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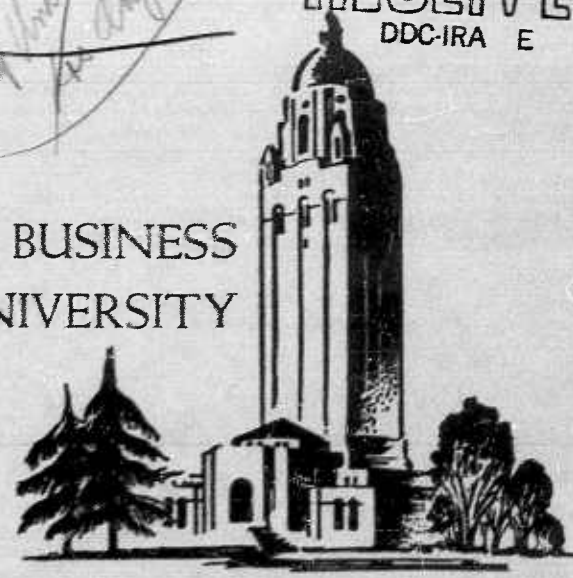
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A PARAMETRIC PROGRAMMING SOLUTION TO THE
VECTOR MAXIMUM PROBLEM, WITH APPLICATIONS
TO DECISIONS UNDER UNCERTAINTY,

(10) By

Arthur M. Geoffrion .

(11) Feb 1965,

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ABSTRACT

This work begins with a study of individual decision-making under uncertainty, a problem which we formulate as

$$(1) \qquad \text{Maximize}_{\underline{x}} \ f(\underline{x},\underline{\beta}) \text{ subject to } g_i(\underline{x},\underline{\beta}) \geq 0, \quad i = 1, \ldots , m ,$$

where $\underline{x}$ is a decision n-vector, $\underline{\beta}$ is a b-vector of exogenous variables and parameters of the decision model, f is an objective function to be maximized, and the g_i are constraint functions which determine the set of feasible decisions. The source of uncertainty is $\underline{\beta}$, which is known only to lie in a given set B. We also consider the case in which a probability distribution over B is given.

Several methods for circumventing uncertainty in the constraints are briefly reviewed, and several decision criteria for circumventing uncertainty in the objective function are discussed. Particular attention is devoted to the demonstration of certain relationships between these criteria. It is concluded that vector maximum reformulations of the problem (1) play a prominent role in dealing with uncertainty in such decision problems.

A vector maximum problem is of the form

$$(2) \qquad \text{"Maximize"}_{\underline{x}} \ f_1(\underline{x}), \ldots , f_r(\underline{x})$$

$$\text{subject to } g_i(\underline{x}) \geq 0, \quad i = 1, \ldots , m .$$

The quotation marks signify that it is desired to find <u>all</u> <u>efficient</u>

<u>decisions</u>, i.e., all decision vectors satisfying the constraints such that it is impossible to achieve an increase in any one objective function without violating the constraints or decreasing at least one of the other objective functions. In Chapter II we discuss two methods for transforming a vector maximum problem into an equivalent parametric programming problem. Existing computational methods for the latter problems are briefly surveyed.

The principal contribution of this work is presented in Chapter III: a class of algorithms for solving parametric concave programming problems of the form

(3)
$$\text{Maximize}_{\underline{x}} \quad \alpha f_1(\underline{x}) + (1-\alpha)f_2(\underline{x})$$

$$\text{subject to } g_i(\underline{x}) \geq 0, \quad i = 1, \ldots, m$$

for each fixed value of α in the closed interval $[0,1]$, where f_i $(i = 1,2)$ are strictly concave functions, g_i $(i = 1,\ldots,m)$ are concave functions, and certain additional regularity assumptions are made. Under these assumptions it is shown that (2) (with $r = 2$) and (3) are equivalent in the sense that $\underline{x}^o$ is efficient in (2) if and only if $\underline{x}^o$ solves (3) for some value of α in the unit interval. The present class of algorithms is not "simplex-like" or "gradient" in nature, but proceeds by maintaining a solution of the Kuhn-Tucker Conditions as α varies by small increments (under our assumptions these conditions are necessary and sufficient for an optimal solution of (3)). The main algorithm given herein displays quadratic convergence at each increment of α. A simple modification for handling linear equality constraints is indicated.

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This problem (3) also subsumes the standard (non-parametric) concave programming problem when a feasible solution is known. Thus the present algorithms provide a deformation method of concave programming. Since many of the results of this chapter hold for much more general parametric problems than (3), moreover, the present algorithms are pertinent to sensitivity analysis applications.

The final chapter presents a numerical example which illustrates the solution of a decision problem under uncertainty by means of the techniques discussed in the preceding chapters.

TABLE OF CONTENTS

Notation

$\underline{x} \equiv (x_1, \ldots, x_n)$ is a decision vector in E^n (n-dimensional Euclidean space), and is under the control of the decision-maker

$\underline{\beta} \equiv (\beta_1, \ldots, \beta_b)$ is an uncertain vector in E^b representing exogenous variables and model parameters, and is *not* under the control of the decision-maker

$f(\underline{x}, \underline{\beta})$ is a real-valued criterion function which is to be maximized; if there is no dependence on $\underline{\beta}$, we write $f(\underline{x})$; if there are several criterion functions, we write $\underline{f}(\underline{x})$ for $(f_1(\underline{x}), \ldots, f_r(\underline{x}))$

$\underline{g}(\underline{x}, \underline{\beta}) \equiv (g_1(\underline{x}, \underline{\beta}), \ldots, g_m(\underline{x}, \underline{\beta}))$ is a real vector-valued constraint function; if there is no dependence on $\underline{\beta}$, we write $\underline{g}(\underline{x})$

$\{z \in Z: z$ has property $P\}$ denotes the set of all elements z in the set Z which have property P; when Z is omitted, it is implicitly understood to be the pertinent universal set

X is a subset of E^n consisting of the feasible decisions; often X represents $\{\underline{x}: \underline{g}(\underline{x}) \geq \underline{0}\}$

B (in Chapter I) is a subset of E^b which is known to contain the "true realization" of $\underline{\beta}$

$\underline{x} \geq (>) \underline{0}$ signifies $x_i \geq (>) 0$ $(i = 1, \ldots, n)$

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$\underline{x} \geq \underline{0}$ signifies $\underline{x} \geqq \underline{0}$ but $\underline{x} \neq \underline{0}$

μ denotes a probability distribution over B

$C \subseteq (\subset) D$ signifies that the set C is a (proper) subset of D

$$N_r(\underline{x}^0) \triangleq \left\{ \underline{x} \colon \left[\sum_{i=1}^{n} (x_i - x_i^0)^2 \right]^{1/2} < r \right\}, \quad \text{an open neighborhood of } \underline{x}^0$$

of radius r

$\bar{F}(\alpha)$ denotes the maximum α-fractile criterion (see problem (4.5) of Chapter I)

$A(M)$ denotes the aspiration criterion with aspiration level M (see problem (4.6) of Chapter I)

$[a,b) \triangleq \{t \in E^1 \colon a \leq t < b\}$

$(P\alpha)$ denotes the parametric programming problem considered in Chapter III; the parameter α may vary in this notation (there is no relation between this usage of α and that of Chapter I)

$f(\underline{x};\alpha) \triangleq \alpha f_1(\underline{x}) + (1-\alpha) f_2(\underline{x})$

$$\nabla_{\underline{x}} f(\underline{x}) \triangleq \left(\frac{\partial f(\underline{x})}{\partial x_1}, \ldots, \frac{\partial f(\underline{x})}{\partial x_n} \right), \quad \text{the gradient of } f(\underline{x})$$

S denotes a subset of constraint indices; $S \subseteq M$, where M is the set of the first m positive integers

$\underline{u} = (u_1, \ldots, u_m)$ denotes the dual variables associated with the Kuhn-Tucker conditions

(KT-1),...,(KT-4) are, collectively, one version of the Kuhn-
Tucker conditions associated with (Pα)

(=S)α is a more complete notation for the equations (KT-1) and
(KT-2); S and α may vary in this notation

($\underline{x}$*(α), $\underline{u}$*(α)) is the optimal solution and dual variables of (Pα)
as functions of α

($\underline{x}^S(\alpha)$, $\underline{u}^S(\alpha)$) is a solution of (=S)α as a function of α

$\nabla_{\underline{x}}^2 f(\underline{x})$ denotes the matrix of second partial derivatives (i.e., the
hessian) of $f(\underline{x})$

$< \underline{x}^\nu > \rightarrow \underline{x}^0$ means that the (infinite) sequence $\underline{x}^1, \underline{x}^2, ..., \underline{x}^\nu, ...$
converges to $\underline{x}^0$

C-D denotes the points in the set C which are not in the set D

A$\alpha \overset{\triangle}{=}$ {i ϵ M: $u_i^*(\alpha) > 0$}, the set of active constraints at α; α
may vary in this notation

B$\alpha \overset{\triangle}{=}$ {i ϵ M: $g_i(\underline{x}^*(\alpha)) = 0$}, the set of binding constraints at
$\underline{x}^*(\alpha)$; α may vary in this notation

α_j'(j = 1,...,N) are the points of change of Aα or of Bα in the
unit interval; α' is a generic term for a point of change

α'+ is an arbitrary point strictly between two points of change

$\overline{I}\alpha' \overset{\triangle}{=} [\alpha' - \overline{l}, \alpha' + \overline{l}]$, where $\overline{l}$ is defined immediately above Theorem 4.2,
Chapter III

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CHAPTER I

On the Relevance of the Vector Maximum
Problem to Decision-Making Under Uncertainty

1. Introduction

This chapter addresses a problem of individual decision-making under uncertainty of the form

$$(1) \qquad \text{Maximize} \quad f(\underline{x},\underline{\beta}) \quad \text{subject to} \quad \underline{g}(\underline{x},\underline{\beta}) \geq \underline{0} \,,$$
$$\underline{x}$$

where $\underline{x} = (x_1,\ldots,x_n)$ is the decision vector, $\underline{\beta} = (\beta_1,\ldots,\beta_b)$ is a vector of exogenous variables and parameters of the model, f is the objective (or criterion or payoff) function to be maximized, and $\underline{g} = (g_1,\ldots,g_m)$ is a vector-valued constraint function which determines the set of feasible decisions. We assume that the functions f and $\underline{g}$ are known, but that $\underline{\beta}$ is known only to lie in a given set $B \subseteq E^b$, where E^b is b-dimensional Euclidean space. Often we shall make the additional assumption that $\underline{\beta}$ may be regarded as a random variable with a known probability distribution over B. A choice of $\underline{x}$ must be made before $\underline{\beta}$ is found out, if, indeed, it ever is revealed to the decision-maker. Throughout this chapter, no experimentation is permitted in order to reduce uncertainty about $\underline{\beta}$.

If $\underline{\beta}$ were known exactly, then (1) would be a well-defined problem (providing that the desired maximum exists, of course). But we have assumed that $\underline{\beta}$ is uncertain, and so (1) is not well-defined.

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There are two distinct aspects of the difficulties arising from uncertainty in $\underline{\beta}$: the set of feasible decisions is uncertain, and the objective function is uncertain. Maximization cannot be performed until the constraints and objective function are reformulated so as to be independent of $\underline{\beta}$. We shall discuss a variety of such reformulations, and it will be seen that quite frequently vector maximum reformulations play a prominent role.

The Vector Maximum Problem

A vector maximum problem arises whenever there is more than one objective function to be extremized. Consider the problem

$$(2) \qquad \text{"Maximize"} \;\; \underline{f}(\underline{x}) \; , \atop \underline{x} \in X$$

where $\underline{f}(\underline{x}) = (f_1(\underline{x}),\ldots,f_r(\underline{x}))$ is a vector-valued objective function (each component of $\underline{f}$ represents an objective, usually non-additive with the others, which the decision-maker wants to maximize), and $X \subseteq E^n$ is a set of feasible decisions. In the fortunate event that each component of the objective function reaches its maximum simultaneously, as in Figure 1, then (2) is said to have a perfect solution. In general, however, an improvement of one objective beyond a certain point can only be obtained at the expense of worsening another. Suppose that for a feasible decision $\underline{x}^o$ there exists no other feasible decision $\underline{x}^1$ such that[1] $\underline{f}(\underline{x}^1) \geq \underline{f}(\underline{x}^o)$. Then $\underline{x}^o$ is termed an

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[1] In this work we adopt the convention that $\underline{x} \geqq \underline{0}$ signifies $x_i \geq 0$ ($i = 1,\ldots,n$), $\underline{x} \geq \underline{0}$ signifies $x_i \geq 0$ ($i = 1,\ldots,n$) and $x_i > 0$ for at least one i, and $\underline{x} > \underline{0}$ signifies $x_i > 0$ ($i = 1,\ldots,n$).

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efficient solution[2/] of (2). The quotation marks in (2) signify
that it is desired to find all efficient solutions. When they are
all found, the vector maximum problem (2) has been solved.

When $\underline{f}$ has only two or three components, we envision determining
the entire set of efficient solutions and presenting the corresponding
outcomes in graphical form to the decision-maker, who would then
subjectively determine a trade-off between conflicting objectives
and thus make the final selection of a decision. Figures 1 and 2
illustrate the graph of attainable outcomes for two hypothetical cases
involving two objective functions. The efficient outcomes are denoted
by the heavy line and dot.
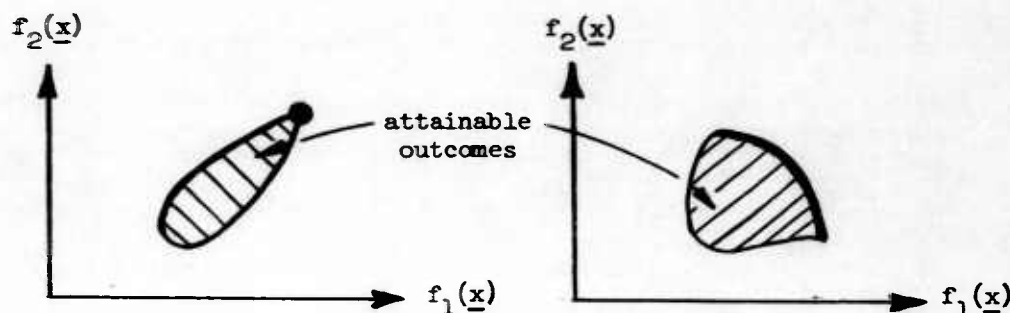
.Figure 1   Figure 2

In many applied decision problems, even in the absence of uncer-
tainty, there are several objective functions which naturally present
themselves to the decision-maker. In such situations, the relevance

[2/] The notion of an efficient solution is essentially the same as
the notion of "undominated" or "admissible" decisions in decision
theory, and the notion of "Pareto optimality" in game theory (see
Luce and Raiffa, 1957, p. 287 and p. 118).

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of the vector maximum problem is obvious, and need not be emphasized further. What we do wish to emphasize is that in the presence of uncertainty even a single-criterion-function problem such as (1), which we would accept as the "correct" formulation if β were known exactly, tends to explode into vector maximum reformulations when one attempts to turn it into a well-defined problem.

Plan of Discussion

Because uncertainty in the constraints is fundamentally different from uncertainty in the objective function of (1), we split our discussion into two parts: in section 2 we consider ways of reformulating the constraints so as to be independent of β, and in section 3 we consider ways of reformulating the objective function so as to be independent of β (this is usually known as invoking a decision criterion). These two steps must be accomplished in order to convert (1) into a well-defined problem. The conversion usually can be accomplished in several ways, reflecting various compromises which may be made to uncertainty in β, realism in the final model, and computational considerations.

In section 2, three reformulations of the constraints will be discussed: permanent feasibility, the penalty function reformulation, and probabilistic constraints. The first two do not require a probability distribution over B, while the last does. The last two reformulations sometimes lead to a vector maximum problem.

In section 3 we consider several decision criteria, and some relations between them are noted. We suggest that a given decision

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problem should be attacked by several decision criteria rather than by only one. The result is, of course, a vector maximum problem. Two examples are presented which demonstrate the usefulness of considering two criteria simultaneously. The second example is a one-period inventory model, and an argument is given for deviating from the now classical solution.

2. Treating Uncertainty in the Feasibility Constraints

This section is essentially a review of some of the existing ways of circumventing uncertainty in the constraints, and is included mainly for completeness. Mixtures and variations of these basic approaches can be improvised to cover most particular applications.

The Permanent Feasibility Reformulation

To be absolutely sure of choosing a feasible decision, choice must be limited to those values of $\underline{x}$ which are feasible for all $\underline{\beta} \in B$. That is, restrict attention to the set[3] $\bigcap_{\underline{\beta} \in B} \{\underline{x}: \ \underline{g}(\underline{x},\underline{\beta}) \geq \underline{0}\}$ (see Madansky, 1962 and 1963).

An obvious difficulty with this reformulation is that when B is "large," the permanently feasible set is apt to be "small," and even may be empty. When the maximization operation is performed subsequently, there may be little opportunity to achieve a satisfactorily high value of the objective function.

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[3] We adopt the notation of using braces to denote sets in this work. The symbol $\emptyset$ denotes the empty set.

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The Penalty Function Reformulation

The above reformulation does not admit the possibility of ever choosing a decision which is infeasible. What does it mean to say that a decision $\underline{x}'$ is "infeasible" when, say, $\underline{\beta}'$ obtains? Mathematically, we have $\underline{g}(\underline{x}',\underline{\beta}') \ngeq \underline{0}$, which means that either $(\underline{x}',\underline{\beta}')$ is physically impossible, or is physically possible but "undesirable" (we are distinguishing between those constraints which are dictated by the physical limitations of the system and those which are imposed at the model-maker's discretion). In the second case, it may be possible to take additional action in order to make the outcome less "undesirable," or at least to pay a price for being "infeasible." Denote this "price" by $\underline{p}(\underline{x}',\underline{\beta}')$, not necessarily measured in dollars. Note that $\underline{p}$ is, in general, a vector-valued function, reflecting the fact that violations of different constraints may imply different dimensions of disutility. For example, consider an investment portfolio optimization model which has as its objective the maximization of portfolio worth at the end of a specified horizon. One constraint may specify a desired level of diversification (e.g., a maximum of 30% of the portfolio in defense industries), and another constraint may specify a lower bound on the average Standard and Poor's quality rating of the securities. Violation of each of these constraints would be measured in different units from the unit of measurement of the objective function.

The penalty function reformulation of (1) results, in general, in a vector maximum problem of the form

$$(3) \qquad \text{"Maximize"} \; f(\underline{x},\underline{\beta}), \; -\underline{p}(\underline{x},\underline{\beta}) .$$
$$\underline{x}$$

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An important special case arises when p has but one component, and this component is additive with f. This reformulation then becomes[4]

(3.1) $$\text{Maximize}_{\underline{x}} \; [f(\underline{x},\underline{\beta}) - p(\underline{x},\underline{\beta})] \; .$$

All of the two-stage "stochastic programming" problems (see, e.g., Dantzig, 1955, Madansky, 1962, and Mangasarian and Rosen, 1964) can be thought of as penalty function reformulations. The basic idea of these problems is to append a second stage to the original problem to "correct for" possible infeasibility of the original decision; p then represents the minimum cost of correcting for an infeasible $\underline{x}$, as affected by the then known actual value of $\underline{\beta}$. The usual example of a situation in which the two-period formulation may be appropriate is the case of a manufacturer who is committed to produce to satisfy an unknown demand $\underline{\beta}$ for his perishable products. If all of the demand is not satisfied, then he purchases the difference on the open market.

