylindrical coordinates ρ, Z, and φ for the electric field $E = E_z$ is

$$\frac{\partial^2 E}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E}{\partial \varphi^2} + \frac{\left(\omega^2 - \omega_p^2\right)}{c^2} E = 0 \qquad (1)$$

$$\omega_p = \sqrt{\frac{4\pi n_e e^2}{m_e}} = \text{plasma frequency}$$

Letting

$$y = \rho/\rho_o$$

$$K^2 = \left(\omega^2 - \omega_{p\infty}^2\right)\frac{\rho_o^2}{c^2}$$

$$\beta = \left(\omega_{po}^2 - \omega_{p\infty}^2\right)\frac{\rho_o^2}{c^2}$$

$$g(y) = \frac{n_e - n_{e\infty}}{n_{eo} - n_{e\infty}}$$

Eq. (1) becomes

$$\frac{\partial^2 E}{\partial y^2} + \frac{1}{y}\frac{\partial E}{\partial y} + \frac{1}{y^2}\frac{\partial^2 E}{\partial \varphi^2} + \left[K^2 - \beta g(y)\right] E = 0 \qquad (2)$$

The characteristic radius of the scattering region is ρ_o, and the subscripts 0 and ∞ on n_e and ω_p refer to $\rho = 0$ and $\rho = \infty$, respectively. The dimensionless parameters K and β are related to the wave number k and the parameter α of Part I by

$$K = k\rho_o$$
$$\beta = k^2 \rho_o^2 \alpha = K^2 \alpha \qquad (3)$$

The parameter β is not a function of the frequency of the radiation. For monotonic electron-density distributions, it depends upon the difference between the maximum and minimum electron densities as well as on the size of the scattering region. It is a more convenient parameter than α for small K, since it remains finite as $K \to 0$. The small-K limit refers to $\sqrt{1 - \left(\omega_{p\infty}/\omega\right)^2}\left(\omega\rho_o/c\right) < 1$. Note that this situation can occur even if $\omega\rho_o/c$ is large, provided that ω is not much larger than $\omega_{p\infty}$. For example, for the ionosphere, ω_p is of the order of 10^{+8} rad/sec ($n_e \sim 3 \times 10^5/cm^3$). Large negative values of β can occur for electron-density "holes" such as the partially evacuated region behind a hypersonic body moving through the ionosphere if the body radius is larger than a few meters. For K to be less than one under these conditions, we require radiation very close in frequency to the ionospheric plasma frequency, i.e., of the order of 20 Mc/sec.

Expanding E in a Fourier series, we get

$$E = \sum_{m=0}^{\infty} \epsilon_m \cos (m\varphi)\, \psi_m(y)$$

$$\epsilon_m = 1 \qquad m = 0 \tag{4}$$

$$\epsilon_m = 2 \qquad m \neq 0$$

The function $\psi_m(y)$, which is related to $u_m(y)$ of Part I by $\psi_m(y) = \frac{u_m(y)}{\sqrt{Ky}}$, satisfies the differential equation

$$\psi_m''(y) + \frac{1}{y}\psi_m'(y) + \left[K^2 - \beta g(y) - \frac{m^2}{y^2}\right]\psi_m(y) = 0 \tag{5}$$

with the boundary conditions

$$\psi_m(o) \text{ is finite}$$

$$\psi_{ms} = \psi_m - \psi_{mi} \rightarrow \text{constant} \times \frac{e^{iKy}}{\sqrt{y}}$$

$$\text{as } y \rightarrow \infty$$

The subscripts s and i refer to the scattered and incident wave, respectively. If the asymptotic form of ψ_m is written as

$$\psi_m \rightarrow \left(\frac{2}{\pi K y}\right)^{1/2} e^{im\pi/2}\, e^{-i\eta_m} \cos\left(Ky - \frac{m\pi}{2} - \frac{\pi}{4} - \eta_m\right) \tag{6}$$

where η_m is the phase shift of the m^{th} partial wave, the scattered field E_s for large y becomes

$$E_s = \frac{e^{i(Ky-\pi/4)}}{\sqrt{2\pi Ky}} \sum_{m=0}^{\infty} \epsilon_m \cos m\varphi \left(e^{-2i\eta_m} - 1\right) \equiv f(\varphi) \frac{e^{iKy}}{\sqrt{y\rho_o}} \tag{7}$$

The differential-scattering cross section $|f(\varphi)|^2$ then is given by

$$|f(\varphi)|^2 = \frac{\rho_o}{2\pi K} \left| \sum_{m=0}^{\infty} \epsilon_m \cos (m\varphi) \left(e^{-2i\eta_m} - 1\right) \right|^2 \tag{8}$$

It was shown in Section IV of Part I that the smaller the K, the fewer the phase shifts which contribute to the scattering. As $K \to 0$, we can generally expect all the phase shifts except η_o to be negligible. The resultant scattering will be isotropic, with $|f(\varphi)|^2$ given by

$$|f(\varphi)|^2 \simeq \frac{2\rho_o}{\pi K} \sin^2 \eta_o \tag{9}$$

As we will see, exceptions can arise when it is necessary to consider one or more higher phase shifts, even as $K \to 0$. The exceptions occur when either $|\eta_o|$ is an integral multiple of π, causing $\sin \eta_o$ to vanish, or when one of the higher m partial waves is in resonance, causing a phase shift which is an odd multiple of $\pi/2$.

If K is small but finite and we do not have a resonance situation for $m > 1$, we need consider only η_o and η_1. The differential cross section $|f(\varphi)|^2$ becomes

$$|f(\varphi)|^2 \simeq \frac{2\rho_o}{\pi K} \Big[\sin^2 \eta_o + 4 \cos^2 \varphi \sin^2 \eta_1 + 4 \cos \varphi \left(\sin^2 \eta_o \sin^2 \eta_1 + \sin \eta_o \cos \eta_o \sin \eta_1 \cos \eta_1\right)\Big] \tag{10}$$

We can see from Eq. (10) that the ratio of the contribution of the $m = 1$ term to the $m = 0$ term to the total cross section

$$\int_0^{2\pi} \left| f(\varphi) \right|^2 d\varphi$$

is proportional to $\sin^2 \eta_1$. However, the ratio of the largest angle-dependent term to the constant term in the differential cross section is proportional to $\sin \eta_1$. Thus, the partial wave $m = 1$ generally manifests itself in the differential cross section at a smaller value of K than that at which it becomes significant in the total cross section. For example, if $\eta_o = 45$ deg and $\eta_1 = 4.5$ deg, the $m = 1$ wave contributes only about 2.4 per cent to the total scattering, while it makes the forward scattering ($\varphi = 0$) about twice as large as the backward scattering ($\varphi = \pi$).

III. SMALL-K PHASE SHIFTS

FINITE-RADIUS CYLINDER SOLUTIONS

Let us consider functions g(y) which vanish for y > 1. For such cases, the Eq. (5) solutions inside and outside the cylinder are matched at the boundary; for the T.M. mode, both $\psi_m(y)$ and $\psi'_m(y)$ are continuous at y = 1. The phase shifts η_m are then given by

$$\cot \eta_m = \frac{KN'_m(K) - \gamma_m N_m(K)}{\gamma_m J_m(K) - KJ'_m(K)}$$

$$\gamma_m = \left[\frac{1}{\psi_m}\frac{d\psi_m}{dy}\right]_{y=1} \tag{11}$$

The functions $J_m(K)$ and $N_m(K)$ are Bessel and Neumann functions, respectively. The γ_m's are found from the solutions to Eq. (5) inside the cylinder; for each m, the required solution is that which is finite at y = 0.

For K << 1, we have approximately

$$\cot \eta_o \cong \frac{2}{\pi} \frac{\left[1 + \gamma_o \, \ell n\left(\frac{2}{Ke^C}\right)\right]}{\gamma_o + \frac{K^2}{2}} \tag{12a}$$

$$C = \text{Euler's constant} \tag{12b}$$

$$\cot \eta_m = \frac{-(m + \gamma_m)N_m(K) + KN_{m-1}(K)}{(\gamma_m - m)J_m(K) + KJ_{m+1}(K)} \qquad m \geq 1 \tag{12c}$$

$$\cot \eta_m \cong \frac{m!(m-1)!}{\pi} \frac{(\gamma_m + m)}{(\gamma_m - m)} \left(\frac{2}{K}\right)^{2m} \tag{12d}$$

We will discuss two cases for which we can obtain analytic expressions for the γ_m's.

Homogeneous Cylinder

Equation (5) can be solved analytically for a step-function electron-density distribution (homogeneous cylinder):

$$g(y) = 1 \qquad y \leq 1$$
$$= 0 \qquad y > 1$$

The solutions are the Bessel functions $J_m\left(\sqrt{K^2 - \beta}\, y\right)$ regular at the origin. The γ_m's are given by

$$\gamma_m = \sqrt{K^2 - \beta}\, \frac{J'_m\left(\sqrt{K^2 - \beta}\right)}{J_m\left(\sqrt{K^2 - \beta}\right)} \tag{13}$$

Using the recurrence relationships of the Bessel functions, we get for $m \geq 1$

$$\frac{\gamma_m + m}{\gamma_m - m} = - \frac{J_{m-1}\left(\sqrt{K^2 - \beta}\right)}{J_{m+1}\left(\sqrt{K^2 - \beta}\right)} \tag{14}$$

so that an approximate expression for $\cot \eta_m$ for $K \ll 1$ and $m \geq 1$ is

$$\cot \eta_m \cong - \frac{m!(m-1)!}{\pi} \left(\frac{2}{K}\right)^{2m} \frac{J_{m-1}\left(\sqrt{K^2 - \beta}\right)}{J_{m+1}\left(\sqrt{K^2 - \beta}\right)} \tag{15}$$

For a given small K and $\beta < K^2$, there are maxima and minima for $\sin^2 \eta_m$ ($m \geq 1$) which occur at the zeroes of $J_{m-1}\left(\sqrt{K^2 - \beta}\right)$ and $J_{m+1}\left(\sqrt{K^2 - \beta}\right)$, respectively.

Quadratic Distribution

Another electron-density distribution for which we can obtain an analytic solution to Eq. (5) is the so-called quadratic distribution

$$g(y) = 1 - y^2 \qquad y \leq 1$$
$$= 0 \qquad y \geq 1$$

The solution inside the cylinder for this case is

$$\psi_m = e^{i\sqrt{\beta}\,\frac{y^2}{2}}\, y^m\, {}_1F_1\left(a_m;\ c_m;\ -\,i\sqrt{\beta}\, y^2\right) \tag{16}$$

$$a_m = \frac{m+1}{2} - \frac{i}{4}\,\frac{\left(K^2 - \beta\right)}{\sqrt{\beta}}$$

$$c_m = m + 1$$

The function ${}_1F_1(a;\ b;\ Z)$ is the confluent hypergeometric function

$${}_1F_1(a;\ b;\ Z) = 1 + \frac{a}{b}\,\frac{Z}{1!} + \frac{a(a+1)}{b(b+1)}\,\frac{Z^2}{2!} + \ldots \tag{17}$$

The γ_m's can be found from the recursion formulas for the hypergeometric functions and are given by

$$\gamma_m = -\,i\sqrt{\beta}\left[m + 1 + 2\left(\frac{a_m}{c_m} - 1\right)\frac{{}_1F_1\left(a_m;\ c_m + 1;\ -\,i\sqrt{\beta}\right)}{{}_1F_1\left(a_m;\ c_m;\ -\,i\sqrt{\beta}\right)}\right] \tag{18}$$

EVALUATION OF η_o FOR ARBITRARY ELECTRON-DENSITY DISTRIBUTIONS

With certain exceptions to be discussed later, Eq. (9) represents a fair approximation to the small-K cross section. It would be especially useful, therefore, to develop a technique for calculating η_o for arbitrary functions g(y) (subject to the restriction that g(y) falls off faster than $1/y^2$ for large y). We will do this by analogy with a technique used for obtaining an approximation to the s-wave phase shift for the scattering from spherically symmetric scattering regions.

