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PASCUAL JORDAN

FINAL REPORT III, 1964

CONTRIBUTIONS TO ACTUAL PROBLEMS OF GENERAL RELATIVITY

With contributions of W. Beiglböck, K. Bichteler,
W. Budich, W. Kundt M. Trümper

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Preface

This is the third and last part of the Final Report 1964 concerning my mentioned contract. The first and second part of the Final Report are concerned with two quite different topics: 1) The empirical evidence in favour of Dirac's gravitational hypothesis - this evidence coming in a highly interesting manner from those branches of natural sciences which are busied with the study of the Earth and its history. 2) A chapter of abstract algebra, developed by the author and several mathematicians. This mathematical theory of "skew lattices", though going beyond the frame of those chapters of mathematics which have been dealt with hitherto, seems to me to be a really fascinating object of mathematical research.

The own work of the signer as the principal investigator in connection with this contract, during the last three years, has been chiefly devoted to the two mentioned topics of research. Concerning the actual problems of general relativity, my activity during these three years in the first line took the form of encouragement of the research work of my young friends in my HAMBURG SEMINAR FOR GENERAL RELATIVITY . Considerable parts of the results of this endeavour have been published already by my coworkers in different journals or books; but the greater part of these results being still unpublished (or published only in the forms of short indications): it was natural to take the opportunity of this Final Report for a presentation of parts of this material. It being impossible to present all of it, a selection has been made in such a manner that characteristic examples of the activities of our mentioned Seminar will be shown here.

The table of contents at the end of this Report III shows together with the enumeration of chapters and paragraphs the authors who kindly contributed to this Report, delivering a picture of the activities of my young coworkers, whose earnest scientific interest and endeavour has been to me a constant source of joy.

I am also glad to have here the occasion to emphasize our indebtedness to AEROSPACE RESEARCH for sponsoring these investigations.

P. Jordan

Chapter I.

On the Consistency of Abraham's Electromagnetic Energy-Momentum-Tensor with Relativistic Non-Equilibrium Thermodynamics in Continuous Media

§1. Definitions. Basic physical assumptions.

We assume a piece of continuously distributed matter locally in thermal equilibrium; that means: every volume element large enough to apply statistical mechanics but small compared to the whole piece of matter may be regarded as a cell with rigid walls. Within this cell we assume a thermodynamically homogenuous state, but exchange of matter, entropy etc. with neighbouring cells will be admitted. By this at every point the extensive and the intensive thermodynamic variables are well defined by the well known equilibrium thermodynamics; we assume all the quantities to be continuous. These are the only assumptions for the validity of non-relativistic thermodynamics of irreversible processes; they are fulfilled if the system is "close to thermal equilibrium".

Going over to a relativistic formulation we have to postulate that the above assumptions are valid in the rest space of an observer moving with the average velocity of the medium. In analogy to non-relativistic notation we call this observer barycentric although up to now in General Relativity we have no possibility to define a center of mass. Whether this postulate is physically correct may be proven by the results of the theory, namely the transport equations.

In the following treatment, matter should be composed of K components; in the case of a conductor the free electrons are one of them.

We assume the first law of thermodynamics (Gibbs law) in the form

$$(1)\quad d\tilde{e} = T\,ds - p\,d\left(\frac{1}{\varrho}\right) + \sum_{i=1}^{K} \overset{i}{\mu}\, d\overset{i}{c} + E_a\, d\left(\frac{D^a}{\varrho}\right) + H_a\, d\left(\frac{B^a}{\varrho}\right)$$

where $\tilde{e} = \tilde{e}\,(s, \frac{1}{\varrho}, \overset{i}{c}, D^a, B^a)$ is the internal energy per mass unit, s the specific entropy, $\frac{1}{\varrho}$ the specific volume, $\overset{i}{c} \equiv \overset{i}{\varrho}/\varrho$ the concentration of the i-th component. In the case of non-zero free current density s^a, in (1) we will understand by $\overset{1}{\mu}$ the electron affinity and by $\overset{1}{c}$ the electron concentration. The electromagnetic quantities used in (1) are defined by

$$(2)\qquad E_a \equiv F_{ab}u^b \qquad\qquad D_a \equiv G_{ab}u^b$$
$$B_a \equiv -\overset{*}{F}_{ab}u^b \qquad\qquad H_a \equiv -\overset{*}{G}_{ab}u^b \quad +)$$

where F_{ab}, G_{ab} are given by Maxwell's equations

$$(3)\qquad \overset{*}{F}{}^{ab}{}_{;b} = 0$$

$$(4)\qquad G^{ab}{}_{;b} = s^a\ .$$

All the 4-vectors defined by (2) are orthogonal to the normalized barycentric 4-velocity u^a, $u_a u^a = -1$. Our form of $\tilde{e}$ includes a functional dependence between the electromagnetic vectors E^a, D^a, H^a, B^a; for simplicity we restrict ourselves to the isotropic

+) $\overset{*}{F}_{ab} \equiv \frac{1}{2}\,\eta_{abcd}F^{cd}$ is the dual tensor to F_{ab}.

case in adding to the Maxwell equations the phenomenological relations

$$(5) \quad D^a = \varepsilon(s,\tfrac{1}{\varrho},\overset{i}{c})\, E^a \qquad B^a = \mu(s,\tfrac{1}{\varrho},\overset{i}{c})\, H^a$$

We shall hint at the fact that the expression (1) for $d\tilde{e}$ is not generally accepted; we use the formulation of (L 2).

The second law of thermodynamics is taken in the form

$$(6) \qquad S^a{}_{;a} \equiv \sigma \geq 0$$

where S^a is the entropy flow and σ the production rate of the entropy-density.

Further we assume $\tilde{e}$ to consist of two terms,

$$\tilde{e} = u(s,\tfrac{1}{\varrho},\overset{i}{c}) + e(s,\tfrac{1}{\varrho},\overset{i}{c},D^a,B^a),$$

the mechanical (=u) and the electromagnetic (=e) specific energy. Being aware of this fact we will separate the total energy-momentum tensor T^{ab} into

$$(7) \qquad T^{ab} = \overset{m}{T}{}^{ab} + \overset{e}{T}{}^{ab} .$$

For the mechanical term $\overset{m}{T}{}^{ab}$ we use the well known symmetric form

$$(8) \qquad \overset{m}{T}{}^{ab} = (\varrho u)u^a u^b + 2\, q^{(a}u^{b)} + p^{ab}$$

$$q^a u_a = 0, \quad p^{ab}u_b = 0 .$$

We call q^a the thermal flow, p^{ab} the pressure tensor containing the viscosity-stress tensor π^{ab} and the hydrostatic pressure p in the form $p^{ab} = \pi^{ab} + ph^{ab}$; $\frac{1}{3}\pi^a{}_a$ is the difference between the dynamical and the static pressure. As an abbreviation we introduced the projection tensor into the space section orthogonal to u^a defined by $h_{ab} \equiv g_{ab} + u_a u_b$.

As an additional mechanical law we use the conservation of baryons in the form

(9) $$\dot{\varrho} + \varrho\theta = 0$$

where ϱ is the number-density of baryons multiplied with the average nuclear mass of the baryons. (The dot means the substantial derivative in the direction of u^a, i.e. $(t^{ab..}_{r..})^{\cdot} \equiv (t^{ab..}_{r..})_{;c}u^c$.)

The electromagnetic energy per mass unit is +)

(10) $$e = \frac{1}{2\varrho}(E_a D^a + H_a B^a) .$$

This is the 44-component of the Abraham tensor in the barycentric coordinates: $u^a \equiv \delta^a_4$; therefore we have $e = u_a \overset{A}{T}{}^{ab} u_b$ where

(11) $$\overset{A}{T}{}^{ab} \equiv \overset{M}{T}{}^{ab} + (j^a - k^a)u^b$$

(12) $$\overset{M}{T}{}^{ab} \equiv F^{ar}G^b{}_r - \frac{1}{4} g^{ab} F^{rs}G_{rs}$$

(13) $$j^a \equiv - u_c \overset{M}{T}{}^{cd} h_d{}^a \qquad (\Rightarrow j^a u_a = 0).$$

j^a is the Poynting-vector; the quantity $k^a \equiv - h^a{}_c \overset{M}{T}{}^{cd} u_d$ equals $\varepsilon\mu j^a$ in the isotropic case as will be seen explicitely below. x)

§2. Some useful formulae. Poynting's theorem.

By a purely algebraic calculation we get, starting from (2), the explicit formulae for the electromagnetic tensors:

(14) $$F_{ab} = 2\, u_{[a}E_{b]} + \eta_{ab}{}^{cd} u_c B_d$$

$$G_{ab} = 2\, u_{[a}D_{b]} + \eta_{ab}{}^{cd} u_c H_d$$

+) In the terminology of thermodynamics, (10) is a free energy.

x) $\overset{M}{T}{}^{ab}$ is Minkowski's tensor.

(12),(13),(14) give the well known expression for Poynting's vector

$$(15) \qquad j^a = - \eta^{abcd} u_b E_c H_d$$

Analogeously we get $k^a = - \eta^{abcd} u_b D_c B_d$.

Definitions (12) and (14) result in the equation

$$(16) \qquad \overset{M}{T}{}^{ab} = E_r D^r u^a u^b + u^a j^b + k^a u^b - E^a D^b + \\ + \eta^{arcd} \eta^{b}{}_{rst} u_c u^s B_d H^t - \tfrac{1}{2} g^{ab} (H_r B^r - E_r D^r)$$

This proves, together with (11), that we may replace definition (13) by $j^a = - u_c \overset{A}{T}{}^{cd} h_d{}^a$. In the isotropic case one sees immediately that Abraham's tensor is symmetric whilst Minkowski's tensor is not.

For the entropy balance we will need a generalization of Poyntings theorem. To get this, we start from the definition (13) and calculate

$$(17) \qquad j^a{}_{;a} = - (u_c \overset{A}{T}{}^{ca})_{;a} - (u_c \overset{A}{T}{}^{cd} u_d)^{\cdot} - (u_c \overset{A}{T}{}^{cd} u_d)\theta$$

To get the right hand side more explicitely we will use the following decomposition:

$$(18) \qquad u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - \dot{u}_a u_b \qquad {}^{+)}$$

Equations (2),(3),(4),(12),(13) result in

$$(19) \qquad u_a \overset{M}{T}{}^{ab}{}_{;b} = E_r s^r + \dot{u}_a (j^a - k^a) + \dot{B}_r H^r + \dot{D}_r E^r + \\ - (u_a \overset{M}{T}{}^{ab} u_b)^{\cdot}$$

With (17),(18),(19) we get the final form of Poynting's theorem:

$$(20) \qquad j^a{}_{;a} + E_r i^r + \tfrac{2}{3}\theta (E_r D^r + H_r B^r) + j^b \dot{u}_b + \\ + E_r \dot{D}^r + H_r \dot{B}^r + (\omega_{ab} + \sigma_{ab}) \overset{M}{T}{}^{ab} = 0.$$

+) A detailed discussion of the kinematic quantities is given e.g. in (L3)

Because of $F_r u^r = 0$ we have replaced s^r by the current $i^r \equiv h^r{}_s s^s$; i^r is the relativistic analogon of the usual currentdensity $\vec{i}$.

§ 3. The entropy balance.

We start the calculation of the entropy balance using the conservation law $T^{ab}{}_{;b} = 0$, identifying $\overset{e}{T}{}^{ab}$ with Abraham's tensor; we write, being aware of (7):

$$(21) \qquad (\overset{m}{T}{}^{ab} + \overset{A}{T}{}^{ab})_{;b} = 0.$$

This equation expresses the fact that there are no exterieur forces acting on the system except electromagnetic and gravitational ones.

We will use (1) in the form of a substantial balance:

$$(22) \quad T\dot{s} = \dot{u} + \dot{e} + p\frac{\theta}{\varrho} - \sum_{i=1}^{K} \overset{i}{\mu}\overset{i}{\nu} I_{ch} + \sum_{i=1}^{K} \overset{i}{\mu}\overset{i}{I}{}^{a}{}_{;a} + - \frac{1}{\varrho}(F_r \dot{D}^r + H_r \dot{B}^r) - \frac{\theta}{\varrho} (F_r D^r + H_r B^r)$$

where we have written the substantial derivative of the concentration in the following way:

$$(23) \qquad (\overset{i}{c})^{\cdot} = \overset{i}{\nu} I_{ch} - \overset{i}{I}{}^{a}{}_{;a} \qquad (\overset{i}{I}{}^{a} u_a = 0)$$

$\overset{i}{I}{}^{a}$ is the diffusion flow of the i-th component, $\overset{i}{\nu}$ are the stöchiometric coefficients, and I_{ch} is the chemical production rate.

(12),(13),(11),(18) result in

$$u_a \overset{A}{T}{}^{ab}{}_{;b} = - \varrho\dot{e} - j^a \dot{u}_a - j^a{}_{;a} - (\sigma_{ab} + h_{ab}\tfrac{\theta}{3})\overset{A}{T}{}^{ab}$$

and (8),(9),(18) in

$$u_a \overset{m}{T}{}^{ab}{}_{;b} = - \varrho\dot{u} - q^a \dot{u}_a - q^a{}_{;a} - (\sigma_{ab} + \tfrac{\theta}{3} h_{ab}) p^{ab}$$

These expressions together with (20),(21),(22) give, after some simple calculations, the final form of the entropy balance:

$$(24)\quad \rho\dot{s} = \left\{\frac{1}{T}\Big(-q^a + \sum_{i=1}^{K} \overset{i}{I}{}^a \overset{i}{\mu}\Big)\right\}_{;a} - \frac{1}{T}\left\{\Big[\frac{1}{3}\pi^a{}_a + \frac{1}{2}(E_r D^r + H_r B^r)\Big]\theta + \sum_{i=1}^{K} \overset{i}{\nu} I_{ch} \overset{i}{\mu} + \pi^{ab}\sigma_{ab}\right\} + q^a\Big[\Big(\frac{1}{T}\Big)_{,a} - \frac{\dot{u}_a}{T}\Big] + i^a \frac{E_a}{T} - \sum_{i=1}^{K} \overset{i}{I}{}^a \Big(\frac{\overset{i}{\mu}}{T}\Big)_{,a} .$$

Before discussing this formula, we will give a more explicit form of (6). We write as a definition:

$$(25)\qquad S^a = (\rho s)u^a + \sigma^a \quad , \qquad \sigma^a u_a = 0$$

where σ^a is now the analogon to the nonrelativistic entropy-flow, and s is the specific entropy assumed as the fourth component of a 4-vector (L 4). With (9) and (25) the second law reads :

$$(26)\qquad \rho\dot{s} = -\sigma^a{}_{;a} - \sigma \quad .$$

Comparing (24) and (26), we are tempted to write for the entropy-flow, in agreement with the non-relativistic result (L 1,2) :

$$(27)\qquad \sigma^a = \frac{1}{T}\Big(q^a - \sum_{i=1}^{K} \overset{i}{\mu} \overset{i}{I}{}^a \Big) \quad .$$

Let us assume that every component of our system has the specific charge $\overset{i}{e}$ with $\sum_{i=1}^{K} \overset{i}{e} = \overset{t}{e}$ (= total specific charge); then we may write:

$$(28)\qquad s^a = \rho \overset{t}{e} u^a + \sum_{i=1}^{K} \rho \overset{i}{e} \overset{i}{I}{}^a \quad .$$

With this, the dissipation function σ of (24) becomes:

$$(29)\qquad \sigma = \sum_{i=1}^{K}\left[\overset{i}{\varrho_e}\frac{F^a}{T} - \left(\frac{\overset{i}{\mu}}{T}\right)_{,c} h^c{}_a\right]{}^i I_a - \frac{q^a}{T}\left(\dot{u}_a + \frac{T_{,a}}{T}\right) +$$
$$- \frac{1}{T}\pi^{ab}\sigma_{ab} - \frac{1}{T}\sum_{i=1}^{K}\overset{i}{\mu}\overset{i}{\nu}\, I_{ch} - \frac{1}{T}\theta\left(\frac{1}{3}\pi^a{}_a + \right.$$
$$\left. + \frac{1}{2}(E_r D^r + H_r B^r)\right) .$$

(29) essentially agrees with the non-relativistic form of the dissipation function, but we find now two relativistic corrections; namely the term $\frac{1}{T}q^a\dot{u}_a$ expresses the fact, that the thermal flow has inertia according to the equivalence principle, whilst the last term in (29) expresses that electromagnetic energy produces inertial effects too.(L 4,6)

At the end of this section we shall ~~hint~~ at the following fact: Even when taking into account the inequality $\sigma \geq 0$,the separation of (24) into a divergence and a rest will not be unique (L 1). Which one of the various possibilities will be the correct one may only be justified by experimental evidence.

§ 4.The transport equations.

Usually the fluxes are assumed to be linearely dependent on the thermodynamic forces. As a consequence,the local entropy production rate becomes a quadratic form in the thermodynamic forces and therefore fulfills $\sigma \geq 0$. When completing this assumption by the Onsager-Casimir postulate on the coefficients of the quadratic form (L 5), it closely agrees with the experimental experience. In this paper we will only use the fact that the coefficients connecting quantities of different tensorial character vanish in the isotropic case (Curie-law) as a consequence of symmetry considerations (L 2, p.57 ff).

We will write down explicitely the transport equations derived from (29) by Onsager's theory. The experimental verification of these equations indirectly proves the class of energy-momentum-tensors consistent with thermodynamics.

We start with the scalar equations:

(30) $$I_{ch} = \sum_{i=1}^{K} \overset{i}{n}_1 \overset{i}{\mu}\overset{i}{\nu}\left(\frac{1}{T}\right) + n_2 \frac{\theta}{T}$$

(31) $$\psi = \sum_{i=1}^{K} \overset{i}{n}_3 \overset{i}{\mu}\overset{i}{\nu}\left(\frac{1}{T}\right) + n_4 \frac{\theta}{T}$$

where $\psi \equiv \frac{1}{3}\pi^a{}_a + \frac{1}{2}(F_r D^r + H_r B^r)$.

We define the chemical production density of the i-th component $\overset{i}{\nu} I_{ch} \equiv \Gamma^i$, the chemical viscosities $n_2 \overset{i}{\nu} \equiv \lambda^i$ and $-\Lambda^{ik} \equiv n_1 \overset{i}{\nu}\overset{k}{\nu}$; then we get instead of (30) the form

$$\Gamma^i = \frac{\lambda^i}{T} u^a{}_{;a} - \sum_{k=1}^{K} \Lambda^{ik} \frac{\overset{k}{\mu}}{T}$$

This agrees with the nonrelativistic relation in (L1). (31) expresses the inertia of energy and possesses no non-relativistic analogon.

The vectorial equation reads:

(32) $$\overset{i}{e}\overset{i}{I}^a = \sum_{k=0}^{K}\left[(L^i{}_k)^{ab} \frac{\overset{k}{e}}{T} F_b + (M^i{}_k)^{ab}\left(\frac{\overset{k}{\mu}}{T}\right)_{,b}\right] + (N^i)^{ab} \frac{1}{T} \dot{u}_b$$

where now $i,k = 0,1,\dots,K$ with $\overset{o}{e} \equiv 1$, $\overset{o}{\mu} \equiv 1$; by this we have included the transport equation for the thermal flow $q^a \equiv \overset{o}{I}^a$ in (32).

We discuss this equation in various steps. First we specialize (L 4): $(L^i{}_k)^{ab} = (L^i{}_k)h^{ab}$, $(M^i{}_k)^{ab} = (M^i{}_k)h^{ab}$, and $(N^i)^{ab} = (N^i)h^{ab}$.

1.No electromagnetic effects ($\Rightarrow F_a = 0$): Then ($M^k{}_o$) describes the thermal diffusion and ($M^o{}_k$) the diffusion thermoeffects. The last term is purely relativistic and states that inertia influences the thermal and the diffusion flow (L 3,4,6).

2.For simplicity we assume in the discussion of $F_a \neq 0$ that the system consists of two components only; a heavy component that moves with barycentric velocity ($\Rightarrow {}^1I^a = 0$) - as an example take the atoms in a crystal lattice - and a component that moves with ${}^2I^a$ relatively to the barycentric observer and has the charge e - free electrons in a metal. Then (32) becomes:

$$(33) \qquad i^a = L^2 \frac{F^a}{T} + M^2 \frac{\mu'^a}{T} - \bar{M}^2 \frac{T'^a}{T^2} + N^2 \frac{\dot{u}^a}{T}$$

$$(34) \qquad q^a = L^o \frac{F^a}{T} + M^o \frac{\mu'^a}{T} - \bar{M}^o \frac{T'^a}{T^2} + N^o \frac{\dot{u}^a}{T} \quad .$$

(33) is the relativistic form of the generalized Ohm-law (non-relativistic form in L 2) with:

$R = \frac{T}{L^2}$ is the isothermal resistivity

(35) $\lambda = \frac{\bar{M}^o L^2 + \bar{M}^2 T}{L^2 T}$ is the heat conductivity coefficient for $i^a = 0$

$\pi = \frac{L^o T}{L^2}$ is the Peltier-coefficient

$\eta = - \frac{\bar{M}^2}{L^2 T}$ is the Thompson-coefficient

When not specializing to two components we have the relativistic form of Ohm's law for an ionized plasma in an exterieur magnetic field (L 9); in this we have to include the galvanomagnetic effects too.