Probabilistic Constraints

Assume that $\underline{\beta}$ may be regarded as a random variable, and that its probability distribution over B is known.

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[4] Note that (1) can be written equivalently in this form if p is taken to be arbitrarily large for infeasible combinations of $\underline{x}$ and $\underline{\beta}$, and equal to zero for feasible combinations. For example,

$$\text{Maximize}_{\underline{x}} \; [\; \underset{\underline{u} \geq \underline{0}}{\text{Inf}} \; [f(\underline{x}) + \sum_i u_i g_i(\underline{x},\underline{\beta})]] \; .$$

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The notion of permanent feasibility may be relaxed if one requires merely that each or all of the constraints must hold with at least some prescribed probability. For example, consider

$$\text{Maximize}_{\underline{x}} \quad f(\underline{x}, \underline{\beta})$$

$$\text{subject to} \quad \text{Prob}[g_i(\underline{x}, \underline{\beta}) \geq 0] \geq a_i , \qquad i = 1, \ldots , m ,$$

where $0 \leq a_i \leq 1$ ($i = 1, \ldots, m$). Charnes and Cooper (1959, 1963) refer to this as "chance-constrained" programming. Note that when each a_i is nearly one, this reformulation approaches the permanent feasibility reformulation.

Another probabilistic constraint reformulation is

$$\text{Maximize}_{\underline{x}} \quad f(\underline{x}, \underline{\beta})$$

$$\text{subject to} \quad E[\underline{g}(\underline{x}, \underline{\beta})] \geq \underline{0} ,$$

where "E" denotes expectation.

As an alternative to the formulations above, one may incorporate some or all of the probabilistic constraints in the objective function, e.g.,

$$\text{"Maximize"}_{\underline{x}} \quad f(\underline{x}, \underline{\beta}) , \quad \text{Prob}[g_1(\underline{x}, \underline{\beta}) \geq 0]$$

$$\text{subject to} \quad \text{Prob}[g_i(\underline{x}, \underline{\beta}) \geq 0] \geq a_i , \qquad i = 2, 3, \ldots , m .$$

The efficient solutions to the resulting vector maximum problem show clearly the available trade-offs between the original objective function and assurance that various of the constraints will be met.

3. Treating Uncertainty in the Objective Function

In section 2 we discussed several ways of reformulating the constraints so as to be independent of $\underline{\beta}$. Here we assume that this has been accomplished, and discuss several ways of reformulating the objective functions so as to be independent of $\underline{\beta}$. For the sake of simplicity of discussion, we shall treat the case of but a single objective function, so that the problem to be considered in this section can be rewritten as

$$(4) \qquad \begin{array}{c} \text{Maximize} \\ \underline{x} \in X \end{array} \quad f(\underline{x},\underline{\beta}) \ .$$

As before, $\underline{\beta}$ is known to lie in a given set B, and X is the set of feasible decisions.

Since it is necessary to choose a decision $\underline{x}$ before $\underline{\beta}$ is revealed (if it is ever revealed), $f(\underline{x},\underline{\beta})$ must be replaced by a known function of $\underline{x}$ alone. That is, (4) must be reformulated as

$$(4.0) \qquad \begin{array}{c} \text{Maximize} \\ \underline{x} \in X \end{array} \quad \widetilde{f}(\underline{x})$$

where $\widetilde{f}$ is a known function to be chosen. The choice of $\widetilde{f}$ in a given situation is equivalent to what is customarily known as the choice of a decision criterion. If a decision is an optimal solution of (4.0), it is said to satisfy the decision criterion which produces $\widetilde{f}(\underline{x})$ from $f(\underline{x},\underline{\beta})$.

After first discussing two alternative restatements of (4), we shall briefly summarize the admissibility criterion, the maxmin payoff criterion, the estimate criterion, and the Principle of Insufficient

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Reason. The difficulty of finding a single ideal decision criterion is well-known, and so we take the position that it may be more useful to select two criteria, each with distinct merits of its own, and recast (4) as a vector maximum problem (each component of the vector-valued objective function is derived from one decision criterion). An example is presented to illustrate the possible advantages of such a procedure.

We then shall assume that a probability distribution over B is given. The concept of stochastic admissibility is introduced as a generalization of the ordinary concept of admissibility. Next we examine three decision criteria for reducing (4) to a well-defined problem with heavy emphasis on a geometric motivation for each in order to gain insight and understanding. These are the maximum expected payoff criterion, the maximum α-fractile criterion (maximize the α-fractile of the distribution of $f(\underline{x},\underline{\beta})$ under the probability distribution of $\underline{\beta}$, for some preselected α), and an aspiration criterion (maximize the probability of achieving at least some pre-scribed level of payoff). Several propositions are proved which relate these criteria to each other and to the previously mentioned criteria which do not involve probabilities. Finally, a one-period inventory example is presented to illustrate the ideas of this section and to support the suggestion that several criteria, rather than a single one, should be selected to embody the conflicting aims of the decision-maker. The resulting vector maximum problem should then be solved in place of (4).

Alternative Problem Statements

In some situations the objective function of (4) can be written as $f(\underline{x},\underline{\beta}) = F_1(\underline{x}) + F_2(\underline{x},\underline{\beta})$. If F_1 and F_2 each represent a quantity which the decision-maker wants to maximize, one may reformulate (4) as a two-component vector maximum problem

$$\text{"Maximize"}_{\underline{x} \in X} \quad F_1(\underline{x}), \quad F_2(\underline{x},\underline{\beta}) \ ,$$

so as to quarantine the part depending on $\underline{\beta}$. The advantage of this formulation is that the decision-maker gains a clearer understanding of how his objectives are influenced by uncertainty. As an example, let F_1 represent the immediate payoff of a multistage decision problem, and let F_2 represent the present worth of the future payoffs, where $\underline{\beta}$ represents the future values of exogenous variables.

Another restatement of (4) is obtained by using regret in place of payoff. Assume that $[\ \underset{\underline{x} \in X}{\text{Max}}\ f(\underline{x},\underline{\beta})\ ]$ is achieved for each $\underline{\beta} \in B$. The regret due to making decision $\underline{x}$ and then observing $\underline{\beta}$ is defined to be

$$r(\underline{x},\underline{\beta}) = [\ \underset{\underline{x} \in X}{\text{Max}}\ f(\underline{x},\underline{\beta})\ ] - f(\underline{x},\underline{\beta}) \ .$$

Stating problems in terms of regret rather than payoff has the advantage of highlighting the consequences of uncertainty in $\underline{\beta}$ dramatically. In addition, regret may have more tractable mathematical properties than payoff (assuming that the indicated maximization operation is not overly difficult), due to non-negativity and sometimes symmetry.

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When $\underline{\beta}$ is known exactly, maximizing payoff is, of course, exactly equivalent to minimizing regret. When $\underline{\beta}$ is uncertain, however, and various criteria are applied in order to arrive at a decision, it is well-known that different decisions often result depending on whether payoff or regret is used.

In this work the discussion will be carried on primarily in terms of payoff, but with the obvious modifications each criterion can be applied to regret as well.

3.1 Reformulations not Involving Probabilities

We shall briefly review a few classical decision criteria which do not involve probabilities. An example is given to illustrate that it can be more useful to consider several criteria simultaneously rather than to search for a single ideal criterion.

Admissibility Criterion

Consider (4). A decision $\underline{x}'$ is said to be admissible (with respect to X and B) if $\underline{x}' \in X$ and if there exists no other decision $\underline{x}'' \in X$ such that $f(\underline{x}'',\underline{\beta}) \geq f(\underline{x}',\underline{\beta})$ for all $\underline{\beta} \in B$, with strict inequality holding for some value of $\underline{\beta} \in B$. If such a decision $\underline{x}''$ did exist, it would be said to dominate $\underline{x}'$ (one may also define weak dominance by dropping the proviso that strict inequality must hold for some value of $\underline{\beta}$). The admissibility criterion requires that one choose an admissible decision. In other words, if $a(\underline{x})$ is defined to be equal to 0 if $\underline{x}$ is admissible and equal to -1 if $\underline{x}$ is inadmissible, (4) is reformulated as:

(4.1)
$$\text{Maximize} \quad a(\underline{x}) .$$
$$\underline{x} \in X$$

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The difficulties with this criterion are twofold: the set of admissible decisions may be onerous to determine computationally, and this set may be quite a large subset of X.

Maxmin Payoff Criterion

A conservative decision-maker might invoke the maxmin payoff criterion, which yields

$$(4.2) \qquad \underset{\underline{x} \, \in \, X}{\text{Maximize}} \quad [\underset{\underline{\beta} \, \in \, B}{\text{Inf}} \; f(\underline{x},\underline{\beta})] \, .$$

The corresponding criterion in terms of regret is known, of course, as the minmax regret criterion.

Estimate Criterion

The estimate criterion requires that one pick a value for $\underline{\beta}$, say $\hat{\underline{\beta}}$, and then act as though $\hat{\underline{\beta}}$ were the true value of $\underline{\beta}$.[5] That is, solve

$$(4.3) \qquad \underset{\underline{x} \, \in \, X}{\text{Maximize}} \quad f(\underline{x},\hat{\underline{\beta}}) \, .$$

Since $\hat{\underline{\beta}}$ may be chosen to be any point in B, we see that we really have a whole family of criteria.

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[5] This criterion is included in order to formalize the common practice of using judgmental or engineering approximations to costs and other parameters of decision models. The notion of an estimate is related to the idea of a certainty equivalent, which will be discussed at the end of subsection 3.2. It should be noted that this criterion may also be invoked when $\underline{\beta}$ is regarded as a random variable, and in fact, the expected value of $\underline{\beta}$ is a popular estimate.

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The computational advantages of this approach are obvious. It is not so obvious that there exists a "good" estimate in B, or how to find one.

The Principle of Insufficient Reason

Assume that B consists of a finite number (k) of elements, each denoted by $\underline{\beta}^i$. Then the Principle of Insufficient Reason asserts that one should replace (4) by

$$(4.4) \qquad \underset{\underline{x} \in X}{\text{Maximize}} \left[\frac{1}{k} \sum_{i=1}^{k} f(\underline{x}, \underline{\beta}^i) \right].$$

Comparison of Criteria

The above decision criteria are representative of the methods which have been proposed in an effort to circumvent uncertainty in the objective function in the absence of probabilities. The difficulties of selecting one criterion which satisfies all of a comprehensive set of intuitively appealing desiderata for "rational" decision-making are well-known (see, e.g., Luce and Raiffa, 1957, Chapter 13), and suggest the futility of seeking an ideal criterion. One possible way out of this dilemma is to consider several criteria at once, and thus to reformulate (4) as a vector maximum problem. The actual choice of a decision would be made on an ad hoc basis from the set of efficient solutions.

Table 1 defines a decision problem in which there are four possible values of $\underline{\beta}$, and five possible decisions. The entries give the values of $f(\underline{x}^i, \underline{\beta}^j)$ and the consequences of each possible decision in terms of average payoff (on which the Principle of

14
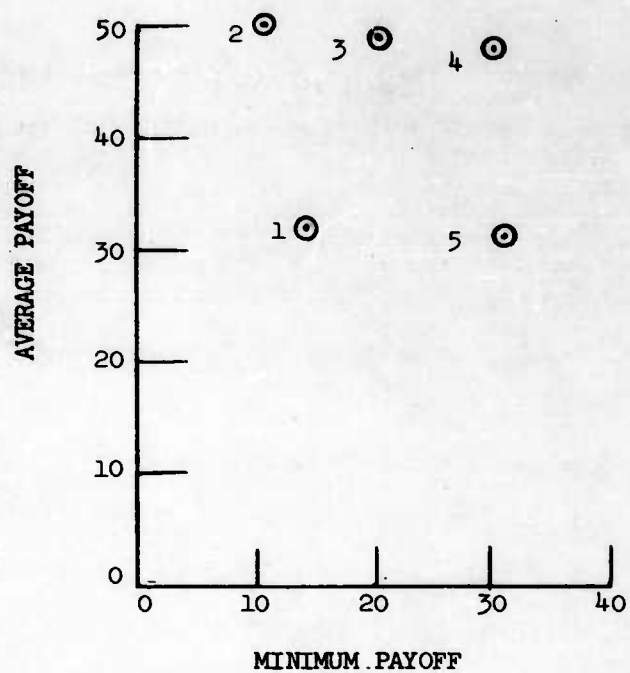
Figure 3

	Value of β				AVG PAYOFF	MIN PAYOFF
	β^1	β^2	β^3	β^4		
x^1	15	40	53	20	32	15
x^2	10	60	50	80	50	10
x^3	20	40	45	91	49	20
x^4	60	58	30	44	48	30
x^5	31	31	31	31	31	31

TABLE 1

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Insufficient Reason is based) and in terms of minimum payoff (on which the maxmin payoff criterion is based). Figure 3 graphs these consequences.

All decisions are admissible. The Principle of Insufficient Reason would lead to the choice of decision number two, while the maxmin payoff criterion leads to the fifth decision. However, it seems reasonable to favor the fourth decision over any of the others because it comes very close to satisfying both of the above criteria.

We submit that by judicious choice of two criteria the resulting vector maximum reformulation of (4) can be expected to lead to a more satisfactory decision than a single criterion.

3.2 Reformulations Involving Probabilities

With the additional assumption that β may be regarded as a random variable, one may choose to regard (4) as a continuous game in normal form. This viewpoint, and the consequent game-theoretic solutions, will not be considered here. Instead it will be assumed that β has a known probability distribution μ over B and so (4) may be regarded as a game against a neutral "Nature." That is, we are in what is sometimes known as a situation of individual decision-making under "risk."

The principal tenet of utility theory (an excellent summary is given in Luce and Raiffa, 1957, Chapter 2) is that for a "rational" decision-maker there exists a utility transformation of f, which we denote by $u(f)$, such that the most preferred decision is an

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optimal solution of:

$$\text{Maximize} \quad E[u(f(\underline{x},\underline{\beta}))] \; .$$
$$\underline{x} \in X$$

If one accepts any of the sets of axioms of rational behavior leading
to this result, then the maximum expected utility criterion is justified
provided that the required utility transformation is at hand.

Unfortunately it may be very tedious actually to determine $u(f)$.
For this reason (and also because of certain reservations which we
have with regard to the axioms of utility theory), we shall consider
other criteria which can be applied directly to $f(\underline{x},\underline{\beta})$ without the
need for a utility transformation. We begin by introducing a natural
analog of the admissibility criterion.

Stochastic Admissibility Criterion

For fixed $\underline{x}$, μ induces a probability distribution on f which
may be plotted in cumulative form as in Figure 4 (each curve represents
the cumulative distribution function of f corresponding to different
values of $\underline{x}$). Loosely speaking, one wishes to perform (4) by choosing
an $\underline{x}$ which determines a c.d.f. that is uniformly as low (or, equiva-
lently, as far to the right) as possible. In Figure 4 it is clear that
the c.d.f. determined by $\underline{x}_2$ must be strictly preferred to that of
$\underline{x}_1$, while $\underline{x}_2$ need not be preferred to $\underline{x}_3$. Observe that although
the probability density functions determined by $\underline{x}_1$ and $\underline{x}_2$ overlap,
the c.d.f.'s do not.

We formalize the above ideas in terms of the concept of stochastic
dominance. A decision $\underline{x}^o$ is said to stochastically dominate $\underline{x}'$

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Figure 4

18
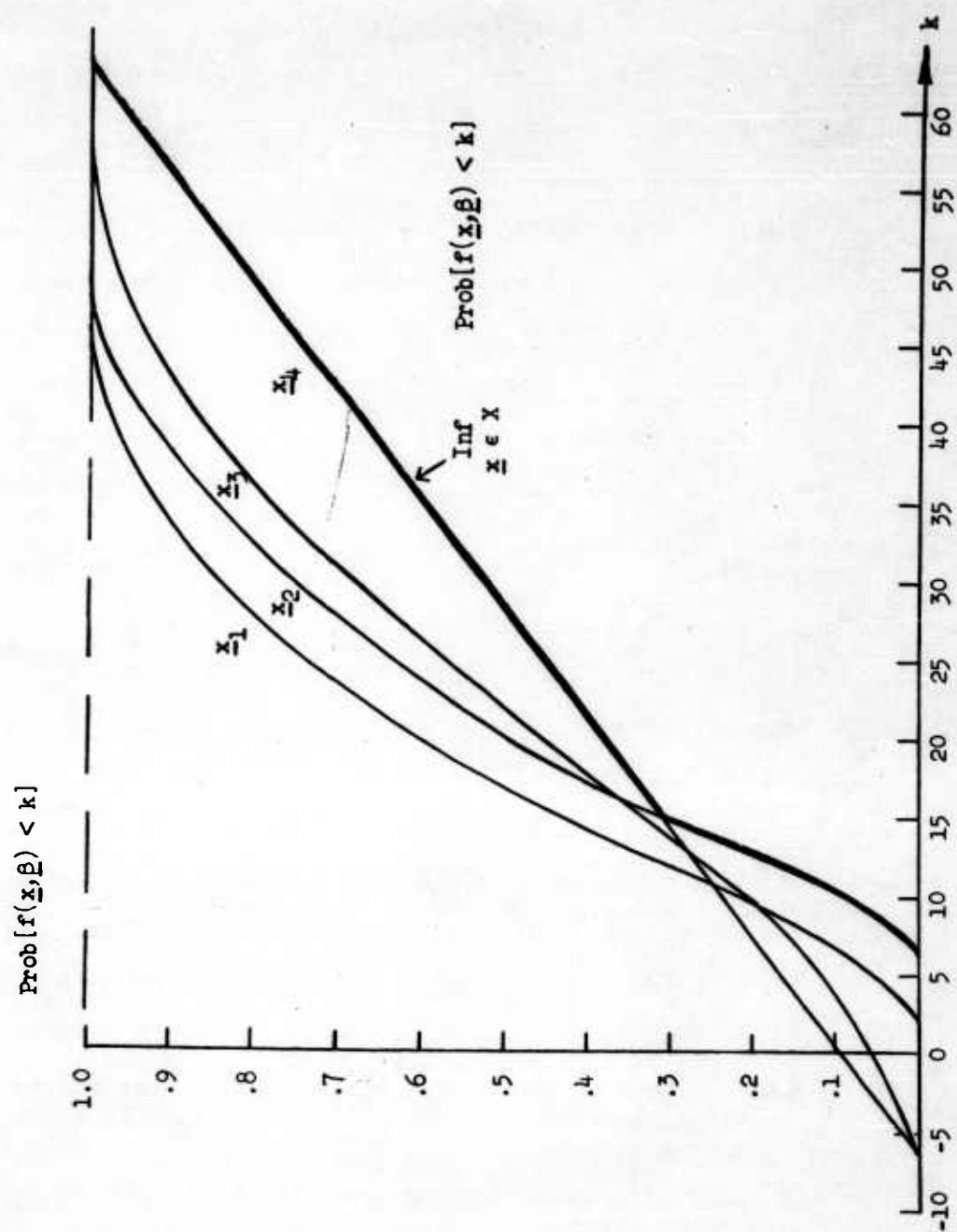
if $\text{Prob}[f(\underline{x}^o,\underline{\beta}) < k] \leq \text{Prob}[f(\underline{x}',\underline{\beta}) < k]$ for all real k, with

strict inequality holding for at least one value of k (if we drop

the proviso that strict inequality must hold for at least one value

of k, then we use the term weak stochastic dominance). If a feasible

decision is not stochastically dominated by any other feasible decision,

it is said to be stochastically admissible.[6] The stochastic admissi-

bility criterion requires that one choose a stochastically admissible

decision (this criterion can be written in a form similar to (4.1)).