Let us rewrite Eq. (5) for m = 0 as follows

$$\frac{d}{dy}\left(y\,\frac{d\psi_{oK}}{dy}\right) + \left(K^2 - \beta g(y)\right) y\psi_{oK} = 0 \qquad (19)$$

with boundary conditions

$$\psi_{oK}(0) \text{ finite}$$

$$\psi_{oK}(y) \to \frac{\cos\left(Ky - \frac{\pi}{4} - \eta_o\right)}{\sqrt{y}} \qquad \text{as } y \to \infty$$

The subscript K simply refers to the solution for a particular K. The function $\psi_{oo}(y)$ would be the solution to Eq. (19) (finite at the origin) with K = 0.

Let us also consider functions $W_k(y)$ which have the same asymptotic behavior as $\psi_{oK}(y)$ and which satisfy Eq. (19) with $\beta = 0$

$$\frac{d}{dy}\left(y\,\frac{dW_K}{dy}\right) + K^2 y W_K = 0$$

$$W_K(y) \to \psi_{oK}(y) \qquad \text{as } y \to \infty \qquad (20)$$

Equation (28) can be solved immediately, and we get

$$W_K(y) = B\left[\cot \eta_o J_o(Ky) + N_o(Ky)\right] \quad \text{for } K \neq 0$$

$$W_o(y) = \frac{2}{\pi} B\left[\ln y + \frac{1}{\gamma}\right] \tag{21}$$

The factor $\frac{2}{\pi}$ is put in the function $W_o(y)$ to cause $yW_o'(y)$ and $yW_K'(y)$ to be equal at y = 0. The parameter γ can only be found from the solution of Eq. (19) with K = 0. Since γ is equal to $\frac{1}{W_o}\frac{dW_o}{dy}$ at y = 1, it is the same as γ_o of Eq. (11) for K = 0 and for those functions g(y) which vanish for y > 1. The amplitude B is as yet unspecified, since we have not specified the value of ψ_{oK} at y = 0. We will choose B by requiring, for convenience, that $\psi_{oK}(0) = 1$. From the differential Eq. (19), we then find that $\left[\psi_{oo}'(y)/y\right] = \beta/2$. Obtaining numerical solutions of Eq. (19) with K = 0 is a straightforward problem for functions g(y) which fall off faster than $\frac{1}{y^2}$ for large y. From the solutions, we obtain the parameters B and γ as functions of β.

Let us now see how we can obtain $\cot \eta_o$ for arbitrary small K from the function $\psi_{oo}(y)$. Let us consider the following two equalities:

$$\psi_{oo}\frac{d}{dy}\left(y\frac{d\psi_{oK}}{dy}\right) - \psi_{oK}\frac{d}{dy}\left(y\frac{d\psi_{oo}}{dy}\right) + K^2 y\psi_{oo}\psi_{oK} = 0 \tag{22a}$$

$$W_o\frac{d}{dy}\left(y\frac{dW_K}{dy}\right) - W_K\frac{d}{dy}\left(y\frac{dW_o}{dy}\right) + K^2 yW_oW_K = 0 \tag{22b}$$

Subtracting Eq. (22b) from Eq. (22a), integrating over y from 0 to ∞, and inserting the boundary conditions, we get

$$\left[yW_o'(y)(W_o - W_K)\right]_{y=0} = \frac{2}{\pi} B^2\left[\frac{2}{\pi}\left(\ell n \frac{2}{Ke^C} + \frac{1}{\gamma}\right) - \cot \eta_o\right]$$

$$= K^2 \int_o^\infty \left(W_o W_K - \psi_{oo}\psi_{oK}\right) y \, dy \qquad (23)$$

From this exact equation, an approximate expression for $\cot \eta_o$ correct to first order in K^2 may be obtained by replacing W_K by W_o and ψ_{oK} by ψ_{oo} in the integral

$$\cot \eta_o \cong \frac{2}{\pi}\left(\ell n \frac{2}{Ke^C} + \frac{1}{\gamma}\right) - K^2 A$$

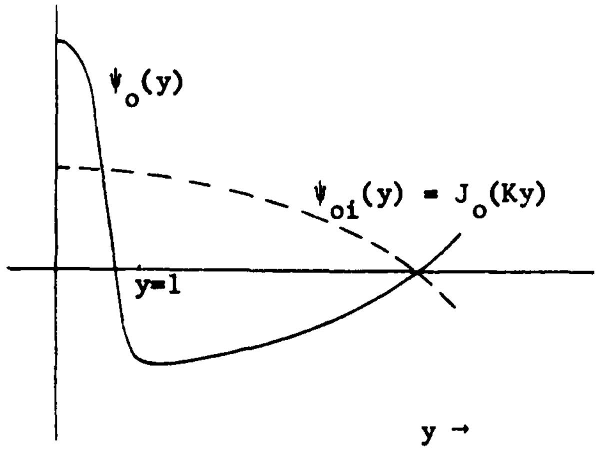
$$A = \frac{\pi}{2} \int_o^\infty \left[\frac{4}{\pi^2}\left(\ell n\ y + \frac{1}{\gamma}\right)^2 - \frac{\psi_{oo}^2(y)}{B^2}\right] y \, dy \qquad (24)$$

Equation (24) can be expected to break down as

$$\gamma \to 0 \left(\left|\cot \eta_o\right| \to \infty\right)$$

For values of β which are such that we are near a zero of γ, we cannot replace the wave functions for finite K by those for K = 0, since small changes in K have large effects on the wave functions.

The zeros of $\sin \eta_o$ for small K will cause the cross section to exhibit deep minima. If β is negative, the smallest value of $|\beta|$ for which $\sin \eta_o = 0$ will be that which will cause the m = 0 wave to be pulled in by just half a cycle, giving a phase shift η_o of $-\pi$ as illustrated in the following sketch.



The effect is a diffraction of the wave around the cylinder, in which the wave inside the cylinder is distorted in such a way that it fits on smoothly to an undistorted wave outside. This particular phenomenon with K small is associated only with negative β; for positive β, η_o is π (m = 0 wave pulled out by half a cycle) only for fairly large K, so that the higher m phase shifts are not negligible. Such minima in the cross section are well known in the analogous situation of the scattering of low-energy particles from spherically symmetric attractive potentials. In particular, an extremely low minimum known as the Ramsauer-Townsend effect occurs in the scattering cross section of low-energy electrons by rare-gas atoms.

When we are not at such a minimum, we can see from Eq. (24) that for finite β, as $K \to 0$

$$\cot \eta_o \to \frac{2}{\pi} \ln\left(\frac{1}{K}\right) \tag{25}$$

so that

$$|f(\varphi)|^2 \rightarrow \frac{\pi\rho_o}{2K}\left(\ell n \frac{1}{K}\right)^2 \qquad (26)$$

For small but finite K, it is necessary to use the complete Eq. (24) to evaluate cot η_o.

The parameter γ and the integral A were computed on the IBM 7090 for a Gaussian distribution $g(y) = e^{-y^2}$ for a variety of values of positive and negative β. In Fig. 1, we have shown $\frac{1}{\gamma}$ and $\frac{2}{\pi}$ A versus β for the Gaussian distribution for positive β. In Fig. 2, we have plotted $\frac{1}{\gamma}$ versus β for negative β for the Gaussian distribution, homogeneous cylinder, and quadratic distribution for those values of β for which $\left|\frac{1}{\gamma}\right| < 2$. In Fig. 3, we have plotted $\left|\frac{1}{\gamma}\right|$ versus β for the homogeneous cylinder and Gaussian distribution in the region in which $\gamma \rightarrow 0$. The integral A for the Gaussian distribution is shown as a function of β for negative β in Fig. 4.

The approximate cot η_o, given by Eq. (24), and the exact cot η_o were calculated for a homogeneous cylinder for β = -4, -14.2, and -14.6. For $\beta = -4\left(\frac{1}{\gamma} = -0.194\right)$, it was found that the error involved in the approximation was less than 1 per cent for K up to unity, and the results were not plotted. For β = -14.2 and -14.6, $|\gamma|$ is small, and one can expect the approximation given by Eq. (24) to break down as $K \rightarrow 1$. In Fig. 5, we have plotted both the approximate and exact values of cot η_o versus K for β = -14.2 and -14.6. For $\beta = -14.2\left(\frac{1}{\gamma} \sim 5\right)$, the approximation is surprisingly good for K near unity; the error in cot η_o is only 20 per cent for K = 1. For $\beta = -14.6\left(\frac{1}{\gamma} \approx 22.3\right)$, the approximation breaks down badly as $K \rightarrow 1$ but is, nevertheless, excellent for small K; for K = 0.5, the error is only 8.5 per cent.