To do this, we have only to assume that $(L^{i}_{k})^{ab}$, $(M^{i}_{k})^{ab}$, $(N^{i})^{ab}$ are skew symmetric tensors orthogonal to u^a, then (32) gives the galvanomagnetic effect like for instance: Hall effect, Ettingshausen-effect etc. A detailed discussion of these effects is given in (L 2), p.357 ff.

Finally we discuss the tensorial equation

$$(36) \quad \Pi^{ab} = \frac{1}{T} \underset{1}{R}{}^{ab}{}_{cd}\, \sigma^{cd} + \frac{1}{T}(\theta \underset{2}{R}{}^{ab}{}_{cd} + \sum_{i=1}^{K} \overset{i}{\mu} \overset{i}{\nu} \underset{3}{\overset{i}{R}}{}^{ab}{}_{cd}) h^{cd}$$

When using (18) and putting $\underset{\varkappa}{R}{}^{ab}{}_{cd} = \underset{\varkappa}{R}\, h^{(a}_{(c} h^{b)}_{d)}$ for all the above coefficients, (36) will become more familiar. We define: $(2T)^{-1}\underset{1}{R} \equiv -\tau$ (shear viscosity), $-(T)^{-1}\underset{2}{R} \equiv \zeta$ (volume viscosity) and $\underset{3}{\overset{i}{R}}\overset{i}{\nu} \equiv \lambda^i$ (chemical viscosity); Then we get

$$(37) \quad p^{ab} - ph^{ab} = -2\tau u^{(a;b)} + h^{ab}\Big[-(\zeta - \frac{2}{3}\tau)u^{c}{}_{;c} + - \sum_{i=1}^{K} \frac{1}{T} \overset{i}{\mu} \overset{i}{\lambda}\Big] - 2\tau \dot{u}^{(a}u^{b)}$$

This is very analogous to the non-relativistic formula (L 1): $(\alpha, \beta = 1,2,3)$

$$p_{\alpha\beta} - p\,\delta_{\alpha\beta} = -2\tau v_{(\alpha,\beta)} - \delta_{\alpha\beta}\Big[(\zeta - \frac{2}{3}\tau) v^{\gamma}{}_{,\gamma} + - \sum_{i=1}^{K} \frac{1}{T} \lambda^{i} \overset{i}{\mu}\Big] .$$

§ 5. Concluding remarks. Literature.

This work was stimulated by the paper of Kluitenberg (L 4). But instead of Kluitenberg's purely formal arguings on the admissible forms of the electromagnetic terms in (1) and the energy-momentum-tensor, we start with a physically reasonable Gibbs law (L 8, p.87) and prove its consistency with the Abrahamtensor by calculating the transport equations.

Another approach for the justification of Abrahams tensor was given by Marx & Györgyi (L 7). They proved that it is consistent with the classical non-relativistic ponderomotoric force. But this method is very problematic because of the ambiguity of the electromagnetic fields within the continuous matter distribution.

The present work was done in close collaboration with Dr. J. Ehlers (present adress: Div.Math. and Th. Phys. - South West center for Adv.Studies, Dallas)

The following list of books and papers is anounced in the text by an L; e.g. (L 3).

1. J. Meixner, H.G. Reik — Handb.f.Phys. III/2 (1959)
2. S.R. de Groot, P. Mazur — Non-Equilibrium Thermodynamics Amsterdam (1963)
3. J. Ehlers — Ak.Wiss.Lit.Mainz(math.nat.Kl.) 11 (1961)
4. G.A. Kluitenberg — Dissertation Leiden (1954)
   -"-, S.R. de Groot — Physica 19 (1953), 20 (1954)
5. L. Ohnsager — Phys.Rev. 37, 38 (1931)
   H.B.G. Casimir — Rev.Mod.Phys. 17 (1945)
6. C. Eckart — Phys.Rev. 58 (1940)
7. Marx — Acta phys.hung. 3, (1953)
   -"- & Györgyi — Acta phys.hung. 3, (1953)
8. R. Becker, F. Sauter — Theorie der Elektrizität Stuttgart 1957
9. J. Brandstatter — An Intr. to Waves, Rays and Radiation in Plasma Media McGraw Hill 1963.

CHAPTER II. EXTERIOR DIFFERENTIAL CALCULUS FOR SPINOR-FORMS IN GENERAL RELATIVITY.

Introduction.

The spinor formalism, introduced into General Relativity by PENROSE (1), has become a very important tool for studying radiation properties of gravitational fields. This is due to the fact that the light cone and information about the PETROV type and null congruences is treated most easily in terms of spinors. Nevertheless, they have not yet been used for explicit calculations, e.g. of the curvature spinors of a given field or for the construction of a field with given radiation properties.

We shall show here how the spinor formalism can also be used for problems of the latter kind. The method will be analogous to the well known CARTAN formalism: We generalize the notion of a spinor field to that of a spinor form, and the notion of the covariant derivative to that of the covariant differential of an arbitrary spinor form. To this end, we have to recall the notion of a spinor in General Relativity and its covariant derivative; in fact, we shall do this in some detail, in order to compare the formalism developed here later on with some other formalisms, which have a similar base. We shall then give a new technique for calculating the covariant derivative and the curvature spinors of a given field in a purely spinorial manner. We shall describe a canonical way to construct coordinates adapted to certain information about a null congruence.

The second half of this chapter, beginning with §6, will contain some applications of the technique to radiation problems. We present the connection form of the general radiation metric, the curvature spinors of the pure radiation field with normal ray and treat some special radiation fields in order to show how the technique presented here can be used for questions concerning the connection between features of a null congruence, PETROV type, motion group, inner geometry of the wave fronts etc. The chapter is closed by a discussion about the definition of radiation in matter.

§1. Spinor algebra and spinor forms.

In order to generalize the notion of a spinor field to that of a spinor form, let us recall the introduction of spinors into General Relativity, which is possible because of the local isomorphy of the group u_2 of unimodular complex 2x2-matrices $u^A{}_B$ to the proper orthochronous LORENTZ group $L^{\uparrow}_{+}$. We do it in some detail in order to compare it later on with the introduction of complex 3-vectors into General Relativity done by DEBEVER, CAHEN and DEFRISE, which leads to a form-formalism similar to the one presented here.

u_2 admits of the following fundamental (complex) 2-dimensional representations:

$$(1.1)\quad \begin{aligned} u^A{}_B &: \varkappa^A \longrightarrow u^A{}_B \varkappa^B \\ u^A{}_B &: \varkappa_A \longrightarrow \varkappa_B u^{-1B}{}_A \\ u^A{}_B &: \varkappa^{A'} \longrightarrow \bar{u}^{A'}{}_{B'} \varkappa^{B'} \\ u^A{}_B &: \varkappa_{A'} \longrightarrow \varkappa_{B'} \bar{u}^{-1B'}{}_{A'} \end{aligned} ,$$

All capital indices (dashed and undashed ones) taking the values 0,1 in the sequel.

where u^{-1} is the inverse of u, which exists since u has determinant unity, and $\bar{u}$ is the matrix complex conjugate (c.c.) to u.

Let ε_{AB}, ε^{AB} be the skew symmetric systems

$$(1.2)\quad \varepsilon_{AB} = -\varepsilon_{BA} = \varepsilon^{AB} = -\varepsilon^{BA} \quad , \quad \varepsilon_{01} = 1$$

and define lowering and raising of indices by

$$(1.3)\quad \varkappa^A = \varepsilon^{AB}\varkappa_B \quad , \quad \varkappa_A = \varkappa^B \varepsilon_{BA}.$$

One then easily shows that (1.3) provides a vectorspace isomorphism and its inverse, which commutes with the first two of the representations (1.1): it is an equivalence of these representations. By an analogous definition of $\varepsilon_{A'B'}$, $\varepsilon^{A'B'}$, one gets the same result for the dashed representations in (1.1). These facts may be expressed by saying that the representations (1.1) leave invariant the **symplectic** metric (1.2) (and dashed). It is easy to see that a dashed representation (1.1) is not equivalent to an undashed one.

Actually, we are not only interested in the representations (1.1), but in all tensorial products of them:

$$(1.4) \quad u^A{}_B : S^{A..A'..}{}_{B..B'..} \rightarrow u^A{}_{C}..\bar{u}^{A'}{}_{C'}..S^{C..C'..}{}_{D..D'..}u^{-1D}{}_{B}.\bar{u}^{-1D'}$$

where $S^{..}{}_{..}$ is a spinor, an element of the corresponding tensorial product of the complex 2-dimensional spaces appearing in (1.1).

In order to find all structures of the spinor algebra, let us look for mappings of it into itself, which commute with the representation (1.4) of u_2. There is the tensorial product ($(S^A, T_{A'B}) \rightarrow S^A T_{A'B}$), the contraction of indices of the same kind (with respect to dashes) ($S^{A..}{}_{B..} \rightarrow S^{A..}{}_{A..}$), symmetrization and antisymmetrization of indices or pairs of indices of the same kind, raising and lowering of indices according to (1.3) and the transition to the c.c. spinor ($S^{AA'}{}_B \rightarrow \bar{S}^{A'A}{}_{B'} =: \overline{S^{AA'}{}_B}$). Consideration along usual lines of these mappings yields the following **three** main properties of the spinor algebra, which we shall make frequent use of in the sequel:

1.) let $T_{AA'BB'}$ be a real spinor. Then for its part skew in the (underlined) pairs of indices $T_{[\underline{AA'}\,\underline{BB'}]} = 1/2(T_{AA'BB'} - T_{BB'AA'})$, we have

$$(1.5) \quad T_{[\underline{AA'}\,\underline{BB'}]} = T_{AB}\varepsilon_{A'B'} + \bar{T}_{A'B'}\varepsilon_{AB} \quad \text{with } T_{AB} = 1/2T_{(AA'B)}{}^{A'}.$$

(What is meant by this statement is exactly that the decomposition (1.5) is preserved by the representation (1.4), if we treat all T's as spinors. This would of course not be true, if we had considered all linear mappings instead of u_2 only. Thus decomposition properties like (1.5) are particular for the representation (1.4) rather than for the spinor algebra itself, and all formulae like (1.5) are meant in this sense.)

2.)

$$(1.6) \quad S_{[\underline{AA'}\,\underline{BB'}\,\underline{CC'}]} = o \text{ and } S^{AA'}{}_{[\underline{AB'}\,\underline{CA'}]} = o. \text{ are equivalent.}$$

This equivalence may be obtained by applying formula (1.6) of (1) for spinor equivalent of the η^{abcd}-pseudotensor to S.

3.) The real spinors with two indices $k^{AA'}$ form a real vector space T, which, with the metric (1.2) is isometric to MINKOWSKI space of signature (+---). The representation (1.4) induces proper orthochronous LORENTZ transformations in it, preserving orientation and time orientation suitably defined. Due to the local isomorphism of u_2 and $L_+^{\uparrow}$, any proper orthochronous LORENTZ transformation is obtained by (1.4).

Since PENROSE (1), one makes use of the facts under 3.) to introduce spinors into General Relativity in the following fashion:

Let us assume the V^4 considered there to be orientated and time oriented and to have signature (+---). Since the tangent space T_x in a point x of V^4 is itself a MINKOWSKI vector space of signature (+---), one may choose any orientations preserving isometry: $k^a(x) \rightarrow k^{AA'}(x) =: \sigma^{AA'}{}_a(x) k^a(x)$ of T_x on T and extend it to the whole tensor algebras over both spaces:

$$T^{a..}{}_{b..}(x) \rightarrow T^{AA'..}{}_{BB'..}(x) =: \sigma^{AA'}{}_a(x) .. T^{a..}{}_{b..}(x) \sigma_{BB'}{}^b(x) .. \quad ,$$

thus describing each geometrical quantity by a real spinor field. (def. of $\sigma_{AA'}{}^a$ by the metrics in T_x and T).

Now, the choice of the isometry σ above is in the same way completely arbitrary as is the choice of a coordinate system in the usual descriptions of RIEMANNian geometry. But, if we choose another isometry , there is, due to property 3.), - a $u^A{}_B$ u_2 with

$$(1.7) \quad \tilde{\sigma}^{AA'}{}_a = u^A{}_B \bar{u}^{A'}{}_{B'} \sigma^{BB'}{}_a .$$

The spinor components T of the same tensor, calculated for the new isometry may be expressed in the old ones by (e.g.):

$$(1.8) \quad T^{AA'..}{}_{BB'..} = u^A{}_C \bar{u}^{A'}{}_{C'} T^{CC'..}{}_{DD'..} u^{-1D}{}_B \bar{u}^{-1D'}{}_{B'} ..$$

Thus, to any tensor field there corresponds a real spinor field, which for any change of the isometry (1.7) transforms according to (1.8). Starting from this transformation property we define now an arbitrary (non-real) spinor field as a mapping of V^4 into the spinor algebra, which transforms according to (1.7), (1.8).

This definition is slightly different from the one initially given by PENROSE (1), who set up an explicit correspondence of spinors (e.g. T_{AB}) and tensors by formulae like (1.5) followed by a translation by σ's, whereas we leave the geometrical interpretation of non-real spinor fields to a supplementary consideration, using only the formal properties of them, which are expressed by the transformation rules. We need this definition here in order to generalize it a bit farther to that of a s p i n o r i a l f o r m: a spinorial form (spinor form) is a form on V^4 with values in the spinor algebra, which for a change of isometry (1.7) transforms analogously to (1.8):

$$(1.9) \quad \tilde{\phi}^{A.A'.}{}_{B.B'.} = u^A{}_C \bar{u}^{A'}{}_{C'} \cdots \phi^{C.C'.}{}_{D.D'.} u^{-1D}{}_B \bar{u}^{-1D'}{}_{B'} \cdots$$

It is a"spinor, the components of which are forms".

Spinor forms may be treated exactly like spinor fields, if one replaces ordinary multiplication by the wedge product "$\wedge$". The statements made above on mappings, which commute with (1.1) remain untouched, especially the properties 1.) to 3.) remain true. It is clear what a real spinor form is.

Let us give some examples for spinor forms, which will prove important in the sequel:

There are at first the spinorial o-forms, which are by definition identical with the spinor fields. The wedge product coincides with the ordinary one, if one of the factors is a o-form.

Next, there is the f u n d a m e n t a l f o r m $\theta^{AA'} =: \sigma^{AA'}{}_a dx^a$ (x^a the coordinate system). With its aid, the metrical fundamental form may be put into the form

$$(1.10) \quad Q = g_{ab} dx^a dx^b = \varepsilon_{AB} \varepsilon_{A'B'} \theta^{AA'} \theta^{BB'} = 2\theta^{00'}\theta^{11'} - 2\theta^{01'}\theta^{10'}.$$

On the other hand, once the fundamental form is put into this form, one knows the $\theta^{AA'}$, and the knowlege of them is sufficient for all calculations to come. $\theta^{AA'}$ is a real spinor form, the components of which are a basis of the space of 1-forms. Every spinor form may be expressed by $\theta^{AA'}$, e.g.

$$(1.11)\quad \phi^{AA'}{}_{B} = F^{AA'}{}_{BCC'DD'}\,\theta^{CC'}\wedge\theta^{DD'} ,$$

the coefficients $F^{\cdot\cdot}$ beeing a spinor again.

Decomposing $\theta^{AA'}\wedge\theta^{BB'}$ according to (1.5)

$$(1.12)\quad \theta^{AA'}\wedge\theta^{BB'} = \theta^{AB}\varepsilon^{A'B'} + \bar{\theta}^{A'B'}\varepsilon^{AB} \quad , \quad \theta^{AB} = 1/2\,\theta^{A}{}_{A'}\wedge\theta^{BA'} ,$$

the structural form gives rise to the spinorial 2-form θ^{AB}, which is symmetric in AB and which, together with its c.c. $\bar{\theta}^{A'B'}$, is a basis for the 6-space of two-forms. Applying again equ. (1.6) of (1) to the equation
$\theta^{AA'}\wedge\theta^{BB'}\wedge\theta^{CC'}\wedge\theta^{DD'} = i\theta\eta^{AA'BB'CC'DD'}$, $\theta = \theta^{00'}\wedge\theta^{01'}\wedge\theta^{10'}\wedge\theta^{11'}$,
one easily proves

$$(1.13)\quad \theta^{AB}\wedge\theta^{CD} = -\varepsilon^{A(C}\varepsilon^{D)B}\theta \quad , \quad \theta^{AB}\wedge\bar{\theta}^{A'B'} = 0$$

Another example for a spinor form follows in the next paragraph.

§2 Spinor-analysis.

Since spinors are here used only as equivalent expressions for tensors, it is at least clear for real spinor fields, for which the correspondence to tensors has already been defined, how the covariant derivative must be defined: namely, by translation of the tensorial covariant derivative:

(2.1) $T^{AA'..}{}_{..} \rightarrow T^{a.}{}_{.} \rightarrow T^{a.}{}_{.;c} \rightarrow T^{AA'..}{}_{..;CC'}$.

The aim of this paragraph is to generalize the definition of the covariant differentiation to arbitrary spinor forms and to calculate the differential operator,so that the translations are avoided and a compact expression for the covariant derivation of an arbitrary spinor form is derived, which is analogous to the one used in ordinary CARTAN formalism (cf.(2)(3)(4)).

To this end let us first define the covariant differential for real spinor fields with the aid of (2.1) by

(2.2) $DT^{AA'..}{}_{..} = T^{AA'..}{}_{..;BB'}\theta^{BB'}$

it is a spinor form with the same transformation rule as T. Let us extend this definition at once to real spinorial forms,

e.g. $\phi^{AA'..}{}_{..} = F^{AA'..}{}_{..BB'CC'}\theta^{BB'}\wedge\theta^{CC'}$ by

(2.3) $D\phi^{AA'..}{}_{..} = DF^{AA'..}{}_{..BB'CC'}\wedge\theta^{BB'}\wedge\theta^{CC'}$.

It is then a matter of straightforward translation of well known tensorial rules, using (2.1) to (2.3), to check that the covariant differential of real spinorial forms has the following six properties: (S, T stand for sets of spin-indices; recall that if one of the forms in a wedge product is o-form - spinor field -, the wedge multiplication coincides with the ordinary one)

1) The sum and product rules
$D(\phi^S+X^T)=D\phi^S+DX^T$, $D(\phi^S{\wedge}X^T)=D\phi^S{\wedge}X^T+(-1)^{\deg\phi}\phi^S{\wedge}X^T$

2) D commutes with the contraction.

3) for scalar functions f, Df = df.

4) D commutes with the transition to the c.c. spinor form. (The statement is empty for real spinor forms, for which the covariant differential is only defined up to now; it will be a property of the extended D to be defined below.)

5) D is torsionfree, i.e. $D\theta^{AA'} = o$ and $D\phi = d\phi$ for scalar forms ϕ.

6) D commutes with raising and lowering of indices:
$D\varepsilon_{AB} = D\varepsilon_{A'B'} = D\varepsilon^{AB} = D\varepsilon^{A'B'} = o$.

On tensor forms, properties 1) to 6) fully characterize the covariant differential (cf. (8), chap.I, §9). We shall show now that there is a unique extension of D to all spinor forms such that 1) to 6) still hold, and give a compact expression for it.