Remark: Although we do not choose to do so in this paper, it is possible

to strengthen the stochastic admissibility criterion somewhat

by permitting randomized decisions over X. One would say

that the feasible decision $\underline{x}'$ is stochastically inadmissible

under a randomized decision strategy if there exists a proba-

bility distribution λ on X not involving $\underline{x}'$ such that

$\text{Prob}_{\mu,\lambda}[f(\underline{x},\underline{\beta}) < k] \leq \text{Prob}_{\mu}[f(\underline{x}',\underline{\beta}) < k]$ for all k, with

strict inequality holding for at least one value of k. For

example, in Figure 4, $\underline{x}_3$ is stochastically dominated by

the randomized strategy which chooses $\underline{x}_2$ and $\underline{x}_4$ each with

a probability of one-half, even though neither $\underline{x}_2$ nor $\underline{x}_4$

stochastically dominate $\underline{x}_3$ alone. Randomized decision

rules have the effect of taking vertically convex combina-

tions of the c.d.f.'s. It is clear that the set of

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[6] Since stochastic admissibility is defined in terms of X and the
particular distribution μ, to be precise we should qualify stochastic
admissibility as being "with respect to X and μ." We omit this
qualification for the sake of brevity, since no confusion is likely to
result in our discussion.

stochastically admissible decisions allowing randomized strategies is contained in the set of stochastically admissible decisions allowing only pure strategies.

We now explore the relationship between ordinary and stochastic admissibility.

Proposition 1:

Let μ vanish outside of B. If $\underline{x}^o$ weakly dominates $\underline{x}'$, then $\underline{x}^o$ weakly stochastically dominates $\underline{x}'$.

Proof: We must show that for all real k, $\text{Prob}[f(\underline{x}^o,\underline{\beta}) < k] \leq \text{Prob}[f(\underline{x}',\underline{\beta}) < k]$. By the definition of (non-stochastic) weak dominance, we have $f(\underline{x}',\underline{\beta}) \leq f(\underline{x}^o,\underline{\beta})$ for all $\underline{\beta} \in B$. Thus for any fixed value of k, $f(\underline{x}^o,\underline{\beta}) < k$ implies $f(\underline{x}',\underline{\beta}) < k$, and so for each k we have

$$\{\underline{\beta} \in B: \quad f(\underline{x}^o,\underline{\beta}) < k\} \subseteq \{\underline{\beta} \in B: \quad f(\underline{x}',\underline{\beta}) < k\} \ .$$

The proposition follows.

Remark: To see that the converse of this proposition need not hold, consider the following example. $X = \{x^o, x^1\}$, $B = \{\beta^1, \beta^2\}$, $f(x^o,\beta^1) = f(x^1,\beta^2) = 1$, $f(x^o,\beta^2) = f(x^1,\beta^1) = 2$, $\text{Prob}[\beta = \beta^1] = .2$, $\text{Prob}[\beta = \beta^2] = .8$. Then x^o stochastically dominates x^1, but x^o does not weakly dominate x^1.

With additional hypotheses, one may strengthen Proposition 1.

<u>Proposition 2</u>:

Let $f(\underline{x},\underline{\beta})$ be continuous on B for each $\underline{x} \in X$, and let μ be positive[7] everywhere on and vanish outside of B. If $\underline{x}^o$ dominates $\underline{x}'$, then $\underline{x}^o$ stochastically dominates $\underline{x}'$.

<u>Proof</u>: From Proposition 1 we have that $\underline{x}^o$ weakly stochastically dominates $\underline{x}'$. It remains to show that $\text{Prob}[f(\underline{x}^o,\underline{\beta}) < k^*] < \text{Prob}[f(\underline{x}',\underline{\beta}) < k^*]$ for some k^*. Since $\underline{x}^o$ dominates $\underline{x}'$, there exists $\underline{\beta}^* \in B$ such that $f(\underline{x}^o,\underline{\beta}^*) > f(\underline{x}',\underline{\beta}^*)$. Put $k^* = 1/2(f(\underline{x}^o,\underline{\beta}^*) + f(\underline{x}',\underline{\beta}^*))$. By the continuity of f there is a neighborhood N^* of $\underline{\beta}^*$ such that $f(\underline{x}^o,\underline{\beta}) \geq k^* > f(\underline{x}',\underline{\beta})$ for all $\underline{\beta} \in N^* \cap B$, and so by the positivity of μ on B we have $\text{Prob}[f(\underline{x}^o,\underline{\beta}) \geq k^* > f(\underline{x}',\underline{\beta})] > 0$. This fact, with the definition of $\underline{x}^o$, yields

$$\text{Prob}[f(\underline{x}',\underline{\beta}) < k^*] = \text{Prob}[f(\underline{x}',\underline{\beta}) < k^* \leq f(\underline{x}^o,\underline{\beta})] +$$

$$\text{Prob}[f(\underline{x}',\underline{\beta}) < k^* > f(\underline{x}^o,\underline{\beta})]$$

$$= \text{Prob}[f(\underline{x}',\underline{\beta}) < k^* \leq f(\underline{x}^o,\underline{\beta})] + \text{Prob}[f(\underline{x}^o,\underline{\beta}) < k^*]$$

$$> \text{Prob}[f(\underline{x}^o,\underline{\beta}) < k^*] .$$

---

[7] A probability distribution is said to be <u>positive</u> everywhere on B if for each $\underline{\beta}^o \in B$ then for every (b-dimensional) neighborhood N^o of $\underline{\beta}^o$ the event $[N^o \cap B]$ has a non-zero probability. A neighborhood of $\underline{\beta}^o$ of radius ρ is defined as $\left\{\underline{\beta}: \sqrt{\sum_{i=1}^{b} (\beta_i^o - \beta_i)^2} < \rho\right\}$, and is denoted by $N_\rho(\underline{\beta}^o)$ when a complete notation is desired.

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Proposition 2 shows that, under the given assumptions, the set of stochastically admissible decisions is contained in the set of admissible decisions, as one would expect and hope. To see that the set of stochastically admissible decisions can be considerably smaller than the set of admissible decisions, consider the example

$$\text{Maximize} \underset{x \in R^1}{} \quad [10 - (\beta - x)^2] \, ,$$

where μ is the Normal distribution with mean $\bar{\beta}$ and variance σ^2, and $B \equiv R^1$. Viewing the objective function as a family of functions of β indexed by x, this family is seen to consist of concave parabolas which are identical except for the axis of symmetry, which occurs at $\beta = x$. Clearly every $x^o \in R^1$ is admissible, for $f(x^o, \beta = x^o) = 10 > f(x, \beta = x^o)$ for all $x \neq x^o$. It is also clear that $x' \neq \bar{\beta}$ is stochastically inadmissible, for $\text{Prob}[f(\bar{\beta}, \beta) < k] <$ $\text{Prob}[f(x', \beta) < k]$ for all k. To see this assertion, observe that $\{\beta: \ f(x, \beta) \geq k\}$ is an interval of width $2(10-k)^{1/2}$ centered at $\beta = x$. By the symmetry and unimodality of the Normal distribution, the interval centered at $\beta = \bar{\beta}$ must include the greatest probability for any k, and hence $\text{Prob}[f(\bar{\beta}, \beta) \geq k] > \text{Prob}[(x', \beta) \geq k]$ when $x' \neq \bar{\beta}$, which is equivalent to the assertion that $x' \neq \bar{\beta}$ is stochastically inadmissible. Since $x = \bar{\beta}$ is stochastically admissible, we see that only $x = \bar{\beta}$ is stochastically admissible, whereas all x are admissible.

The Maximum α-Fractile and the Aspiration Criteria

In terms of Figure 4, we would like to choose a decision which

achieves the lower envelope of c.d.f.'s everywhere. In general this is impossible, but we can attempt to achieve it at a single point and hope that this one point will "pin down" a c.d.f. so that it is close to the lower envelope. The point may be specified in terms of its ordinate or abcissa value, whichever seems most natural in a given problem context. The criteria implied by this idea are, respectively and loosely:

> Criterion F: Choose an $\underline{x}$ which corresponds to a c.d.f. which approaches the lower envelope of c.d.f.'s at an ordinate value of $\alpha(0 < \alpha < 1)$.

> Criterion A: Choose an $\underline{x}$ which corresponds to a c.d.f. which approaches the lower envelope at an abcissa value of $M(-\infty < M < \infty)$.

It is evident that we have two entire families of criteria here, indexed by α and M respectively. Criterion F with $\alpha = 0.1$ would lead to the choice of $\underline{x}_2$ in Figure 4, and Criterion A with M = 20 would lead to the choice of $\underline{x}_4$.

Criterion F is equivalent to maximizing the α-fractile[8] of the distribution of $f(\underline{x},\underline{\beta})$ under μ. That is, it maximizes the payoff level below which there is at most an α probability of falling.[9]

---

[8] We define the α-fractile of a (possibly mixed) cumulative distribution function $F(y) = Prob[Y < y]$ as

$$Sup\{k:\ F(k) \leq \alpha\}\ .$$

[9] See Kataoka (1963) for a linear programming model of this type. It is one of the few published references to this criterion.

It corresponds, for fixed $0 \leq \alpha < 1$, to:

$$\text{Maximize } k$$
$$k, \underline{x}$$

(4.5)   subject to $\underline{x} \in X$

$$\text{Prob}[f(\underline{x}, \underline{\beta}) < k] \leq \alpha .$$

When α is small, say less than 0.1, this criterion should appeal to conservative decision-makers because it tends to control the lower tail of the distribution of payoffs. When $\alpha = 1/2$, (4.5) maximizes the median of the distribution of payoffs, of course. We sometimes use the mnemonic notation $F(\alpha)$ for this criterion.

Criterion A is equivalent to maximizing the probability of exceeding a prescribed "aspiration" level M of payoff (see Charnes and Cooper, 1963, for an application to linear programming). It corresponds to:

(4.6) $\underset{\underline{x} \in X}{\text{Minimize}} \qquad \text{Prob}[f(\underline{x}, \underline{\beta}) < M] .$

We sometimes use the notation $A(M)$ for this criterion.

Remark: It is to be noted that all cumulative distribution functions in this subsection are written as $\text{Prob}[f(\underline{x}, \underline{\beta}) < k]$ rather than as $\text{Prob}[f(\underline{x}, \underline{\beta}) \leq k]$ (regard $\underline{x}$ as being fixed). This convention is followed in order to avoid some minor difficulties which would be encountered by these two criteria if the opposite convention were adopted and the c.d.f.'s were discontinuous.

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We introduced these two criteria together because of their intimate mathematical relationship, as well as their common graphical motivation. When the lower envelope is attained by some $\underline{x}$ at every point, and is continuous and strictly increasing, it is geometrically clear that the F and A criteria are complementary in the sense that for every α there is an M which leads to the same set of decisions, and conversely. Without such assumptions, however, the complementarity is weakened, as we shall see in the following two easy propositions.

Proposition 3:

(i) Assume that criterion $F(\alpha^o)$ is satisfied by at least one decision. Then the set of decisions which satisfy criterion $F(\alpha^o)$ contains the set of decisions which satisfy criterion $A(M^o)$, where M^o is the maximum α^o-fractile.

(ii) Assume that criterion $A(M^o)$ is satisfied by at least one decision. Then the set of decisions which satisfy criterion $A(M^o)$ contains the set of decisions which satisfy criterion $F(\alpha^o)$, where $\alpha^o \overset{\triangle}{=} \underset{\underline{x} \in X}{\text{Min}} \ \text{Prob}[f(\underline{x},\underline{\beta}) < M^o]$.

Proof: (i). Let $\underline{x}^*$ satisfy $F(\alpha^o)$, and let M^o be the maximum α^o-fractile. If $\underline{x}^o$ satisfies $A(M^o)$, then $\text{Prob}[f(\underline{x}^o,\underline{\beta}) < M^o] \leq \text{Prob}[f(\underline{x}^*,\underline{\beta}) < M^o] \leq \alpha^o$, and so $\underline{x}^o$ must also satisfy $F(\alpha^o)$.

(ii). Let $\underline{x}^*$ satisfy $A(M^o)$, and let $\alpha^o \overset{\triangle}{=} \underset{\underline{x} \in X}{\text{Min}}$ $\text{Prob}[f(\underline{x},\underline{\beta}) < M^o] = \text{Prob}[f(\underline{x}^*,\underline{\beta}) < M^o]$. If $\underline{x}^o$ satisfies $F(\alpha^o)$, then there exists $k^o \geq M^o$ such that $\text{Prob}[f(\underline{x}^o,\underline{\beta}) < k^o] \leq \alpha^o$; since

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$k^o \geq M^o$, we have $\mathrm{Prob}[f(\underline{x}^o, \underline{\beta}) < M^o] \leq \mathrm{Prob}[f(\underline{x}^o, \underline{\beta}) < k^o] \leq \alpha^o$, from which it follows that $\underline{x}^o$ must satisfy $A(M^o)$.

Proposition 4:

(i) If $\underline{x}^o$ satisfies criterion $F(\alpha^o)$ uniquely, then it satisfies criterion $A(M^o)$ uniquely, where M^o is the maximum α^o-fractile.

(ii) If $\underline{x}^o$ satisfies criterion $A(M^o)$ uniquely, then it satisfies criterion $F(\alpha^o)$ uniquely, where
$$\alpha^o = \mathrm{Prob}[f(\underline{x}^o, \underline{\beta}) < M^o].$$

Proof: (i). Suppose that $\underline{x}^o$ does not satisfy $A(M^o)$ uniquely. Then there exists $\underline{x}' \in X$, $\underline{x}' \neq \underline{x}^o$, such that $\mathrm{Prob}[f(\underline{x}', \underline{\beta}) < M^o] \leq \mathrm{Prob}[f(\underline{x}^o, \underline{\beta}) < M^o]$, which contradicts the fact that $\underline{x}^o$ satisfies $F(\alpha^o)$ uniquely.

(ii). Suppose that $\underline{x}^o$ does not satisfy $F(\alpha^o)$ uniquely. Then there exist $k^o \geq M^o$ and $\underline{x}' \in X$, $\underline{x}' \neq \underline{x}^o$, such that $\mathrm{Prob}[f(\underline{x}', \underline{\beta}) < k^o] \leq \alpha^o = \mathrm{Prob}[f(\underline{x}^o, \underline{\beta}) < M^o]$. Since $k^o \geq M^o$, we have $\mathrm{Prob}[f(\underline{x}', \underline{\beta}) < M^o] \leq \mathrm{Prob}[f(\underline{x}', \underline{\beta}) < k^o]$, and so $\mathrm{Prob}[f(\underline{x}', \underline{\beta}) < M^o] \leq \mathrm{Prob}[f(\underline{x}^o, \underline{\beta}) < M^o]$. This contradicts the fact that $\underline{x}^o$ satisfies $A(M^o)$ uniquely.

It is possible for criteria $F(\alpha)$ and $A(M)$ to lead to stochastically inadmissible decisions. The next proposition is of interest in this regard.

<u>Proposition 5:</u>

(i) If $\underline{x}^o$ satisfies criterion $F(\alpha^o)$ uniquely, then $\underline{x}^o$ also satisfies the stochastic admissibility criterion.

(ii) If $\underline{x}^o$ satisfies criterion $A(M^o)$ uniquely, then $\underline{x}^o$ also satisfies the stochastic admissibility criterion.

<u>Proof:</u> (i). In view of part (i) of Proposition 4, to prove (i) it is sufficient to prove (ii).

(ii). Let $\underline{x}^o$ satisfy $A(M^o)$ uniquely, so that $\text{Prob}[f(\underline{x}^o,\underline{\beta}) < M^o] < \text{Prob}[f(\underline{x},\underline{\beta}) < M^o]$ for all $\underline{x} \in X$, $\underline{x} \neq \underline{x}^o$. Suppose that $\underline{x}^o$ were stochastically inadmissible. Then there would exist $\underline{x}' \in X$, $\underline{x}' \neq \underline{x}^o$, such that $\text{Prob}[f(\underline{x}',\underline{\beta}) < k] \leq \text{Prob}[f(\underline{x}^o,\underline{\beta}) < k]$ for all k. Letting $k = M^o$, one would obtain a contradiction.

Now we turn to the relationship between the maxmin payoff criterion and the maximum α-fractile criterion with $\alpha = 0$. It is not at all surprising that under mild assumptions these criteria are in fact equivalent, i.e., the same decisions satisfy both.

<u>Proposition 6:</u>

Assume that $f(\underline{x},\underline{\beta})$ is upper semicontinuous[10] on B for each $\underline{x} \in X$, and that μ is positive on and vanishes outside of B. Then the maxmin payoff criterion is equivalent to the maximum 0-fractile criterion.

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[10] Let $\underline{x}$ be fixed in X. Then $f(\underline{x},\underline{\beta})$ is upper semicontinuous at $\underline{\beta}^o \in B$ if for each $\epsilon > 0 \; \exists \; \delta > 0$ (depending on $\underline{\beta}^o$ and ϵ) such

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Proof: We shall rewrite (4.2) and (4.5) in such a way as to emphasize their similarity, and then show that they are in fact identical.

The maxmin payoff criterion can be written[11]

$$\text{Maximize} \quad [\text{Sup}\{k: \ f(\underline{x},\underline{\beta}) \geq k, \ \forall \ \underline{\beta} \in B\}] ,$$
$$\underline{x} \in X$$

and the maximum 0-fractile criterion can be written

$$\text{Maximize} \quad [\text{Sup}\{k: \ \text{Prob}[f(\underline{x},\underline{\beta}) \geq k] = 1\}] .$$
$$\underline{x} \in X$$

Define $S_1(\underline{x})$ and $S_2(\underline{x})$ to be the sets appearing in the first and second problems, respectively, for fixed $\underline{x}$. Clearly $S_1(\underline{x}) \subseteq S_2(\underline{x})$, $\forall \ \underline{x} \in X$, for μ vanishes outside of B. The proof will be complete when we show that $S_2(\underline{x}) \subseteq S_1(\underline{x})$, $\forall \ \underline{x} \in X$.