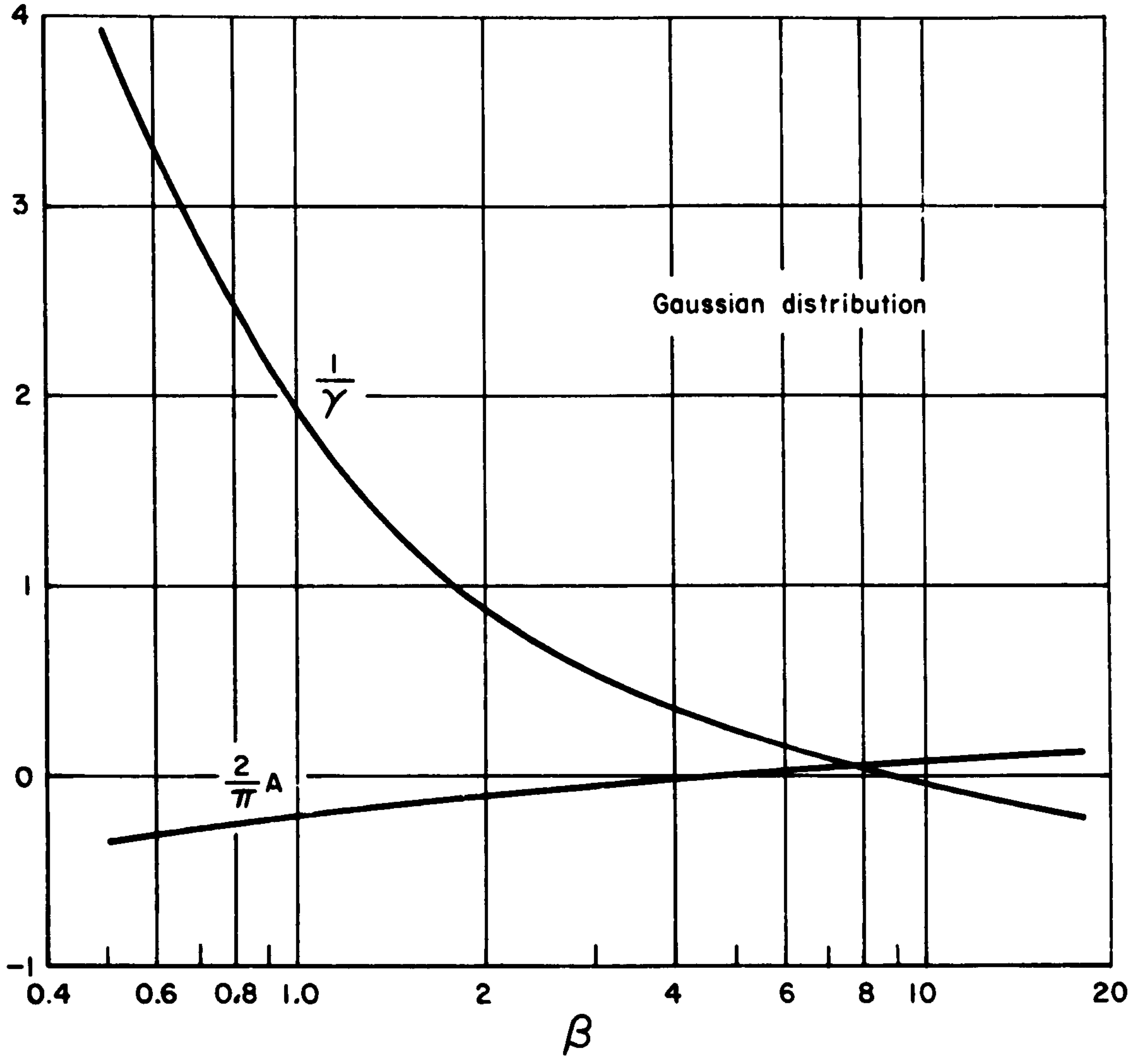


Fig. 1—Curves of $\frac{1}{\gamma}$ and $\frac{2}{\pi}$ A versus β for β positive

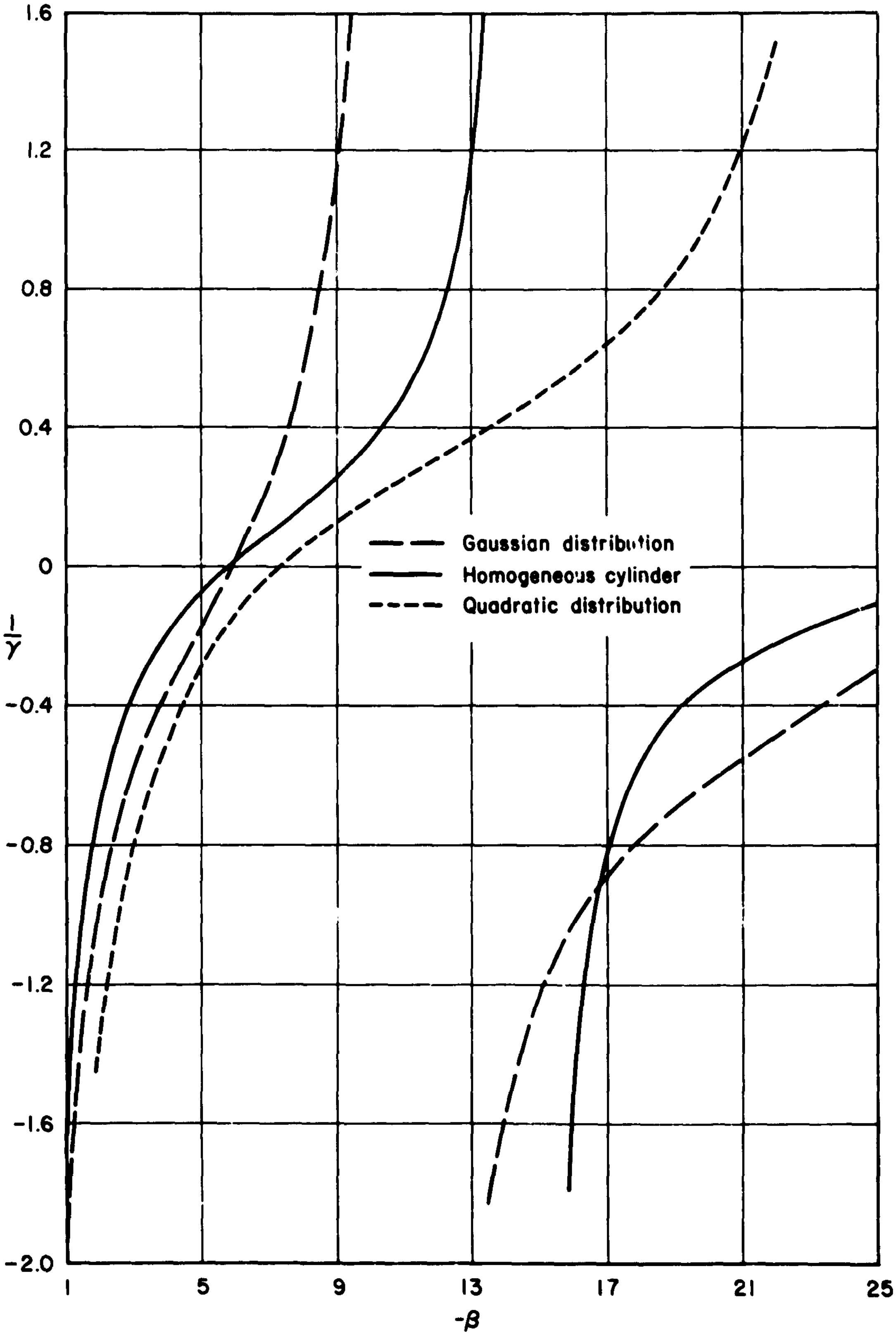


Fig. 2—Curves of $\frac{1}{\gamma}$ versus β for negative β ($|\frac{1}{\gamma}| < 2$)

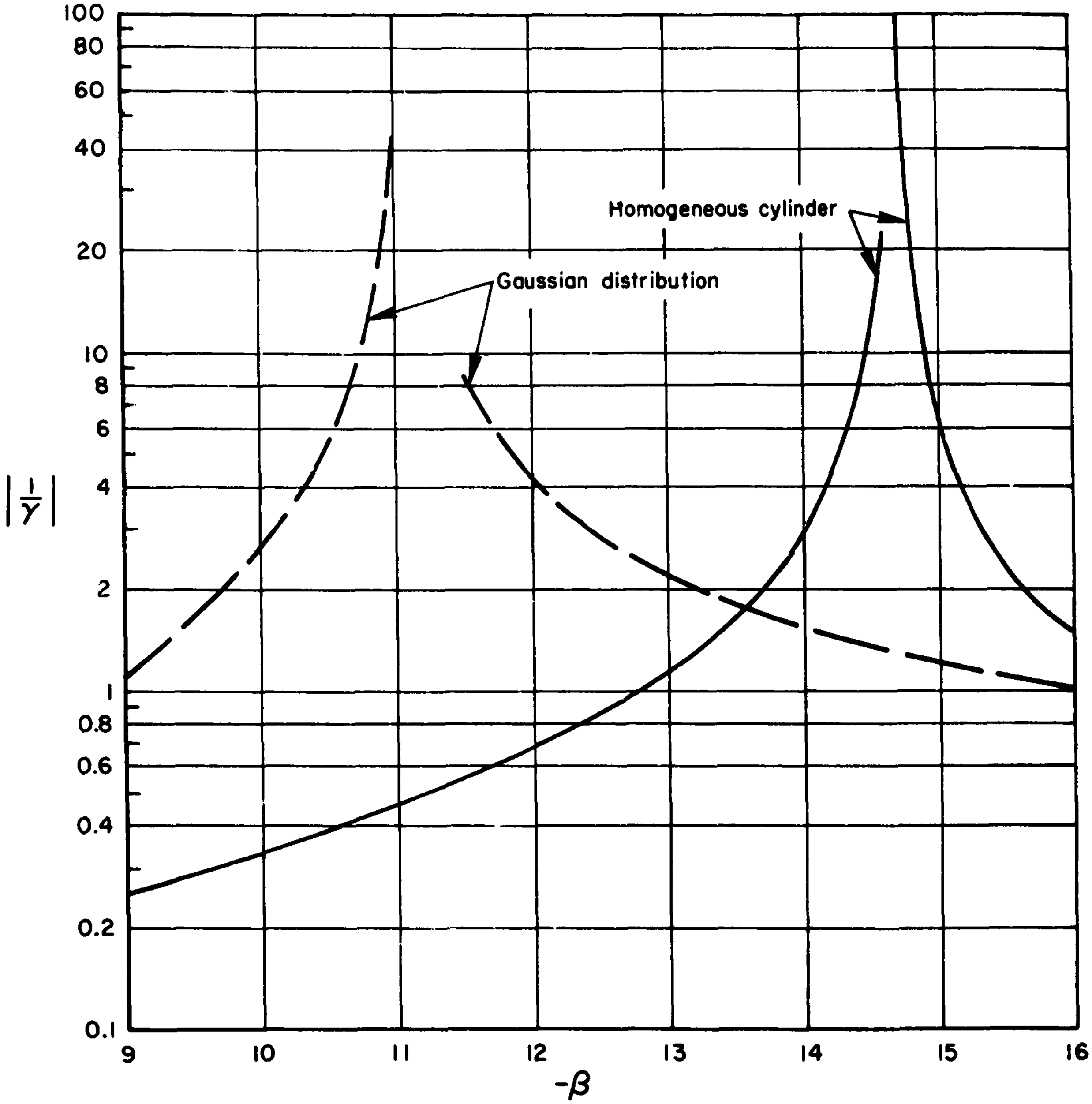


Fig. 3—Curves of $\left|\frac{1}{\gamma}\right|$ versus β for negative β near $\gamma \rightarrow 0$