Theorem. There is one and only one operator D on all spinor forms which increases the degree of a form by one and obeys the rules 1) to 6). It is of the form

$$(2.4)\quad D\phi^{A..A'..}{}_{B..B'..} = d\phi^{A..A'..}{}_{B..B'..} + \omega^{A}{}_{C}{\wedge}\phi^{C..A'..}{}_{B..B'..} + \dots + \bar{\omega}^{A'}{}_{C'}{\wedge}\phi^{A..C'..}{}_{B..B'..} + \dots - \omega^{D}{}_{B}{\wedge}\phi^{A..A'..}{}_{D..B'..} - \dots - \bar{\omega}^{D'}{}_{B'}{\wedge}\phi^{A..A'..}{}_{B..D'..}\quad ,$$

Where $\omega^A{}_B$ is the unique 1-form solution of the equations

$$(2.5)\quad d\theta^{AA'} + \omega^A{}_B \wedge \theta^{BA'} + \bar\omega^{A'}{}_{B'} \wedge \theta^{AB'} = o = \omega^A{}_A$$

__remark__. With (2.4), (2.5) may be rewritten as $D\theta^{AA'} = o = D\varepsilon^{AB}$.

__Proof.__

a) Let us first show that every D with the properties of the theorem admits of the representation (2.4) on spinor fields. To this end consider the operator D-d. Because of 3) it obeys $(D-d)(fS^{A..}{}_{..}) = f(D-d)S^{A..}{}_{..}$ for arbitrary funktions f, so it is linear in every point:

$$(2.6)\quad (D-d)S^{A..}{}_{B..} = \omega^{A..}{}_{B..;C..}{}^{D..}S^{C..}{}_{D..}\quad ,$$

where $\omega^{A..}{}_{B..;C..}{}^{D..}$ are 1-forms with the doubled set of indices. By means of the other properties 1) to 6) we shall now put into correspondence the ω's belonging to different representations. For the sake of simpler notation we shall confine ourselves to a sample spinor field $S^{AB'}{}_B$ from now on. The general case follows similarly. Applying 1) to (2.6) we get

$$(D-d)(k^A l^{A'} m_B) = \omega^{AA'}{}_{B;CC'}{}^D k^C l^{C'} m_D$$

$$= \omega^A{}_{;C} k^C l^{A'} m_B + \omega^{A'}{}_{;C'} k^A l^{C'} m_B + \omega_{B;}{}^D k^A l^{A'} m_D \;.$$

Taking account of the arbitrariness of k,l,m, we see that every ω in (2.6) admits of a reduction to one of the forms $\omega^A{}_{;B'}$, $\omega^{A'}{}_{;B'}$, $\omega_{A;}{}^B$, $\omega_{A';}{}^{B'}$. Now, because of 2), 3)

$(D-d)(k^A l_A) = \omega^A{}_{;B} k^B l_A + \omega_{A;}{}^B k^A l_B = o$ and we get

$\omega^{A}{}_{;B} = -\omega_{B;}{}^{A}$ and analogously $\omega^{A'}{}_{;B'} = -\omega_{B';}{}^{A'}$.
By 4) $\omega^{A'}{}_{;B'} = \overline{\omega^{A}{}_{;B}}$ = c.c. of $\omega^{A}{}_{;B}$.

Putting all this information together one sees that all the ω's of (2.6) may be reduced to $\omega^{A}{}_{;B} = \omega^{A}{}_{B}$ and that (2.4) holds for spinor fields.

b) By 6) $D\varepsilon^{AB} = \omega^{A}{}_{C}\varepsilon^{CB} + \omega^{B}{}_{C}\varepsilon^{AC} = \omega^{BA} - \omega^{AB} = 0$, thus ω^{AB} is symmetric and $\omega^{A}{}_{A} = 0$.

c) Applying 5) and (2.6) to the scalar form $\phi = k_{AA'}\theta^{AA'}$ we find $(D-d)\phi = k_{AA'}(d\theta^{AA'} + \omega^{A}{}_{B}\wedge\theta^{BA'} + \bar{\omega}^{A'}{}_{B'}\wedge\theta^{AB'}) = 0$, so $\omega^{A}{}_{B}$ obeys equ. (2.5).

d) In order to show that (2.4) holds also for spinor forms, one develops the spinor form in question in terms of the $\theta^{AA'}$, the coefficients being a spinor field again, and applies the result obtained above. It will be sufficient to calculate the covariant differential for a 1-form $\phi^{A} = F^{A}{}_{BB'}\theta^{BB'}$ to show the scheme:

$$D\phi^{A} = DF^{A}{}_{BB'}\wedge\theta^{BB'} = dF^{A}{}_{BB'}\wedge\theta^{BB'} + \omega^{A}{}_{C}\wedge F^{C}{}_{BB'}\theta^{BB'}$$

$$- F^{A}{}_{CB'}\omega^{C}{}_{B}\wedge\theta^{BB'} - F^{A}{}_{BC'}\bar{\omega}^{C'}{}_{B'}\wedge\theta^{BB'} =$$

using (2.5)
$$= dF^{A}{}_{BB'}\wedge\theta^{BB'} + F^{A}{}_{BB'}d\theta^{BB'} + \omega^{A}{}_{C}\wedge\phi^{C} =$$

$$= d\phi^{A} + \omega^{A}{}_{C}\wedge\phi^{C}\ .$$

We know now that if there is a D with the properties 1) to 6), it is of the form given in the theorem. On the other hand, it is obvious that if we define a D by (2.4), (2.5) it has these properties. Thus, the proof of the theorem is reduced to showing that there is one and only one system of PFAFFian forms $\omega^{A}{}_{B}$ satisfying (2.5). We shall not calculate the the solution of

(2.5) in order to prove its existence and uniqueness - although it would be equally easy - , we shall instead transform (2.5) into an equivalent equation for which we do the calculation. We thus give two formulae (2.5) and (2.7) for the $\omega^A{}_B$. To this end let us make the following remark: If there is a solution of (2.5) and we define with its aid by (2.4) a differential D, this then satisfies 1) to 6). Then by (1.12) $D\theta^{AB} = o$, so $\omega^A{}_B$ obeys the equations

$$(2.7) \quad d\theta^{AB} + \omega^A{}_C \wedge \theta^{CB} + \omega^B{}_C \wedge \theta^{AC} = o, \qquad \omega^A{}_A = o\ .$$

On the other hand, let $\omega^A{}_B$ be a solution of (2.7) and define with it a differential D by (2.4). Then obviously D has properties 1) to 4) and 6), and by (1.12) we have $o = D(\theta^{AA'} \wedge \theta^{BB'})$, by 1) $D\theta^{AA'} \wedge \theta^{AA'} = o$, so $D\theta^{AA'} = \emptyset \wedge \theta^{AA'}$, where $\emptyset$ is a 1-form, which by reasons of covariance does not depend on AA' anymore. Putting this into $o = D(\theta^{AA'} \wedge \theta^{BB'})$, one finds $\emptyset = o$. So we have the result that any solution of (2.7) is a solution of (2.5), thus the solutions of these equations are identical, and the proof the theorem is now reduced to the proof of the first statement of the following

<u>Corollary</u> (2.7) has a unique solutions of 1-forms $\omega^A{}_B$. Inserted into (2.4), they give the covariant differential.

<u>proof</u> : Let $d\theta^{AB} = T^{AB}{}_{CC'DD'EE'}\theta^{CC'} \wedge \theta^{DD'} \wedge \theta^{EE'}$ and

$$(2.8) \quad \omega^A{}_B = \Gamma^A{}_{BCC'}\theta^{CC'} ,$$

where T is skew in the pairs DD' and EE' of indices. Then the first equation of (2.7) reads

$$(T^{AB}{}_{CC'DD'EE'} + 1/2\,\delta^B_E \Gamma^A{}_{DCC'}\varepsilon_{D'E'} + 1/2\,\delta^A_E \Gamma^B{}_{DCC'}\varepsilon_{D'E'})\theta^{CC'} \wedge \theta^{DD'} \wedge \theta^{EE'} = o.$$

Applying (1.6) one gets $2T^{ABDE'}{}_{DD'EE'} = \delta^{(B}_{E}\Gamma^{A)D}{}_{D D'} + \Gamma^{(A\ B)}{}_{E D'}$.

Contraction with δ^{E}_{B} and use of the second equation of (2.7) yields

(2.9) $\Gamma^{A}{}_{[BC]C'} = 1/2T^{AEDE'}{}_{DC'EE'}\varepsilon_{BC}$

and because of $\Gamma^{(A\ B)}{}_{C\ C'} = \Gamma_{C}{}^{(AB)}{}_{C'}$, one then gets

(2.10) $\Gamma^{A}{}_{(BC)C'} = 2T_{BC}{}^{DD'}{}_{DC'}{}^{A}{}_{D'} + \delta^{A}_{(C}T_{B)}{}^{EDD'}{}_{DC'ED'}$

Thus, the theorem and the corollary are proven now.

The proof above gives immediately a way to calculate the solutions of (2.5), (2.7): one calculates the $T^{\cdot\cdot}{}_{\cdot\cdot\,\cdot\cdot\,\cdot\cdot}$, which have only to be skew in the last pair of indices, and applies formulae (2.9), (2.10). In special cases, when the metrical fundamental form and consequently the θ^{AB} have a simple form , one gets the solution easier in the following fashion: By wedge multiplication of the three complex 3-form equations (2.7) by the coordinate differentials dx^{a}, one obtains twelve complex equations $D\theta^{AB}\wedge dx^{a} = 0$ (AB = (01),(10),(11); a = 1...4). Sometimes it is easier to solve these equations than to calculate the $T^{\cdot\cdot}{}_{\cdot\cdot\cdot\cdot\cdot\cdot}$.

$\omega^{A}{}_{B}$ is called the connexion form (3), (4). It is not a spinorial form, but under a change of isometry (1.7) it transforms according to

(2.11) $\tilde{\omega}^{A}{}_{B} = u^{A}{}_{C}\omega^{C}{}_{D}u^{-1D}{}_{B} - du^{A}{}_{D}u^{-1D}{}_{B}$,

as may be seen from any of the equations (2.4), (2.5), (2.7): from $D(u^{A}{}_{B}\varkappa^{B}) = u^{A}{}_{B}D\varkappa^{B}$ it follows $d(u^{A}{}_{B}\varkappa^{B}) + \tilde{\omega}^{A}{}_{C}u^{C}{}_{B}\varkappa^{B} =$ $= u^{A}{}_{B}(d\varkappa^{B} + \omega^{B}{}_{C}\varkappa^{C})$ and $(du^{A}{}_{B} + \tilde{\omega}^{A}{}_{C}u^{C}{}_{B} - u^{A}{}_{C}\omega^{C}{}_{B})\varkappa^{B} = 0$.

It is clear now, how one defines the covariant derivative $T^{AB}{}_{B';CC'}$ of an arbitrary spinor field $T^{AB}{}_{B'}$:

$$(2.12)\quad DT^{AB}{}_{B'} = T^{AB}{}_{B';CC'}\theta^{CC'} \quad .$$

By the remark preceding the theorem one sees that for real spinor fields (2.12) leads back to the original definition (2.1).

Thus, once the equations (2.5) or (2.7) are solved, we are able to calculate the covariant derivative of any spinor field by (2.4), (2.12). Every differential property of a gravitational field (1.1o), e.g. concerning KILLING vectors or completeness or congruences of curves, may be calculated in a purely spinorial manner. The work to be done to evaluate a covariant differential equation this way is surely not more than proceeding along a tensorial way, rather less because of the complex unification of real quantities.
However, what renders really useful the introduction of spinors into General Relativity is the fact that the curvature spinors - the analogues of the irreducible parts of the RIEMANN tensor - are of a particularly simple form and provide a most adequate means for studying the PETROV type and the radiation properties of a gravitational field.
So in the next paragraph we shall be concerned with calculating the curvature spinors. We shall give some very simple formulae for them, using only definition (2.12) and the theorem above. We shall end up with a very simple prescription how to calculate in a purely spinorial manner the irreducible curvature spinors of a given metric (1.1o) - avoiding entirely the complicated translations by σ's, which generally have been used up to now. We should perhaps stress that with the theorem of this paragraph the work is done; all what follows are only simple applications of it.

§3 The curvature spinors

Let us now apply D twice to a spinor form $\phi^{AA'}{}_{B..}$; we get

$$(3.1)\quad DD\phi^{AA'}{}_{B..} = \Omega^{A}{}_{C}\wedge\phi^{CA'}{}_{B..} + \bar{\Omega}^{A'}{}_{C'}\wedge\phi^{AC'}{}_{B..} - \Omega^{C}{}_{B}\wedge\phi^{AA'}{}_{C..} - \ldots,$$

where

$$(3.2)\quad \Omega^{A}{}_{B} = d\omega^{A}{}_{B} + \omega^{A}{}_{C}\wedge\omega^{C}{}_{B} \quad .$$

The 2-form $\Omega^{A}{}_{B}$ is called the curvature form. From (3.1) we see that it is a spinorial form - remember that $\omega^{A}{}_{B}$ is not! -. We now develop $\Omega^{A}{}_{B}$ in terms of the basis θ^{AB}, $\bar{\theta}^{A'B'}$ of the space of 2-forms:

$$(3.3)\quad \Omega^{A}{}_{B} = -1/2X^{A}{}_{BCD}\theta^{CD} - 1/2\Sigma^{A}{}_{BA'B'}\theta^{A'B'} \qquad \text{with}$$

$$X^{A}{}_{B[CD]} = \Sigma^{A}{}_{B[A'B']} = 0,$$

thus getting two spinor fields X and Σ, which are second order differential expressions in the metric coefficients. In fact, it is easy to see that these spinor fields contain all information about the curvature:

Let $k^{AA'}$ be a real spinor field corresponding to a vector field k^{a}. Then, by (2.5), (2.12)

$$DDk^{AA'} = D(k^{AA'}{}_{;BB'}\theta^{BB'}) = k^{AA'}{}_{;BB'CC'}\,\theta^{CC'}\wedge\theta^{BB'}.$$

On the other hand, (3.1), (3.3) and (1.12) yield

$$DDk^{AA'} = 1/4(\delta^{A'}_{D'}X^{A}{}_{DBC}\varepsilon_{B'C'} + \delta^{A}_{D}\bar{X}^{A'}{}_{D'B'C'}\varepsilon_{BC}$$

$$+\delta^{A'}_{D'}\Sigma^{A}{}_{DB'C'}\varepsilon_{BC} + \delta^{A}_{D}\bar{\Sigma}^{A'}{}_{D'BC}\varepsilon_{B'C'})k^{DD'}\theta^{BB'}\wedge\theta^{CC'}.$$

Comparing these equations we find

$$(3.4)\quad k^{AA'}{}_{;[\underline{BB'}\,\underline{CC'}]} = 1/4(\delta^{A'}_{D'}X^{A}{}_{DBC}\varepsilon_{B'C'} + \delta^{A}_{D}\bar{X}^{A'}{}_{D'B'C'}\varepsilon_{BC}$$

$$+\delta^{A'}_{D'}\Sigma^{A}{}_{DB'C'}\varepsilon_{BC} + \delta^{A}_{D}\bar{\Sigma}^{A'}{}_{D'BC}\varepsilon_{B'C'})k^{DD'}.$$

Appearently, this equation is the translation to spinors of the equation $k^{a}{}_{;[bc]} = 1/4R^{a}{}_{dbc}k^{d}$, which defines the RIEMANN tensor.[1] Thus, the bracket on the right hand side of (3.4) is the spinor equivalent of the RIEMANN tensor. In fact, in (3.4) the RIEMANN spinor appears already in the decomposition which defines - following PENROSE (1) - the dualsymmetric and dualskewsymmetric "curvature spinors" X and Σ.

With the formalism developed thus far, we shall now derive very briefly the well known properties of the curvature spinors (1).

a) RICCI identity: As above we compare

$$DD\kappa^{A} = \kappa^{A}{}_{;BB'CC'}\theta^{BB'}\wedge\theta^{CC'} = \kappa^{A}{}_{;(B\;C)B'}{}^{B'}\theta^{CD} - \kappa^{A}{}_{;B(B'\;C')}{}^{B}\bar{\theta}^{B'C'} \quad\text{and}$$

$$DD\kappa^{A} = \Omega^{A}{}_{B}\kappa^{B} = -1/2X^{A}{}_{BCD}\kappa^{B}\theta^{CD} - 1/2\Sigma^{A}{}_{BB'C'}\kappa^{B}\bar{\theta}^{B'C'} \quad\text{and get}$$

$$(3.5)\quad \kappa^{A}{}_{;(BB'C)}{}^{B'} = 1/2X^{A}{}_{DBC}\kappa^{D} \;;\; \kappa^{A}{}_{;B(B'\;C')}{}^{B} = 1/2\Sigma^{A}{}_{DB'C'}\kappa^{D}.$$

This proof is considerably shorter than the one initially given by PENROSE (1).

b) Symmetry properties of X and Σ: Applying D to $D\theta^{AA'} = 0$ we get

$$\Omega^{A}{}_{B}\wedge\theta^{BA'} + \bar{\Omega}^{A'}{}_{B'}\wedge\theta^{AB'} = 0,$$

and inserting (1.12) and (3.3)

$$(\delta^{A'}_{B'} X^{A}{}_{BCD}\varepsilon_{C'D'} + \delta^{A'}_{B'}\Sigma^{A}{}_{BC'D'} + \delta^{A}_{B}\bar{X}^{A'}{}_{B'C'D'}\varepsilon_{CD} + \delta^{A}_{B}\bar{\Sigma}^{A'}{}_{B'CD}\varepsilon_{C'D'}) \times$$
$$\times \theta^{CC'} \wedge \theta^{BB'} \wedge \theta^{DD'} = 0$$

To the spinor in brackets we apply (1.6) and get

$$X_{AC}{}^{C}{}_{D}\varepsilon_{C'A'} + \varepsilon_{AD}\bar{X}_{A'D'}{}^{D'}{}_{C'} - \Sigma_{ADA'C'} + \bar{\Sigma}_{A'C'AD} = 0$$

Remember now that X and Σ are symmetric in the first two and in the second two indices, because of their definition (3.2), (3.3). Thus , considering the symmetric and skewsymmetric parts of the last equation,we get

$$(3.6)\quad \Sigma_{ABA'B'} = \bar{\Sigma}_{A'B'AB}\ ,\ X_{(A}{}^{C}{}_{B)C} = 0\ ,\ \Lambda =: X_{AB}{}^{AB} = \bar{X}_{A'B'}{}^{A'B'} = \bar{\Lambda}\ .$$

Thus, Σ is a real and irreducible spinor field, whereas X allows for the reduction

$$(3.7)\quad X_{ABCD} = \Gamma_{ABCD} - \Lambda/3\, \varepsilon_{A(C}\varepsilon_{D)B}\quad ,$$

where Γ is totally symmetric. In fact, inserting (3.7) into (3.4), we see that Γ, Σ, Λ, is the WEYL spinor, the reduced RICCI spinor,and one quarter of the curvature scalar respectively, in the definition of PENROSE (1).

c) Calculation of the curvature spinors: Multiplying $\Omega^{A}{}_{B}$ by θ^{AB}, $\bar{\theta}^{A'B'}$ and taking account of (1.13) we get the very simple formulae

$$(3.8)\quad -2\Omega^{A}{}_{B}\wedge\theta_{CD} = X^{A}{}_{BCD}\theta,\ 2\Omega^{A}{}_{B}\wedge\bar{\theta}^{A'B'} = \Sigma^{A}{}_{BA'B'}\theta,\ \Lambda\theta = -2\Omega^{AB}\wedge\theta_{AB}.$$

Here the form-formalism provides a very simple means for solving the equations (3.3) for the unknowns X and Σ.

Let us recall briefly the procedure how to find the curvature spinors for a given metric field. One first puts it into the shape (1.1o) getting the $\theta^{AA'}$, and by (1.12) the θ^{AB}. Then one solves equations (2.5) or (2.7) for the $\omega^A{}_{B'}$, calculates $\Omega^A{}_B$ by (3.2), and the curvature spinors by (3.8).

d) <u>BIANCHI-identity</u>: We apply the operator D to the RICCI-identity $DD\xi^A = \Omega^A{}_B\,\xi^B$ and get on one hand with the product rule

$$DDD\xi^A = \Omega^A{}_B \wedge D\xi^B + D\Omega^A{}_B\,\xi^B$$

and on the other hand with (3.1)

$$DDD\xi^A = \Omega^A{}_B \wedge D\xi^B \quad , \text{ so } D\Omega^A{}_B = o.$$

Using (3.3), (1.12), (2.12) we get

$$(X^A{}_{BCD;EE'}\varepsilon_{C'D'} + \Sigma^A{}_{BC'D';EE'}\varepsilon_{CD})\theta^{EE'} \wedge \theta^{CC'} \wedge \theta^{DD'} = o$$

With (1.6) we get the BIANCHI-identity

$$(3.9) \qquad X^A{}_{BCD;}{}^D{}_{B'} = \Sigma^A{}_{BB'D';C}{}^{D'} \qquad .$$

e) <u>Conformal invariance of the WEYL spinor</u> : After a conformal transformation $Q \rightarrow e^{2U}Q \Longleftrightarrow \theta^{AB} \rightarrow e^{2U}\theta^{AB} =: \tilde{\theta}^{AB}$, we denote the new connection by $\tilde{\omega}^A{}_B = \omega^A{}_B + \pi^A{}_B$. According to the theorem of §2, $\tilde{\omega}^A{}_B$ is determined by the equations $\tilde{\omega}^A{}_A = o = \tilde{D}(e^{2U}\theta^{AB})$. Since $D\theta^A{}_B = o$, we derive for $\pi^A{}_B$ the equation

$$2dU\wedge\theta^{AB} + \pi^A{}_C\wedge\theta^{CB} + \pi^B{}_C\wedge\theta^{AC} = o.$$

(2.9) and (2.1o) give immediately

$$\pi^A{}_B = 1/4(\delta^A_B dU + \delta^A_D U_{;BB'}\theta^{DB'} + 3U_{;}{}^A{}_{B'}\varepsilon_{DB}\theta^{DB'}) .$$

The new curvature form is (cf.(3.2))

$$\tilde{\Omega}^A{}_B = \Omega^A{}_B + D\pi^A{}_B + \pi^A{}_C\wedge\pi^C{}_B .$$

Developing $D\pi^A{}_B$ in terms of θ^{CD} we see that every coefficient so obtained contains ε's, so its part totally symmetric in ABCD vanishes. A little calculation shows the same for $\pi^A{}_C\wedge\pi^C{}_B$. Thus, the WEYL-form $\Gamma^A{}_B =: \Gamma^A{}_{BCD}\theta^{CD}$ remains unchanged. The WEYL spinor itself takes a factor e^{-2U}, of course.