We consider a fixed $\underline{x}$, and drop the $\underline{x}$ arguments from S_1 and S_2. We may assume that S_2 is not empty, for if it is empty then S_1 is also empty, and the proof is complete. Take $k' \in S_2$. Suppose that $k' \notin S_1$. Then there exists $\underline{\beta}' \in B$ such that $f(\underline{x},\underline{\beta}') < k'$. But by the upper semicontinuity of $f(\underline{x},\underline{\beta})$ there exists a neighborhood N' of $\underline{\beta}'$ such that $f(\underline{x},\underline{\beta}) < k'$ for all

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that $f(\underline{x},\underline{\beta}) \leq f(\underline{x},\underline{\beta}) + \epsilon$ whenever $\underline{\beta} \in N_\delta(\underline{\beta}^\circ)$. If f is continuous, then f is upper semicontinuous. Also, recall that if B is a finite point set in E^m, then $f(\underline{x},\underline{\beta})$ is automatically continuous on B.

[11] This problem follows from the definition of 'inf' as the greatest lower bound.

$\underline{\beta} \in N' \cap B$. By the positivity of μ on B, this contradicts the fact that $k' \in S_2$.

The F and A criteria have the interesting property that one may perform a continuous monotonic transformation on $f(\underline{x}, \underline{\beta})$ without altering the decisions which satisfy these criteria. This certainly is not true of the next criterion we shall discuss, the expected value criterion. We emphasize this point in

Proposition 7:

Let $g(t)$ be any strictly increasing and continuous function defined from R^1 into R^1. Then (i) the set of decisions which satisfy criterion $F(\alpha)$ does not alter if $f(\underline{x}, \underline{\beta})$ is replaced by $g(f(\underline{x}, \underline{\beta}))$, and (ii) the set of decisions which satisfy criterion $A(M)$ does not alter if $f(\underline{x}, \underline{\beta})$ is replaced by $g(f(\underline{x}, \underline{\beta}))$ and M is replaced by $g(M)$.

Proof: Observe that $f(\underline{x}, \underline{\beta}) < k$ if and only if $g(f(\underline{x}, \underline{\beta})) < g(k)$, since g is invertible and strictly increasing. Hence $\{\underline{\beta}: f(\underline{x}, \underline{\beta}) < k\} = \{\underline{\beta}: g(f(\underline{x}, \underline{\beta})) < g(k)\}$, and so $\text{Prob}[f(\underline{x}, \underline{\beta}) < k] = \text{Prob}[g(f(\underline{x}, \underline{\beta})) < g(k)]$. This yields (ii). To see (i), write

$$\text{Sup}\{k: \text{Prob}[f(\underline{x}, \underline{\beta}) < k] \le \alpha\}$$

$$= \text{Sup}\{k: \text{Prob}[g(f(\underline{x}, \underline{\beta})) < g(k)] \le \alpha\}$$

$$= \text{Sup}\{g^{-1}(g(k)): \text{Prob}[g(f(\underline{x}, \underline{\beta})) < g(k)] \le \alpha\}$$

$$= g^{-1}(\text{Sup}\{g(k): \text{Prob}[g(f(\underline{x}, \underline{\beta})) < g(k)] \le \alpha\})$$

$$= g^{-1}(\text{Sup}\{t: \text{Prob}[g(f(\underline{x}, \underline{\beta})) < t] \le \alpha\}) .$$

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Finally,

$$\underset{\underline{x} \in X}{\text{Max}} \ [\text{Sup}\{k: \ \text{Prob}[f(\underline{x},\underline{\beta}) < k] \leq \alpha\}]$$

$$= g^{-1}(\underset{\underline{x} \in X}{\text{Max}} \ [\text{Sup}\{k: \ \text{Prob}[g(f(\underline{x},\underline{\beta})) < k] \leq \alpha\}]) \ .$$

Maximum Expected Payoff Criterion

The F and A criteria are designed to achieve the lower envelope of the family of c.d.f.'s $\{\text{Prob}[f(\underline{x},\underline{\beta}) < k]\}_{\underline{x} \in X}$ at a single point, in an attempt to "pin down" a c.d.f. to lie "close" to the lower envelope. Another approach would be to use the area above the lower envelope and below a candidate c.d.f. as a measure of "closeness."

> Criterion E: Choose an $\underline{x} \in X$ which determines the c.d.f. with the least area below it and above the lower envelope.

We shall show now that this geometrically motivated criterion is equivalent to the maximum expected payoff criterion:

(4.7) $$\underset{\underline{x} \in X}{\text{Maximize}} \ E[f(\underline{x},\underline{\beta})] \ .$$

Proposition 8:

Criterion E is equivalent to the maximum expected payoff criterion.

Proof: The proof is a simple consequence of the geometric interpretation of the mean of a random variable in terms of the graph of its cumulative distribution function. In Figure 5, the mean of the random variable Y is area 1 minus area 2 (see Parzen, 1960, p. 211).

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Denote by $A(\underline{x})^+$ the area corresponding to area 1 of Figure 5 for the c.d.f. $\text{Prob}[f(\underline{x},\underline{\beta}) < k]$, and by $A(\underline{x})^-$ the area corresponding to area 2. Similarly, denote by A^+ and A^- the areas above and below the lower envelope of all such c.d.f.'s. The the maximum expected payoff criterion may be written

$$\underset{\underline{x} \in X}{\text{Maximize}} \quad [A(\underline{x})^+ - A(\underline{x})^-] \, ,$$

and Criterion E may be written

$$\underset{\underline{x} \in X}{\text{Minimize}} \quad [(A(\underline{x})^- - A^-) + (A^+ - A(\underline{x})^+)] \, .$$

Clearly these two problems lead to the same decisions.
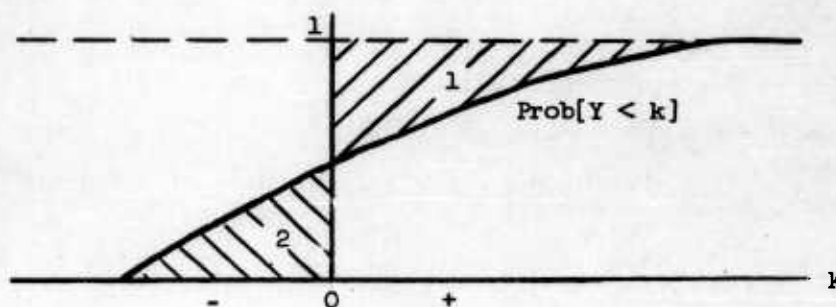
Figure 5

There is an obvious and fortunate relationship between the maximum expected payoff criterion and the estimate criterion which sometimes permits one to choose an estimate in a simple way so that the estimate criterion is satisfied by the same set of decisions as the expected payoff criterion.

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Proposition 9:

Assume that $f(\underline{x},\underline{\beta})$ can be written as

$$f(\underline{x},\underline{\beta}) = F_1(\underline{x}) + F_2(\underline{\beta}) + \sum_i H_i(\underline{x})\beta_i \ .$$

Then the estimate criterion with $\underline{\hat{\beta}} = E[\underline{\beta}]$ is satisfied by the same set of decisions as the maximum expected payoff criterion.

Proof: The maximum expected payoff criterion gives

$$\underset{\underline{x} \in X}{\text{Maximize}} \quad \mathbf{E}\left[F_1(\underline{x}) + F_2(\underline{\beta}) + \sum_i H_i(\underline{x})\beta_i\right] , \quad \text{or}$$

$$\underset{\underline{x} \in X}{\text{Maximize}} \quad \left[F_1(\underline{x}) + E[F_2(\underline{\beta})] + \sum_i H_i(\underline{x}) E[\beta^i]\right] .$$

The estimate criterion with $\underline{\hat{\beta}} = E[\underline{\beta}]$ gives

$$\underset{\underline{x} \in X}{\text{Maximize}} \quad \left[F_1(\underline{x}) + F_2(E[\underline{\beta}]) + \sum_i H_i(\underline{x}) E[\beta^i]\right] .$$

Since the F_2 terms of each problem do not contain $\underline{x}$, they may be deleted, and hence the two criteria lead to identical sets of decisions.

When the above proposition applies, we say that the estimate $\underline{\hat{\beta}} = E[\underline{\beta}]$ is a certainty equivalent with respect to the maximum expected payoff criterion. Other results in the same vein are given by Reiter (1957), Simon (1956), and Theil (1964).

It is easy to see from Proposition 8 that any decision which satisfies the maximum expected payoff criterion must be stochastically admissible.

It is also worth noting that the expected value criterion leads to the same decisions when applied to payoff as when applied to regret. In general this is not true for criteria $A(M)$ and $F(\alpha)$.

3.3 An Example

We present a simple inventory model as an illustration of the ideas of this section and as a vehicle for further discussion. Consider a firm stocking and selling a single commodity for a single period of time. We use the notation

x = number of units to be ordered in advance of the demand

β = unknown demand level during the period

c = cost per unit

r = revenue per unit $(r > c)$

v = salvage value per unit left at end of period $(v < c)$

$f(x,\beta)$ = total profit

$X = [0,\infty)$

$B = [0,\beta_{MAX}]$, where β_{MAX} is chosen sufficiently large to account for the largest likely demand

The payoff and regret are given by

$$f(x,\beta) = \begin{cases} (r-c)\beta - (x-\beta)(c-v) & \text{if } \beta \leq x \\ (r-c)x & \text{if } \beta \geq x \end{cases}$$

$$r(x,\beta) = \begin{cases} (c-v)(x-\beta) & \text{if } \beta \leq x \\ (r-c)(\beta-x) & \text{if } \beta \geq x \;. \end{cases}$$

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First we examine the criteria not involving probabilities over the set of possible demand levels. All choices for $x \in X$ are readily seen to be admissible. The maxmin payoff criterion leads to the decision to order zero units, since $\underset{\beta \in B}{\text{Min}} \; f(x,\beta) = -(c-v)x$. When this criterion is applied to regret, however, it (minmax regret) leads to the decision to order $[(r-c)/(r-v)]\beta_{MAX}$. This is the same decision that the Principle of Insufficient Reason would give if we interpret it as putting a uniform distribution over $[0,\beta_{MAX}]$. The estimate criterion leads to a trivial maximization problem once an estimate $\hat{\beta}$ is chosen, and indicates that we should order exactly $x = \hat{\beta}$.

Next we examine the criteria involving probabilities over the set of possible demand levels. In order to plot the cumulative distributions of payoff for various candidate x's, we need to know the set of β's for which the payoff is less than k.

$$\{\beta : \; \beta \geq 0, \; f(x,\beta) < k\} = \begin{cases} \emptyset & \text{if} \quad k < -(c-v)x \\[2mm] \left[0, \; \dfrac{k + (c-v)x}{r-v}\right) & \text{if} \quad -(c-v)x \leq k \leq (r-c)x \\[2mm] [0,\infty) & \text{if} \quad k > (r-c)x \; . \end{cases}$$

Using the fact that x is non-negative, we have for $k \geq 0$

$$\text{Prob}[\,f(x,\beta) < k\,] = \begin{cases} 1 & \text{if} \quad x < \dfrac{k}{(r-c)} \\[3mm] 1 - \displaystyle\int_{\frac{k+(c-v)x}{(r-v)}}^{\infty} d\mu & \text{if} \quad x \geq \dfrac{k}{(r-c)} \end{cases}$$

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For k < 0,

$$\text{Prob}[f(x,\beta) < k] = \begin{cases} 0 & \text{if } x < \frac{-k}{(c-v)} \\[2em] 1 - \int_{\frac{k+(c-v)x}{(r-v)}}^{\infty} d\mu & \text{if } x \geq \frac{-k}{(c-v)} \end{cases}$$

The lower envelope may be obtained by solving, for all real k, the problem

$$\begin{array}{c} \text{Minimize} \quad \text{Prob}[f(x,\beta) < k] \ . \\ x \geq 0 \end{array}$$

This problem has a very simple solution for this example. For k < 0, the minimum is zero and is achieved for $0 \leq x < |k|/(c-v)$. For $k \geq 0$, the minimum is $1 - \int_{k/(r-c)}^{\infty} d\mu$ and is achieved for x = k/(r-c).

Assume for computational simplicity that the demand is exponentially distributed with mean 10, that (c-v) = 1/2, and that (r-c) = 3/2. Then for $k \geq 0$, the lower envelope has height $[1 - \exp[-.066\overline{6}\ k]]$, and is achieved at $x = 2k/3$.[12] Figure 6 illustrates the lower envelope and a few sample c.d.f.'s. Observe that each c.d.f. jumps to the value 1 as soon as it attains the lower envelope, and that every $x \geq 0$ is stochastically admissible.

We are now in a position to read off the "optimal" decisions corresponding to criteria A(M) and $F(\alpha)$ for any choice of M or α. $A(M^o)$ leads to the unique choice of $x = M^o/(r-c)$, and $F(\alpha^o)$ leads to the unique choice $x = -10 \ln(1-\alpha_o)$. In this

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[12] Note that the lower envelope is the c.d.f. of an exponential distribution with mean 15.
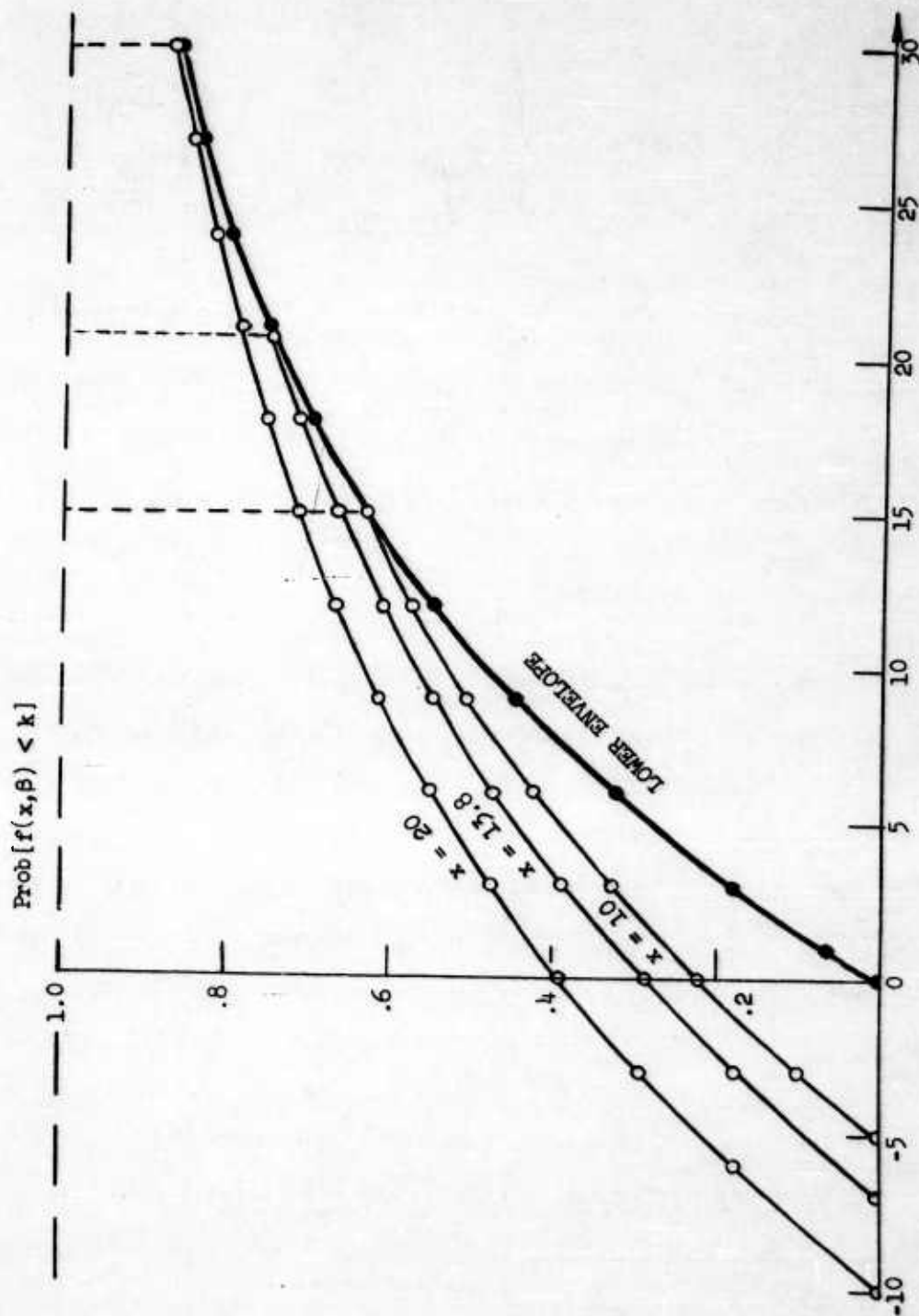
Figure 6

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particular example, these criteria do not fulfill their promise of "pinning down" a c.d.f. to lie close to the lower envelope, because each c.d.f. is discontinuous at the point at which it achieves the lower envelope.

The maximum expected payoff criterion may be applied by setting the derivative of $E[f(x,\beta)]$ equal to zero and solving for x. This computation leads to the well-known (Dvoretzsky, Kiefer, and Wolfowitz, 1952) result that one should choose the value of x corresponding to the $(r-c)/(r-v)$-th fractile of μ. That is, x^* should satisfy $\int_0^{x^*} d\mu = (r-c)/(r-v)$. For the data assumed above, $x^* = 13.8$. It is interesting to observe that if μ were uniform on $[0,\beta_{MAX}]$, then the minmax regret criterion would lead to exactly the same action as would the maximum expected payoff criterion.

Next we carry out a parallel analysis in terms of regret rather than payoff. It will be seen that $A(M)$ and $F(\alpha)$ are more appealing when applied to the regret distributions. An argument will be presented for choosing a value of x other than that which minimizes expected regret (which, of course, is equivalent to maximizing the expected payoff, the now classical solution to this problem).

We have, for $k \geq 0$,

$$\{\beta: \quad \beta \geq 0, \quad r(x,\beta) \leq k\} = \begin{cases} \left[0, \quad x + \dfrac{k}{(r-c)}\right] & \text{if} \quad x < \dfrac{k}{(c-v)} \\[4mm] \left[x - \dfrac{k}{c-v}, \quad x + \dfrac{k}{r-c}\right] & \text{if} \quad x \geq \dfrac{k}{(c-v)} \end{cases}.$$

Since we are dealing in terms of regret, rather than payoff, we seek the <u>upper</u> envelope rather than the lower envelope. It is obtained by

maximizing, for all $k \geq 0$, $\text{Prob}[r(\underline{x}, \underline{\beta}) \leq k]$:

$$\underset{x \geq 0}{\text{Maximize}} \int_{\text{Max}\left\{0, \ x - \frac{k}{c-v}\right\}}^{x + \frac{k}{r-c}} d\mu \ .$$

Since the exponential distribution is monotone decreasing, the maximum is easily seen to be achieved at $x = k/(c-v)$. The height of the upper envelope is therefore equal to $\text{Prob}[\beta \leq k/(c-v) + k/(r-c)]$. For the data given previously, this quantity is computed to be $[1-\exp(-0.266\bar{6} \ k)]$, and the upper envelope is achieved for $x = 2k$.