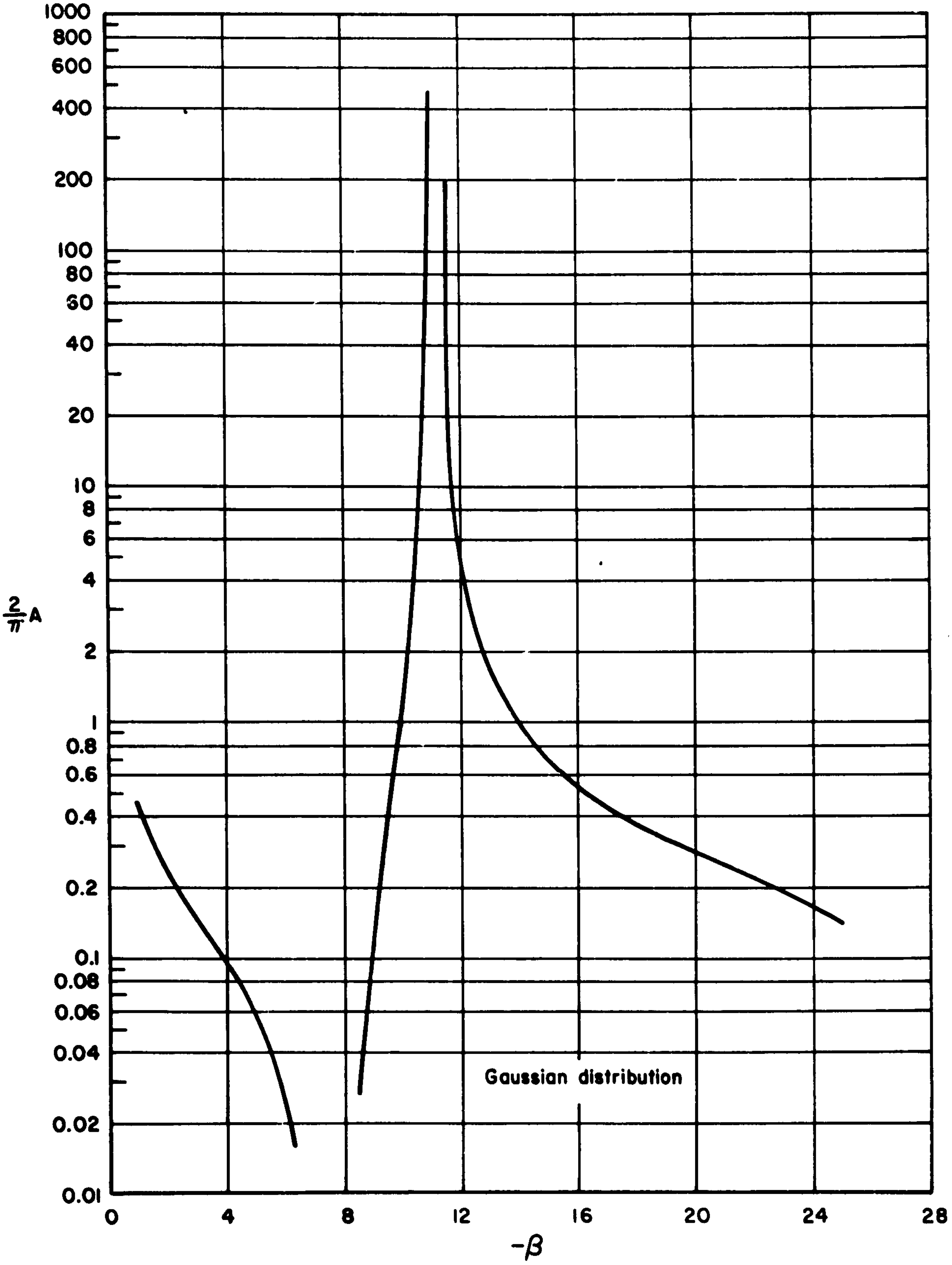


Fig. 4—Curves of $\frac{2}{\pi}$ A versus β for negative β

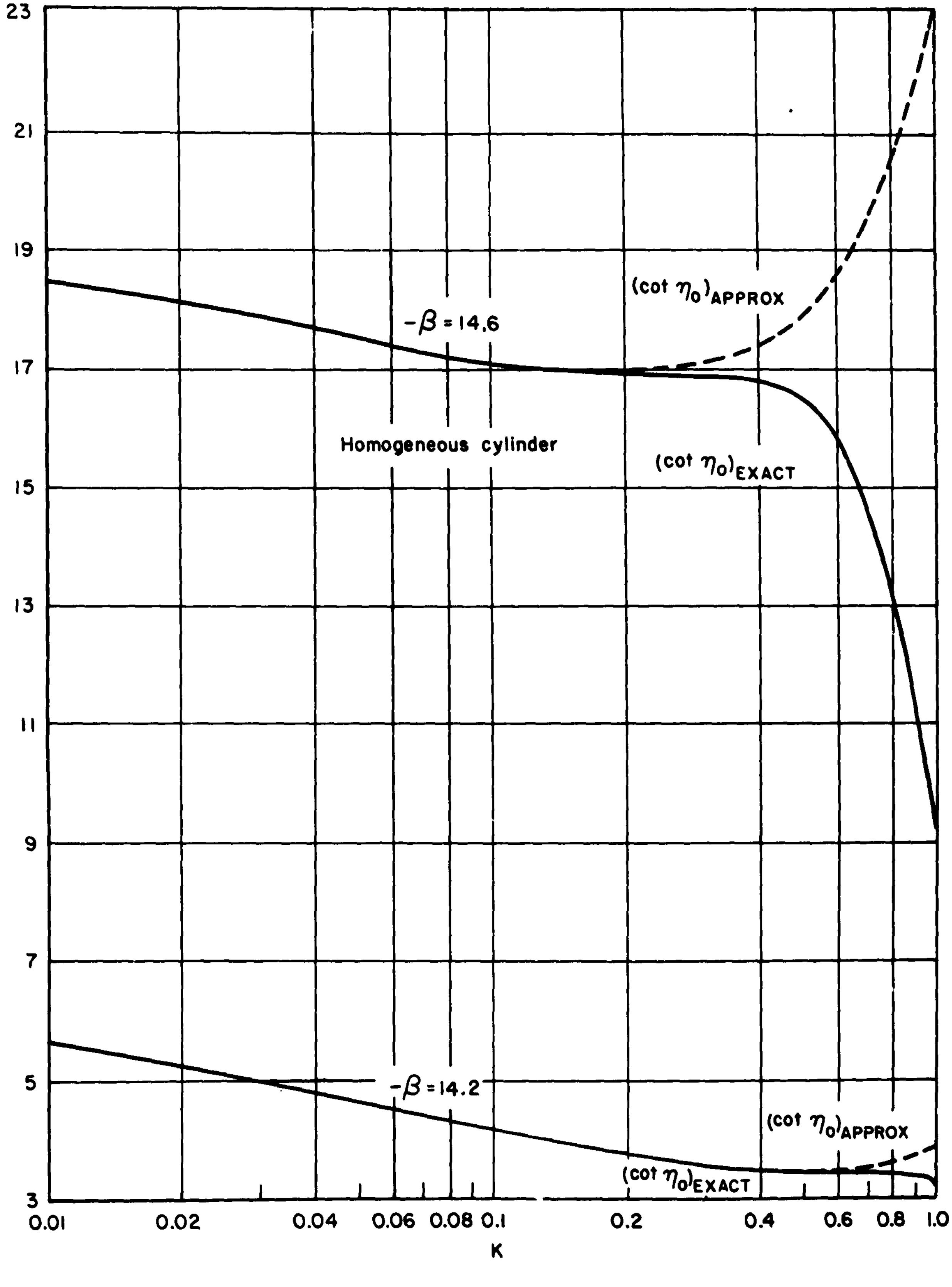


Fig. 5—Comparison of exact and approximate values of cot η_o for $|\frac{1}{\gamma}|$ large

IV. RESONANCE SCATTERING

We can see from Eq. (11) for the finite-radius cylinder that when $\gamma_m = K \frac{N'_m(K)}{N_m(K)}$, cot η_m will be zero. Whenever cot $\eta_m = 0$, the m^{th} partial wave is said to be in resonance.

Small-K resonances for $m \geq 1$ cannot arise if β is positive. We can easily prove this for finite-radius cylinders by showing that for $\beta > 0$, $m \geq 1$, and $K < 1$, γ_m is always positive. Since $\frac{N'_m(K)}{N_m(K)} < 0$ for small K, a resonance can only occur if γ_m is negative.

Let us rewrite Eq. (5) as follows

$$\frac{d}{dy}\left(y \frac{d\psi_m}{dy}\right) = y\psi_m\left[\beta g(y) + \frac{m^2}{y^2} - K^2\right] \qquad (27)$$

Integrating both sides of Eq. (27) over y from 0 to 1, we get

$$\left(\frac{d\psi_m}{dy}\right)_{y=1} = \int_0^1 y\psi_m(y)\left[\beta g(y) + \frac{m^2}{y^2} - K^2\right]dy \qquad (28)$$

$$\equiv \gamma_m \psi_m(1)$$

If $\beta > 0$, $m \geq 1$, and $K < 1$, the quantity $\left[\beta g(y) + \frac{m^2}{y^2} - K^2\right]$ is positive throughout the range of integration, $\frac{d\psi_m}{dy}$ always has the same sign as ψ_m, and γ_m is positive.

On the other hand, if β is negative and sufficiently large in magnitude, the quantity $\left[\beta g(y) + \frac{m^2}{y^2} - K^2\right]$ may be negative for part of the range of integration, and the solution $\psi_m(y)$ may be of the periodic type $\left(\psi_m(y) \text{ and } \frac{d\psi_m}{dy} \text{ of opposite sign}\right)$ for y near one, causing γ_m to be negative. For a given g(y), m, and small K we can expect an infinite sequence of values β_{mn} of negative β for which γ_m will have precisely the right negative value to cause $\eta_m = -(2n-1)\frac{\pi}{2}$, where n is a positive integer. For example, for the homogeneous cylinder, the β_{mn}'s are found from the roots of the equation $J_{m-1}\left(\sqrt{K^2 - \beta_{mn}}\right) = 0$. Considered as a function of m, the lowest root $\sqrt{K^2 - \beta_{m1}}$ will increase with m.

In a small-K resonance situation for $m \geq 1$, the scattering will usually be the result of the interference between the wave in resonance and the $m = 0$ wave. The differential cross section in the vicinity of an $m \geq 1$ resonance will be approximately

$$|f(\varphi)|^2 \simeq \frac{2\rho_o}{\pi K}\Big[\sin^2\eta_o + 4\cos^2 m\varphi \sin^2\eta_m + 4\cos m\varphi(\sin^2\eta_o \sin^2\eta_m + \sin\eta_o \sin\eta_m \cos\eta_o \cos\eta_m)\Big] \quad (29)$$

The angular distribution $S(\varphi)$ at the peak of the resonance will be given by

$$S(\varphi) \equiv \frac{|f(\varphi)|^2}{\frac{1}{2\pi}\int_o^{2\pi}|f(\varphi)|^2 d\varphi} = \frac{\sin^2\eta_o + 4\cos^2 m\varphi + 4\cos m\varphi \sin^2\eta_o}{\sin^2\eta_o + 2} \quad (30)$$

Illustrations of resonant scattering are shown in Figs. 6 to 9. In Fig. 6, we have shown $|\sin\eta_1|$ and $|\sin\eta_o|$ as functions of K near an $m = 1$ resonance for a homogeneous cylinder. In Fig. 7, we have plotted the radar cross section $|f(\pi)|^2$ near the resonance in comparison with the small-K cross section $\frac{2\rho_o}{\pi K}\sin^2\eta_o$ in the absence of an $m \geq 1$ resonance. Figure 8 shows the angular distribution $S(\varphi)$ at the peak of this particular $m = 1$ resonance and compares it to $2\cos^2\varphi$, the angular distribution of a pure m = 1 wave. In Fig. 9, we have shown the angular distribution at the peak of an m = 1 resonance for the quadratic distribution in comparison with the 2 $\cos^2 \varphi$ distribution. Figures 8 and 9 show the effects of the interference between the m = 0 and m = 1 waves on the angular distribution. The distortion of the 2 $\cos^2 \varphi$ distribution by the m = 0 wave is more pronounced in Fig. 9 than in Fig. 8 simply because $|\sin \eta_o|$ is larger in the former case.