§4. Comparison with other formalisms.

a) The tensorial CARTAN formalism.

Let us first describe briefly how the usual CARTAN formalism may be introduced into differential geometry in exactly the same manner as the spinorial one in the preceding pragraphs. We shall confine ourselves to orthogonal n-beins, but leave the dimension n arbitrary.

Given any orthogonal n-bein in a point x of a RIEMANNian V^n, we may express every tensor by its n-bein-components, thus defining an isometry of the tensor algebra over T_x onto the tensor algebra over MINKOWSKI space. Changing the n-bein and consequently the isometry, we get a transformation law analogous to (1.7), (1.8), where the u_2 is replaced by the LORENTZ group (cf. (19)). We may now define a tensor by this transformation law and extend the definition to tensorial forms as we did for spinor forms in §1. The covariant differential is then defined by an analogue of (2.1), (2.2), and again properties 1) to 6) hold. Out of these, one shows with the same calculations as in §2 that D is of the form $D = d + \omega^a{}_b$, $\omega^a{}_b$ being the unique solution of

(4.1) $$\omega_{(ab)} = o = d\theta^a + \omega^a{}_b \wedge \theta^b.$$

θ^a is the tensorial structure form defined by $Q = \eta_{ab}\theta^a\theta^b$, $\eta_{ab} = \mathrm{diag}(+---)$.

Applying D twice to a vector as in (3.1), one proves that the RIEMANN tensor is obtained by the formulae

(4.2) $d\omega^a_b + \omega^a_c \wedge \omega^c_b = \Omega^a_b = R^a{}_{bcd}\theta^c \wedge \theta^d$.

Up to now, the analogy is incomplete in two respects. First there is only one formula (4.1) to calculate the connection form ω^a_b, in comparison to the two different formulae (2.5), (2.7). We shall fill this lack by the following theorem which, to our knowledge, does not appear yet in the literature.

Theorem: A covariant differential $D = d + \omega^a_b$ on a manifold V^n is torsionfree ($D\theta^a = o$, θ^a the structure form) if and only if one of the equations

(4.3) $D(\theta^{a_1} \wedge \dots \wedge \theta^{a_r}) = o$, $r \leq n-2$,

holds. In particular, the RIEMANNian connection on a RIEMANNian manifold $(V^n, g_{ab}(x))$ is uniquely determined by one of the equs.

(4.4) $D(\theta^{a_1} \wedge \dots \wedge \theta^{a_r}) = o = Dg_{ab}$.

Proof: If $D\theta^a = o$, then (4.3) holds because of the product rule. Now let $D(\theta^{a_1} \wedge \dots \wedge \theta^{a_r}) = o$. The product rule yields

(4.5) $\sum_{i=1}^{r} (-1)^{i+1}\theta^{a_1} \wedge \dots \theta^{a_{i-1}} \wedge D\theta^{a_i} \wedge \theta^{a_{i+1}} \wedge \dots \wedge \theta^{a_r} = o$

By multiplication with θ^i we get $D\theta^i \wedge \theta^{a_1} \wedge \dots \wedge \theta^{a_r} = o$ for all combinations $(a_1, \dots, a_r)$ of indices containing i. Thus $D\theta^i \wedge \theta^i = o$ and $D\theta^i = \emptyset \wedge \theta^i$, where $\emptyset$ is a 1-form which by reasons of covariance does not depend on i anymore. Inserting this into (4.5) we get $r\emptyset \wedge \theta^{a_1} \wedge \dots \wedge \theta^{a_r} = o$, thus $\emptyset = o$, and the first statement of the theorem is proven. The second one is then immediate.

Secondly, in (4.2) one has to expand $\Omega^a_{\ b}$ in terms of $\theta^c \wedge \theta^d$, which in general is a tedious work, whereas for spinors the simple multiplications (3.8) give the curvature spinors, and even better: at once split into their irreducible parts Γ, Σ, Λ. Actually, there is a weak analogue of (3.8) for tensors:

If we define θ^{ab}, $\overset{*}{\theta}{}^{ab}$ and θ by

(4.6) $\theta^{ab} =: \theta^a \wedge \theta^b$, $\overset{*}{\theta}{}^{ab} =: \eta^{ab}{}_{cd}\theta^{cd}$ (2η is the totally skew pseudotensor)

$\theta\,\eta^{abcd} := \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$

respectively, and take account of the well known property of the duality-rotation

(4.7) $\eta^{ab}{}_{cd}\eta^{cd}{}_{ef} = -\delta^a_{[e}\delta^b_{f]}$,

we find the following formulae

(4.8)
$$-\Omega^{ab} \wedge \overset{*}{\theta}{}^{cd} = R^{abcd}\theta$$
$$\Omega^{ab} \wedge \theta^{cd} = R^{*abcd}\theta$$
$$-\overset{*}{\Omega}{}^{ab} \wedge \overset{*}{\theta}{}^{cd} = {}^{*}R^{abcd}\theta$$
$$\overset{*}{\Omega}{}^{ab} \wedge \theta^{cd} = {}^{*}R^{*abcd}\theta$$
$$-\Omega^a{}_c \wedge \overset{*}{\theta}{}^{bc} = R^{ab}\theta \quad .$$

Here the star $*$ indicates the right- respectively left-duality-rotated tensor, in the notation of (18). Out of these equations one may construct a lot of similar ones, where the irreducible parts of the RIEMANN tensor appear as coefficients of θ.

b) NEWMAN-PENROSE formalism (5).

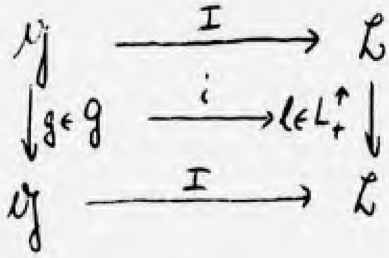
As we remarked above (cf. (1.11)), all forms may be developed into the four basic forms $\theta^{AA'}$. If we do this, every form-equation will be transformed into a scalar equation. The set of equations obtained this way out of our equations (2.5), (2.12), (2.6), (2.8), (3.2), (3.3) is exactly the content of the formalism of NEWMAN-PENROSE (5). There several skew-symmetrizations are carried out, which correspond to the wedge multiplication used here. As an example, we calculate the covariant derivative of a spinor field ξ^A in the way of NEWMAN-PENROSE. According to (2.8), (2.12), (2.4) we get

$$\xi^A{}_{;BB'} = \xi^A{}_{,BB'} + \Gamma^A{}_{CBB'}\xi^C \quad ,$$

where $\Gamma^A{}_{CBB'}$ is the solution of the scalar equivalent of (2.5) and $\xi^A{}_{,BB'}$ the PFAFFian derivative of ξ^A for the form base $\theta^{AA'}$. In general, the latter is quite tedious to calculate. In our opinion, the form-formalism has the advantage that at any step one has the choice of expanding the forms in question into the $\theta^{AA'}$ or the coordinate differentials dx^a , as is convenient. In the examples of §6 and §7 we shall see that for explicit calculations it is adequate to take the dx^a as a form-base. On the other hand, the lack of a compact notation, as the use of forms provides, forces NEWMAN and PENROSE to write down explicitely a set of very useful equations for the components of the curvature spinors in terms of the connection coefficients $\Gamma^A{}_{BCC'}$ and their PFAFFian derivatives. These equations may be obtained from (3.2), (3.3) here.

c) Complex bivector formalism of DEBEVER-CAHEN-DEFRISE (DCD)(6).

We shall discuss the connection of this formalism with the other ones in terms of fibre bundles[1]. In General Relativity, one may restrict oneself to the fibre bundle of orthogonal tetrads. It is a principal fibre bundle (p.f.b.) with the LORENTZ group as its structure group. We assume (V^4, g_{ab}) to be orientated and time-orientated and restrict ourselves to the fibre bundle $\mathcal{L}$ of orientated and time-orientated tetrads, which is again a p.f.b. with group $L_+^\uparrow$. The connection is given by a distribution of horizontal tangent spaces $\mathcal{H}$ in each point of $\mathcal{L}$, the null spaces of the connection form ω, which has its values in the LIE-algebra $dL_+^\uparrow$ of $L_+^\uparrow$.
Consider now a local isomorphism $i: G \rightarrow L_+^\uparrow$ of a certain group G onto $L_+^\uparrow$ (G equals u_2 in our treatment and $SO_3(C)$ in the DCD-case). Under certain restrictions on V^4, we may construct out of these data in a canonical way another p.f.b. $\mathcal{G}$ with group G, and extend i to a local homeomorphism I of $\mathcal{G}$ onto $\mathcal{L}$ in such a way that the following diagram is commutative (The vertical arrows indicate the action of the groups on their bundles):



1 The language of fibre bundles has been avoided up to now, since it is not familiar to all physicists. It would have shortened only the proof of §2. All other formulae are explicit. For the discussion here, it seems to be the only possible means. (cf. (3))

The interest in the other fibre bundles $\mathcal{G}$ lies in the fact, that some representations of G are sometimes of a much simpler form than the equivalent tensorial ones of $L_+^\uparrow$. E.g., the representation of $L_+^\uparrow$ to which the WEYL tensor belongs, is equivalent to the representation of u_2 in the space of totally symmetric 4-spinors Γ_{ABCD}, which is quite easy to overlook. The DCD-formalism makes use of the fact that this representation is also equivalent to the natural representation of $SO_3(C)$ in the space of complex 3x3-matrices.

We may now define a connection $\mathcal{H}' = (dI)^{-1}\mathcal{H}$ on $\mathcal{G}$ correspon-ding to a connection form

(4.9) $\omega_{\mathcal{G}} = di \circ \omega \circ dI$ on $\mathcal{G}$ onto dG.

Now, the aim of all the formalisms discussed in this paragraph is to characterize the connection form $\omega_{\mathcal{G}}$ in such a way that the translations (4.9) are avoided. We did this in §2 for the spinorial case and under a) for the tensorial case, giving two equations for the ω's, each of which determines them entirely. Now, to every representation r of $L_+^\uparrow$ there belongs a representation r_G of G in a prescribed manner. Thus, once having the $\omega_{\mathcal{G}}$, one knows how to derivate the equivalent of any tensor field on $\mathcal{L}$ by means of the connection $\omega_{\mathcal{G}}$ on $\mathcal{G}$.

DCD took G = $SO_3(C)$, making use of the fact that the bivectorial representations of $L_+^\uparrow$ have particularly simple equivalents for this group. The tensorial structure form $\theta^a \wedge \theta^b$ on $\mathcal{L}$ gives rise to the DCD-structure form θ^α (α = 1,2,3) on $\mathcal{G}$, and the connection $\omega_{\mathcal{G}}{}^\alpha{}_\beta$ is determined by the equation $D_{\mathcal{G}}\theta^\alpha = d\theta + \omega^\alpha_\beta \wedge \theta^\alpha = o$.

Unfortunately, the simplest representation of $L_{+}^{\uparrow}$ on R^4 has an equivalent for $SO_3(C)$ which is quite difficult to describe. Thus DCD have to confine themselves to the treatment of bivectors and their tensorial products (to which the RIEMANN tensor belongs). As for the analysis of ordinary vector fields, they go back to the LORENTZ bundle $\mathcal{L}$. This is the reason why they do not succeed for example to derive compact expressions for the BIANCHI identities in their formalism.
But they get the same technique for calculating the curvature as we got here. One has only to replace the pairs of indices AB in (2.7),(3.2), (3.3) according to the rule

$$\begin{array}{cccc} AB = & 00 & 01 & 11 \\ & \downarrow & \downarrow & \downarrow \\ \alpha = & 1 & 2 & 3 \end{array}$$

in order to get their formulae.

§5. Adapted coordinates. Explicit formulae.

Given a metric $Q = g_{ab}dx^a dx^b$, we have shown now how to study its analytic properties in a purely spinorial manner; for the curvature it is said explicitly in §3c. As for KILLING equations, completeness etc. it is a matter of straightforward translation of tensorial methods. More often, the problem is given the other way round. We are not given the metric Q, but know some properties of a field and have to construct the fundamental form Q and to find the other properties.

We shall show in this paragraph how this situation can be treated systematically within the formalism presented here. We shall assume that we are given certain information about a null congruence C. We shall show how to adapt a coordinate system to the information in a systematic way.

Let k^a be a vector field tangent to the null congruence given. Then there exists a spinor field $\varkappa^A$ such that $\varkappa^A \bar{\varkappa}^{A'}$ corresponds to k^a by means of the isometry σ of §1. We choose now another spinor field μ^A such that $\varkappa_A \mu^A = 1$ and call the pair $(\varkappa^A, \mu^A) = :(\underset{0}{\varkappa}^A, \underset{1}{\varkappa}^A)$ a " spindyad adapted to the null congruence C".
Having this, we imagine a change (1.7), (1.8) of isometry to be carried out such that the components of the dyad are given by

$$(5.1) \qquad \underset{A}{\varkappa}^B = \delta_A^B , \qquad \text{so } d\underset{A}{\varkappa}^B = 0 .$$

From (2.4), (2.8), (2.12), (5.1) we get

$$(5.2) \qquad \varkappa_{AD;EE'} \varkappa^D_B \varkappa^E_C \bar{\varkappa}^{E'}_{C'} = \Gamma_{ABCC'}$$

The functions (5.2) are called spin-coefficients (5). Some of them have the meaning of optical scalars of the congruences spanned by $\varkappa^A$ and μ^A (cf. § 6). Following the notation of ( ), (9), we shall denote them according to the following scheme:

$$(5.3) \qquad \omega^0{}_0 = \mu\theta^{00'} + \kappa\theta^{01'} + \tilde{\kappa}\theta^{10'} + \tilde{\mu}\theta^{11'}$$

$$\omega^0{}_1 = -\tilde{\Omega}\theta^{00'} + \tilde{z}\theta^{01'} + \tilde{\sigma}\theta^{10'} + \tilde{k}\theta^{11'}$$

$$\omega^1{}_0 = -k\theta^{00'} - \sigma\theta^{01'} - z\theta^{10'} + \Omega\theta^{11'}$$

E.g., $\tilde{\Omega} = -\mu_{B;CC'}\mu^B \varkappa^C \bar{\varkappa}^{C'}$, ...

Up to now, the spindyad attached to C is by no means unique. Any other dyad $(\varkappa'^A, \mu'^A)$ which is connected with $(\varkappa^A, \mu^A)$ by

$$(5.4) \qquad \underline{\varkappa', \mu'} = \underline{\varkappa, \mu} \begin{pmatrix} \lambda & \Lambda \\ 0 & \bar{\lambda}^{-1} \end{pmatrix} \qquad \lambda, \Lambda \text{ complex functions}$$

is also a dyad adapted to C. The transformation of the spin-coefficients corresponding to (5.4) is given by (2.11). Since we need it for the applications to follow, we shall write it down in the notation (5.3). We introduce the abbreviations

$$\begin{matrix} 00' & 01' & 10' & 11' \\ \downarrow & \downarrow & \downarrow & \downarrow \\ m & \bar{t} & t & k \end{matrix} \qquad \text{and} \quad F_k =: F_{,11'}$$

for the pairs of indices and the PFAFFian derivatives respectively.

(5.5)
$$\begin{pmatrix} \mu' & \alpha' & -\tilde{\Omega}' & \tilde{z}' \\ \tilde{\alpha}' & \tilde{\mu}' & \tilde{\sigma}' & \tilde{\kappa}' \\ -\kappa' & -\sigma' & -\mu' & -\alpha' \\ -z' & \Omega' & -\tilde{\alpha}' & -\tilde{\mu}' \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & -\lambda\Lambda & 0 \\ \bar\lambda^{-1}\Lambda & \lambda^{-2} & -\Lambda^2 & -\bar\lambda^{-1}\Lambda \\ 0 & 0 & \bar\lambda^{2} & 0 \\ 0 & 0 & \lambda\Lambda & 1 \end{pmatrix} \begin{pmatrix} \mu & \alpha & -\tilde{\Omega} & \tilde{z} \\ \tilde{\alpha} & \tilde{\mu} & \tilde{\sigma} & \tilde{\kappa} \\ -\kappa & -\sigma & -\mu & -\alpha \\ -z & \Omega & -\tilde{\alpha} & -\tilde{\mu} \end{pmatrix} \begin{pmatrix} \lambda\bar\lambda & \lambda\bar\Lambda & \bar\lambda\Lambda & \Lambda\bar\Lambda \\ 0 & \lambda\bar\lambda^{-1} & 0 & \bar\lambda^{-1}\Lambda \\ 0 & 0 & \bar\lambda\lambda^{-1} & \lambda^{-1}\bar\Lambda \\ 0 & 0 & 0 & \lambda^{-1}\bar\lambda^{-1} \end{pmatrix} +$$

$$\begin{pmatrix} 1 & 0 & -\lambda\Lambda & 0 \\ \lambda^{-1}\Lambda & \lambda^{-2} & -\Lambda^2 & -\bar\lambda^{-1}\Lambda \\ 0 & 0 & \bar\lambda^{2} & 0 \\ 0 & 0 & \lambda\Lambda & 1 \end{pmatrix} \begin{pmatrix} \lambda_m & \lambda_{\bar t} & \Lambda_m & \Lambda_{\bar t} \\ \lambda_t & \lambda_\kappa & \Lambda_t & \Lambda_\kappa \\ 0 & 0 & -\lambda^{-2}\lambda_m & -\lambda^{-2}\lambda_{\bar t} \\ 0 & 0 & -\bar\lambda^{-2}\lambda_t & -\lambda^{-2}\lambda_\kappa \end{pmatrix} \begin{pmatrix} \bar\lambda & \bar\Lambda & 0 & 0 \\ 0 & \bar\lambda^{-1} & 0 & 0 \\ 0 & 0 & \bar\lambda & \bar\Lambda \\ 0 & 0 & 0 & \bar\lambda^{-1} \end{pmatrix}$$

We have

(5.6)
$$\begin{pmatrix} \theta^{m'} & \theta^{\bar t'} \\ \theta^{t'} & \theta^{\kappa'} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & -\Lambda \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \theta^{m} & \theta^{\bar t} \\ \theta^{t} & \theta^{\kappa} \end{pmatrix} \begin{pmatrix} \bar\lambda^{-1} & 0 \\ -\bar\Lambda & \bar\lambda \end{pmatrix} .$$

(2.5) reads with the notation (5.3) as follows

$$
\text{(5.7)}\quad
\begin{aligned}
\text{a)}\quad & d\theta^m + (\alpha + \bar{\hat{\alpha}} + \bar{\tilde{\Omega}})\theta^{\bar t}\wedge\theta^m + (\bar\alpha + \breve{\alpha} + \tilde{\Omega})\theta^t\wedge\theta^m \\
& \quad + (\tilde{\mu} + \bar{\hat{\mu}})\theta^k\wedge\theta^m \qquad + (\tilde{z} - \bar{\tilde{z}})\theta^{\bar t}\wedge\theta^t \\
& \quad + \bar{\tilde{k}}\theta^k\wedge\theta^{\bar t} \qquad + \tilde{k}\theta^k\wedge\theta^t \qquad = o
\end{aligned}
$$

$$
\begin{aligned}
\text{b)}\quad & d\theta^t + (\bar\mu - \mu + z)\theta^m\wedge\theta^t + (\tilde\mu - \bar{\hat\mu} + \bar z)\theta^t\wedge\theta^k \\
& \quad + (\Omega + \hat{\tilde{\Omega}})\theta^k\wedge\theta^m \qquad + (\alpha - \bar{\tilde\alpha})\theta^t\wedge\theta^{\bar t} \\
& \quad + \sigma\theta^m\wedge\theta^{\bar t} \qquad + \bar{\tilde\sigma}\theta^{\bar t}\wedge\theta^k \qquad = o
\end{aligned}
$$

$$
\begin{aligned}
\text{c)}\quad & d\theta^k + (\alpha + \bar{\tilde\alpha} + \Omega)\theta^k\wedge\theta^{\bar t} + (\bar\alpha + \hat{\tilde\alpha} + \bar\Omega)\theta^k\wedge\theta^t \\
& \quad + (\mu + \bar\mu)\theta^k\wedge\theta^m \qquad + (z - \bar z)\theta^{\bar t}\wedge\theta^t \\
& \quad + k\theta^{\bar t}\wedge\theta^m \qquad + \bar k\theta^t\wedge\theta^m \qquad = o
\end{aligned}
$$

We evaluate these equations with the aid of the theorem of FROBENIUS (3), (20), stating that for r 1-forms $\theta^1 \ldots \theta^r$ there exist locally r functions $x^1 \ldots x^r$ with $\theta^{r'} = a^{r'}_s dx^s$ (r', s from 1 to r), if and only if $d\theta^s \wedge \theta^1 \wedge \ldots \wedge \theta^r = o$ (s = 1..r). The more of the combinations of spincoefficients appearing in (5.7) are zero, or may be made ~~to~~ zero by a gauge (5.5), the simpler is the fundamental form (1.1o).