Figure 7 is the counterpart of Figure 6. Note that the c.d.f.'s are continuous, so that $A(M)$ and $F(\alpha)$ are more effective in their endeavor to "pin up" a c.d.f. to lie near the upper envelope.

For a given value of x, it is a straightforward matter to calculate the expected regret and the α-fractile. This has been done for $\alpha = .95$ and some representative values of x in Figure 8. The striking feature of this graph is that large relative changes in .95-fractile are available with only small relative changes in expected regret, with the result that it becomes attractive to deviate from the ordinary minimum expected regret solution to the problem. For example, consider $x = 13.8$ (which yields the minimum expected regret) in comparison with $x = 20$. The former has an expected regret of 6.9 and a .95-fractile of 24.1, whereas the latter has an expected regret of 7.7 and a .95-fractile of 14.8. That is, by choosing $x = 20$ instead of 13.8, one may achieve a 38.5% decrease in .95-fractile at the expense of only 11.6% increase in expected regret; for $x = 18$ instead of 13.8, the percentages become 26.1% and 5.9%.

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Figure 7

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Figure 8

40
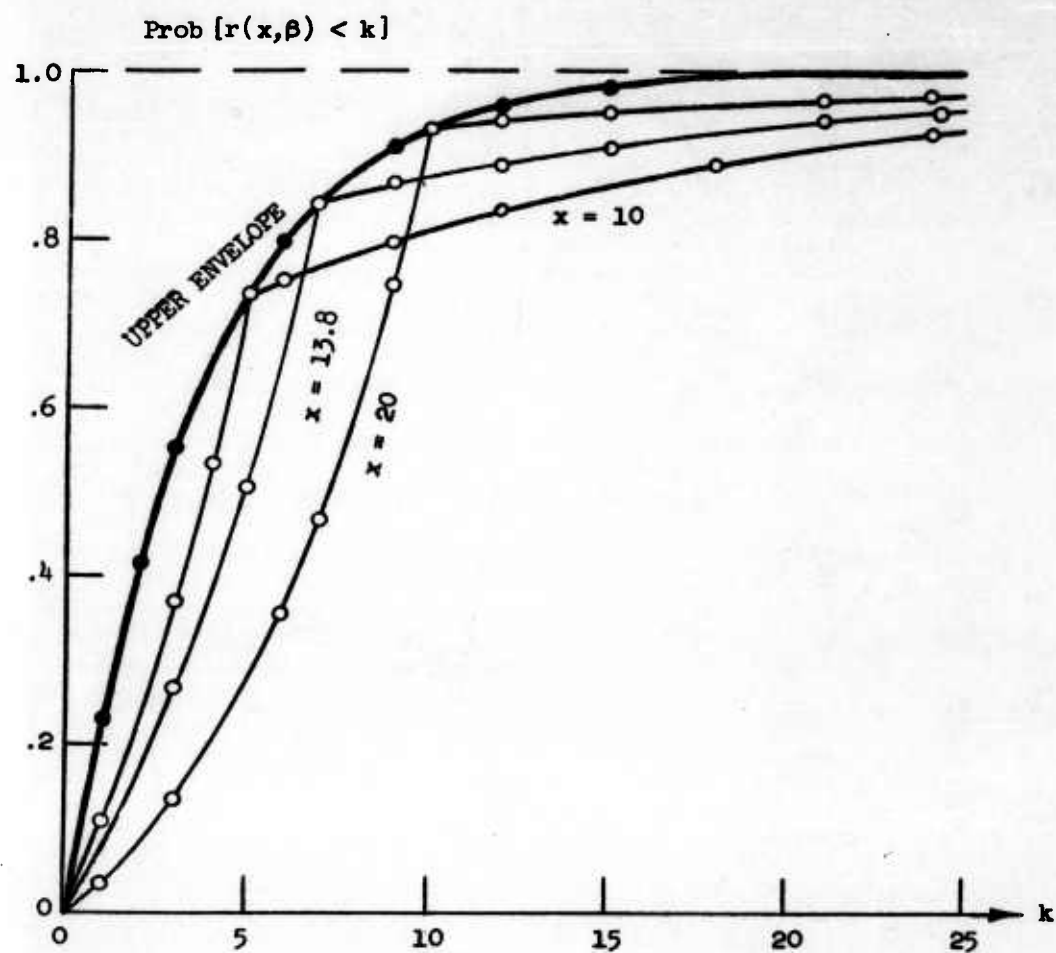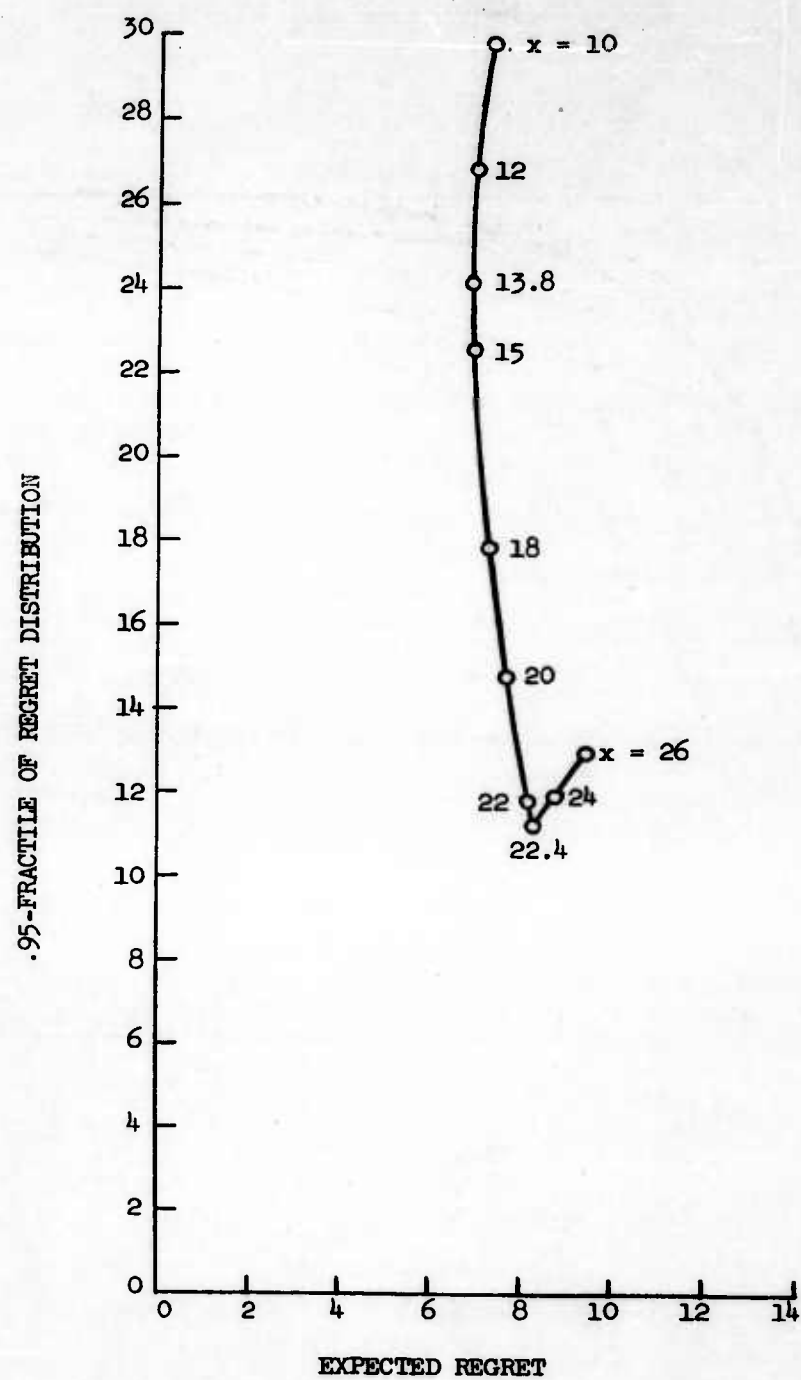
This example shows a special instance of what is likely to be a quite general situation: in the neighborhood of the decision indicated by the maximum expected payoff criterion, it is possible to substantially improve the α-fractile or aspiration levels of payoff or regret without lowering the expected payoff very much. Such possibilities ought to be investigated and exploited when found to be relevant to the decision-maker's objectives.

3.4 Vector Maximum Reformulations

The "ideal" decision criterion is analagous to the much-sought philosophers' stone of medieval times, and seems about as likely to exist. We suggest that one might profitably consider, in a given application, two or even three plausible criteria (not necessarily the ones discussed herein) and reformulate (4) as a vector maximum problem. The solution of this vector maximum problem would reveal clearly the tradeoffs involved between the criteria, and a decision may be chosen in an ad hoc manner from the efficient candidates. For example, if a situation such as Figure 9 occurs, one would probably choose an efficient solution nearer to point B than to point A, for a large gain in criterion 2 can be achieved at the expense of a relatively small loss in criterion 1.

Figure 9

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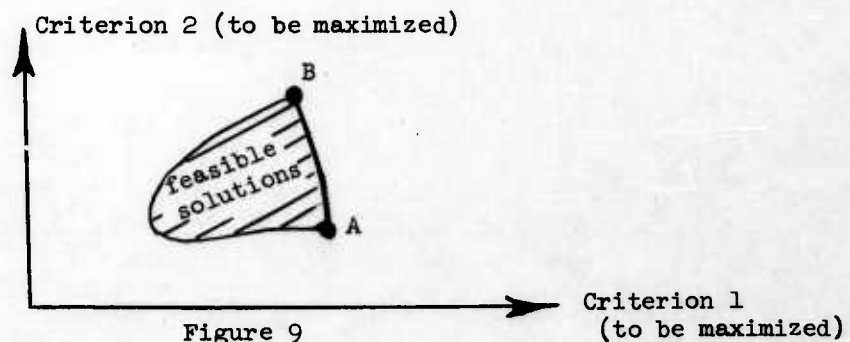
One combination of criteria which seems particularly plausible when a probability distribution over B is available is the α-fractile criterion with the expected value criterion. With α small, the first criterion tends to control the lower tail of the distribution of payoffs, while the second tends to control the mean. Such a combination might be used to program a mutual investment fund, for example, for the possibility of ruin or large losses seems to loom as a separate dimension of utility from the average growth rate. Markowitz (1956) had precisely this viewpoint in mind for his well-known portfolio problem, except that he used variance in place of the α-fractile.

Hodges and Lehmann (1952) proposed essentially this combination of criteria, except that they took α equal to zero. Letting α rise above zero seems to avoid some of the excessive conservatism in their formulation, while keeping the aim of protection against large losses.

CHAPTER II

Reducing a Vector Maximum Problem to a

Parametric Programming Problem

In this chapter it is assumed that uncertainty has been removed from a decision problem by means of devices such as those discussed in the first chapter, and that it is desired to solve the vector maximum problem.

(1)
$$\text{"Maximize"} \ \underset{\underline{x} \, \in \, X}{\underline{f}(\underline{x})} \ ,$$

where $\underline{f}(\underline{x}) = (f_1(\underline{x}),\ldots,f_r(\underline{x}))$, $\underline{x}$ is an n-vector, and X is a given set of feasible decisions. Recall that "solving" (1) means finding all <u>efficient</u> decisions, where a feasible decision $\underline{x}^o$ is called efficient if there exists no feasible decision $\underline{x}'$ such that $\underline{f}(\underline{x}') \geq \underline{f}(\underline{x}^o)$.[1] We shall discuss two ways of reducing (1) to a parameterized family of ordinary (one criterion function) mathematical programming problems, or "parametric" programming problems. Existing computational methods for these problems will be indicated.

This chapter is intended to serve as a bridge between the study of decision problems under uncertainty, which was the topic of the first chapter, and the study of a class of algorithms for parametric programming, which is the topic of the third chapter.

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[1] Recall that by this notation we mean $f_i(\underline{x}') \geq f_i(\underline{x}^o)$ $(i = 1,\ldots,r)$ with $f_i(\underline{x}') > f_i(\underline{x}^o)$ for some i (see Footnote 1, Chapter I).

1. Reducing (1) to a Problem Parametric in the Constraints

From the definition of an efficient decision for (1), it is easy to see that a feasible decision $\underline{x}^o$ is efficient if and only if $\underline{x}^o$ is an optimal solution to each of the r problems

$$
\text{(2i)} \quad \begin{array}{l} \text{Maximize} \quad f_i(\underline{x}) \\ \underline{x} \in X \end{array}
$$

$$
\text{subject to} \quad f_j(\underline{x}) \geq f_j(\underline{x}^o), \quad j = 1, \ldots, r \text{ but } j \neq i,
$$

$i = 1, \ldots, r$. It follows immediately that the following assertion holds.

Proposition 1:

Let $1 \leq i_o \leq r$ be fixed. If $\underline{x}^o$ is efficient in (1), then there exists an $(r-1)$-vector $\underline{\delta}$ such that $\underline{x}^o$ is an optimal solution of $(3i_o)$, where (3i) is given by

$$
\text{(3i)} \quad \begin{array}{l} \text{Maximize} \quad f_i(\underline{x}) \\ \underline{x} \in X \end{array}
$$

$$
\text{subject to} \quad f_j(\underline{x}) \geq \delta_j, \quad j = 1, \ldots, r \text{ but } j \neq i.
$$

This proposition suggests a method for finding all efficient decisions. Taking $r = 2$ and $i_o = 1$, for example, we find the set of all efficient decisions among the totality of optimal solutions to

$$
\text{(3)} \quad \begin{array}{l} \text{Maximize} \quad f_1(\underline{x}) \\ \underline{x} \in X \end{array}
$$

$$
\text{subject to} \quad f_2(\underline{x}) \geq \delta.
$$

as δ varies over $(-\infty, +\infty)$. Often $f_2(\underline{x})$ is bounded from above

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on X, and so the interval of parametric variation does not extend
to $+\infty$. Likewise when $f_2(\underline{x})$ is bounded from below on X, or when
the maximum of $f_1(\underline{x})$ on X is achieved for some value of $\underline{x}$, the
interval of parametric variation need not extend to $-\infty$.

This method yields not only all efficient decisions, but possibly
some inefficient ones as well, since it may be possible to increase
$f_2(\underline{x})$ without decreasing $f_1(\underline{x})$ below its maximum value for a parti-
cular value of δ. A similar remark holds a fortiori for $r > 2$.
Culling out the inefficient decisions when $r = 2$ is easily done,
in principle, by viewing the graph of $(f_1(\underline{x}), f_2(\underline{x}))$ for all candi-
date decisions generated by the method. For $r > 2$, graphical analysis
rapidly becomes impractical, and one must rely on sufficient conditions
such as those given in

Proposition 2:

Let $1 \leq i_o \leq r$ and the $(r-1)$-vector $\underline{\delta}_o$ be fixed, and let
$\underline{x}^o$ be an optimal solution to $(3i_o)$ with $\underline{\delta} = \underline{\delta}_o$. If any of
the following three conditions are satisfied, then $\underline{x}^o$ is
efficient in (1).

(i) $\underline{x}^o$ is also an optimal solution of the $r-1$ problems
     $(3i)$, $i \neq i_o$, with $\delta_j = f_j(\underline{x}^o)$, $j = 1, \ldots, r$.

(ii) $\underline{x}^o$ is the unique optimal solution to $(3i_o)$ with
      $\underline{\delta} = \underline{\delta}_o$.

(iii) $\underline{x}^o$ is the unique optimal solution to $(3i_o)$ with
       $\delta_j = f_j(\underline{x}^o)$, $j \neq i_o$.

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<u>Proof</u>: If (i) is satisfied, $\underline{x}^o$ is efficient in (1) by the opening remark of this section.

Assume that (ii) is satisfied, and suppose that $\underline{x}^o$ is not efficient. Then there exists $\underline{x}' \in X$ such that $\underline{f}(\underline{x}') \geq \underline{f}(\underline{x}^o)$, which implies that $\underline{x}'$ is feasible and optimal in $(3i_o)$ with $\underline{\delta} = \underline{\delta}_o$, thus contradicting the unique optimality of $\underline{x}^o$. Hence $\underline{x}^o$ is efficient.

Since $\underline{x}^o$ also is an optimal solution of $(3i_o)$ with $\delta_j = f_j(\underline{x}^o)$, the argument apropos (ii) applies.

Under additional hypotheses, Propositions 1 and 2 can be combined to give

<u>Proposition 3</u>:

Let $1 \leq i_o \leq r$ be fixed. Assume that f_{i_o} is strictly concave, $f_j (j \neq i_o)$ is concave, and X is convex.[2] Then $\underline{x}^o$ is efficient in (1) if and only if $\underline{x}^o$ solves $(3i_o)$ for some $(r-1)$-vector $\underline{\delta}$.

<u>Proof</u>: Necessity was proven in Proposition 1. To prove sufficiency, apply A.2 of Appendix A and part (ii) of Proposition 2.

2. <u>Reducing (1) to a Problem Parametric in the Objective Function</u>

We shall give some conditions under which (1) can be reduced to

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[2] See Appendix A for definitions of convex sets and concave functions, and some properties thereof which will be used freely in the sequel.

a family of problems of the form

$$(4) \qquad \underset{\underline{x} \in X}{\text{Maximize}} \ \sum_{i=1}^{r} v_i f_i(\underline{x}) \ ,$$

where $\underline{v} \geq \underline{0}$ is a vector-valued parameter.

Proposition 4:

(i) If $\underline{v} > \underline{0}$ and $\underline{x}^o$ is an optimal solution to (4), then $\underline{x}^o$ is efficient in (1).

(ii) If $\underline{v} \geq 0$ and $\underline{x}^o$ is the unique optimal solution of (4), then $\underline{x}^o$ is efficient in (1).

Proof: Suppose that (i) is false. Then there exists $\underline{x}' \in X$ such that $\underline{f}(\underline{x}') \geq \underline{f}(\underline{x}^o)$; since $\underline{v} > \underline{0}$, this implies that $\sum v_i f_i(\underline{x}') > \sum v_i f_i(\underline{x}^o)$, thus contradicting the optimality of $\underline{x}^o$ in (4). This proves (i).

Suppose that (ii) is false. Then there exists $\underline{x}' \in X$, $\underline{x}' \neq \underline{x}^o$, such that $\underline{f}(\underline{x}') \geq \underline{f}(\underline{x}^o)$; since $\underline{v} \geq \underline{0}$, this implies that $\sum v_i f_i(\underline{x}') \geq \sum v_i f_i(\underline{x}^o)$, thus contradicting the unique optimality of $\underline{x}^o$ in (4). This proves (ii).

Proposition 5:[3]

Let X be convex, let $f_i(\underline{x})$ be concave, $i = 1,\ldots,r$, and let $\underline{x}^o$ be efficient in (1). Then there exists an r-vector $\underline{v}^o \geq \underline{0}$ such that $\underline{x}^o$ is an optimal solution of (4) with $\underline{v} = \underline{v}^o$.