In Fig. 10, we have shown $|\sin \eta_o|$, $|\sin \eta_1|$, and $|\sin \eta_2|$ near an m = 2 resonance for the homogeneous cylinder. It happens that this particular resonance occurs near a minimum of $|\sin \eta_o|$, so that at the peak of this m = 2 resonance we have an almost pure m = 2 wave. This result is not surprising. We can see from Eq. (15) that for the homogeneous cylinder, $\cot \eta_2 \approx 0$ when $J_1\left(\sqrt{K^2 - \beta}\right) \approx 0$. However, from Eq. (12) we see that for small K, $|\cot \eta_o| \to \infty$ when

$$\gamma_o = - \frac{\sqrt{K^2 - \beta}\, J_1\left(\sqrt{K^2 - \beta}\right)}{J_o\left(\sqrt{K^2 - \beta}\right)}$$

is very small in magnitude or when $J_1\left(\sqrt{K^2 - \beta}\right)$ is near zero. We have not plotted the angular distribution at the peak of the particular m = 2 resonance being considered here, since it is essentially a pure 2 $\cos^2 2\varphi$ distribution.

So far we have discussed only resonance maxima for the m ≥ 1 partial waves. We may also have small-K resonances for m = 0; they occur in the vicinity of $\frac{1}{\gamma} = - \ln \frac{2}{Ke^C}$ (see Eq. (24)).. If K is very small, the m = 0 resonance maxima will occur for γ negative and fairly small in magnitude; however, we have seen that the minima in $\sin^2 \eta_o$ occur for $|\gamma|$ very small. Therefore, for a given small K, if we consider $\sin^2 \eta_o$ as a function of β for negative β in a region where γ passes through zero, we can expect a zero and a maximum in $\sin^2 \eta_o$ to be fairly close together. We have shown this effect in Fig. 11 by plotting $\sin^2 \eta_o$ versus β for a homogeneous cylinder for K = 0.06 and $-\beta$ between 13 and 16. We can see that a minimum occurs for $-\beta$ near 14.7 and a maximum for $-\beta$ near 15.4. For K smaller, the minimum and maximum would be still closer together.