<u>Example</u>. We consider a field which admits of a hypersurface-orthogonal and shearfree null congruence. Taking $\varkappa^A$ as its tangent spinor field, this is equivalent to $k = \mathrm{Im}(z) = \sigma = o$ (cf. (9)). From $k = o = z - \bar z$ and (5.7c) we get at once

$d\theta^k \wedge \theta^k = o$, and the theorem of FROBENIUS gives $\theta^k = Fdu$. The gauge ((5.4), $\Lambda = o$, $\lambda = F^{-1/2}$) gives $\theta^k = du$. From $\sigma = o$ and (5.7b) we get $d\theta^t \wedge \theta^k \wedge \theta^t = o$, thus with the FROBENIUS theorem $\theta^t = pe^{2ir}(dz + Bdu)$, where p and r are real, z and B complex functions. We now make the gauge ((5.4), $\Lambda = o$, $\lambda = e^{ir}$), and get with (5.6) $\theta^t = p(dz + Bdu)$.

On the other hand, the inspection of (5.7b) shows that this form for θ^t can only be achieved for $\sigma = o$. Let s be a fourth coordinate besides u, $x = Re(z)$, $y = Im(z)$. Then this result means that the shadow-cast-mapping of two wave-fronts $u = c$, $s = c'$ and $u = c$, $s = c''$ is conformal if and only if $\sigma = o$.

We now make $\tilde{z} - \bar{\tilde{z}}$ ~~into~~ zero by a gauge ((5.4), $\lambda = 1$, $\Lambda =$ real), and get with (5.7a) and the theorem of FROBENIUS $\theta^m = e^{-C}ds + hdu$. We now make a last gauge ((5.4), $\lambda = 1$, $\Lambda = p^{-1}\int e^C C,_z ds$) getting $d\theta^m \wedge \theta^k = o$, $\theta^m = dr + hdu$. ($C,_z$ is the PFAFFian derivative in the form-base du, dz, $d\bar{z}$, ds). Taking account of (1.1o) we summarize (cf. (14), (15)):

<u>Theorem</u>: The fundamental form of a gravitational field admitting a normal shearfree null congruence can be put into the form

$$Q = -2p^2 \left| dz + Bdu \right|^2 + 2drdu + 2hdu^2 . \qquad (5.8)$$

This example shows very clearly that equtns. (5.7) may be looked at as integrability conditions for the fundamental form Q. Obviously, one could treat eqns. (4.1) for the tensorial case in exactly the same manner, transforming information about a time-like congruence, say, into a simple form for Q.

§ 6. The general metric adapted to a normal null congruence

In this paragraph we shall obtain the connexion form of the general ~~radiation~~ metric. The calculations will provide an example of the application of the theory developed in § 2.
Given a twist-free null congruence - it always exists in a suitable neighbourhood of some event[1] - we can introduce two space-like coordinates x^A with $x^A{}_{,a}u,{}^a=0$, i.e., the gradients are orthogonal to the rays l_a (the tangent vector l_a is the gradient of the phase or "retarded time" u). We get for the metric form

$$Q = g_{AB}dx^A dx^B + 2m_a dx^a du, \qquad (6.1)$$

$$(x^a)=(x^A,s,u);\ A,B=1,2;\ a,b=1,..,4.$$

We call this metric the general radiation metric.[2] The special coordinates introduced above which are adapted to the congruence are used in accordance with (10) and (11).
The wave hypersurfaces, i.e., the (lightlike) hypersurfaces orthogonal to the rays of the congruence, are given by u = const.; they contain the wave surfaces, i.e., the space-like 2-surfaces orthogonal to the rays which are given by u = const., s = s(x^A).
In order to determine the connexion form, we have to choose the structural form $\Theta^{AB'}$ first,

[1] Cf. (12), Anhang D.

[2] This notation may be misunderstandable : we mean the general normal hyperbolic metric adapted to a normal congruence of rays.

$$\theta^{00'} = du,\ \theta^{01'} = Adx+edy,$$
$$\theta^{11'} = gdx+hdy+jds+kdu, \tag{6.2}$$

where $(x,y)=x^A$, $(g,h,j,k)=m_a$, $e=1/2(-2g_{22})^{1/2}$,

$Re(A)=c=-g_{12}(-2g_{22})^{-1/2}$, $Im(A)=if=i(\frac{\det(g_{AB})}{-2g_{22}})^{1/2}$.

If we agree to put

	l	t	$\bar{t}$	m
for	00'	01'	10'	11'

this choice of the structural form means an adaptation to the geometry, for we get

$$\theta^l=du,\ \theta^m=m_a dx^a,\ \theta^t\theta^{\bar{t}}=-1/2g_{AB}dx^A dx^B,$$

on the other hand (1.10) holds for the metric form,

$$Q = -2\theta^t\theta^{\bar{t}}+2\theta^l\theta^m.$$

This shows that with the aid of the structural form (6.2) we have constructed the metric form (6.1) in an adapted way. Of the components of the metric tensor we demand $g_{22}<0$, $m_3\neq 0$, furthermore $\det(g_{AB})>0$. The next step is to calculate the 2-form

$$\theta^{AB} = 1/2\ \theta^A{}_{C'}\wedge\theta^{BC'}.$$

We get

$$\left.\begin{aligned}
\Theta^{00} &= \theta^{l}\wedge\theta^{t} = -(A\,dx\wedge du+e\,dy\wedge du),\\
\Theta^{01} &= 1/2(\theta^{l}\wedge\theta^{m}+\theta^{\bar t}\wedge\theta^{t}) = -(ief\,dx\wedge dy+\\
&\quad +1/2g\,dx\wedge du+1/2h\,dy\wedge du+1/2j\,ds\wedge du),\\
\Theta^{11} &= \theta^{\bar t}\wedge\theta^{m} = (\bar A h-eg)dx\wedge dy+\bar A j\,dx\wedge ds+\\
&\quad +\bar A k\,dx\wedge du+ej\,dy\wedge ds+ek\,dy\wedge du.
\end{aligned}\right\}\qquad (6.3)$$

The connexion form is the solution of equations (2.7); if we substitute $-\omega^{0}{}_{0}$ for $\omega^{1}{}_{1}$, there remain the three equations $D\Theta^{AB}=0$,

$$(dA+2A\omega^{0}{}_{0}+g\omega^{0}{}_{1})\wedge dx\wedge du+(de+2e\omega^{0}{}_{0}+h\omega^{0}{}_{1})\wedge dy\wedge du+ \qquad (6.4)$$

$$+2ief\omega^{0}{}_{1}\wedge dx\wedge dy+j\omega^{0}{}_{1}\wedge ds\wedge du = 0,$$

$$-(d(\bar A h-eg)-2(\bar A h-eg)\omega^{0}{}_{0}-2ief\omega^{1}{}_{0})\wedge dx\wedge dy+$$

$$+(d(\bar A j)-2\bar A j\omega^{0}{}_{0})\wedge dx\wedge ds+(d(\bar A k)-2\bar A k\omega^{0}{}_{0}-g\omega^{1}{}_{0})\wedge$$

$$\wedge dx\wedge du+(d(ej)-2ej\omega^{0}{}_{0})\wedge dy\wedge ds+(d(ek)-2ek\omega^{0}{}_{0}-$$

$$-h\omega^{1}{}_{0})\wedge dy\wedge du-j\omega^{1}{}_{0}\wedge ds\wedge du = 0,$$

$$(-id(ef)+(\bar{A}h-eg)\omega^0{}_1)\wedge dx\wedge dy+(-1/2dg+\bar{A}k\omega^0{}_1-$$
$$-A\omega^1{}_0)\wedge dx\wedge du+(-1/2dh+ek\omega^0{}_1-e\omega^1{}_0)\wedge dy\wedge du-$$
$$-1/2dj\wedge ds\wedge du+\bar{A}j\omega^0{}_1\wedge dx\wedge ds+ej\omega^0{}_1\wedge dy\wedge ds = 0.$$

Now we shall form the wedge product of these expressions and the coordinate differentials dx^a and thus obtain the twelve complex equations $D\Theta^{AB}\wedge dx^a = 0$. This will provide an easier way to get the solution $\omega^A{}_B$. Using the notation $\gamma^A{}_{Bx^a} = \gamma^A{}_{Ba}$, $\omega^A{}_B = \gamma^A{}_{Ba}dx^a$ we find

$$\gamma^0{}_{1s} = 0, \tag{6.5}$$

$$-A_y-2A\gamma^0{}_{0y}-g\gamma^0{}_{1y}+e_x+2e\gamma^0{}_{0x}+h\gamma^0{}_{1x}+2ief\gamma^0{}_{1u}=0, \tag{6.6}$$

$$-A_s-2A\gamma^0{}_{0s}+j\gamma^0{}_{1x} = 0, \tag{6.7}$$

$$-e_s-2e\gamma^0{}_{0s}+j\gamma^0{}_{1y} = 0, \tag{6.8}$$

$$(\bar{A}h-eg)_s-2(\bar{A}h-eg)\gamma^0{}_{0s}-2ief\gamma^1{}_{0s}-(\bar{A}j)_y+2\bar{A}j\gamma^0{}_{0y}+ \\ +(ej)_x-2ej\gamma^0{}_{0x} = 0, \tag{6.9}$$

$$-(\bar{A}h-eg)_u+2(\bar{A}h-eg)\gamma^0{}_{0u}+2ief\gamma^1{}_{0u}+(\bar{A}k)_y-2\bar{A}k\gamma^0{}_{0y}-$$
$$-g\gamma^1{}_{0y}-(ek)_x+2ek\gamma^0{}_{0x}+h\gamma^1{}_{0x} = 0, \qquad (6.10)$$

$$(\bar{A}j)_u-2\bar{A}j\gamma^0{}_{0u}-(\bar{A}k)_s+2\bar{A}k\gamma^0{}_{0s}+g\gamma^1{}_{0s}-j\gamma^1{}_{0x} = 0, \qquad (6.11)$$

$$(ej)_u-2ej\gamma^0{}_{0u}-(ek)_s+2ek\gamma^0{}_{0s}+h\gamma^1{}_{0s}-j\gamma^1{}_{0y} = 0, \qquad (6.12)$$

$$-i(ef)_s-\bar{A}j\gamma^0{}_{1y}+ej\gamma^0{}_{1x} = 0, \qquad (6.13)$$

$$2i(ef)_u-2(\bar{A}h-eg)\gamma^0{}_{1u}-g_y+2\bar{A}k\gamma^0{}_{1y}-2A\gamma^1{}_{0y}+h_x-$$
$$-2ek\gamma^0{}_{1x}+2e\gamma^1{}_{0x} = 0, \qquad (6.14)$$

$$g_s+2A\gamma^1{}_{0s}-j_x+2\bar{A}j\gamma^0{}_{1u} = 0, \qquad (6.15)$$

$$-h_s-2e\gamma^1{}_{0s}+j_y-2ej\gamma^0{}_{1u} = 0, \qquad (6.16)$$

where the factor $dx \wedge dy \wedge ds \wedge du$ has been dropped from each term.

Even in this general case the equations can be separated into six groups of only three or less equations; besides (6.5) there are the following five groups each of which allows for the determination of three, two, or one γ^A_{Ba}, simply by solving a set of as many linear equations: (6.15) and (6.16) yield γ^0_{1u} and γ^1_{0s}; (6.7), (6.8) and (6.13) yield γ^0_{0s}, γ^0_{1x} and γ^0_{1y}; (6.6) and (6.9) yield γ^0_{0x} and γ^0_{0y}; (6.11), (6.12) and (6.14) yield γ^0_{0u}, γ^1_{0x} and γ^1_{0y}; (6.10) finally yields γ^1_{0u}.

The connexion form of the general radiation metric, as obtained by these calculations, is

$$\gamma^0_{0x} = (4efj)^{-1}(2icj(c_y-e_x)+i(-2ch+eg)c_s+icge_s+ +2if(jf_y-hf_s)+e(ic-2f)(g_s-j_x)- -i\,A^2(h_s-j_y)),$$

$$\gamma^0_{0y} = (4efj)^{-1}(2iej(c_y-e_x)-iehc_s+i(-ch+2eg)e_s+ +ie^2(g_s-j_x)-e(ic+2f)(h_s-j_y)),$$

$$\gamma^{0}{}_{0s} = i(4f)^{-1}e(ce^{-1})_s,$$

$$\gamma^{0}{}_{0u} = (4e^2f^2j)^{-1}(ie^3fj(ce^{-1})_u - 1/2iefj(g_y - h_x) + 1/2e(ch-eg)(g_s - j_x) + 1/2(|A|^2h - ceg)(h_s - j_y) + 2e^2f^2(j_u - k_s)),$$

$$\gamma^{0}{}_{1x} = i(2jef)^{-1}(\bar{A}ec_s - Ace_s + 2eff_s),$$

$$\gamma^{0}{}_{1y} = (2fj)^{-1}(iec_s + (-ic+2f)e_s),$$

$$\gamma^{0}{}_{1s} = 0,$$

$$\gamma^{0}{}_{1u} = i(4efj)^{-1}(-e(g_s - j_x) + A(h_s - j_y)),$$

$$\Gamma^1_{0x} = (4e^2f^2j)^{-1}(-2iAe^2f(jc_u-kc_s)+2i\bar{A}cef(je_u-ke_s)+ +4ie^2f^2(kf_s-jf_u)+i\bar{A}efj(g_y-h_x)+ce(\bar{A}h-eg)(g_s- -j_x)-|A|^2(\bar{A}h-eg)(h_s-j_y)),$$

$$\Gamma^1_{0y} = (4ef^2j)^{-1}(2ie^2f(kc_s-jc_u)+2ef(ic+2f)(ke_s+je_u)+ +iefj(g_y-h_x)+e(\bar{A}h-eg)(g_s-j_x)-c(\bar{A}h-eg)(h_s-j_y)),$$

$$\Gamma^1_{0s} = i(4ef)^{-1}(-e(g_s-j_x)+\bar{A}(h_s-j_y)),$$

$$\Gamma^1_{0u} = i(4efj)^{-1}(-3ek(g_s-j_x)+2ej(g_u-k_x)+3\bar{A}k(h_s-j_y)- -2\bar{A}j(h_u-k_y)+2(\bar{A}h-eg)(j_u-k_s)).$$

The optical scalars or spin coefficients (5) (9)

$$\Gamma_{ABCD'} \stackrel{*}{=} \varkappa_{A\,E;FG'}\varkappa_B^{\;E}\varkappa_C^{\;F}\bar{\varkappa}_{D'}^{\;G'} .$$

are easily obtained from the connexion form by means of (2.8) (cf. § 5). We calculate some of them,

$$k = \Gamma^0{}_{111'} = 0,$$

$$\omega = -i\,\mathrm{Im}(z) = -i\,\mathrm{Im}(\Gamma^0{}_{101'}) = 0,$$

$$\Theta = \mathrm{Re}(z) = \mathrm{Re}(\Gamma^0{}_{101'}) = (2jef)^{-1}(ef)_s =$$

$$= \overset{1}{\nabla}\ln(\det(g_{AB}))^{1/4},\ (\text{cf. } (11)),$$

$$\sigma = \Gamma^0{}_{110'} = -(2jef)^{-1}(f^2(ef^{-1})_s + ie^2(ce^{-1})_s),$$

$$\Omega = -\Gamma^0{}_{100'} = -\gamma^0{}_{1u}.$$

$k = 0$ means, the congruence is geodetic. $\omega = 0$ is the vanishing of the twist. Both properties are characteristics of the general radiation metric. If the congruence is nonexpanding and distortion-free, $\Theta = 0 = \sigma$, we can find from the spin coefficients calculated above $g_{AB,3}=0$ as an equivalent condition. The equations $z = \sigma = \Omega = 0$ characterize congruences with recurrent ray vector (parallel propagation of the ray vector does not change the

ray direction): $l_{[a}l_{b];c} = 0$ (cf. (13) p.126) resp. $\varkappa_A D\varkappa^A = 0$ (cf. (19), p.46).
From $\Omega = -\gamma^0{}_{1u}$ we get the equivalence

$$\Omega = 0 \leftrightarrow m_{A,3} = m_{3,A}, \text{ where } A = 1,2.$$

Having arrived at the connexion form of the general radiation metric, we should like to be able to give the RIEMANN spinor in full generality. However the results for the components are extremely long expressions that are not likely to yield worthwhile information; therefore it is preferable to make use of the general connexion form and then calculate the special RIEMANN spinor, if we wish to study a special radiation metric.
In §§ 7 to 9 we shall consider special fields, that we shall assume to be shear-free.

The general radiation metric is studied in (11); in particular, the pure radiation fields (empty EINSTEIN-MAXWELL-space-times which contain a distortion-free geodetic null congruence (a so-called ray congruence) that is at the same time eigencongruence of the MAXWELL bivector) are considered in (11) table 7.1, also in (10) table 1. If, as a special case, the vacuum field equations $R_{ab} = 0$ are satisfied, we find there, too, a column containing the functions that can be chosen arbitrarily, but cannot be restricted furthermore by means of gauge transformations. Besides, the distortion-free fields are studied in (19) and in (14); in (14) exclusively and in (19) preferably the case $\Theta \neq 0$ (expanding fields). The case $\Theta = 0$ is dealt with more thoroughly in (15) chapter III, especially as regards combined EINSTEIN-MAXWELL-fields in empty space (the case $\emptyset_{ab}\emptyset^{ab} = 0$, $\emptyset_{ab}$ = MAXWELL bivector, is treated in (16)). The special case of pp-waves (plane-fronted waves with parallel rays) is considered most thoroughly in (17), also in (18).

§ 7. Distortion-free congruences of null geodesics.

In § 5 it has been shown, that with the additional assumption of vanishing distortion the metric form (6.1) can be written simply as

$$Q = -2p^2 \left| dz+Bdu \right|^2 + 2dsdu + 2hdu^2,$$

where B is complex, $x^a = (x,y,s,u)$, $z = x+iy$.