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[3] The earliest statement and proof of a theorem of this type seems to be due to Kuhn and Tucker (1951). An elegant proof of this proposition has been given by Karlin (1959, p. 217). For the sake of completeness we record a slightly different version of that proof here.

<u>Proof</u>: Put $P = \{\underline{p} \in E^r: \underline{p} \geq \underline{f}(\underline{x}^0)\}$. Clearly P is convex.
Put $Z = \{\underline{z} \in E^r: \underline{z} \leq \underline{f}(\underline{x})$ for some $\underline{x} \in X\}$. Z is convex, for
let $\underline{z}', \underline{z}'' \in Z$ and let $0 < \lambda < 1$. By the definition of Z there
exist $\underline{x}', \underline{x}'' \in X$ such that $\underline{z}' \leq \underline{f}(\underline{x}')$ and $\underline{z}'' \leq \underline{f}(\underline{x}'')$. Hence

$$(\lambda \underline{z}' + (1-\lambda)\underline{z}'') \leq \lambda \underline{f}(\underline{x}') + (1-\lambda)\underline{f}(\underline{x}'') \leq \underline{f}(\lambda \underline{x}' + (1-\lambda)\underline{x}'') ,$$

where the last inequality follows from the concavity of $\underline{f}(\underline{x})$. Since
$(\lambda \underline{x}' + (1-\lambda)\underline{x}'') \in X$ by the convexity of X, $(\lambda \underline{z}' + (1-\lambda)\underline{z}'') \in Z$.
This shows that Z is convex.

Because $\underline{x}^0$ is efficient, $Z \cap P$ is the single point $\underline{f}(\underline{x}^0)$,
so that Z and P have no interior points in common. Hence we may
apply the well-known Theorem of the Separating Hyperplane (see A.7,
Appendix A) to assert the existence of an r-vector $\underline{v}^0 \neq \underline{0}$ and a
scalar c such that

$$\sum v_i^0 z_i \leq c \leq \sum v_i^0 p_i , \quad \forall \underline{z} \in Z, \ \underline{p} \in P .$$

The right-hand inequality and the definition of P imply that
$\underline{v}^0 \geq \underline{0}$, for otherwise the sum $\sum v_i^0 p_i$ would be unbounded from
below. By the definition of Z, the left-hand inequality yields
$\sum v_i^0 f_i(\underline{x}) \leq c$, $\forall \underline{x} \in X$. Taking $\underline{p} = \underline{f}(\underline{x}^0)$, we have $\sum v_i^0 f_i(\underline{x}) \leq$
$\sum v_i^0 f_i(\underline{x}^0)$, $\forall \underline{x} \in X$, which is equivalent to the assertion that
$\underline{x}^0$ is an optimal solution of (4) with $\underline{v} = \underline{v}^0$.

When the hypotheses of Proposition 5 hold, one is sure to find
all efficient decisions for (1) among the totality of optimal decisions
for (4) as $\underline{v}$ ranges over all non-negative values. Notice that

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without loss of generality one may take $\sum v_i = 1$ in (4), since for fixed $\underline{v} \geq \underline{0}$ the objective function of that problem can be scaled by a factor of $1/\sum v_i$ without affecting the set of optimal solutions. Hence $\underline{v}$ is really only an $(r-1)$-dimensional parameter. When $r = 2$, for example, (4) reduces to the parametric problem

(4.1) $\quad\quad \underset{\underline{x} \in X}{\text{Maximize}} \quad vf_1(\underline{x}) + (1-v) f_2(\underline{x}) \quad$ for each $\quad 0 \leq v \leq 1$.

By strengthening the hypotheses of Proposition 5, the last two propositions can be combined to give

Proposition 6:

Let X be convex, and let $f_i(\underline{x})$ $(i = 1,\ldots,r)$ be strictly concave. Then $\underline{x}^o$ is efficient in (1) if and only if $\underline{x}^o$ solves (4) for some $\underline{v} \geq \underline{0}$.

Proof: Necessity was proven in Proposition 5. To prove sufficiency, apply A.2, A.4, and part (ii) of Proposition 4.

3. Computational Methods for Parametric Problems

A very common approach for a decision-maker to take, when faced with solving a multi-criterion problem such as (1), is to reformulate (1) in the form of (3i) or (4) (or possibly a combination of the two) with $\underline{\delta}$ or $\underline{v}$ fixed at some value of particular interest. Problem (3i) corresponds to selecting and retaining the most important criterion function and putting the rest in as constraints so that the remaining criteria each meet at least some minimally acceptable level.[4]

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[4] For an early and important example of this, see Neyman and Pearson (1933), who employed this device as a cornerstone of their theory of statistical hypothesis testing.

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Problem (4) corresponds to maximizing a weighted combination of criteria which is designed to reflect the relative importance of each. Such an approach offers computational simplicity in comparison with a complete solution of (1), since just one ordinary maximization problem has to be solved. After (3i) or (4) has been solved for the selected $\underline{\delta}^{o}$ or $\underline{v}^{o}$, the value of $\underline{\delta}$ or $\underline{v}$ may be varied in a neighborhood of $\underline{\delta}^{o}$ or $\underline{v}^{o}$ in order to ascertain how the corresponding optimal decisions and payoff function vary. This is a type of "sensitivity analysis." The above propositions relate this type of sensitivity analysis to the partial solution of (1) in the vector maximum sense.

Whether for purposes of sensitivity analysis or of solving (1), solution methods are required for the parametric problems associated with (3i) and (4). Since analytic methods can be expected to have very limited applicability—if experience with non-parametric mathematical programming is any guide—numerical methods must be employed. In this regard, we are obliged to limit our consideration to problems for which X is convex and $f_i(\underline{x})$ (i = 1,...,r) is concave, for most known programming algorithms[5] require at least convexity of the feasible region and concavity of the objective function. We shall further limit our consideration to the important case r = 2, because the vastness of the parameter space increases so rapidly with r as to preclude the reasonable hope of solving parametric problems even to reasonable approximation when r is much larger than 2 or 3.

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[5] For surveys of (nonlinear) programming algorithms, see, e.g., Dorn (1963), Hadley (1964), Saaty and Bram (1964, Chapter 3), Wolfe (1962), and Zoutendijk (1960).

We now indicate some existing computational methods, and point out the need for the developments of the next chapter.

If X is a convex polyhedron (i.e., the feasible region is determined by a set of linear equalities or inequalities), then several efficient parametric programming algorithms are available for certain special classes of criterion functions: when f_1 and f_2 are both linear functions, parametric versions of (3) and (4.1) can be solved by parametric linear programming (Gass, 1955); when f_1 is linear and f_2 is a quadratic polynomial,[6] the algorithms of Houthakker (1960), Markowitz (1956), and Wolfe (1959) are available;[7] when f_1 and f_2 are both quadratic polynomials, an algorithm of Zahl (1964) essentially solves (4.1), although it seems possible to improve upon the efficiency of his procedure by utilizing the developments of the next chapter. Little if anything appears to have been done to devise efficient algorithms for parametric problems involving more general classes of criterion functions or feasible regions other than convex polyhedra. The class of algorithms developed in Chapter III is intended as a contribution in this direction. At the present state of the art of parametric programming, however, one must fall back upon more rudimentary methods.

In principle, if an algorithm is available which will solve (3i) or (4) for any particular value of the parameter, then by

---

[6] That is, $f_2(\underline{x}) = \underline{x}^t Q \underline{x} + \underline{c}^t \underline{x}$, where t denotes transpose and Q is a negative semidefinite matrix.

[7] See also Boot (1963a, 1963b).

employing a suitably fine grid of parameter values one can obtain a discrete approximation to the optimal solutions of the parametric problem. This is a very straightforward approach, and for many problems it may be fairly practical, since the optimal solution for one parameter value can be expected to provide a nearly optimal solution at the next parameter value on the grid. Because most programming algorithms may be viewed as gradient methods, this approach should provide roughly first order convergence between optimal solutions at adjacent pairs of grid points.

In the next chapter we offer an alternative to the last approach under quite general assumptions on the criterion functions and the feasible region. We shall develop a class of algorithms for solving (4.1), a main member of which exhibits <u>second</u> order[8] convergence between adjacent pairs of grid points.

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[8] A sequence $< x^\nu >$ which converges to $\underline{x}^o$ exhibits first (second) order convergence if the norm of the error at the n-th step is asymptotically proportional to the (square of the) norm of the error at the n-1st step (see Appendix C, section 1).

CHAPTER III

A Class of Algorithms for Parametric Concave Programming

1. Introduction and Preliminaries

In this chapter we present a class of algorithms for solving parametric concave programming problems of the form

$$\text{Maximize} \quad \alpha f_1(\underline{x}) + (1-\alpha)f_2(\underline{x})$$

$(P\alpha)$ $\qquad \underline{x}$

$$\text{subject to} \quad \underline{g}(\underline{x}) \geq \underline{0}$$

for each $\alpha \in [0,1]$, where $\underline{x}$ is an n-vector, $f_i(\underline{x})$ (i = 1,2) is strictly concave,[1]/ and each component function of $\underline{g}(\underline{x}) = (g_1(\underline{x}),\ldots,g_m(\underline{x}))$ is concave. Certain additional regularity requirements are detailed in subsection 2.1.

Since our topic is _parametric_ programming, rather than ordinary (non-parametric) mathematical programming, we shall further assume that an optimal solution of $(P\alpha)$ is available for some value of α in the unit interval. This assumption is in fact not restrictive, for it is shown in subsection 1.1 that a parametric programming algorithm for $(P\alpha)$ which requires an optimal solution for some value of α in order to "get started" can itself be used to generate such an optimal solution.

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[1]/ The algorithms to be given still apply if (in the following, $\epsilon > 0$ is arbitrarily small): (a) f_1 is strictly concave and f_2 is (non-strictly) concave and $[0,1]$ is replaced by $[\epsilon,1]$, or (b) f_1 is concave and f_2 is strictly concave and $[0,1]$ is replaced by $[0,1-\epsilon]$, or (c) $\alpha f_1 + (1-\alpha)f_2$ is strictly concave for each fixed $\alpha \in (0,1)$ and $[0,1]$ is replaced by $[\epsilon,1-\epsilon]$.

The remainder of this section motivates (Pα) and the present class of algorithms: in subsection 1.1 it is noted that (Pα) subsumes the vector maximum problem for two criterion functions and also the standard (non-parametric) concave programming problem, and in subsection 1.2 the Kuhn-Tucker Theorem for nonlinear programming is presented in slightly unconventional form so as to display clearly the foundation upon which the present class of algorithms is built. Section 2 is devoted to presenting and proving a Basic Conceptual Algorithm for solving (Pα) for each value of α in the unit interval. Three graphical examples are given in Appendix B. The development of this conceptual algorithm into a Basic Computational Algorithm, via the use of Newton's method for solving the relevant systems of equations, is the subject of section 3. Some necessary computational devices are recorded in Appendix C. Section 4 hosts a modification (more accurately, a completion) of the algorithms aimed at improving their efficiency. Two extensions are indicated in section 5: the adaptation of the present algorithms to handle linear equality constraints, and the possibility of solving more general kinds of parametric problems than (Pα).

1.1 Motivation of (Pα)

One motive for studying (Pα) was given in Chapter II. From Proposition 6 of that chapter, which applies because of the above assumptions, solving (Pα) for all $0 \leq \alpha \leq 1$ is exactly equivalent to solving the vector maximum problem

(1)   "Maximize" $f_1(\underline{x})$, $f_2(\underline{x})$ subject to $\underline{g}(\underline{x}) \geq \underline{0}$.
           $\underline{x}$

That is, every efficient decision for (1) is an optimal solution of (Pα) for some $0 \leq \alpha \leq 1$, and conversely.

Another reason for studying (Pα) is that it subsumes the standard problem of concave programming. Suppose that it is desired to solve

(2) $\qquad\qquad$ Maximize $F(\underline{x})$ subject to $\underline{g}(\underline{x}) \geq \underline{0}$,
$\qquad\qquad\quad \underline{x}$

where $F(\underline{x})$ is strictly concave and the constraint functions are all concave. If $\underline{x}^o$ is any <u>feasible</u> decision whatsoever of (2), put (Pα) equal to

(3α)
$$\text{Maximize } \alpha F(\underline{x}) + (1-\alpha)(-1) \sum_{1}^{n} (x_i - x_i^o)^2$$
$$\underline{x}$$
$$\text{subject to } \underline{g}(\underline{x}) \geq \underline{0} .$$

Then $\underline{x}^o$ clearly is the <u>optimal</u> solution of (3o), and (3α) satisfies the assumptions required of (Pα) in the opening paragraph. Applying an algorithm for parametric concave programming to (3α) beginning with $\alpha = 0$ and increasing α until $\alpha = 1$, one obtains the optimal solution to (3_1), which is identical to (2). Hence a parametric algorithm for (Pα) provides a "deformation" method of concave programming.

Problem (3α) is capable of an interesting interpretation, which we shall now sketch briefly. Consider an enterprise currently "operating" at the (feasible) point $\underline{x}^o$, with a single criterion function $F(\underline{x})$ and a feasible operating region $\{\underline{x}: \underline{g}(\underline{x}) \geq \underline{0}\}$. Due to conservatism, or a desire to avoid disrupting the operations of the enterprise

radically, or to a desire to hedge against the risk of a faulty decision model, assume that the managers of the enterprise prefer to adjust the operating point _gradually_ from $\underline{x}^o$ toward $\underline{x}^*$, where $\underline{x}^*$ is optimal in (2). If the managers have a quadratic loss function $\sum (x_i - x_i^o)^2$ associated with deviations from $\underline{x}^o$, the optimal solution to (3α) as α varies from 0 to 1 gives an optimum path from $\underline{x}^o$ to $\underline{x}^*$.

Since $(P\alpha)$ for fixed α is of the form (2), the device represented by (3α) can be used to find a starting optimal solution to $(P\alpha)$ if one exists (providing that a feasible decision is known), so that the assumption stated in the introduction is not restrictive, as asserted.

Of course, in place of (3α) one could use

(4α)
$$\text{Maximize } \alpha F(\underline{x}) + (1-\alpha)H(\underline{x})$$
$$\underline{x}$$
$$\text{subject to } \underline{g}(\underline{x}) \geq \underline{0} \, ,$$

where $H(\underline{x})$ is a strictly concave function with a known maximum over the feasible region.

1.2 Theoretical Foundation

The standard problem of concave programming can be written in the form of $(P\alpha_o)$ with α_o fixed. For simplicity of notation, we write $f(\underline{x};\alpha)$ for $\alpha f_1(\underline{x}) + (1-\alpha)f_2(\underline{x})$. Hence $(P\alpha_o)$ may be written as

$(P\alpha_o)$
$$\text{Maximize } f(\underline{x};\alpha_o) \text{ subject to } \underline{g}(\underline{x}) \geq \underline{0} \, .$$
$$\underline{x}$$

Fundamental theoretical results concerning this problem have been given by Kuhn and Tucker (1951). A version of their Theorem 3 is recorded here without proof.

Theorem (Kuhn-Tucker):

Consider $(P\alpha_o)$ with α_o fixed. Let $f(\underline{x};\alpha_o)$ and $g_i(\underline{x})$ $(i = 1,\ldots,m)$ be differentiable on the feasible region $\{\underline{x}: \underline{g}(\underline{x}) \geq \underline{0}\}$, let $f(\underline{x};\alpha_o)$ be concave on the feasible region, and let $g_i(\underline{x})$ $(i = 1,\ldots,m)$ be concave on E^n. Assume that the constraint functions satisfy the Kuhn-Tucker Constraint Qualification (see the remark following the statement of the theorem).

Then $\underline{x}^o$ is an optimal solution of $(P\alpha_o)$ if and only if there exist real m numbers λ_i^o such that $(\underline{x}^o, \underline{\lambda}^o)$ satisfies the following (Kuhn-Tucker) conditions[2] at $\alpha = \alpha_o$:

(5) $$\nabla_x f(\underline{x};\alpha) + \sum_1^m \lambda_i \nabla_x g_i(\underline{x}) = \underline{0}$$

(6) $$g_i(\underline{x}) \geq 0 , \qquad i = 1, \ldots , m$$

(7) $$g_i(\underline{x}) \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \ \text{ implies } \ \lambda_i \left\{ \begin{matrix} = \\ \geq \end{matrix} \right\} 0 , \qquad i = 1, \ldots , m .$$

Remark: For a statement and discussion of the Kuhn-Tucker Constraint Qualification, see Kuhn and Tucker (1951, p. 483) or Arrow, Hurwicz, and Uzawa (1961). It has been shown, for example, that if all the constraints are linear then this qualification

[2] The symbol ∇ denotes the gradient of a function of several variables, e.g., $\nabla_x f(\underline{x}) \equiv \left(\dfrac{\partial f(\underline{x})}{\partial x_1} , \ldots , \dfrac{\partial f(\underline{x})}{\partial x_n} \right)$.

is satisfied; and that the existence of an interior point of the feasible region is also sufficient for the qualification to be satisfied. The sufficient condition which will be of direct use in the sequel is: if $\underline{x}^*(\alpha_o)$ is an optimal solution of $(P\alpha_o)$, then the matrix whose rows are $\nabla_x g_i(\underline{x}^*(\alpha_o))$, i such that $g_i(\underline{x}^*(\alpha_o)) = 0$, is of maximal rank (see Arrow, Hurwicz, and Uzawa, 1961).

Direct analytical or numerical attempts to satisfy these conditions have proven quite difficult, in general.

We shall find the following equivalent version of the Kuhn-Tucker Theorem more suitable for our purposes.

Theorem (Kuhn-Tucker, an alternate version):

Assume that the hypotheses of the Kuhn-Tucker Theorem are satisfied. Then $\underline{x}^o$ is an optimal solution of $(P\alpha_o)$ if and only if there exist m real numbers u_i^o and a subset S^o of constraint indices such that $(\underline{x}^o, \underline{u}^o, S^o)$ satisfies the following conditions at $\alpha = \alpha_o$:

$$(=S)\alpha \begin{cases} \text{(KT-1)} \quad \nabla_x f(\underline{x};\alpha) + \sum_S u_i \nabla_x g_i(\underline{x}) = \underline{0} \\[2mm] \text{(KT-2)} \quad g_i(\underline{x}) = 0, \ \forall \ i \in S \\[2mm] \qquad\qquad u_i = 0, \ \forall \ i \notin S \\[2mm] \text{(KT-3)} \quad g_i(\underline{x}) \geq 0, \ \forall \ i \notin S \\[2mm] \text{(KT-4)} \quad u_i \geq 0, \ \forall \ i \in S \ . \end{cases}$$

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Equations (KT-1) and (KT-2) appear so often together in the sequel that we introduce the special symbol $(=S)\alpha$ to denote them (in this notation, S and α may vary). We also denote the set of the first m positive integers by M.