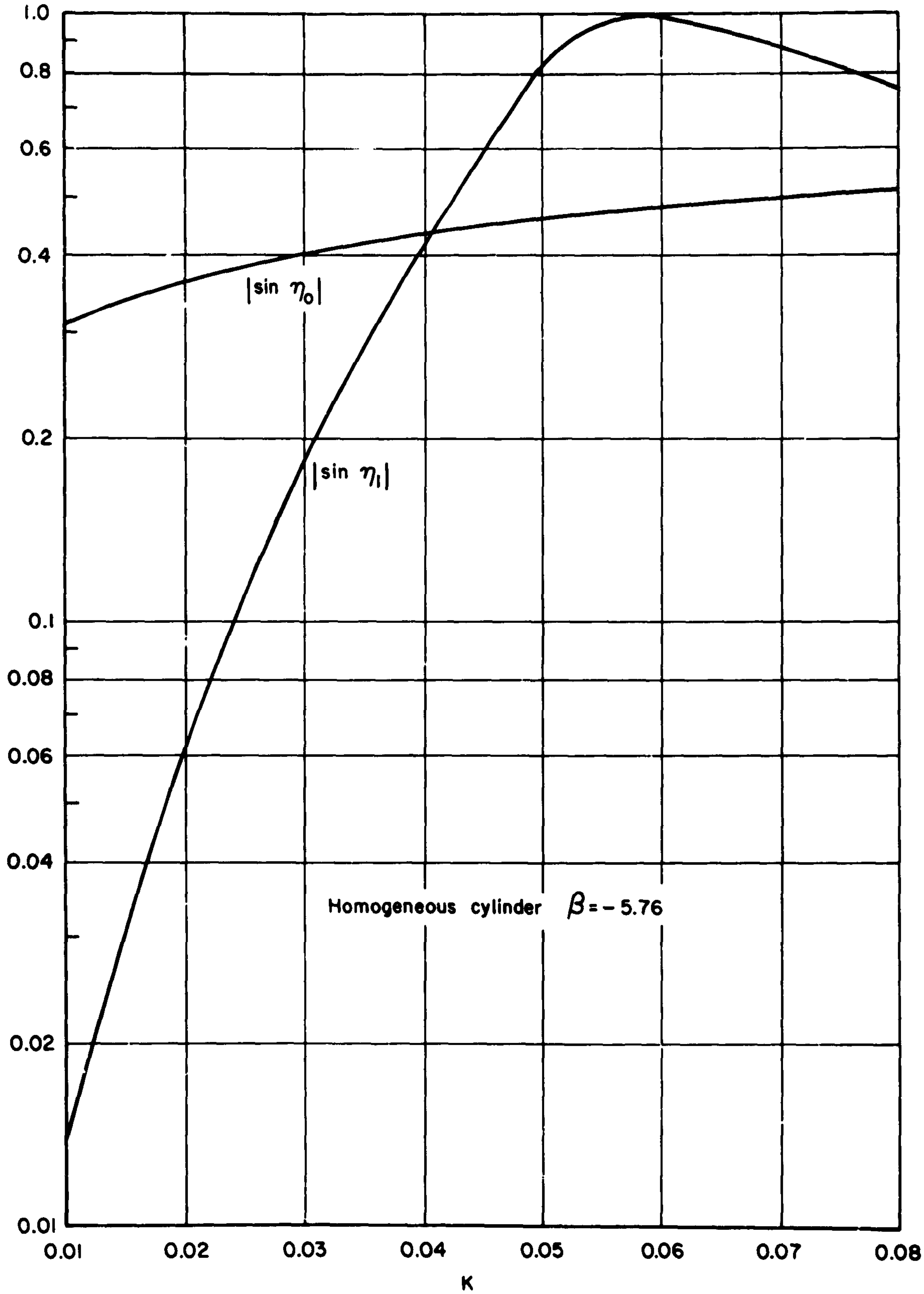


Fig. 6—Curves of m = 0 and m = 1 phase shifts near an m = 1 resonance

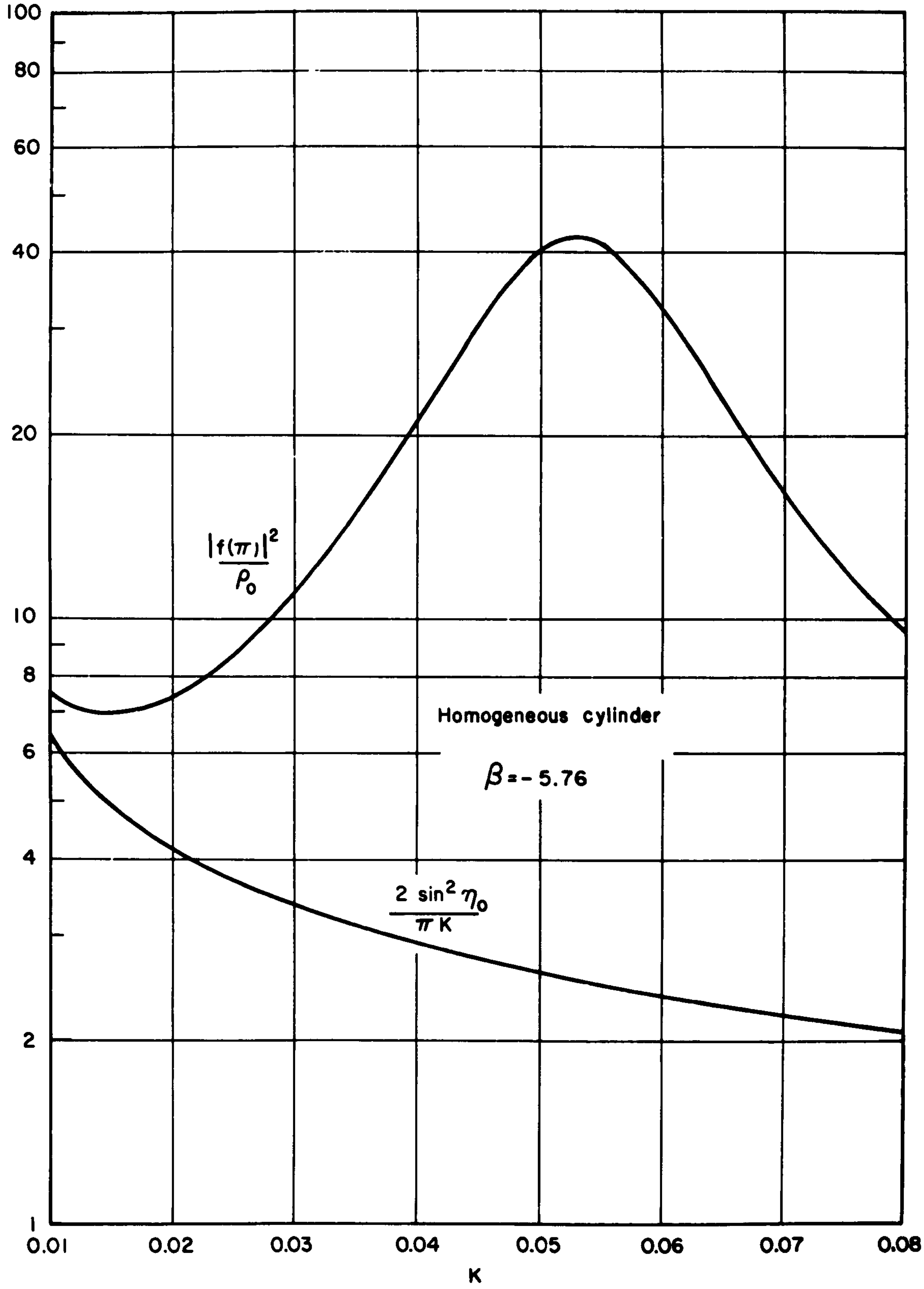


Fig. 7— Effect of an m = 1 resonance on the radar cross section

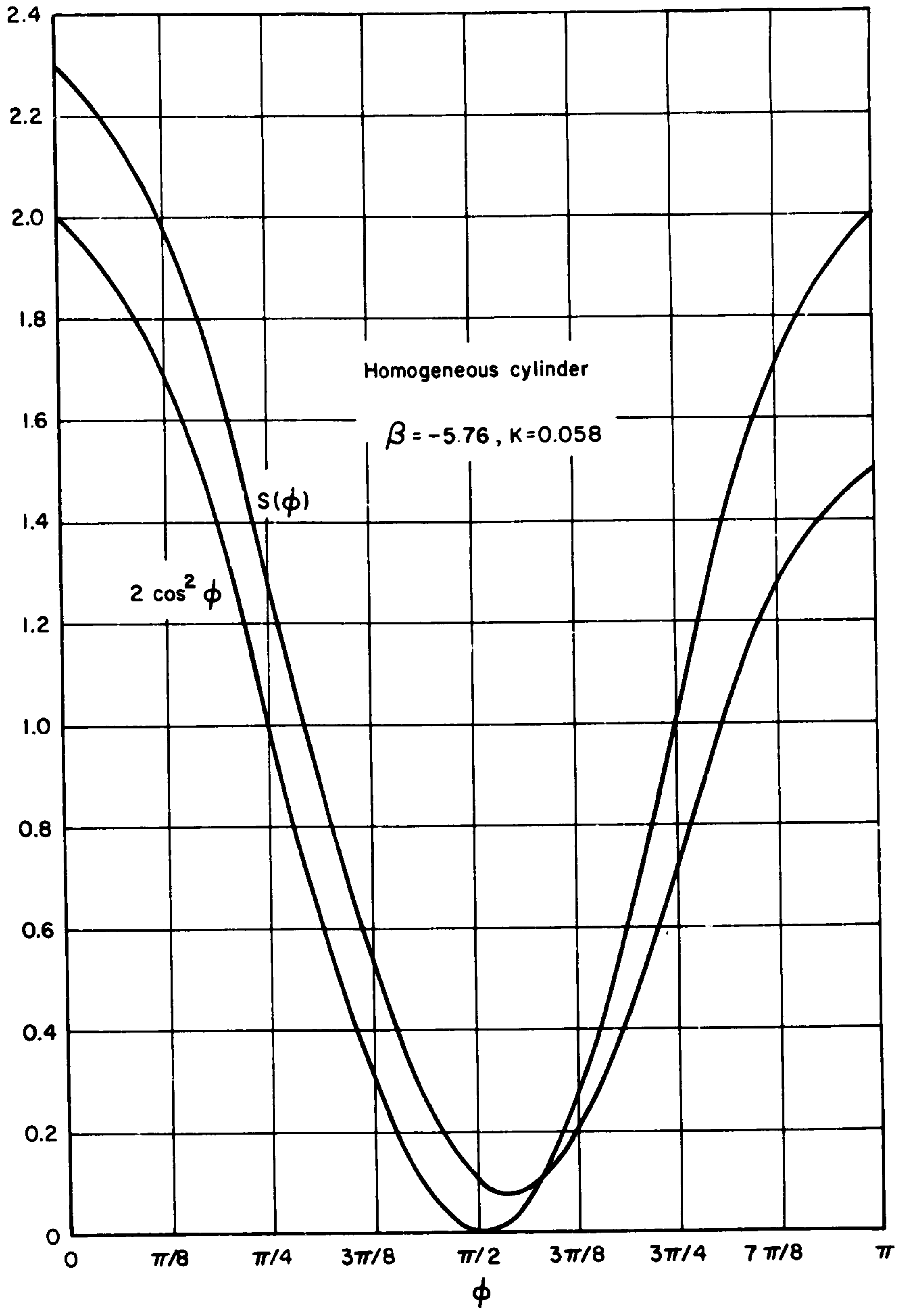


Fig. 8—Angular distribution at the peak of an m = 1 resonance (homogeneous cylinder; β = -5.76, K = 0.058)

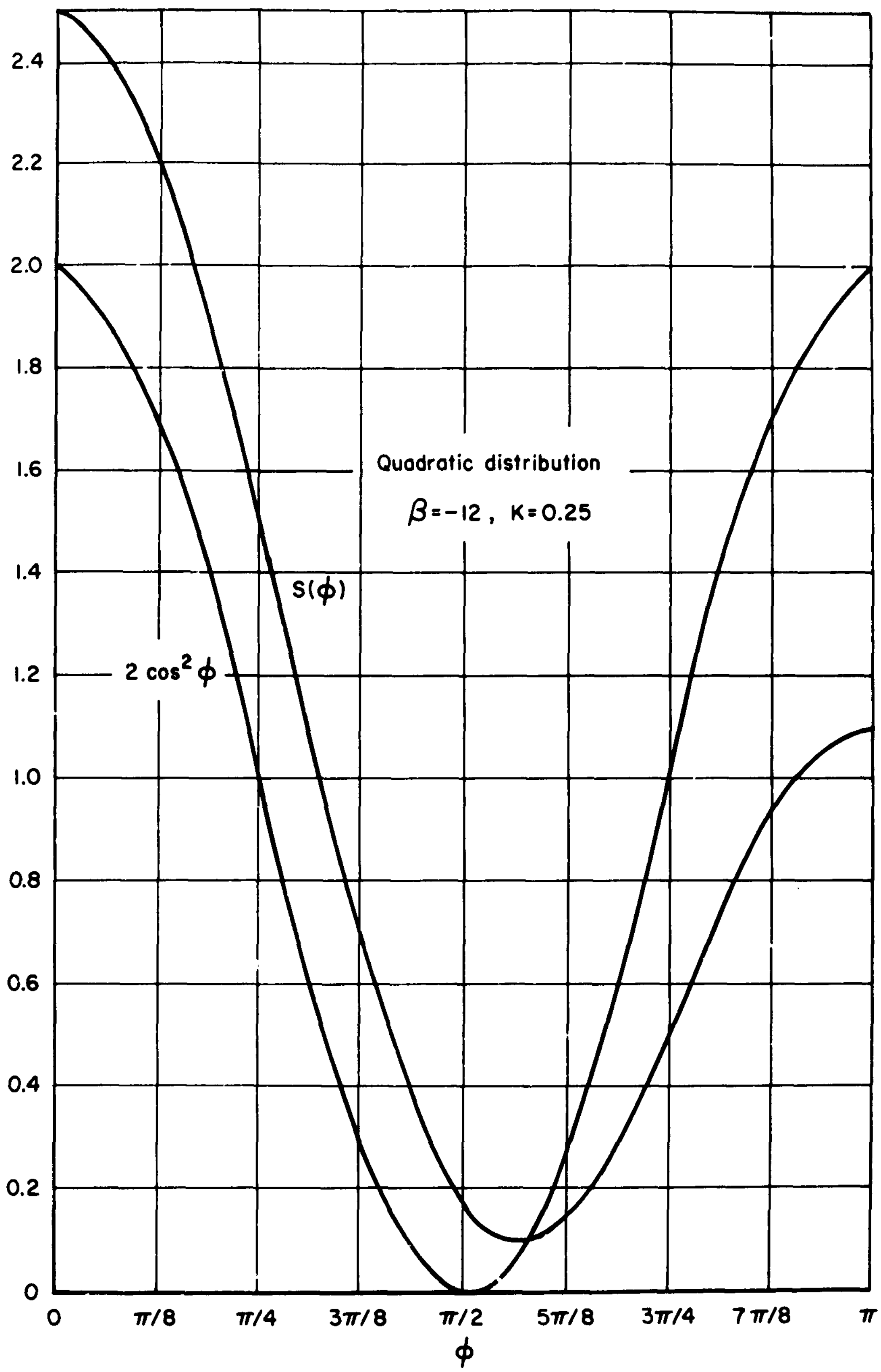


Fig. 9—Angular distribution at the peak of an $m = 1$ resonance (quadratic distribution; $\beta = -12$, $K = 0.25$)