The metric tensor of the general radiation metric is thereby specified as follows,

$$g_{11} = -2p^2 \qquad m_1 = -2p^2 Re(B)$$

$$g_{22} = -2p^2 \qquad m_2 = 2ip^2 Im(B)$$

$$g_{12} = 0 \qquad m_3 = 1$$

$$m_4 = h-p^2 \; B^2$$

The structural form obtained in § 5 was

$$\theta^{1} = du,\ \theta^{t} = p(d\bar{z}+\bar{B}du),\ \theta^{m} = ds+hdu,$$

where the convention

	1	$\bar{t}$	t	m
stands for	11'	01'	10'	00'

was valid. Using these coordinates we get the connexion form

$$\omega^{0}{}_{1} = (p^{-1}h_{\bar{z}}-p(\bar{B}B_{\bar{z}}+1/2hB_{s})+B(p_{u}-hp_{s}-$$
$$-p^{-1}\mathrm{Re}((p^{2}B)_{z})du+(p_{u}-hp_{s}-$$
$$-p^{-1}\mathrm{Re}((p^{2}B)_{z})dz-pB_{\bar{z}}d\bar{z}-1/2pB_{s}ds,$$

$$\omega^{1}{}_{0} = (2p)^{-1}(p^{2}\bar{B})_{s}du+p_{s}d\bar{z},$$

$$\omega^{1}{}_{1} = 1/4(p^{2}(B\bar{B})_{s}-2h_{s}-2p^{-2}\mathrm{Im}((p^{2}B)_{z}))du+$$
$$+1/4(p^{2}\bar{B}_{s}-2(\ln p)_{z})dz+1/4(p^{2}B_{s}+$$
$$+2(\ln p)_{\bar{z}})d\bar{z}.$$

We shall calculate the RIEMANN spinor of the distortion-free fields next. As has been shown in § 3, the first step is to get the curvature form Ω_{AB} of the connexion $\omega^A{}_B$, (cf. (3.2)),

$$\Omega^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B.$$

Calculations lead to

$$\begin{aligned}\Omega^0{}_0 = {} & (G_s - 1/4B_s\bar{C}_s)ds\wedge du + I_s ds\wedge dz + (J_s - 1/2pp_sB_s)ds\wedge d\bar{z} + \\ & +(I_u - G_z - 1/2p^{-1}f\bar{C}_s)du\wedge dz + (J_u - G_{\bar{z}} + Ep_s + \\ & +1/2B_{\bar{z}}\bar{C}_s)du\wedge d\bar{z} + (J_z - I_{\bar{z}} + fp_s)dz\wedge d\bar{z},\end{aligned}$$

$$\begin{aligned}\Omega^0{}_1 = {} & (E_s + 1/2(pB_s)_u + pGB_s)ds\wedge du + (f_s + 1/2(pB_s)_z + \\ & +pIB_s)ds\wedge dz + (1/2(pB_s)_{\bar{z}} - (pB_{\bar{z}})_s + pJB_s)ds\wedge d\bar{z} + \\ & +(f_u - E_z + 2fG - 2EI)du\wedge dz - (E_{\bar{z}} + (pB_{\bar{z}})_u + 2pGB_{\bar{z}} + \\ & +2EJ)du\wedge d\bar{z} - (f_{\bar{z}} + (pB_{\bar{z}})_z + 2pIB_{\bar{z}} + 2fJ)dz\wedge d\bar{z},\end{aligned}$$

$$\begin{aligned}\Omega^1{}_0 = {} & 1/2(p^{-1}\bar{C}_s)_s ds\wedge du + p_{ss}ds\wedge d\bar{z} + (p^{-1}I\bar{C}_s - \\ & -1/2(p^{-1}\bar{C}_s)_z)du\wedge dz + (p_{su} - 1/2(p^{-1}\bar{C}_s)_{\bar{z}} - 2Gp_s + \\ & +p^{-1}J\bar{C}_s)du\wedge d\bar{z} + (p_{sz} - 2Ip_s)dz\wedge d\bar{z}.\end{aligned}$$

Here we have introduced (temporarily)

$$C = p^2 B,\quad E = p^{-1}h_{\bar z} - p(\bar B B_{\bar z} + 1/2\, h B_s) + Bf,$$

$$f = p_u - hp_s - p^{-1}\mathrm{Re}(C_z),\quad G = (2p)^{-2}(2p^2 h_s + 2\mathrm{Im}(C_z) - p^4(B\bar B)_s),\quad I = 1/4(2(\ln p)_z - p^2\bar B_s),$$

$$J = -1/4(2(\ln p)_{\bar z} + p^2 B_s).$$

By means of (3.7) and (3.8), the following components of the conformal tensor result,

$$\Gamma_{1111} = 2p^{-2}(B(p^2 B_{\bar z})_z + \bar B(p^2 B_{\bar z})_{\bar z}) + 2B_{\bar z}(p^{-1}(2hp_s - 2p_u) + 2\bar B_{\bar z} - h_s) + 2(hB_s - B_u - p^{-2}h_{\bar z})_{\bar z}, \tag{7.1}$$

$$\begin{aligned}\Gamma_{1110} = {} & -(2p^3)^{-1}(p^2 B_{\bar z})_z + (2p)^{-1}(2B(\ln p)_{z\bar z} + 2(p^{-1}p_{\bar z}\bar B)_{\bar z} + \bar B_{\bar z\bar z} - 2(\ln p)_{u\bar z} + 2(hp^{-1}p_s)_{\bar z} + \\ & + (p^2\bar B B_s)_{\bar z} + B(p^2 B_s)_z - (p^2 B_s)_u + h(p^2 B_s)_s) - \\ & - (p^{-1}h_{\bar z})_s + 3/2\, p\bar B_s B_{\bar z},\end{aligned} \tag{7.2}$$

$$\begin{aligned}\Gamma_{1100} = {} & 1/3(p^{-2}(2(p^2\bar B_s)_{\bar z} - 2(\ln p)_{z\bar z} - (p^2 B_s)_z) + \\ & + 4\mathrm{Re}(B(\ln p)_{sz}) - h_{ss} + 2(hp^{-1}p_s)_s - \\ & - 2(\ln p)_{su} + 2p^2 B_s\bar B_s),\end{aligned} \tag{7.3}$$

$$\Gamma_{1000} = p_s\bar{B}_s + 1/2p\bar{B}_{ss} - p^{-1}(\ln p)_{sz}, \qquad (7.4)$$

$$\Gamma_{0000} = 0. \qquad (7.5)$$

They were published in (4) and are given here with the correction: brackets in the term $(p^2\bar{B}B_s)_{\bar{z}}$ of (7.2) include the factor $\bar{B}$).

For $\Lambda = X_{AB}{}^{AB} = R/4$ (R = curvature scalar) we have

$$\Lambda = 2p^{-3}\mathrm{Re}(2p^3B(\ln p)_{sz} + (p(p^2B)_z)_s) + +2p^{-2}(\ln p)_{z\bar{z}} + h_{ss} - 4p^{-3/2}(p^{1/2}(p_u - -hp_s))_s - 1/2p^2B_s\bar{B}_s. \qquad (7.6)$$

We also want to write down the reduced RICCI spinor, that will be used in the next paragraph and is taken here from (4),

$$\Sigma_{111'1'} = 2(\bar{B}_z B_{\bar{z}} + p^{-1}(f_u + fh_s - hf_s)) - -2\mathrm{Re}(p^{-1}(2Bf_z + fB_z) - B_s h_z) - 2p^{-2}h_{z\bar{z}}, \qquad (7.7)$$

$$\Sigma_{011'1'} = p^{-1}(g_u + 1/2\bar{B}_{\bar{z}} - 1/2B_z - \bar{B}g_{\bar{z}} - Bg_z - h_s)_z + +1/2p(B\bar{B}_z - B_u)_s + 1/2p(\bar{B}\bar{B}_{s\bar{z}} + h\bar{B}_{ss}) + +p^{-1}(2\bar{B}_z g_{\bar{z}} - hg_{sz}) + p\bar{B}_s(2\bar{B}g_{\bar{z}} + 2hg_s + +2Bg_z + 1/2\bar{B}_{\bar{z}} + B_z - 2g_u), \qquad (7.8)$$

$$\Sigma_{001'1'} = 2g_s\bar{B}_z+\bar{B}_{sz}+1/2p^2\bar{B}_s^2, \tag{7.9}$$

$$\Sigma_{010'1'} = 1/4(4p^{-2}g_{z\bar{z}}+3p^2B_s\bar{B}_s-2h_{ss}- -4p^{-2}fp_s), \tag{7.10}$$

$$\Sigma_{010'0'} = p^{-1}g_{s\bar{z}}+2p_sB_s+1/2pB_{ss}, \tag{7.11}$$

$$\Sigma_{000'0'} = 2p^{-1}p_{ss}, \tag{7.12}$$

where $f=p_u-hp_s-p^{-1}\mathrm{Re}((p^2B)_z)$, and $g=\ln p$ hold for all six of the components.

For further investigations of the distortion-free fields we prefer to distinguish the two cases of expansion-free and expanding fields. The former ones will become subject of § 8, the latter ones will be studied in § 9.

§ 8. Classification of the expansion-free fields according to the PETROV type of the conformal tensor.

Assuming $\Theta = 0$ (vanishing expansion), we now specialize s as affine parameter,

$$l^a = u,^a = \frac{\partial x^a}{\partial s}, \qquad (8.1)$$

and call it r. From the connexion form $\omega^A{}_B$ for instance, we are able to infer, by means of (2.8), $\Theta = \mathrm{Re}(z) = -\mathrm{Re}(\Gamma^1{}_{010'}) = -(\ln p)_r$, therefore $\Theta = 0$ amounts to

$$p_r = 0. \qquad (8.2)$$

With that we are ready to classify the fields according to PETROV's type of the conformal tensor. Before we start to do so, however, let us restate our assumptions. The fields considered are to contain a twist-free, non-expanding "ray congruence", i.e., ~~a~~ distortion-free, geodetic null congruence; on the other hand, no assumptions yet about the field equations.

I. Fields of type I.

Combining (7.4) and (8.2) we receive

$$\Gamma_{1000} = 1/2p\bar{p}_{rr}, \qquad (8.3)$$

and from the connexion form we obtain the complex rotation Ω with the aid of (2.8),

$$\Omega = 1/2p\overline{B}_r. \qquad (8.4)$$

Because of the equivalence of $\Gamma_{0000} = \Gamma_{1000} = 0$ to $\Gamma_{ABCD}\varkappa^B\varkappa^C\varkappa^D = 0$ we can conclude from (8.3) and (8.4)

<u>Theorem 8.1.</u> A field ($\omega=\sigma=\theta=0$, see restatement of assumptions above) is of type $\neq$I, if and only if the complex rotation is constant along the rays.
It can be seen from (7.11),(8.2) and (8.4), that $\Omega_r = 0$ is equivalent to one of the vacuum field equations, $\Sigma_{010'0'} = 0$. The meaning of this equation for the definition of gravitational radiation within matter fields will be explained in § 10.
Because of this theorem, fields are of type I only if they do not satisfy this field equation; the expansionfree radiation fields (cf. (<u>15</u>)) already do so. They are characterized by containing a non-expanding ray congruence l_a as well as fulfilling the field equations $R_{ab} = -T_{cd}m^c m^d l_a l_b$, where $T_{cd}m^c m^d \geq 0$, R_{ab} = RICCI's tensor, T_{ab} = energy-momentum-tensor. According to (<u>15</u>), the equation $\Sigma_{010'0'} = 1/2p\overline{B}_{rr} = 0$ corresponds to $R_{ab}l^a t^b = 0$ simultaneously with $R_{ab}l^a l^b = 0$.
We recollect that on account of $R_{ab}l^a l^b = 0$ the vanishing of the distortion is not an additional assumption in the case of expansion-free radiation fields, but follows from the vanishing of the twist (and v. v., see (<u>11</u>) theorem 2.1).

II. Fields of type II or D.

From (7.3) and (8.2) we get

$$\Gamma_{1100} = 1/3\left(p^{-2}\left(\mathrm{Re}((p^2B_r)_z)-3\mathrm{Im}((p^2B_r)_z)\,i -2(\ln p)_{z\bar{z}}\right)-h_{rr}+2p^2B_r\bar{B}_r\right). \qquad (8.5)$$

We draw the consequence from this, from theorem 8.1 and the equivalence $\Gamma_{ABCD}\varkappa^C\varkappa^D = 0 \Leftrightarrow 0 = \Gamma_{0000} = \Gamma_{1000} = \Gamma_{1100}$ in

Theorem 8.2. A field is of type II or D if and only if the complex rotation is constant along the rays:

$$B_{rr} = 0 \leftrightarrow \Omega_r = 0 \qquad (8.6)$$

and (at least) one of the inequalities

$$2p^2B_r\bar{B}_r-h_{rr}+p^{-2}(-2(\ln p)_{z\bar{z}}+\mathrm{Re}(p^2B_r)_z)\neq 0 \qquad (8.7)$$

$$\mathrm{Im}(p^2B_r)_z\neq 0 \qquad (8.8)$$

is valid.

If we consider the special case $\Omega = 0$, (8.8) is not valid and (8.7) is reduced to $R \neq 0$, i.e.,

Theorem 8.3. A field with recurrent ray vector is of type II or D, if and only if the curvature scalar does not vanish.

This theorem follows from (8.4), theorem 8.2, and

$$\Lambda = h_{rr} - 1/2p^2 B_r \bar{B}_r + 2p^{-2}((\ln p)_{z\bar{z}} + \mathrm{Re}(p^2 B_r)_z), \quad (8.9)$$

where (8.9) was derived from (7.6) and (8.2).

For the fields of special type we shall investigate shortly the vacuum field equations $\Lambda = 0 = \Sigma_{ABC'D'}$.

Two of these are satisfied automatically:

$\Sigma_{000'0'} = 0$ because of (7.12), and (8.2), $\Sigma_{010'0'} = 0$ because of (7.11), (8.2) and (8.5).

From (8.9) we receive, by integrating twice, for $\Lambda = 0$

$$h = 1/4p^2 B^2 - (rp^{-1})^2 (\ln p)_{z\bar{z}} - rp^{-2}\mathrm{Re}(p^2(2B - rB_r))_z + rf_1 + f_2, \quad (8.10)$$

where $f_i = f_i(z, \bar{z}, u)$ is real, $i = 1,2$.

Changing the affine origin,

$$\bar{r} = r + f(x,y,u), \ \bar{z} = z, \ \bar{u} = u, \quad (8.11)$$

(a gauge transformation II, in the notation of (15)) we are able to reduce the third term in (8.10) to $-rp^{-2}\mathrm{Re}(p^2 B)_z$.

In the same way as before we get from (7.10) for

$\Sigma_{010'1'} = 0$

$$h = 3/4\, p^2 |B|^2 + (rp^{-1})^2 (\ln p)_{z\bar{z}} + rf_3 + f_4, \qquad (8.12)$$

f_i as above, $i = 3,4$.

The next field equation (cf. (7.9)) is $\Sigma_{001'1'} = 0$. It reads $B_{r\bar{z}} + 1/2p^2 B_r^2 = 0$ and is empty if $B_r = 0$. In the case $B_r \neq 0$ we divide by B_r^2 and have $(B_r^{-1})_{\bar{z}} = 1/2p^2$, therefore

$$B_r = 2(\int p^2 d\bar{z})^{-1}. \qquad (8.13)$$

Both the remaining field equations are less transparent. Using (7.8), $\Sigma_{011'1'} = 0$ yields a complex equation, from which two real expressions for h, similar in type to (8.10) and (8.12) but considerably longer, can be extracted. $\Sigma_{111'1'} = 0$ is one real equation according to (7.7), it can be regarded as a partial differential equation for h in agreement with our treatment of the other field equations.
For the fields of special type, we also want to give the conformal tensor referring to the case that the tetrad is parallelly propagated along the rays. We shall not be specific about the field equations.

Using the criterion

$$\mathrm{Re}(t_{a;b}l^{b}) = 0 \tag{8.14}$$

for parallel propagation of the tetrad, we proceed as follows: from our structural form $\Theta^{AB'}$ we have $\mathrm{Re}(t_a) \overset{*}{=} p\delta^1_a + p\mathrm{Re}(B)\delta^4_a$, $l_a \overset{*}{=} \delta^4_a$, therefore $l^a \overset{*}{=} \delta^a_3$. With this, (8.14) reduces to

$$\mathrm{Re}(t_{a;3}) = 0 \tag{8.15}$$

A short calculation leads to $\mathrm{Re}(t_{a;3}) \overset{*}{=} 1/2\mathrm{Re}(pB)_{,3}\delta^4_a$ which tells us we have to alter t_a in order to satisfy (8.15); this also shows that t_a can be chosen as

$$t = p(d\bar{z}+1/2\bar{B}du) \quad , \quad t := t_a dx^a , \tag{8.16}$$

if l_a remains unchanged; (8.15) is satisfied then. From (4),(V.3) and (4),(V.5) we find this choice of t amounts to the transformation

$$\begin{aligned} \varkappa_{\bar{A}} &\overset{*}{=} \varkappa_A, \\ \mu_{\bar{A}} &\overset{*}{=} 1/2pB\varkappa_A + \mu_A . \end{aligned} \tag{8.17}$$

Therefore we have for the components of the WEYL-spinor referring to the parallelly propagated tetrad

$$\Gamma_{\overline{0000}} = \Gamma_{0000} = 0, \qquad (8.18)$$

$$\Gamma_{\overline{1000}} = \Gamma_{1000} = 0, \qquad (8.19)$$

$$\Gamma_{\overline{1100}} = \Gamma_{1100} = 1/3(2p^2 B_r \bar{B}_r - h_{rr} + p^{-2}(2(p^2\bar{B}_r)_{\bar{z}} - (p^2 B_r)_z - 2(\ln p)_{z\bar{z}})), \qquad (8.20)$$

$$\begin{aligned}\Gamma_{\overline{1110}} = \Gamma_{1110} + 3/2pB\Gamma_{1100} = \Big[&-p^{-3}\bar{B}p_{\bar{z}}^2 - p^{-2}p_z B_{\bar{z}} + \\ &+\bar{B}p_{\bar{z}}B_r + 2Bp_{\bar{z}}\bar{B}_r + p^{-2}p_{\bar{z}}\bar{B}_{\bar{z}} + 3/2pB_{\bar{z}}\bar{B}_r + \\ &+1/2pB_r\bar{B}_{\bar{z}} + p^3 BB_r\bar{B}_r + p^{-2}\bar{B}p_{\overline{zz}} + \\ &+1/2p^{-1}B_{z\bar{z}} + 1/2p\bar{B}B_{r\bar{z}} + 1/2p^{-1}\bar{B}_{\overline{zz}} + \\ &+pB\bar{B}_{r\bar{z}} - p^{-1}h_{r\bar{z}} - 1/2pBh_{rr}\Big] + \Big[p^{-3}p_{\bar{z}}p_u - \\ &-p_u B_r - p^{-2}p_{u\bar{z}} - 1/2pB_{ru}\Big],\end{aligned} \qquad (8.21)$$

$$
\begin{aligned}
\Gamma_{\overline{1111}} &= \Gamma_{1111}+2pB\Gamma_{1110}+3/2p^{2}B^{2}\Gamma_{1100} = \\
&= \Big[-p^{-2}B^{2}p_{z}p_{\bar z}-2B\bar{B}p_{\bar z}^{2}+2p^{-1}Bp_{z}B_{\bar z}+ \\
&\quad +pB^{2}p_{z}B_{r}+4p^{-1}\bar{B}p_{\bar z}B_{\bar z}+2pB\bar{B}p_{\bar z}B_{r}+ \\
&\quad +2pB^{2}p_{\bar z}\bar{B}_{r}+2p^{-1}Bp_{\bar z}\bar{B}_{\bar z}+4p^{-3}p_{\bar z}h_{\bar z}+ \\
&\quad +4B_{\bar z}\bar{B}_{\bar z}+3p^{2}BB_{\bar z}\bar{B}_{r}+p^{2}BB_{r}\bar{B}_{\bar z}+p^{4}B^{2}B_{r}\bar{B}_{r}- \\
&\quad -2B_{\bar z}h_{r}+2B_{r}h_{\bar z}+p^{-1}B^{2}p_{z\bar z}+2p^{-1}B\bar{B}p_{\overline{zz}}+ \qquad (8.22) \\
&\quad +BB_{z\bar z}-\bar{B}B_{\overline{zz}}+1/2p^{2}B^{2}B_{rz}+(p^{2}B\bar{B}+ \\
&\quad +2h)B_{r\bar z}+B\bar{B}_{\overline{zz}}+p^{2}B^{2}\bar{B}_{r\bar z}-2B_{u\bar z}- \\
&\quad -2p^{-2}h_{\overline{zz}}-2Bh_{r\bar z}-1/2p^{2}B^{2}h_{rr}\Big]+ \\
&\quad +\Big[2p^{-2}Bp_{\bar z}p_{u}-4p^{-1}p_{u}B_{\bar z}-2pBp_{u}B_{r}- \\
&\quad -2p^{-1}Bp_{u\bar z}-p^{2}BB_{ru}\Big].
\end{aligned}
$$

The square brackets are used to separate the terms that are quadratic in, or linear in, or independent of r. The last component, Γ_{1111}, is quadratic in r, Γ_{1110} is linear in r, Γ_{1100} does not depend on the affine parameter, i.e., it does not change along the rays.

III. Fields of type III.

From (7.2) and (8.2) we get for $\Gamma_{1110} = 0$

$$3/2p^2B_{\bar z}\bar B_r - h_{\bar z r} + (\bar B(\ln p)_{\bar z} + 1/2\bar B_{\bar z} - (\ln p)_u)_{\bar z} + \\ +1/2(2B(\ln p)_{z\bar z} + (p^2\bar B B_r)_{\bar z} + B(p^2B_r)_z - \\ -(p^2B_r)_u - p^{-2}(p^2B_{\bar z})_z) = 0. \tag{8.23}$$

With this we have

Theorem 8.4. A field is of type III, if and only if the inequalities (8.7), (8.8) and the equation (8.23) are not satisfied and Ω does not change along the rays.