The equivalence of the two versions of this theorem follows from the easily verified

Proposition 1:

(i) If $(\underline{x}^o,\underline{\lambda}^o)$ satisfies (5) through (7) at α_o, then $(\underline{x}^o,\underline{\lambda}^o,S^o)$ satisfies (KT-1) through (KT-4) at α_o for any S^o satisfying

(8) $$\{i \in M: \ \lambda_i^o > 0\} \subseteq S^o \subseteq \{i \in M: \ g_i(\underline{x}^o) = 0\} \ .$$

(ii) If $(\underline{x}^o,\underline{u}^o,S^o)$ satisfies (KT-1) through (KT-4) at α_o, then $(\underline{x}^o,\underline{u}^o)$ satisfies (5) through (7) at α_o.

The numbers λ_i or u_i will be referred to as dual variables. In view of Proposition 1 it is useless to distinguish between $\underline{\lambda}$ and $\underline{u}$; henceforth we shall use the symbol $\underline{u}$ to refer to the dual variables of either version of the Kuhn-Tucker Theorem.

The concept of a valid set plays a central role in this work. A subset S^o of constraint indices is said to be valid at α_o if and only if there exists $(\underline{x}^o,\underline{u}^o)$ such that $(\underline{x}^o,\underline{u}^o,S^o)$ satisfies (KT-1) through (KT-4) at α_o.

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<u>Proposition 2</u>:

A subset S^o of constraint indices is valid at α_o if and only if S^o satisfies (8) for some $(\underline{x}^o, \underline{\lambda}^o)$ which satisfies (5) through (7) at α_o.

<u>Proof</u>: Assume that S^o is valid at α_o. Then there exists $(\underline{x}^o, \underline{u}^o)$ such that $(\underline{x}^o, \underline{u}^o, S^o)$ satisfies (KT-1) through (KT-4) at α_o, which implies by part (ii) of Proposition 1 that $(\underline{x}^o, \underline{u}^o)$ satisfies (5) through (7) at α_o. By (KT-2) and (KT-4), $\{i \in M: \lambda_i^o > 0\} \subseteq S^o$ holds. By (KT-2), $S^o \subseteq \{i \in M: g_i(\underline{x}^o) = 0\}$ holds. This proves necessity.

Assume now that S^o satisfies (8) for some $(\underline{x}^o, \underline{\lambda}^o)$ satisfying (5) through (7) at α_o. By part (i) of Proposition 1, $(\underline{x}^o, \underline{\lambda}^o, S^o)$ satisfies (KT-1) through (KT-4) at α_o, which shows that S^o is valid at α_o.

The alternate version encourages the important observation that the Kuhn-Tucker Conditions may be viewed as the Lagrange multiplier equations[3]

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[3] The method of Lagrange multipliers (see, e.g., Apostol, 1957, p. 153) gives a set of first order necessary conditions for a point $\underline{x}^o$ to be an optimal solution of the problem

$$\text{Maximize } f(\underline{x}) \text{ subject to } g_i(\underline{x}) = 0, \quad i = 1, \ldots, m.$$
$$\underline{x}$$

Assume that $f(\underline{x})$ and $g_i(\underline{x})$ $(i = 1, \ldots, m)$ are continuously differentiable on some open region containing the feasible region, and that the matrix whose rows are $\nabla_x g_i(\underline{x}^o)$, $i = 1, \ldots, m$, is of maximal rank (note that this last assumption implies that $m \leq n$, where n is the dimension of $\underline{x}$). If $\underline{x}^o$ is an optimal solution of the above problem, then there exist m real numbers λ_i such that $(\underline{x}^o, \underline{\lambda}^o)$ satisfies the (Lagrange multiplier) equations:

$$\nabla_x f(\underline{x}) + \sum_i \lambda_i \nabla_x g_i(\underline{x}) = \underline{0} \text{ and}$$

$$g_i(\underline{x}) = 0, \qquad i = 1, \ldots, m.$$

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applied to a subset S of the constraints, augmented by the inequations (KT-3) and (KT-4). Attention thereby focuses on discovering the identity of a valid set, for if one knew a valid set S* then in principle one could solve $(=S*)\alpha_o$ for all solutions $(\underline{x}', \underline{u}')$, among which at least one would satisfy (KT-3) and (KT-4) and hence solve $(P\alpha_o)$. Indeed, at least one algorithm (see Theil and Van de Panne, 1960, and also Boot, 1961) has already been proposed which is essentially aimed at determining a valid set. However, this approach is probably not very efficient computationally, for although it reduces the concave programming problem to one of solving sets of simultaneous equations, there is a vast number of candidate sets of equations to be tried when a valid set is not known. It seems to be difficult, even for problems of modest size, to know how to order the trials so as to keep the number of erroneous trials at a reasonable level. This combinatorial difficulty is further aggravated by the numerical burden of actually solving $(=S)\alpha_o$. Thus we may expect the customary gradient methods to be more efficient than methods based on the "valid set approach."

Let us turn now to _parametric_ programming. It is perhaps surprising, in view of the immediately preceding comments, that here methods based on the "valid set approach" seem to have the advantage over gradient methods. In fact the parametric programming algorithms (cf. section 3 of Chapter II) of Markowitz (1956), Houthakker (1960), and Zahl (1964) each may be viewed as maintaining the identity of a valid set as a parameter is varied.

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Under appropriate assumptions the optimal solution $\underline{x}^*(\alpha)$ of
$(P\alpha)$ and the associated dual variables $\underline{u}^*(\alpha)$ are unique and con-
tinuous. This fact, coupled with the observation that there is only
a finite number of subsets of constraints, suggests that if S' is
valid at α_o, say, then S' is likely to be valid in some interval
including α_o. If this is the case, then one may derive $\underline{x}^*(\alpha)$ and
$\underline{u}^*(\alpha)$ in that interval by solving $(=S')\alpha$ parametrically, and (KT-3)
and (KT-4) are automatically satisfied. If this is <u>not</u> the case,
then even though $(=S')\alpha$ may have a solution near α_o, either (KT-3)
or (KT-4) will be violated, and it is necessary to find a new valid
set before being able to proceed. Because of continuity, moreover,
a set which is valid near α_o will usually differ by only a few
constraint indices from S'. This approach leads to a decomposition of
$(P\alpha)$ on $[0,1]$ into a chain of parametric subproblems. Each sub-
problem involves the parametric solution of the Lagrange multiplier
equations associated with the constraints specified by a constant valid
set on a subinterval of $[0,1]$. By continuity the optimal terminal
solution to one subproblem is the optimal initial solution to the
next subproblem of the chain, and the valid sets of adjacent sub-
problems are both valid at the transition point between them.

Thus parametric programming can be reduced essentially to the
problem in numerical analysis of solving parameterized (nonlinear,
in general) simultaneous equations. This approach to parametric
programming turns out to be a useful one computationally, since the
systems of equations involved will be shown to be well-behaved. By

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applying Newton's method (see Appendix C), second order convergence
can be achieved as the parameter increases by discrete increments,
whereas gradient methods display roughly first order convergence.

2. A Basic Conceptual Algorithm

In this section we state and prove a Basic Conceptual Algorithm
for solving $(P\alpha)$ for each value of α in the unit interval. We
use the adjective "conceptual" because computational implementation
is not considered at this point of the exposition. The Basic Con-
ceptual Algorithm can be modified and implemented in various ways,
as will be indicated in sections 3 and 4, thus giving rise to an entire
class of computational algorithms.

2.1 Assumptions

We assume that an optimal solution of $(P\alpha)$ is available for some
value of α in the unit interval, say $\alpha = 0$ (in view of the dis-
cussion of subsection 1.1, this assumption is not restrictive).

Throughout this work the following conditions will be imposed
upon $(P\alpha)$. We denote the feasible region $\{\underline{x}: \underline{g}(\underline{x}) \geq \underline{0}\}$ by X.

Condition 1: The functions $f_i(\underline{x})$ $(i = 1,2)$ and $g_i(\underline{x})$
$(i = 1,\ldots,m)$ are analytic on some open region
containing X, and the constraint functions are
concave on E^n.

Condition 2: X is non-empty and bounded.

<u>Condition 3</u>: The hessian matrices[4] $\nabla_x^2 f_i(\underline{x})$ $(i = 1,2)$ are negative definite for all $\underline{x} \in X$.

<u>Condition 4</u>: If $\alpha_o \in [0,1]$ and $\underline{x}^*(\alpha_o)$ is an optimal solution of $(P\alpha_o)$, then the matrix whose rows are the gradients $\nabla_x g_i(\underline{x}^*(\alpha_o))$, i such that $g_i(\underline{x}^*(\alpha_o)) = 0$, is of maximal rank.

A function $f(x_1,\ldots,x_n)$ of n real variables is said to be <u>analytic</u> in a region R if in some neighborhood of every point of R the function is the sum of a convergent power series with real coefficients. The class of all analytic functions includes, for example, all polynomials, and seems amply wide enough to include nearly any continuous function likely to be encountered in applications.

Conditions 1 and 2 imply, by A.1 of Appendix A, that X is convex and compact.

Condition 3 implies, by A.3, that f_1 and f_2 are strictly concave on X. This, in turn, implies by A.4 that $f(\underline{x};\alpha) = \alpha f_1(\underline{x}) + (1-\alpha)f_2(\underline{x})$ is strictly concave on X for each fixed value of $\alpha \in [0,1]$. In the presence of Conditions 1 and 2, this last assertion remains true even on some open interval containing $[0,1]$, as Proposition 3 shows.

<u>Proposition 3</u>:

Assume that Conditions 1, 2, and 3 hold. Then $\nabla_x^2 f(\underline{x};\alpha)$ is

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[4] $\nabla_x^2 f(\underline{x})$ denotes the n by n matrix whose ij-th element is $\dfrac{\partial^2 f(\underline{x})}{\partial x_i \partial x_j}$.

negative definite on X for each fixed value of α in some open interval containing [0,1].

Proof: It is well-known that $\nabla_x^2 f(\underline{x};\alpha)$ is negative definite at $(\underline{x},\alpha)$ if and only if all of its eigenvalues $\xi_\mu(\nabla_x^2 f(\underline{x};\alpha))$ $(\mu = 1,\ldots,n)$ are negative, i.e., if $\underset{\mu}{\text{Max}}\ \xi_\mu(\nabla_x^2 f(\underline{x};\alpha)) < 0$. Assume for the moment that the last-mentioned function is continuous in $(\underline{x},\alpha)$ on some open region containing $X \times [0,1]$, where $\times$ denotes the Cartesian product. Since a positive sum of negative definite matrices is again negative definite, from Condition 3 it follows that $\underset{\mu}{\text{Max}}\ \xi_\mu(\nabla_x^2 f(\underline{x};\alpha)) < 0$ on $X \times [0,1]$. The proposition follows from this fact, the assumed continuity, and the compactness of $X \times [0,1]$.

To see that $\underset{\mu}{\text{Max}}\ \xi_\mu(\nabla_x^2 f(\underline{x};\alpha))$ is continuous on some open region containing $X \times [0,1]$, observe that Condition 1 implies that the elements of $\nabla_x^2 f(\underline{x};\alpha)$ are all continuous on some open region containing $X \times [0,1]$. Since the eigenvalues of a square matrix are continuous functions of its elements (Ostrowski, 1960, p. 192), $\xi_\mu(\nabla_x^2 f(\underline{x};\alpha))$ $(\mu = 1,\ldots,n)$ is therefore continuous on some open region containing $X \times [0,1]$; the same must be true for $\underset{\mu}{\text{Max}}\ \xi_\mu(\nabla_x^2 f(\underline{x};\alpha))$.

Remark: As indicated in Footnote 1 of this chapter, Condition 3 may be weakened to (in the following, $\epsilon > 0$ is arbitrarily small): (a) $\nabla_x^2 f_1(\underline{x})$ $(\nabla_x^2 f_2(\underline{x}))$ is negative (semi-) definite for all $\underline{x} \in X$, if [0,1] is replaced by $[\epsilon,1]$,

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or (b) $\nabla_x^2 f_2(\underline{x})$ $(\nabla_x^2 f_1(\underline{x}))$ is negative (semi-) definite for all $\underline{x} \in X$, if $[0,1]$ is replaced by $[0,1-\epsilon]$, or

(c) $\alpha \nabla_x^2 f_1(\underline{x}) + (1-\alpha) \nabla_x^2 f_2(\underline{x})$ is negative definite for all $\underline{x} \in X$ at each $\alpha \in (0,1)$, if $[0,1]$ is replaced by $[\epsilon, 1-\epsilon]$.

Condition 4 is equivalent to requiring that the gradients $\nabla_x g_i(\underline{x}^*(\alpha_o))$, i such that $g_i(\underline{x}^*(\alpha_o)) = 0$, must be linearly independent; hence at most n constraints can be satisfied with exact equality at an optimal solution of $(P\alpha_o)$. In the remark following the Kuhn-Tucker Theorem, it was noted that this condition implies that the Kuhn-Tucker Constraint Qualification holds. Thus the hypotheses of the Kuhn-Tucker Theorem are satisfied by $(P\alpha_o)$ for each fixed $\alpha_o \in [0,1]$ when Conditions 1, 3, and 4 hold.

2.2 Statement of the Basic Conceptual Algorithm

For convenience we view α as increasing from 0 toward 1.

Step 1: Solve (Po) by any convenient method, so that $(\underline{x}^*(0), \underline{u}^*(0), S^*)$ satisfying (KT-1) through (KT-4) at $\alpha = 0$ is at hand. Put $\alpha^o = 0$, $S^o = S^*$, and $(\underline{x}, \underline{u})^o = (\underline{x}^*(0), \underline{u}^*(0))$.

Step 2: Solve equations $(=S^o)\alpha$ by any convenient method as α increases above α^o for the unique continuous solution[5] $(\underline{x}^{S^o}(\alpha), \underline{u}^{S^o}(\alpha))$ satisfying the left

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[5] Throughout this work we employ the symbol $(\underline{x}^S(\alpha), \underline{u}^S(\alpha))$ to denote a solution of equations $(=S)\alpha$.

end-point value $(\underline{x},\underline{u})^o$ so long as this solution satisfies
(KT-3) and (KT-4); that is, until $\alpha = \alpha'$, where

$$\alpha' \stackrel{\Delta}{=} \text{Max } \{\alpha: \ \alpha^o \leq \alpha \leq 1, \ g_i(\underline{x}^{S^o}(\alpha)) \geq 0, \ \forall \ i \notin S^o,$$

$$u_i^{S^o}(\alpha) \geq 0, \ \forall \ i \in S^o \ \text{ on } \ [\alpha^o, \alpha']\} \ .$$

If $\alpha' = 1$, terminate. Otherwise put $(\underline{x},\underline{u})^o =$
$(\underline{x}^{S^o}(\alpha'), \ \underline{u}^{S^o}(\alpha'))$ and go to Step 3.

Step 3: Solve equations $(=S)\alpha$ by any convenient method as
α increases above α' for the unique continuous
solution $(\underline{x}^S(\alpha), \ \underline{u}^S(\alpha))$ satisfying the left end-point
value $(\underline{x},\underline{u})^o$ for different sets S which satisfy

(8.1) $$\{i \in M: \ u_i^{S^o}(\alpha') > 0\} \subseteq S \subseteq \{i \in M: \ g_i(\underline{x}^{S^o}(\alpha')) = 0\}$$

until for some S', $(\underline{x}^{S'}(\alpha), \ \underline{u}^{S'}(\alpha))$ satisfies (KT-3)
and (KT-4) on $[\alpha',\alpha'+\epsilon]$ for some $\epsilon > 0$. Put
$\alpha^o = \alpha'$, $S^o = S'$, and return to Step 2.

The next subsection is devoted to the development of the theo-
retical results necessary for justifying this conceptual algorithm.
Complete justification requires proof of the following

Theorem (Basic):

Assume that Conditions 1 through 4 hold. Then the following
assertions regarding the Basic Conceptual Algorithm hold:

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(i)   Step 2 is well-defined.

(ii) At each execution of Step 2, $(\underline{x}^{S^o}(\alpha), \underline{u}^{S^o}(\alpha)) = (\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ on $[\alpha^o, \alpha']$.

(iii) Step 3 is well-defined.

(iv) Step 3 will be executed only a finite number of times before termination obtains.

2.3   Theoretical Development

Continuity plays a crucial role in parametric programming.

<u>Theorem 1</u> (Continuity):

(i)   Assume that Conditions 1 through 3 hold. Then $(P\alpha)$ has a unique optimal solution $\underline{x}^*(\alpha)$, and $\underline{x}^*(\alpha)$ is continuous on some open interval containing $[0,1]$.

(ii) Assume that Conditions 1 through 4 hold. Then $(P\alpha)$ has unique dual variables $u_i^*(\alpha)$ $(i = 1,\ldots,m)$ such that $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ satisfies the Kuhn-Tucker Conditions (5) through (7), and $\underline{u}^*(\alpha)$ is continuous, on some open interval containing $[0,1]$.

<u>Proof</u>: First we prove (i). The existence of an optimal solution of $(P\alpha)$ for any fixed value of α follows from the fact that $f(\underline{x};\alpha)$ is a continuous function of $\underline{x}$ on the compact set X. The uniqueness of the optimal solution follows by A.2 from the fact that

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$f(\underline{x};\alpha)$ is strictly concave in $\underline{x}$ over the convex set X for each fixed value of α in some open interval $\mathcal{U}$ containing $[0,1]$. Denote the unique optimal solution by $\underline{x}^*(\alpha)$.

To demonstrate that $\underline{x}^*(\alpha)$ is continuous on $\mathcal{U}$, suppose the contrary. Then there exists a sequence $<\alpha^\nu> \to \overline{\alpha}$ with α^ν, $\overline{\alpha} \in \mathcal{U}$ such that $<\underline{x}^*(\alpha^\nu)> \not\to \underline{x}^*(\overline{\alpha})$. Hence there is an (open) neighborhood $N(\underline{x}^*(\overline{\alpha}))$ of $\underline{x}^*(\overline{\alpha})$ such that $\underline{x}^*(\alpha^\nu) \not\in N(\underline{x}^*(\overline{\alpha}))$ infinitely often, and by taking a subsequence, if necessary, we may assume that this holds for all ν. Since[6] $X-N(\underline{x}^*(\overline{\alpha}))$ is compact we may assume, again taking a subsequence if necessary, that $<\underline{x}^*(\alpha^\nu)> \to \underline{x}' \in \{X-N(\underline{x}^*(\overline{\alpha}))\}$. Thus by the continuity of $f(\underline{x};\alpha)$ with respect to $(\underline{x},\alpha)$, we obtain

$$(9) \qquad <f(\underline{x}^*(\alpha^\nu);\alpha^\nu)> \to f(\underline{x}';\overline{\alpha}) \ .$$

Now $f(\underline{x}^*(\alpha);\alpha) \equiv \underset{\underline{x}}{\text{Max}} \ \{f(\underline{x};\alpha) \text{ subject to } \underline{g}(\underline{x}) \geq \underline{0}\}$ is the supremum of a family of functions linear in α, and therefore is convex in α on $\mathcal{U}$. Using A.5, we obtain

$$(10) \qquad <f(\underline{x}^*(\alpha^\nu);\alpha^\nu)> \to f(\underline{x}^*(\overline{\alpha});\overline{\alpha}) \ .$$

Assertions (9) and (10) imply that $f(\underline{x}';\overline{\alpha}) = f(\underline{x}^*(\overline{\alpha});\overline{\alpha})$; but by construction $\underline{x}' \neq \underline{x}^*(\overline{\alpha})$, so that the unique optimality of $\underline{x}^*(\overline{\alpha})$

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[6] When used with sets, the symbol "-" denotes relative complement. Thus $X-N(\underline{x}^*(\alpha)) \triangleq \{\underline{x} \in X: \ \underline{x} \not\in N(\underline{x}^*(\alpha))\}$.