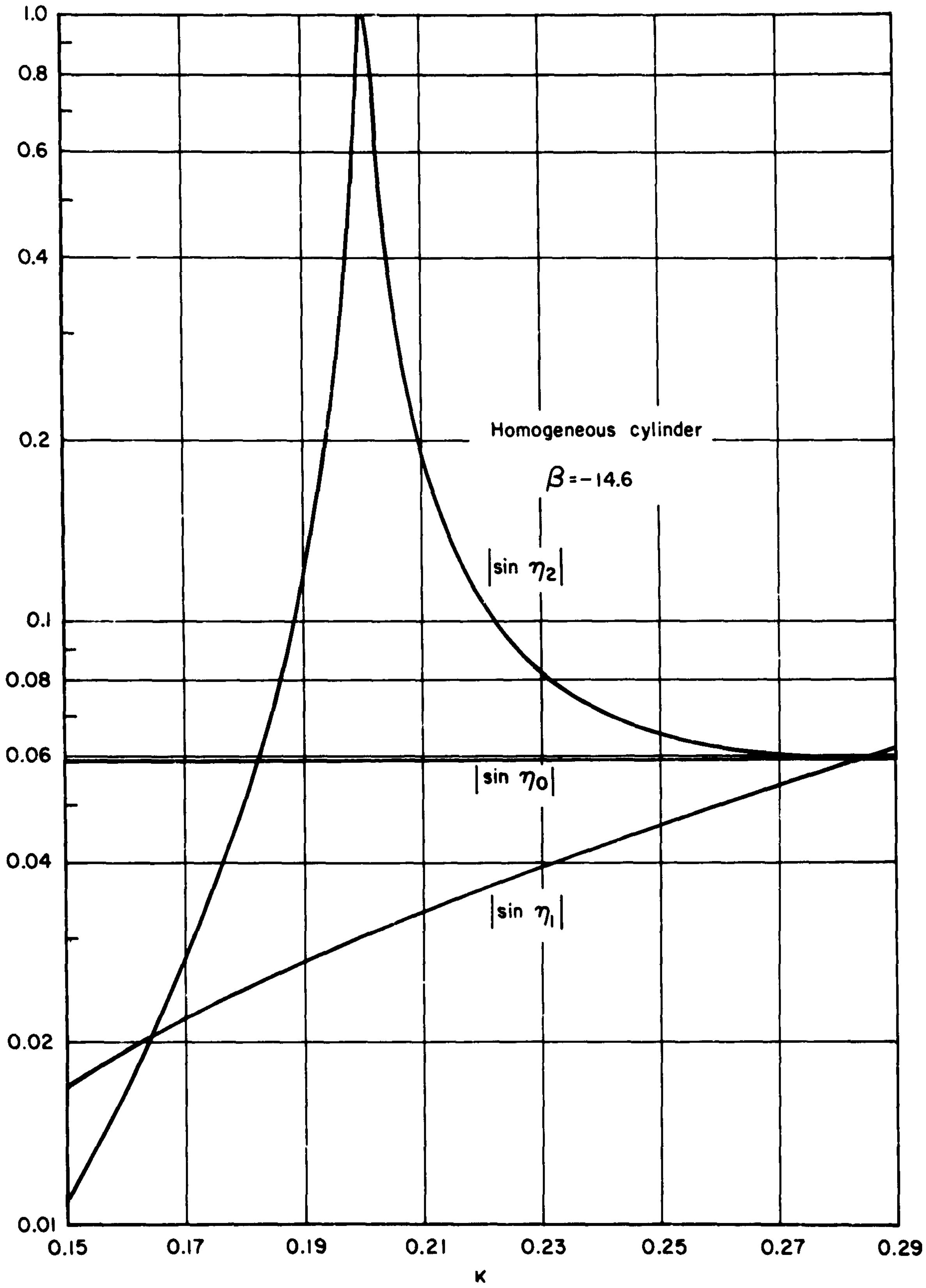


Fig. 10—Phase shifts near an m = 2 resonance

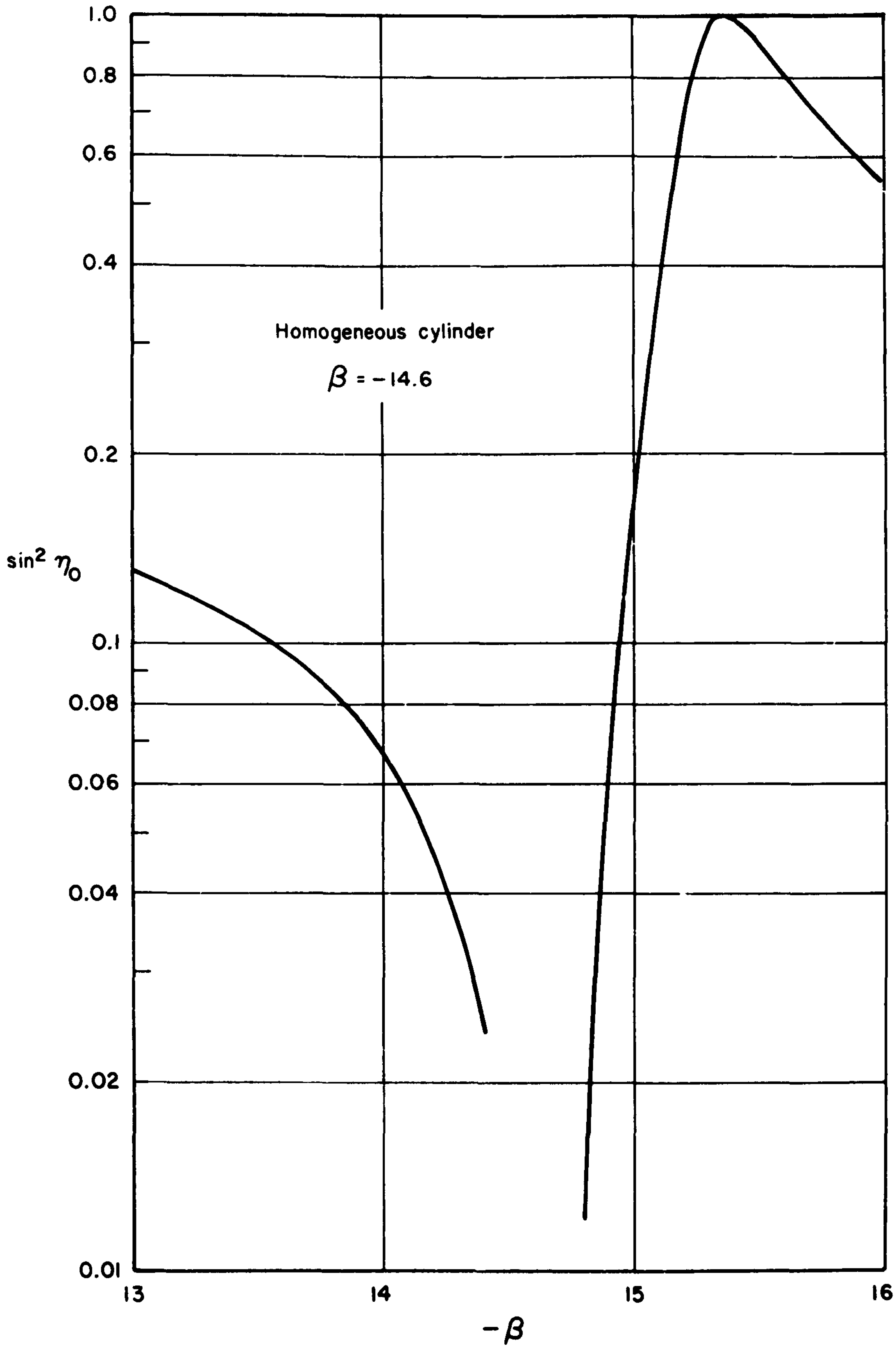


Fig. 11—Curve of $\sin^2 \eta_0$ versus β for fixed K in region where γ passes through zero

V. CROSS SECTION FOR $|\beta| < 1$ AND $K < 1$

Since $|\beta|$ and K are both proportional to ρ_o, in many situations in which K is small, $|\beta|$ will also be less than one. It would be useful, therefore, to develop a simple method for calculating the cross section for $|\beta| < 1$ and $K < 1$.

We know that for K finite, we can use the Born approximation to estimate the cross section or the individual phase shifts for $|\beta| \to 0$. For example, the Born approximation for $\eta_o^{(1)}$ is given by

$$\eta_o \cong \frac{\pi}{2} \beta \int_o^\infty y g(y) J_o^2(Ky) \; dy \tag{31}$$

If we let K go to zero, this gives

$$\eta_o \to \frac{\pi}{2} \beta \int_o^\infty y g(y) \; dy \tag{32}$$

However, we have seen that for β finite, if we let $K \to 0$, $\cot \eta_o \to \frac{2}{\pi} \ln \frac{1}{K}$, so that

$$\eta_o \to \frac{\pi}{2 \ln \frac{1}{K}} \tag{33}$$

Therefore, for $K \to 0$, we cannot use Eq. (32) for finite β, even for $|\beta|$ small. We need a simple approximation to η_o for $K < 1$ and $|\beta| < 1$, which reduces to Eq. (32) for K finite and $|\beta| \to 0$, and to Eq. (33) for $|\beta|$ finite and $K \to 0$.

Since Eq. (24) represents a good approximation to η_o for small K and all finite values of $|\beta|$ (except at the cross-sectional minima), we can obtain the approximation we are seeking by obtaining a power series expansion in β of $1/\gamma$ and A. This can be done by expanding $\psi_{oo}(y)$ in a power series in β

$$\psi_{oo}(y) = \sum_{n=0}^{\infty} \beta^n Y_n(y) \tag{34}$$

From the differential equation for ψ_{oo}, we get the equations for the functions $Y_n(y)$

$$\frac{d}{dy}\left(y \frac{dY_o}{dy}\right) = 0 \tag{35}$$

$$\frac{d}{dy}\left(y \frac{dY_n}{dy}\right) = yg(y)Y_{n-1}(y) \qquad n \geq 1$$

$$Y_o(0) = 1$$

$$Y_n(0) = 0 \qquad n \neq 0$$

$$Y_n' (0) = 0$$

The solutions then are

$$Y_o(y) = 1$$

$$Y_n(y) = \int_o^y y'g(y') \, \ln \frac{y}{y'} \, Y_{n-1}(y') \, dy' \tag{36}$$

$$n \geq 1$$

We can see from Eq. (36) that for $n \geq 1$

$$Y_n(y) \to C_n \ln y + D_n \qquad \text{as } y \to \infty \tag{37}$$

where

$$C_n = \int_0^\infty y' g(y') Y_{n-1}(y')\ dy'$$

$$D_n = -\int_0^\infty y' g(y')\ \ell n\ y'\ Y_{n-1}(y')\ dy'$$

Since $\psi_{oo}(y) \to \frac{2}{\pi} B\ (\ell n\ y + \frac{1}{\gamma})$ as $y \to \infty$, we have

$$\frac{2}{\pi} B = \sum_{n=1}^{\infty} \beta^n C_n \tag{38}$$

$$\frac{2}{\pi}\frac{B}{\gamma} = 1 + \sum_{n=1}^{\infty} \beta^n D_n$$

$$\frac{1}{\gamma} = \frac{1 + \sum_{n=1}^{\infty} \beta^n D_n}{\sum_{n=1}^{\infty} \beta^n C_n}$$

We see that $\beta/\gamma \to \frac{1}{C_1}$ as $\beta \to 0$. The next approximation yields

$$\frac{1}{\gamma} = \frac{1}{\beta C_1}\left[1 + \beta\left(D_1 - \frac{C_2}{C_1}\right)\right] \tag{39}$$

The integral A in Eq. (24) is proportional to $\frac{1}{\beta}$ for small B, and the leading term is

$$A = -\frac{1}{\pi\beta C_1^2}\int_0^\infty y^3 g(y)\ dy \tag{40}$$

Substituting Eqs. (39) and (40) into Eq. (24), we get

$$\cot \eta_o \simeq \frac{2}{\pi}\left[\ln \frac{2}{Ke^C} + \frac{1}{\beta C_1}\left(1 + \beta\left(D_1 - \frac{C_2}{C_1}\right)\right) + \frac{K^2}{2\beta C_1^2}\int_o^\infty y^3 g(y)\, dy\right] \tag{41}$$

If we let β remain fixed, then as $K \to 0$, Eq. (41) reduces to Eq. (33). For K small but finite, as $\beta \to 0$, Eq. (41) becomes

$$\eta_o \to \frac{\pi}{2}\beta C_1\left(1 - \frac{K^2}{2C_1}\int_o^\infty y^3 g(y)\, dy\right) = \frac{\pi}{2}\beta\left[\int_o^\infty y g(y)\, dy - \frac{K^2}{2}\int_o^\infty y^3 g(y)\, dy\right]$$

This is the same as Eq. (31) if we expand $J_o^2(Ky)$ and keep the first two terms.

Thus, we can expect Eq. (41) to represent a good approximation to η_o for $K < 1$ and $|\beta| < 1$ for those functions g(y) which fall off rapidly enough at large distances to insure convergence of the integrals.

Let us note from Eq. (41) that for negative β we can have an $m = 0$ resonance even for $|\beta|$ small. For a resonance to occur for $|\beta| < 1$, K is given by

$$\ln \frac{2}{Ke^C} = -\frac{1}{\beta C_1}\left[1 + \beta\left(D_1 - \frac{C_2}{C_1}\right)\right] \tag{42}$$

In Fig. 12, we have shown the correct radar cross section for $K = 0.06$ for a homogeneous cylinder for small β and compared it to the Born approximation. The peak at $-\beta \sim 0.63$ is due to an $m = 0$

resonance, and it can be seen that the Born approximation is very poor near this resonance. The fact that for the particular case considered the Born approximation cross section happens to be good at $\beta = -1$ is fortuitous.

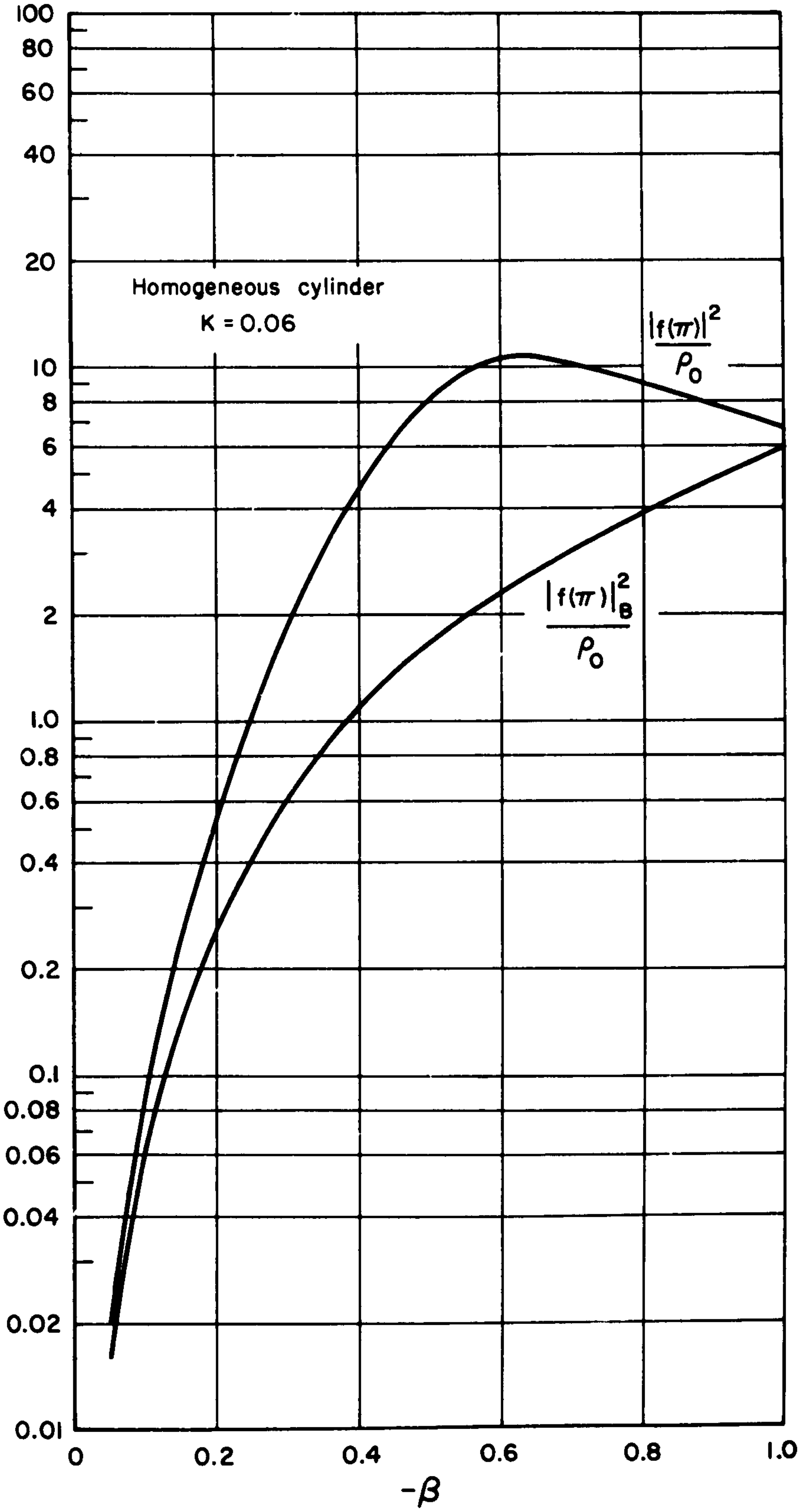


Fig. 12—A comparison of the Born approximation with the correct radar cross section for small $|\beta|$ and K

VI. RÉSUMÉ

In Part I, we described the Born approximation and the geometrical-optics approximation to the scattering cross section and discussed their limits of applicability. We also described methods for approximating the individual phase shifts for those values of large K for which the Born or geometrical-optics approximations are not applicable. In this Memorandum we have confined ourselves to a consideration of the cross section for $K < 1$.