Again we want to consider the case $\Omega = 0$. From theorems 8.4 and 8.3 we infer

Theorem 8.5. A field with recurrent ray vector is of type III, if and only if the curvature scalar vanishes, and (8.23) is not satisfied.

Essentially this theorem is contained in (19). In this paper it is also shown that we can arrive at $p = 1$ and $h_{rr} = 0$ by means of an additional assumption, $\Sigma_{010'1'} = 0$ (cf. 7.10). From (8.21) and (8.22) we are able to conclude that in that case Γ_{1111} is linear in r, and Γ_{1110} does not change along the rays.

<u>IV. Fields of type N.</u>

Substituting (8.2) into (7.1), $\Gamma_{1111} = 0$ reads

$$p^{-2}\left(B(p^2 B_{\bar z})_z + \bar B (p^2 B_{\bar z})_{\bar z}\right) - 2(\ln p)_u B_{\bar z} - \\ - B_{u\bar z} + B_{\bar z}(2\bar B_{\bar z} - h_r) + (hB_r - p^{-2} h_{\bar z})_{\bar z} = 0. \qquad (8.24)$$

We obtain

<u>Theorem 8.6.</u> A field is of type N, if and only if

a) the complex rotation is constant along the rays,

b) the inequalities (8.7), (8.8) do not hold,

c) equation (8.23) holds,

d) equation (8.24) does not hold.

The last condition implies that the N-field does not degenerate. For the fields with recurrent ray vector, condition b) reads "the curvature scalar vanishes", and a) is satisfied automatically.

§ 9. The twistfree expanding radiation fields.

Most of the results presented in this paragraph are already known. We rather want to show here that with the formalism of this chapter all geometrical questions arising in General Relativity can be treated in a way which does not demand more work than the usual ones.

As our example, we treat the twistfree ($\omega = 0$) expanding ($\theta \neq 0$) vacuum ($R_{ab} = 0$) pure radiation ($\sigma = 0$) field. We begin with integrating the vacuum-field equations into a simple form for the metric. Starting from it, we give some sample statements on the connection of PETROV-type, curvature of the wave fronts, and motion groups.

Theorem: The twistfree expanding vacuum radiation fields may be put into the canonical form

$$(9.1) \qquad Q = -2r^2 q^2 |dz|^2 + 2drdu + 2hdu^2$$

where $\quad h = ar^{-1} + rQ_u + K$,

$a = 1$ or 0 , $Q = \ln q$, $r^{-1}K = q^{-2} Q_{z\bar{z}}$ is the curvature of the wave-fronts, and

$$(9.2) \qquad 0 = 3aQ_u + q^{-2} K_{z\bar{z}}$$

The curvature spinors are

$$\Gamma_{0000} = \Gamma_{0001} = 0$$

$$\Gamma_{0011} = -2ar^{-3}$$

(9.3) $$\Gamma_{0111} = q^{-1} K_{\bar{z}} \; r^{-2}$$

$$\Gamma_{1111} = -2 \, (q^{-2} K_{\bar{z}})_{\bar{z}} \, r^{-2} - 2 (q^{-2} Q_{u\bar{z}})_{\bar{z}} r^{-1}$$

<u>Proof:</u> a) That $\sqrt[4]{K}$ is the curvature of the wave fronts $dr = du = 0$ is obvious (cf. (13)) .

5) In § 5 it has been shown that the fields in question may be put into the form

$$Q = -2p^2 \, | dz + Bdu |^2 + 2drdu + 2hdu^2 ,$$

which corresponds to the structure form

(9.4) $$\theta^k = du , \quad \theta^t = p(dz + Bdu), \quad \theta^m = dr + hdu .$$

Now, the vacuum equation $\Sigma_{000\bar{0}'} = 0$ reads (cf.(7.12))
$p_{rr} = 0$, so $p = qr + b$ with $q_r = b_r = 0$. Shifting the origin of r , we get $p = qr$.

From (7.10) we get $(r^4 B_r)_r = 0$, so $B = ar^{-3} + c$ $(a_r = c_r = 0)$. Inserting into (7.11) we receive
$0 = 2r^{-1}(\bar{a}_z r^{-3} + \bar{c}_z) - 3\bar{a}_z r^{-4} + 9/2 \, q^2 r^{-6} a^2$.
This is a polynomial in r^{-1} , the coefficients of which must all vanish: $B_r = B_{\bar{z}} = 0$.

So we find

$$d(dz + Bdu) \wedge (dz + Bdu) = 0$$

and with the theorem of FROBENIUS $dz + Bdu = gdz'$.

We make a gauge as in § 5 in order to make g real .

Thus we arrive at the strueture form

(9.4) with $B = 0$, $p = rq$. From (7.6) and (7.9) we get

$$0 = -2r^{-2}K \quad -h_{rr} - 2r^{-1}Q_u + 2r^{-2}h$$

$$0 = -2r^{-2}K \quad + h_{rr} - 6r^{-1}Q_u + 2r^{-2}h + 4r^{-1}h_r$$

These equations are easily integrated to

$$h = ar^{-1} + rQ_u + K \;, \quad a_r = 0 \;,$$

and (7.8) now gives $a_z = 0$, so $a = a(u)$.

If now $a = 0$, the first halfthof the theorem is proven. If $a \neq 0$, we make the gauge ((5.4), $\lambda = a^{1/6}$, $\Lambda = 0$) and the coordinate transformation $r = va^{1/3}$, $a^{1/3}\,du = d\bar{u}$. The formulae (5.6) give the wanted result.

c) Equations (9.2)(9.3) are now the spezialization of (7.1) to (7.12) to the special form (9.1). The theorem is proven.

We give now a list of consequences

1) From (9.3) one reads easily the peeling-off properties, firstly given by ROBINSON-TRAUTMAN (13).

2) Fields (9.1.) with constant curvature of the wave fronts ($K_z = 0$) are of type D ($a \neq 0$, $\Gamma_{A111} = 0$) or N ($a = 0$, $Q_u \neq 0$). Type - N - fields have constant K.

3) For fields (9.1), the formulae for $\omega^A{}_B$ of § 7 give

$$(9.9)\qquad \omega^0{}_0 = 1/2\ (Q_u - r^{-2}a)du + Q_z dz - Q_{\bar z} d\bar z$$

$$\omega^0{}_1 = q^{-1}(Q_{u\bar z} + r^{-1}K_{\bar z})du + q(-ar^{-1}-K)dz$$

$$\omega^1{}_0 = qd\bar z$$

For fields (9.1) of Type D , ω^A is the second eigenspinor of Γ , because of the corollary and (9.3) . An inspection of (5.3) shows that the second eigencongruence is also normal $(\check z - \bar{\check z} = 0)$ and expanding $(\bar{\check z} + \check z \neq 0)$.

<u>Corollery B:</u> If in a vacuum D-field one of the eigencongruences of the WEYL-tensor is normal and expanding, so is the other.

5) Let $(\xi_{00'}, \xi_{01'}, \xi_{10'}, \xi_{11'}) = (L,T,\bar T,M)$

be a real covariant vector field. With the aid of (9.9), one calculates its covariant differential $D\xi_{AA'}$ to be

$$(9.10)\qquad DL = L_r\theta^m + r^{-1}(q^{-1}L_z - T)\theta^{\bar t}$$

$$+ r^{-1}(q^{-1}L_{\bar z} - \bar T)\ \theta^t$$

$$+(L_u - hL_r - Lh_r)\theta^k$$

$$DT = T_r\theta^m + r^{-1}q^{-1}(T_z - TQ_z)\theta^{\bar t} +$$

$$+ r^{-1}(q^{-1}(T_{\bar z} + TQ_{\bar z}) - M - rLQ_u + Lh)\theta^t$$

$$+ (T_u - hT_r - r^{-1}q^{-1}Lh_z)\theta^k$$

$$D\bar T = \overline{DT}$$

$$DM = M_r\theta^m + r^{-1}(q^{-1}M_z - rTQ_u + Th)\theta^{\bar t} +$$

$$+ r^{-1}(q^{-1}M_{\bar z} - r\bar TQ_u + \bar Th)\theta^t$$

$$+ (M_u - hM_r + Mh_r - r^{-1}q^{-1}(Th_{\bar z} + \bar Th_z))\theta^k .$$

One obtains the following <u>lemma:</u> $\xi_{AA'}$ is a KILLING-vector if and only if

$$L_r = 0;\ T = tqr + q^{-1}L_z \quad \text{mit} \quad t = \text{const. compl.};$$

$$M = Lh - rL_u + m, \quad \text{where} \quad L, m \quad \text{satisfy the equs.}$$

$$0 = m_r = am = aL_z = (q^{-2}L_z)_z = m_z + 2KL_z$$

$$0 = 3aq^2L_u - L_zK_{\bar z} - L_{\bar z}K_z = LQ_{uu} - L_{uu} + L_uQ_u$$

$$-tQ_{u\bar z} - \bar tQ_{uz} = 0 = 2KL_u + LK_u + m_u + tK_{\bar z} + \bar tK_z$$

$$-q^{-2}(L_zQ_{u\bar z} + L_{\bar z}Q_{uz})$$

6) One sees that in the case $a = 1$, i.e. for type D or II we have the linear equation $M - Lh = 0$, so the trajectories have no higher dimension than three:

There are no homogeneous fields (9.1) of types D or II.

7) In case D $(a \neq 0 = K_z)$,

every vector-field $\xi_{AA'} = (L, tqr, Lh)$ is KILLING (L,t constant), so these fields admit of a group with 3-dimensional trajectories. In fact, fields of type D are distinguished by this from those of type II. For the latter, the KILLING equations read $L = \text{const}$, $m = 0$ and

$$0 = LK_u - tK_{\bar{z}} - \bar{t}K_z = LQ_{uu} - tQ_{u\bar{z}} - \bar{t}Q_{uz} .$$

If there are three linearly independent solutions, obviously $K_z = 0$, and we may apply corollary 2) page 75/76.

§ 10 Conform tensor and ray congruence.

Since the early attempts towards a rigorous theoretical treatment of gravitational radiation, two essentially different definitions of "pure radiation" offered themselves by comparision with MAXWELL'S (flat spacetime) electromagnetic theory: a definition (A) based on the algebraic structure of the conform tensor suggested by PIRANI in 1957, (7), and a definition (B) based on the properties of geodesic null lines or "rays" along which the field amplitudes propagate, successfully, introduced by I. ROBINSON in 1960. A year later, GOLDBERG and SACHS (21) succeeded in proving equivalence of both definitions for vacuum fields, and even for a more general class which included combined gravitational and electromagnetic fields. At the same time, wide classes of pure radiation fields were explicitly determined and explored (compare (11) (15)), both of the pure gravitational and electromagnetic type.

But there remained, until today, the unsolved problem of how gravitational waves propagate inside matter. One even does not know what definition to start out with. A step which was meant to help clarifying this problem was a theorem proven by KUNDT and THOMPSON in 1962 (22), and one year later by ROBINSON and SCHILD (23), giving the general field equations for which (A) and (B) are equivalent; they are of the third differential order in the metric.

In this paragraph we will present a modification of the theorem just mentioned which makes a stronger assumption on the congruence of rays considered, and thereby arrives at stronger field equations which are of the second order only. More precisely, we claim:

<u>Theorem:</u> Given some domain of a normal hyperbolic RIEMANNIAN V^4 which contains a normal and non-expanding ray congruence (tangent vector k^a). Its conform tensor is special, with k^a as null eigenvector, (A), if and only if k^a is an eigen vector of the RICCI tensor:

$$k_{[a} R_{b]c} k^c = 0 .$$

Here a "ray congruence" is a distortion-free congruence of null geodesics; it is called "normal" when the rays are orthogonal to hypersurfaces (are not twisted); compare § 6 of this report. The theorem implies that source-free pure radiation as defined by the above congruence is algebraically general unless the RICCI tensor possesses a null eigen vector. Which cannot happen for ordinary matter satisfying EINSTEIN's equations.

In order to prove the theorem, we first copy our equations (7.4)(7.5)(7.11)(7.12), which apply whenever there exists a normal ray congruence (spanned by $k^a \leftrightarrow k^{AA'} = \varkappa^A \bar{\varkappa}^{A'}$), whose "tangent spinor" $\varkappa^A$ is chosen as the 0th basis spinor,

$$\Gamma_{0000} = 0$$

$$\Gamma_{0001} = p_r \bar{B}_r + 1/2p\bar{B}_{rr} - p^{-1}(\ln p)_{rz}$$

$$\Sigma_{000'0'} = 2p^{-1}p_{rr}$$

$$\Sigma_{010'0'} = p^{-1}(\ln p)_{rz} + 2p_r B_r + \frac{1}{2} pB_{rr}$$

In the non-expanding case p_r vanishes, (cf.(8.2)), and we obtain:

$$\Gamma_{0001} = \frac{1}{2} p\bar{B}_{rr} = \overline{\Sigma_{010'0'}}$$

so that the conform tensor is special if and only if $\Sigma_{0B0'0'} = \Sigma_{ABA'B'} \chi^A \bar{\chi}^{A'} \bar{\chi}^{B'}$ vanishes. Now from (3.4) we see that $S_{AA'BB'} = 2 \Sigma_{AA'BB'}$,

$S_{AA'BB'}$ being the Γ-translated reduced RICCI-tensor (cf.§ 1), and (A) holds if and only if

$$S_{ABA'B'} \chi^A \bar{\chi}^{B'} \bar{\chi}^{A'} = 0 , \Longleftrightarrow$$

$$S_{AA'BB'} \chi^A \bar{\chi}^{A'} = \xi_B \bar{\chi}_{B'} \ ;$$

the reality of $S_{ABA'B'}$ implies that ξ_A is a (real) multiple of χ_A , so that A $\Longleftrightarrow$

$$S_{ab} k^b \sim k_a \ , \quad \text{or:} \quad k_{[c} S_{a]b} k^b = 0 \ ;$$

which last equation is equivalent with $k_{[c} R_{a]b} k^b = 0$.

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Chapter III.

A threedimensional formulation of the Bianchi identities for vacuum gravitational fields.

§ 1. Introduction

Einsteins theory of gravitation describes the gravitational field by the curvature of a normal hyperbolic Riemannian 4 - space. The empty space field equations are the conditions that the Ricci tensor is zero. Naturally this purely geometrical formulation of the theory does neither make use of distinguished coordinate systems nor of distinguished observers (i.e. timelike vector fields). In order to relate the geometrical statements of the theory to the measurements made by observers one has to introduce three sorts of congruences, namely

(1) timelike nongeodesic congruences to represent the observers,

(2) timelike geodesic congruences to represent free falling incoherent dust, and

(3) null geodesics to represent light rays.

Our perceptive faculty does not permit the immediate observation of a 4 - dimensional space-time region around the world line we are travelling along. It only allows us to perceive this region by observing the history of our rest space. Therefore it seems to be desirable for the discussion of some problems to pass from the four-dimensional formulation of the theory to a three-dimensional one where the equations describe the change of the field quantities in the course of an eigentime interval.

This idea is not new, it has been put forward by many authors. However, it has not yet been carried through as far as one might wish to go.

In section 2 we will give the three-dimensional formulation of the Bianchi identities for vacuum fields in the case where a timelike congruence is singled out. The method employed is the same as in the case of the electromagnetic field when one passes from the four-dimensional to the three-dimensional form of the equations.

In sections 3 and 4 we will apply the results of section 1 to give general properties of vacuum fields which are type I with either real or imaginary eigenvalues. Such fields correspond formally to purely electric or magnetic fields in electrodynamics.

§2. Induction laws for exterior gravitational fields

The empty space conditions in Einstein's theory of gravitation are

$$R_{ab} = 0. \qquad (1)$$

They are equivalent to the equations

$$R_{abcd} = C_{abcd}. \qquad (2)$$

The conform tensor satisfies the Bianchi identities

$$C^{abcd}{}_{;d} = 0. \qquad (3)$$

We assume that somehow a congruence C of timelike curves is given. The curves of C may be the trajectories of a cloud of test particles. We do not require them to be geodesics because there may be nongravitational forces acting between them. But we will assume that the material which travels along the curves of C does not contribute to the gravitational field under consideration. Sometimes we will use the word "observer" as a synonym for "timelike congruence".

Let u^a be the unit vector field tangent to C. Since we assume the signature to be $+ + + -$ we have

$$u_a u^a = -1. \qquad (4)$$

The kinematical behaviour of the congruence in a first order neighbourhood of an arbitrarily chosen central curve is described by the antisymmetric tensor of rotation ω_{ab}, the symmetric and traceless tensor σ_{ab} of shear, the scalar of expansion θ , and by the vector of acceleration $\dot{u}_a \equiv u_{a;b}u^b$. Since ω_{ab} is a simple bivector it can be replaced by a spacelike vector $\omega^a = \frac{1}{2}\eta^{abcd}\omega_{bc}u_d$, the vector of rotation.

With these definitions we have

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{\theta}{3}h_{ab} - \dot{u}_a u_b, \qquad (5)$$

where $h_{ab} \equiv g_{ab} + u_a u_b$.

Now we form the tensors

$$E_{ac} \equiv - C_{abcd}u^b u^d \qquad (6)$$

and

$$H_{ac} = - {}^{*}C_{abcd}u^b u^d. \qquad (7)$$

They are both symmetric and traceless, they have rank 3, and they lie in the three-space orthogonal to u^a. Their 2×5 components contain the information of the 10 components of the conform tensor. The latter can be expressed by E_{ab} and H_{ab} via

$$(C + i\, C)_{abcd} = -(g + i\eta)_{abpq}(g + i\eta)_{cdrs}u^p u^r (E + iH)^{qs}.$$

Now, using this equation we form the Bianchi identities which may be stated as

$$(C + i{}^{*}C)^{abcd}{}_{;d} = 0. \qquad (8)$$

We decompose the resulting equations into their parts orthogonal and tangential to u^a. Then we have (the symbol $\perp$ denotes the projection of all free indices of the following tensor into the space orthogonal to u^a)

$$\perp \dot{E}_{ab} = -\perp H^{c;d}_{(a}\eta_{b)cdm}u^m - \Theta E_{ab} + E^c_{(a}\omega_{b)c} + E^c_{(a}\sigma_{b)c} + \eta_{arsm}\eta_{bpqn}u^m u^n \sigma^{rp} E^{sq} + 2H^d_{(a}\eta_{b)cdm}\dot{u}^c u^m \,, \tag{9}$$

$$\perp \dot{H}_{ab} = \perp E^{c;d}_{(a}\eta_{b)cdm}u^m - \Theta H_{ab} + H^c_{(a}\omega_{b)c} + H^c_{(a}\sigma_{b)c} + \eta_{arsm}\eta_{bpqn}u^m u^n \sigma^{rp} H^{sq} - 2E^d_{(a}\eta_{b)cdm}\dot{u}^c u^m \tag{10}$$

$$E^b_{a;c}h^c_b + 3H^c_a\omega_c - \eta_{abcd}u^d\sigma^{bm}H^c_m = 0 \tag{11}$$

$$H^b_{a;c}h^c_b - 3E^c_a\omega_c + \eta_{abcd}u^d\sigma^{bm}E^c_m = 0. \tag{12}$$

These equations are equivalent to the complete set of vacuum Bianchi identities. We may state the meaning of these equations by saying that a change of E_{ab} (H_{ab}) in an eigentime interval induces a change of H_{ab} (E_{ab}) in the space orthogonal to u^a. Therefore we may call our equations the laws of gravitational induction. However, the similarity of (9), (10), (11), and (12) to the equations for vacuum electromagnetic fields is not perfect in so far as our equations contain terms involving products of the kinematical quantities with the field intensities E_{ab} and H_{ab}. It is essential for a curved empty space-time that these terms are present since restrictions of the kinematical quantities impose in most cases restrictions on the Riemannian space.

To illustrate this fact we consider the case where $u_{a;b} = 0$, i.e. all the kinematical quantities vanish. Then we have from the Ricci identities $C_{abcd}u^d = 0$ which implies $C_{abcd} = 0$ since u^a is a timelike vector. Therefore the space-time is flat.

Things are different in lorentz covariant electrodynamics where one deals with a Minkowskian space. Since such a space is flat one may choose families of congruences which have constant tangent vector fields. This is the reason why the kinematical quantities of those prefered observers do not appear in Maxwell's equations.

In order to get some insight into the structure of our system of equations we shall now examine those terms which contain derivatives of the field quantities E_{ab} and H_{ab}. For this purpose we consider a Minkowski space and a tensor field C_{abcd} defined on it. The tensor C_{abcd} is supposed to have precisely the same algebraic properties as the conform tensor of a normalhyperbolic Riemannian four-space. As field equations in our model theory we adopt $C^{abcd}{}_{;d} = 0$.† Here and in the following equations we write a semicolon to denote differentiation. This is done to keep the formulae in a form independent of coordinates.