The following briefly summarizes the methods one can use to calculate the cross section for different K values for $|\beta| < 1$ and $\beta > 1$. Since we did not consider specific methods for calculating the cross section for $\beta < -1$ and $K > 1$ in either Part I or Part II (except in the Born-approximation limit), we will not include $\beta < -1$ in this summary.

$|\beta| < 1$

For small $|\beta|$ and $K < 1$, we may use Eq. (24) for the evaluation of cot η_0 and the Born approximation to the higher phase shifts. For larger values of K, the Born approximation to the differential cross section is applicable. Thus, for $|\beta| < 1$, we can get a good estimate of the cross section for all K.

As an example, in Fig. 13, we have shown the radar cross section as a function of K for the Gaussian distribution for $\beta = \frac{1}{2}$. The cross section was computed from Eqs. (9) and (24) for $K \leq \frac{1}{2}$, and from the Born approximation for $K \geq \frac{1}{2}$. For $K = \frac{1}{2}$, the two methods differed by only 10 per cent.

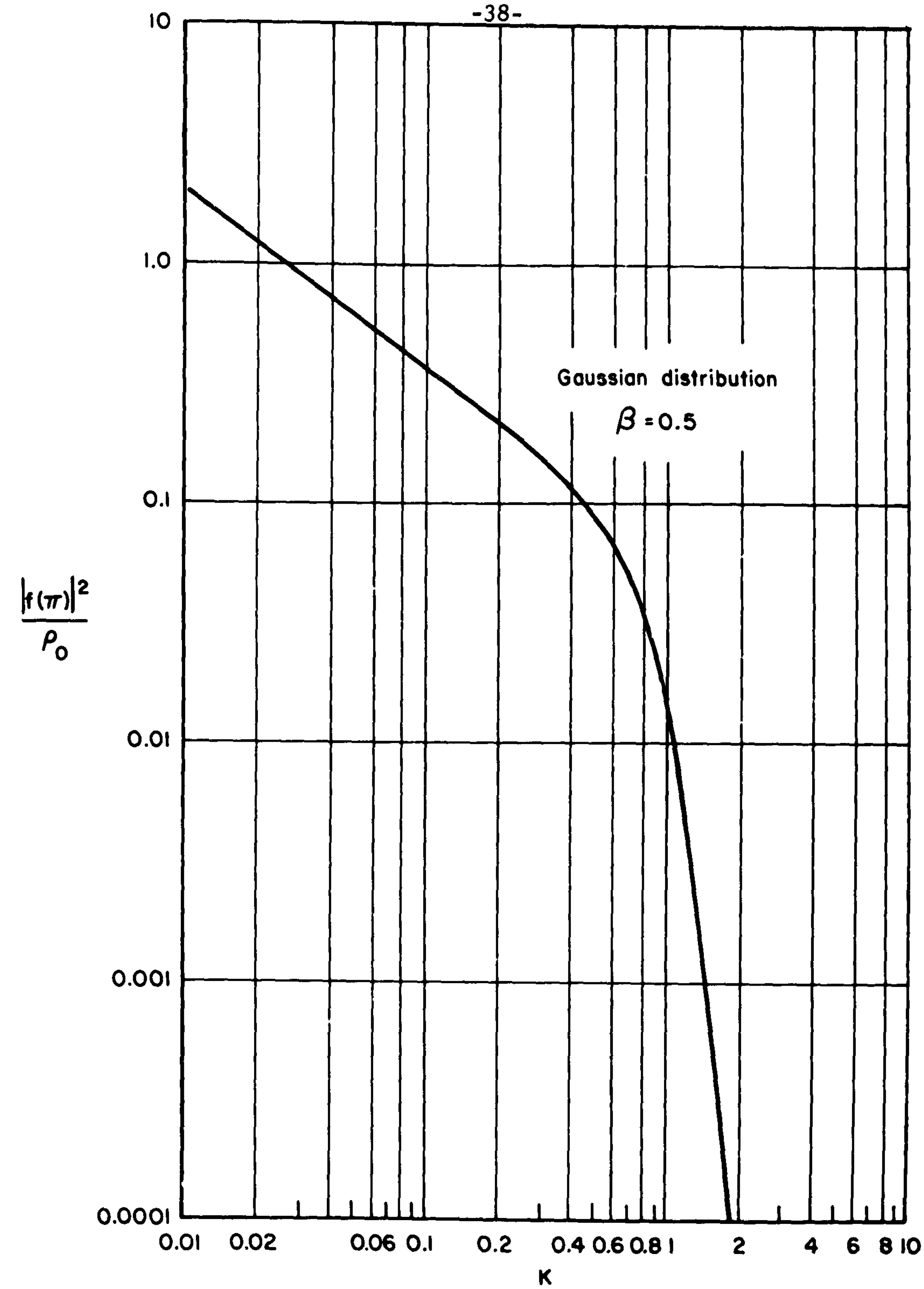


Fig. 13—Radar cross section as a function of K for fixed small positive β

$\beta > 1$

For $\beta > 1$, we can use Eq. (24) to calculate cot η_0 for $K < 1$ and neglect the other phase shifts. For $K > \beta$, we can use the Born approximation to the differential cross section. If β is large, then the geometrical-optics approximation can be applied for $1 < K^2 < \beta$. For all other values of $K > 1$, we can calculate the individual phase shifts by the WKB approximation when the phase shifts are large, and by the small-perturbation (Born) approximation when the phase shifts are small.

As an example, we have calculated the radar cross section as a function of K for the Gaussian distribution for $\beta = 18$. We used Eqs. (9) and (24) for $K < 1$ and the WKB method for $K \geq 1$. The results are shown in Fig. 14. In the same diagram, we have also plotted the geometrical optics radar cross section for $\frac{1}{2} \leq K \leq 4$. This very useful approximation is apparently valid for K as small as unity, provided that β/K^2 is sufficiently large.

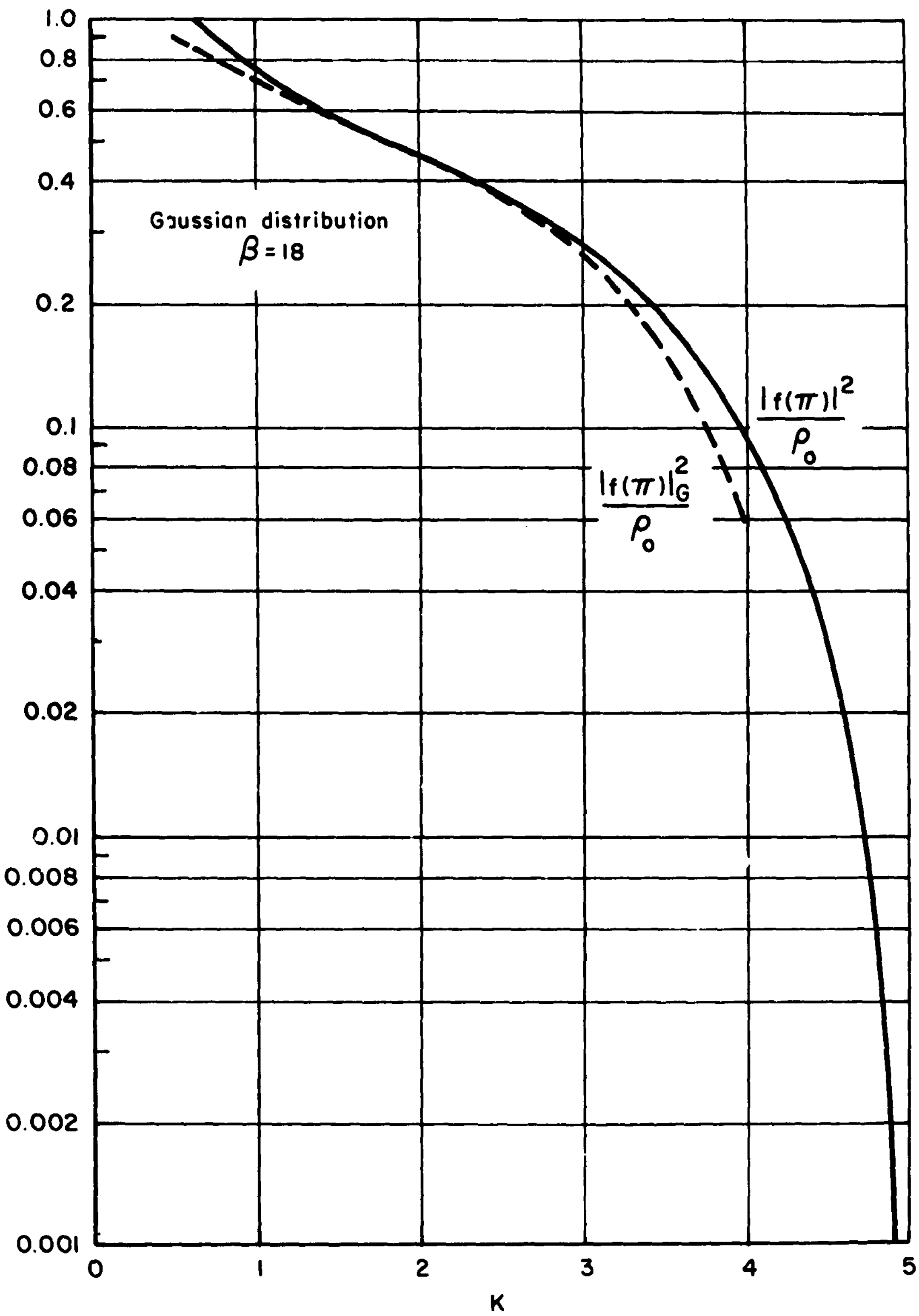


Fig. 14—Radar cross section as a function of K for fixed large positive β