Let u^a be a constant vector field, $u_{a;b} = 0$. We shall limit our considerations to fields which have the properties of static fields, i.e. we assume $\dot{C}_{abcd} = 0$ and $^*C_{abcd}u^bu^d = 0$. Then our

† and $C^{*abcd}{}_{;d} = 0$.

equations reduce to

$$E^{c;d}_{(a}\ \eta_{b)cdm}u^m = 0 \qquad (13)$$

and

$$E^c_{a;c} = 0.$$

Since E^c_a is traceless the last equation can be rewritten as

$$E^{c;d}_{[a}\ \eta_{b]\,cdm}u^m = 0. \qquad (14)$$

Now (13) and (14) yield

$$E^{c;d}_a\ \eta_{bcdm}u^m = 0$$

and, if we write this equation in the form

$$\overset{*}{E}_{a[b;m]}u^m = 0,$$

we see that it is equivalent to

$$E_{a[b;c]} = 0. \qquad (15)$$

In order to investigate the content of (15) we introduce 3 spacelike constant unit vector fields $\underset{\rho}{v}^a$ which are orthogonal to each other and to u^a.
We have

$$g_{ab} = \underset{\rho}{v}_a\underset{\rho}{v}_b - u_a u_b \qquad (\text{sum over } \rho)$$

where $\underset{\rho}{v}_{a;b} = 0$. Then it follows from (15)

$$(E_{ab}\underset{\rho}{v}^a)_{;c} - (E_{ac}\underset{\rho}{v}^a)_{;b} = 0$$

so that we can write $E_{ab}\underset{\rho}{v}^a = \underset{\rho}{\psi}_{,b}$. From this equation we get $E_{ab} = \underset{\rho}{v}_a\underset{\rho}{\psi}_{,b} = (\underset{\rho}{v}_a\psi_\rho)_{;b}$
and if we put $\underset{\rho}{v}_a\underset{\rho}{\psi} \equiv \phi_a$ we have $\phi_{[a,b]} = 0$.
Therefore we conclude

$$E_{ab} = \phi_{;ab}. \qquad (16)$$

The trace of E_{ab} is zero, therefore we have

$$\Delta \phi = 0$$

where $\Delta\phi$ is threedimensional Laplacian of ϕ. In this way we are led to a scalar potential which satisfies the field equation of Newton's theory of gravitation.

§3. Vacuum gravitational fields of the electric type.

In this section we investigate properties of vacuum fields which have Petrov type I with real eigenvalues. Such fields are characterized by the existence of a timelike vector field u^a which satisfies

$$^*C_{abcd}u^b u^d = 0. \tag{17}$$

The only known examples of fields of this sort are the static fields which are characterized by the existence of a timelike hypersurface-orthogonal Killingvector. In this case (17) holds regardless whether field equations are satisfied or not. However, since the stronger conditions $^*R_{abcd}u^b u^d = 0$ hold, the energy-momentum tensor of a possible source is restricted by the conditions $h_a^c R_c^b u_b = 0$. A slightly more general class of fields characterized by a u^a with vanishing shear and rotation also satisfies (17). It has been proven that vacuum fields of this sort are static if one assumes that they are not degenerated (presumably this is also true in the case of degeneracy).

Since (17) is assumed, the geometrical information of the conform tensor is contained in E_{ac}. We can, therefore, reduce the geometry of the conform tensor to properties of E_{ac}.

First we wish to relate the eigen null directions k_a of the conform tensor to the eigenvectors $\underset{\mu}{e}^a$ and the eigenvalues $\underset{\mu}{\lambda}$ of E_{ac}. For this purpose we take the definition of the eigen null directions,

$$k_{[a}C_{b]cd[e}k_{f]}k^ck^d = 0,$$

and replace the conform tensor by

$$C_{abcd} = -(g_{abpq}g_{cdrs} - \eta_{abpq}\eta_{cdrs})u^pu^rE^{qs}.$$

Using the eigenvectors of E_{ac} to analyze this equation we arrive at an explicit expression of the four k^a in terms of the invariants of E_{ac}. The result of this analysis is the following <u>theorem</u> : If in a normal hyperbolic Riemannian 4-space there exists a timelike vector field u^a such that $^*C_{abcd}u^bu^d = 0$ holds, then

A. the four eigen null directions of the conform tensor lie in a hyperplane S,

B. S is spanned by u^a and two eigenvectors of E_{ac} which belong to the eigenvalues with largest absolute value,

C. the four eigen null directions are given by (we have chosen $\underset{1}{e}^a$ to be the eigenvector orthogonal to S)

$$k_{1a} = \cos\Omega\, e_{2a} + \sin\Omega\, e_{3a} + u_a ,$$
$$k_{2a} = \cos\Omega\, e_{2a} - \sin\Omega\, e_{3a} + u_a ,$$
$$k_{3a} = -\cos\Omega\, e_{2a} + \sin\Omega\, e_{3a} + u_a ,$$
$$k_{4a} = -\cos\Omega\, e_{2a} - \sin\Omega\, e_{3a} + u_a ,$$

where $\cos\Omega = \left(\frac{\lambda_1 - \lambda_2}{\lambda_3 - \lambda_2}\right)^{1/2}$, $\sin\Omega = \left(\frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\right)^{1/2}$,

D. the conform tensor is degenerate if and only if $s^a = h^a_c k^c$ for one of the k^a is an eigendirection of E_{ac}. In the case of degeneracy, formally described by $\cos\Omega = 0$, s^a is orthogonal to the degenerate eigenspace of E_{ac}.

Now we turn to the Bianchi identities. Since we assume $H_{ac} = 0$ we get from (9)..(12) immediately

$$\perp\dot{E}_{ab} = -\Theta E_{ab} + E^c_{(a}\omega_{b)c} + E^c_{(a}\sigma_{b)c} + \eta_{arsm}\eta_{bpqn}u^m u^n \sigma^{rp}E^{sq} , \tag{18}$$

$$E^{c;d}_{(a}\eta_{b)cdm}u^m - 2E^d_{(a}\eta_{b)cdm}\dot{u}^c u^m = 0, \tag{19}$$

$$E^b_{a;c}h^c_b = 0, \tag{20}$$

$$3E^c_a\omega_c - \eta_{abcd}u^d\sigma^{bm}E^c_m = 0. \tag{21}$$

Among these equations the last one deserves special attention because it does not contain derivatives of E_{ab}. From (21) we can draw some immediate conclusions. First we mention that no eigenvalue of E_{ab} may be zero. Otherwise the conform tensor would vanish. Therefore the determinant of E_{ab}

(considered as a 3×3 matrix) is always unequal zero. Then we can solve (21) for ω^a and eliminate ω^a from the other equations. This may help in the further investigation of our class of fields.

Also from (21) we conclude that u^a must have non-vanishing shear if we want to obtain more general fields than static ones. Vanishing shear namely implies vanishing rotation as seen from the above remarks.

To analyze equations (18)..(21) we use Ricci coefficients $\underset{\mu\nu\rho}{\gamma} \equiv \underset{\mu}{e}_{a;c}\underset{\nu}{e}^a\underset{\rho}{e}^c$ and derivatives in the non-holonomic system $\underset{\rho}{e}_a$, i.e. we write $S_{\cdot\mu} \equiv S_{,a}\underset{\mu}{e}^a$ for any scalar S and we denote triad components of a tensor σ_{ab}, say, by $\underset{\mu\nu}{\sigma} \equiv \sigma_{ab}\underset{\mu}{e}^a\underset{\nu}{e}^b$.

Then we obtain

$$\dot{\underset{\mu}{\lambda}} + \theta\underset{\mu}{\lambda} - 2\underset{\mu\mu}{\sigma}\underset{\mu}{\lambda} + \underset{\nu\nu}{\sigma}\underset{\nu}{\lambda} + \underset{\rho\rho}{\sigma}\underset{\rho}{\lambda} = 0, \tag{22}$$

$$(\underset{\mu}{\lambda} - \underset{\nu}{\lambda})\dot{\underset{\mu}{e}}_c\underset{\nu}{e}^c + \tfrac{1}{2}(\underset{\mu}{\lambda} - \underset{\nu}{\lambda})\underset{\rho}{\omega} + \tfrac{3}{2}\underset{\rho}{\lambda}\underset{\mu\nu}{\sigma} = 0, \tag{23}$$

$$3\underset{\mu}{\lambda}\underset{\mu}{\omega} + (\underset{\nu}{\lambda} - \underset{\rho}{\lambda})\underset{\nu\rho}{\sigma} = 0, \tag{24}$$

$\mu, \nu, \rho = 1, 2, 3$ and cyclic;

$$(\underset{3}{\lambda} - \underset{1}{\lambda})\underset{312}{\gamma} = (\underset{2}{\lambda} - \underset{3}{\lambda})\underset{231}{\gamma} = (\underset{1}{\lambda} - \underset{2}{\lambda})\underset{123}{\gamma}, \tag{25}$$

$$\underset{\mu}{\lambda}_{\cdot\nu} + (\underset{\mu}{\lambda} - \underset{\nu}{\lambda})\underset{\nu\mu\mu}{\gamma} + (\underset{\mu}{\lambda} - \underset{\rho}{\lambda})\dot{u}_\nu = 0, \tag{26}$$

μ, ν, ρ unequal.

(25) and (26) are exactly the same equations as the Bianchi identities for static fields in the form given by Levi-Civita.

§4. Vacuum gravitational fields of the magnetic type.

In this section we consider vacuum fields which admit a timelike vector field u^a such that

$$C_{abcd}u^b u^d = 0. \qquad (27)$$

We do not know whether such fields exist. Among the known solutions of Einstein's equations there is no field which satisfies (27). Therefore this section is intended to raise interest in the problem of existence and to facilitate the work towards its solution.

We will show how the investigation of this class of fields can be reduced to the previous section. Since we have $E_{ac} = 0$ we can express $^*C_{abcd}$ by H_{ac} according to the formula

$$^*C_{abcd} = -(g_{abpq}g_{cdrs} - \eta_{abpq}\eta_{cdrs})u^p u^r H^{qs}. \qquad (28)$$

We observe that $^*C_{abcd}$ is built of H^{qs} in the same way as C_{abcd} was built of E^{qs} in the case treated in section 3. We assert that $^*C_{abcd}$ and C_{abcd} determine the same eigen null directions. The easiest way to see this fact is to pass to the spinor representation of $(C + i^*C)_{abcd}$. Let the symmetric spinor Γ_{ABCD} be the spinor equivalent of this tensor, then the spinor $\varkappa^A$ which corresponds to an eigen null direction k^a obeys $\Gamma_{ABCD}\varkappa^A\varkappa^B\varkappa^C\varkappa^D = 0$. Since this equation is unchanged when C_{abcd} is replaced by $^*C_{abcd}$ we see that k^a is as well an eigen direction of $^*C_{abcd}$ as of C_{abcd}. From this fact and from (28) we conclude that k_a is related to the eigenvalues and eigenvectors of H_{ab} in the same way as k^a was related to those of E_{ab}

in section 3. Therefore we can transpose the theorem of the previous section to our case.

Theorem: If in a vacuum gravitational field there exists a timelike vector field u^a such that $C_{abcd}u^b u^d = 0$ holds, then

A. the four eigen null directions of the conform tensor lie in a hyperplane T,

B. T is spanned by u^a and two eigenvectors of H_{ac} which belong to the eigenvalues with largest absolute values,

C. the four eigen null directions are given by the formulae in section 3 ($\underset{1}{e}^a$ is the eigenvector which is orthogonal to T)

D. the conform tensor is degenerate if and only if $s^a = h^a_c k^c$ for one of the k^a is an eigendirection of H_{ac}. In the case of degeneracy, formally described by $\cos \Omega = 0$, s^a is orthogonal to the degenerate eigenspace of H_{ac}.

Now we turn to the Bianchi identities which simplify to

$$\perp \dot{H}_{ab} = -\theta H_{ab} + H^c_{(a}\omega_{b)c} + H^c_{(a}\sigma_{b)c} + \eta_{arsm}\eta_{bpqn}u^m u^n \sigma^{rp} H^{sq}, \tag{29}$$

$$H^{c;d}_{(a}\eta_{b)cdm}u^m - 2H^d_{(a}\eta_{b)cdm}\dot{u}^c u^m = 0, \tag{30}$$

$$H^b_{a;c}h^c_b = 0, \tag{31}$$

$$3H^c_a\omega_c - \eta_{abcd}u^d\sigma^{bm}H^c_m = 0. \tag{32}$$

These equations are the same ones as those for E_{ac} in section 3. If we write them in the non-holonomic system of the eigenvectors of H_{ab} we get the set of formulae (22)..(26). We only have to consider $\underset{\rho}{e}^{a}$ and $\underset{\rho}{\lambda}$ as eigenelements of H_{ab}.

If there exists a vacuum solution of the magnetic type, then u^a has necessarily a complicated kinematical behaviour. The vector field u^a may not have vanishing shear and rotation since this property would imply $H_{ab} = 0$. Because of (32) not even the shear alone may vanish, for this would imply that the rotation would vanish too.

Chapter IV.

Temperature equilibrium in stationary spacetimes.

§ 1. Introduction. The physical definition "temperature" is generalized from classical thermodynamics to general relativistic thermodynamics either phenomenologically by keeping the classical (first order) laws in local inertial systems, or statistically by applying the special relativistic laws to the collision of swarms of particles in(first order) small spacetime regions. That is, T is the integrating denominator of the heat quantity differential, or the (inverse) parameter occurring in an equilibrium distribution function; compare (IV). It could then be shown by TOLMAN for a static photon gas, and later on by EHLERS (IV) for an ideal gas, that thermodynamical equilibrium implies spacetime to be stationary, and temperature to vary according to ξT = const where ξ is the norm of the KILLING vector which describes stationarity. The purpose of the present article is to derive this important law from a circular process proposed by R. EBERT which generalizes CARNOT's process to spatially extended domains. This generalized CARNOT process will be highly idealized: one varies the volume of an ideal gas in a laboratory I, the gas acting on a heat machine, then packs it into a perfectly insulating box, shoots it to some different ~~geodesic~~ laboratory II by means of a spaceship which travels geodesically for almost all of its time (acceleration being limited to two spacetime events), unpacks, operates as above, packs again, shoots back to laboratory I, and unpacks again; the circular condition being that initial and final

(IV) Ehlers, J,; Ber. Akad.Wiss. Mainz, Abh. math.-nat. Kl. 1961, No 11.

volume of the ideal gas agree. Of course, such a process has little to do with the real happenings in the convulsive interior, or atmosphere of a hot star (to which one would like to apply the result) where radiative phenomena are dominant. Yet we believe that a refinement of the present considerations could include more realistic processes; in any case, they will provide a better (operational) understanding of the equilibrium condition.

§ 2. Generalized CARNOT process. Such a process is based on two observatories I, II, and two mediating geodesic rockets whose worldlines intersect those of the two observatories in events 2,3,4,5 successively, the pertinent segments being called B,D respectively (see Fig.1). A box of ideal gas undergoes the first halfth of a CARNOT process along I from event 1 to 2

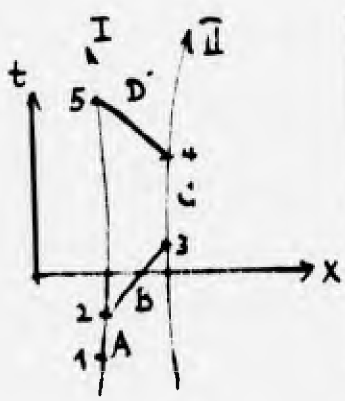

Figure 1

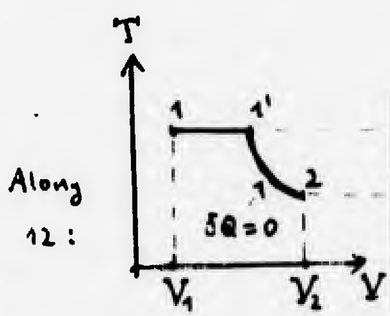

Figure 2

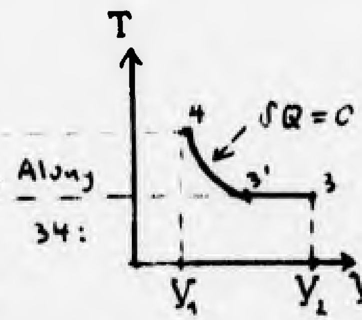

Figure 3

(see Fig. 2) . The second halfth takes place along $3 \; 4$ (see Fig.3) . The acceleration work at 4 is chosen such that the total amount of (mechanical) work performed by II vanishes: $\int_{II} \delta A \overset{!}{=} 0$. The circular condition of the process reads $E_1 \overset{!}{=} E_5 =: E$, where E_j is the energy of the gas (described by its temperature T_j and volume V_j) at event j. Let $Q_I := \int_1^2 \delta Q$ the heat quantity pumped into the gas by laboratory I , $A := \int_I \delta A$ the amount of work performed on the gas by I ; that is our signs are chosen such that the first fundamental theorem of thermodynamics reads $dE = \delta Q + \delta A$. We can now define the degree of efficiency η of this CARNOT - EBERT process as

$$\eta := - A / Q_I \;:$$

A spacetime domain is called "in temperature equilibrium" if no work can be won from test processes of the described kind, that is if $\eta = 0$ throughout.

§ 3. Equilibrium condition. There is no hope for $\eta = 0$ if either the spacetime domain considered is not stationary, or if the observatories I, II are not stationary (i.e. if their world lines are not isometry group orbits): in a non-stationary world, gravitational waves hitting the gas volume shortly before event 5 can change its energy such that $\eta = 0$ is destroyed if otherwise holding true; on the other hand, non-stationary observers (that is observers who see different parts of the world at different times) are in a similar position to quasistationary observers in varying gravitational fields. We avoid a rigorous proof for what is meant to be plausible by now: that temperature equilibrium implies spacetime and observatories to be stationary. And we proceed to the proof of

Theorem: A generalized CARNOT process based on stationary observatories satisfies $\eta = 0$ if and only if $\xi T = \text{const}$; ($\xi^2 := |\xi_a \xi^a|$, where ξ^a is the KILLING vector which describes stationarity).

§ 4. Proof of the Theorem. We begin with some geometric and thermodynamic results whose (simple) proofs are omitted.

1) The scalar product $x := |\xi^a u_a|$ of a KILLING vector ξ^a and the unit tangent vector u_a of a geodesic g is constant along g.

2) The norm of a KILLING vector is constant along its orbits.

3) The energy E of a system of 4-velocity v_a is $E = M\,|v^a u_a|$, where M is the rest energy of the system. Proof immediate in a rest system of v_a. Same assertment for A and Q.

4) For an ideal gas (I.G) and a CARNOT process described by figures 2,3 above we have: (n = number of moles, R = gas constant)

$$Q_I = Q_{11'} \overset{\text{I.G.}}{=} -\int_1^{1'} \delta A = \int_1^{1'} p\,dV \overset{\text{I.G.}}{=} n R T_1 \ln (V_{1'}/V_1) ,$$

$$V_1' / V_1 \overset{\text{I.G.}}{=} V_2 / V_2'$$, whence

(1) $$-Q_I / Q_{II} = T_1 / T_2 \quad .$$

5) The proof of the theorem is now in sight. We first write down the acceleration condition at 4 , using 1), 2), 3), and the fact that $\xi^a = \xi v^a$ both along I and II : $(\Delta M := M_1 - M_2)$.

$$0 = \xi_{II} \int_E \delta A = (M - \Delta M)(x_C - x_B) + \xi_{II} A_{34} + M(x_D - x_C),$$

(2) $$0 = \xi_{II} A_{34} + M(x_D - x_B) + \Delta M(x_B - x_C) \; .$$

In order to calculate η we need:

$$\xi_I A_I = \xi_I A_{12} + (M - \Delta M)(x_B - x_A) + M(x_A - x_D)$$

$$= \xi_I A_{12} + M(x_B - x_D) + \Delta M(x_A - x_B) \; , \quad (2) \Rightarrow$$

$$= \xi_I A_{12} + \xi_{II} A_{34} + \Delta M \underbrace{(x_A - x_C)}_{\xi_I - \xi_{II}}$$

$$= \xi_I (A_{12} - E_{12}) + \xi_{II}(A_{34} - E_{34}) \; ,$$

the latter because of $\Delta M = - E_{12} = E_{34}$ as one operates in the rest system of the gas. That is we have:

$$(\xi A)_I = -(\xi Q)_I - (\xi Q)_{II} \; , \quad \text{or:}$$

$$\eta = 1 + (\xi Q)_{II} / (\xi Q)_I \; ,$$ and from (1) we get:

(3) $$\eta = 1 - (\xi T)_{II} / (\xi T)_I \; ,$$

which is the desired result.

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