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is violated. Hence $\underline{x}^*(\alpha)$ must be continuous on $\mathcal{U}$. This completes the proof of (i).

Now we prove (ii). The existence of $\underline{u}^*(\alpha)$ such that $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ satisfies (5) through (7) on some open interval containing $[0,1]$ would follow from the necessity of the Kuhn-Tucker Conditions if the hypotheses of the Kuhn-Tucker Theorem were satisfied by $(P\alpha)$ on such an interval. It was noted in subsection 2.1 that these hypotheses are satisfied for each value of $\alpha \epsilon [0,1]$. To show that this remains true on some open interval containing $[0,1]$, in view of Condition 1, Proposition 3, and the remark following the statement of the Kuhn-Tucker Theorem, it is enough to show that Condition 4 is still satisfied on some open interval containing each end-point. Consider the left end-point $\alpha = 0$. Denote by $D(\alpha)$ the matrix whose rows are $\nabla_x g_i(\underline{x}^*(\alpha))$, i such that $g_i(\underline{x}^*(0)) = 0$. By Condition 4 applied at $\alpha = 0$, $D(0)$ has rank equal to the number of its rows, which is equivalent to the existence of $[D(0)D^t(0)]^{-1}$, which is equivalent to the determinantal inequality $|D(0)D^t(0)| \neq 0$. Since $|D(\alpha)D^t(\alpha)|$ is a continuous function of α for α sufficiently near 0, it does not vanish in some open interval containing $\alpha = 0$, and so $D(\alpha)$ remains of maximal rank on such an interval. This implies that Condition 4 holds on some open interval containing $\alpha = 0$, for by the continuity of $\underline{x}^*(\alpha)$ and of $g_i(\underline{x})$, and hence of $g_i(\underline{x}^*(\alpha))$, one easily obtains that $\{i: g_i(\underline{x}^*(\alpha)) = 0\} \subseteq \{i: g_i(\underline{x}^*(0)) = 0\}$ for α sufficiently near 0. A similar argument applies to $\alpha = 1$.

To show the uniqueness and continuity of $\underline{u}*(\alpha)$ on some open interval containing $[0,1]$, fix $\alpha_o \in [0,1]$. Since $\underline{x}*(\alpha)$ is unique, from (7) we conclude that $u_i^*(\alpha_o)$ must vanish for each i such that $g_i(\underline{x}*(\alpha_o)) > 0$. By the continuity of $g_i(\underline{x}*(\alpha))$, we have that $g_i(\underline{x}*(\alpha)) > 0$ on some open interval about α_o when $g_i(\underline{x}*(\alpha_o)) > 0$. Hence $u_i^*(\alpha)$ vanishes on some open interval about α_o for each i such that $g_i(\underline{x}*(\alpha_o)) > 0$. Denote $\{i: g_i(\underline{x}*(\alpha_o)) = 0\}$ by B. It remains to consider $u_i^*(\alpha)$, $i \in B$. From (5) and (7) one obtains

$$(11) \qquad \nabla_x f(\underline{x}*(\alpha_o);\alpha_o) + \sum_{i \in B} u_i^*(\alpha_o) \nabla_x g_i(\underline{x}*(\alpha_o)) = \underline{0} .$$

Since by continuity $\{i: g_i(\underline{x}*(\alpha)) = 0\} \subseteq \{i: g_i(\underline{x}*(\alpha_o)) = 0\} = B$ for α sufficiently near α_o, it follows from (5) and (7) that (11) must hold in some open interval about α_o _with_ _the_ _same_ _summation_ _set_. That is,

$$(12) \qquad \nabla_x f(\underline{x}*(\alpha);\alpha) + \sum_{i \in B} u_i^*(\alpha) \nabla_x g_i(\underline{x}*(\alpha)) = \underline{0}$$

holds on some open interval about α_o. Write $\underline{u}_B^*(\alpha)$ for the row vector whose components are $u_i^*(\alpha)$, $i \in B$. Then (12) can be rewritten in matrix notation as

$$(12.1) \qquad \underline{u}_B^*(\alpha)\widetilde{D}(\alpha) = - \nabla_x f(\underline{x}*(\alpha);\alpha) .$$

Repeating a previous argument, one may assert that $[\widetilde{D}(\alpha)\widetilde{D}^t(\alpha)]^{-1}$ exists on some open interval containing α_o. Postmultiplying (12.1)

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by $\widetilde{D}^t(\alpha)[\widetilde{D}(\alpha)\widetilde{D}^t(\alpha)]^{-1}$, one obtains that $\underline{u}_B^*(\alpha)$ must satisfy[7]

(12.2) $\qquad \underline{u}_B^*(\alpha) = - \nabla_x f(\underline{x}^*(\alpha);\alpha)\widetilde{D}^t(\alpha)[\widetilde{D}(\alpha)\widetilde{D}^t(\alpha)]^{-1}$

on some open interval containing α_o. The right-hand side is unique and continuous in α, and therefore $\underline{u}_B^*(\alpha)$ is also unique and continuous on some open interval containing α_o.

It will prove convenient to introduce some special notations. Define $A\alpha$ to be the set of constraint indices corresponding to the constraints which are <u>active</u> at α in the sense that their dual variables are strictly positive:

$$A\alpha \overset{\Delta}{=} \{i \in M: \ u_i^*(\alpha) > 0\} \ .$$

Define $B\alpha$ to be the set of constraint indices corresponding to the constraints which are <u>binding</u> at $\underline{x}^*(\alpha)$:

$$B\alpha \overset{\Delta}{=} \{i \in M: \ g_i(\underline{x}^*(\alpha)) = 0\} \ .$$

The sets $A\alpha$ and $B\alpha$ are well-defined on some open interval containing $[0,1]$ because of the existence and uniqueness of $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ on some such an interval. We can now state two important corollaries of Theorem 1.

Corollary 1.1:

Assume that Conditions 1 through 4 hold. Then for each $\alpha_o \in [0,1]$

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[7] Equation (12.2) is intended only for theoretical and not computational use.

there exists an open interval containing α_0 such that, on this interval,

$$A\alpha_0 \subseteq A\alpha \subseteq B\alpha \subseteq B\alpha_0 .$$

Proof: The outermost relations follow directly from the definitions of $A\alpha$ and $B\alpha$ and the continuity of $\underline{x}^*(\alpha)$ and $\underline{u}^*(\alpha)$. The middle relation follows from (7).

Corollary 1.2:

Assume that Conditions 1 through 4 hold. Then there is an open interval containing $[0,1]$ such that, for each fixed value of α in this open interval, a subset S of constraint indices is valid at α if and only if $A\alpha \subseteq S \subseteq B\alpha$.

Proof: This assertion is an immediate consequence of the uniqueness of $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$, and Proposition 2.

The significance of Corollaries 1.1 and 1.2 is that the totality of valid sets at $\alpha_0 \in [0,1]$ contains the totality of valid sets for α sufficiently near α_0. Hence the optimal solution of $(P\alpha_0)$, which yields $A\alpha_0$ and $B\alpha_0$, gives a strong indication of the identity of a valid set for α near α_0.

The next theorem shows that equations $(=S)\alpha$ can be solved on some open interval about $\alpha_0 \in [0,1]$ if S is valid at α_0.

Theorem 2:

Let $\alpha_0 \in [0,1]$ be fixed, let S be valid at α_0, and assume Conditions 1 through 4 hold.

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Then there exist an open interval $I\alpha_o$ containing and symmetric about α_o, and an open neighborhood $N(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$ containing $(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$, such that on $I\alpha_o$ there is a unique function $(\underline{x}^S(\alpha), \underline{u}^S(\alpha))$ in $N(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$ which satisfies $(=S)\alpha$. Furthermore, $(\underline{x}^S(\alpha), \underline{u}^S(\alpha))$ is analytic on $I\alpha_o$.

Proof: The theorem would follow directly from a version of the Implicit Function Theorem (Bochner and Martin, 1948, p. 39) applied to the equations $(=S)\alpha$ if the following hypotheses of that theorem were satisfied:

(a) $(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$ satisfies $(=S)\alpha_o$.

(b) The left-hand side of each equation of $(=S)\alpha$ is analytic in $(\underline{x}, \underline{u}, \alpha)$ in an open neighborhood of $(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o), \alpha_o)$.

(c) The Jacobian $\dfrac{\partial((=S)\alpha_o)}{\partial(\underline{x}, \underline{u})}$ is non-zero at $(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$.

By the validity of S at α^o, part (i) of Proposition 1 and Corollary 1.2, (a) holds. It follows from Condition 1 that (b) holds. To simplify the task of showing that (c) holds, we regroup the order of partial differentiation, which is equivalent to regrouping the columns of the Jacobian matrix, so that we actually consider the Jacobian $\dfrac{\partial((=S)\alpha_o)}{\partial(\underline{x}; u_i, i \in S; u_i, i \notin S)}$. Writing H for the n by n hessian matrix

$$\nabla^2_x \Big(f(\underline{x}^*(\alpha_o); \alpha_o) + \sum_1^m u_i^*(\alpha_o) \, g_i(\underline{x}^*(\alpha_o)) \Big)$$

and D for the matrix whose rows are $\nabla_x g_i(\underline{x}^*(\alpha_o))$, $i \in S$, one

readily derives that this Jacobian, evaluated at $(\underline{x}*(\alpha_o), \underline{u}*(\alpha_o))$, is the determinant of the matrix (we use dotted line to denote partition)

$$\begin{bmatrix} H & \vdots & D^t & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ D & \vdots & 0 & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & I \end{bmatrix}$$

where 0 and I are zero and identity matrices of the appropriate orders. The determinant is non-zero if and only if

$$\begin{bmatrix} H & \vdots & D^t \\ \cdots & \vdots & \cdots \\ D & \vdots & 0 \end{bmatrix}$$

is invertible, which is true if and only if the matrix equation

(13)
$$\begin{bmatrix} H & \vdots & D^t \\ \cdots & \vdots & \cdots \\ D & \vdots & 0 \end{bmatrix} \begin{pmatrix} \underline{y} \\ \cdots \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \cdots \\ \underline{0} \end{pmatrix}$$

has $\underline{y} = \underline{0}$, $\underline{z} = 0$ as its only solution, where $\underline{y}$ is an n-vector and $\underline{z}$ is a vector with a number of components equal to the number of constraint indices in S. The proof of the theorem will be complete when we show that (13) has only the null solution.

Performing the indicated block multiplications for (13), one obtains

(13.1)
$$H\underline{y} + D^t\underline{z} = \underline{0} \quad \text{and}$$

(13.2)
$$D\underline{y} \qquad = \underline{0} \; .$$

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Now H is negative definite, for it is a positive linear combination of negative semidefinite hessians, at least one of which is known to be negative definite. Hence H is invertible, and (13.1) yields

$$(13.3) \qquad \underline{y} = -H^{-1}D^t\underline{z} \, .$$

Premultiplying (13.3) by D and using (13.2), one obtains

$$(13.4) \qquad D\underline{y} = -DH^{-1}D^t\underline{z} = \underline{0} \, .$$

By Corollary 1.2, $A\alpha_0 \subseteq S \subseteq B\alpha_0$. By Condition 4, therefore, D is of maximal rank, and that rank equals the number of rows of D. Hence $[DH^{-1}D^t]$ is invertible, and (13.4) yields $\underline{z} = \underline{0}$. By (13.3), $\underline{y} = \underline{0}$ also. Thus (13) has only the null solution.

Corollary 2.1:

Let $\alpha_0 \in [0,1]$ be fixed, let S be valid at α_0, and assume that Conditions 1 through 4 hold.

Then there exists an open interval containing α_0 and contained in $I\alpha_0$ such that, for each fixed value of α in this interval, the following three assertions are equivalent:

(i) S is valid at α.

(ii) $(\underline{x}^S(\alpha), \underline{u}^S(\alpha)) = (\underline{x}*(\alpha), \underline{u}*(\alpha))$.

(iii) $g_i(\underline{x}^S(\alpha)) \geq 0, \; \forall \; i \notin S$

$\qquad u_i^S(\alpha) \geq 0, \; \forall \; i \in S$.

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Proof: (i) $\implies$ (ii). By continuity, $(\underline{x}^*(\alpha), \underline{u}^*(\alpha)) \in N(\underline{x}^*(\alpha_o),$ $\underline{u}^*(\alpha_o))$ for all α sufficiently near α_o; by the validity of S at α, part (i) of Proposition 1, and Corollary 1.2, one concludes that $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ satisfies $(=S)\alpha$; since the solution of $(=S)\alpha$ is <u>unique</u> in $N(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$ for $\alpha \in I\alpha_o$, assertion (ii) follows.

(ii) $\implies$ (iii). Because $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ satisfies (5) through (7), (iii) must hold.

(iii) $\implies$ (i). Assertion (iii) and the fact that $(\underline{x}^S(\alpha),$ $\underline{u}^S(\alpha))$ satisfies $(=S)\alpha$ imply by the definition of validity that S is valid at α.

One more result must be established before a complete proof of the Basic Theorem can be given.

Define a <u>point of change</u> of Bα as a point α' with the property that there is no open interval containing α' such that Bα = Bα' everywhere on that interval. A similar definition holds for a point of change of Aα. In the sequel, the phrase "point of change" is used to refer to either a point of change of Aα or of Bα, or possibly of both.

Theorem 3 (Finiteness):

Assume that Conditions 1 through 4 hold. Then Aα and Bα each have a finite number of points of change on [0,1].

Proof: Suppose that Bα has a finite number of points of change on [0,1]. Then there is a cluster point $\bar{\alpha} \in [0,1]$ of

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these points of change. Let $< \alpha^\nu >$, $\alpha^\nu \in [0,1]$, be a sequence of distinct points of change of $B\alpha$ which converges to $\bar{\alpha}$. Applying Corollary 1.1 at α^ν, we see that there exists an open interval containing α^ν such that $A\alpha^\nu \subseteq A\alpha \subseteq B\alpha \subseteq B\alpha^\nu$ on this interval. By the definition of a point of change of $B\alpha$, for each α^ν there exists a number β^ν contained in this interval and in $(\alpha^\nu - \frac{1}{\nu},$ $\alpha^\nu + \frac{1}{\nu})$ such that $A\alpha^\nu \subseteq A\beta^\nu \subseteq B\beta^\nu \subset B\alpha^\nu$ (note that $B\beta^\nu$ is a proper subset of $B\alpha^\nu$). Clearly $< \beta^\nu > \to \bar{\alpha}$. From Corollary 1.1 applied at $\bar{\alpha}$, we see that we have demonstrated the existence of two sequences $< \alpha^\nu > \to \bar{\alpha}$, $< \beta^\nu > \to \bar{\alpha}$, such that $A\bar{\alpha} \subseteq A\alpha^\nu \subseteq A\beta^\nu \subseteq B\beta^\nu \subset B\alpha^\nu \subseteq B\bar{\alpha}$ for all ν sufficiently large. Since there is but a finite number (2^m) of possible sets which $B\beta^\nu$ or $B\alpha^\nu$ could possibly be, we may assume, taking a subsequence if necessary, that there exist sets B' and B'' such that $B\beta^\nu = B'' \subset B\alpha^\nu = B'$ for all ν.

Consider the function $\underline{x}^{B''}(\alpha)$ defined as in Theorem 2 applied at $\bar{\alpha}$. Since B'' is valid at $\bar{\alpha}$ and at all α^ν and β^ν, ν sufficiently large, $\underline{x}^{B''}(\alpha) = \underline{x}^*(\alpha)$ at these points. Take $i_o \in B'-B''$. Then $g_{i_o}(\underline{x}^{B''}(\alpha^\nu)) = 0$ and $g_{i_o}(\underline{x}^{B''}(\beta^\nu)) > 0$, all ν sufficiently large, and $g_{i_o}(\underline{x}^{B''}(\bar{\alpha})) = 0$. In other words, we have shown that $\bar{\alpha}$ is a non-isolated zero of $g_{i_o}(\underline{x}^{B''}(\alpha))$, and that this function is not identically zero on any open interval about $\bar{\alpha}$. But this leads to a contradiction of the well-known fact (Apostol, 1957, p. 518) that the zeros of an analytic function which is not identically zero are isolated, for by Theorem 2 and Condition 1 we have that $g_{i_o}(\underline{x}^{B''}(\alpha))$

is analytic on some open interval about $\bar{\alpha}$. Hence the supposition that $B\alpha$ has an infinite number of points of change on $[0,1]$ is false.

A similar argument shows that $A\alpha$ cannot have an infinite number of points of change on $[0,1]$.

Applying the result of Theorem 3 to a given $(P\alpha)$, define $0 \leq \alpha_1' < \alpha_2' < \cdots < \alpha_N' \leq 1$ to be the collection of all points of change of $A\alpha$ or $B\alpha$ or both. As a matter of convention we take $\alpha_0' = 0$ and $\alpha_{N+1}' = 1$. From Corollaries 1.1 and 1.2 we conclude that any set which is valid at α, $\alpha_j' < \alpha < \alpha_{j+1}'$, is also valid on the entire closed interval $[\alpha_j', \alpha_{j+1}']$. In addition, it may also be valid on other intervals, of course. Among the sets which are valid at α_j' there are all those which are valid on $[\alpha_{j-1}', \alpha_j']$ or on $[\alpha_j', \alpha_{j+1}']$.

We are now in a position to prove the Basic Theorem.

Proof (Basic Theorem): First we prove parts (i) and (ii). At the beginning of each Step 2, $(\underline{x},\underline{u})^o$ and S^o satisfy (KT-1) through (KT-4) at α^o, so that S^o is valid at α^o and $(\underline{x},\underline{u})^o = (\underline{x}^*(\alpha^o), \underline{u}^*(\alpha^o))$. Let J, $1 \leq J \leq N+1$ be the largest integer such that S^o is valid on $[\alpha^o, \alpha_J']$ ($\alpha_J' = \alpha_1' = \alpha^o = 0$ is permissible the first time Step 2 is executed). Applying Theorem 2 at each point of $[\alpha^o, \alpha_J']$, it follows that $(=S^o)\alpha$ has a unique analytic solution $(\underline{x}^{S^o}(\alpha), \underline{u}^{S^o}(\alpha))$ satisfying the left end-point value $(\underline{x},\underline{u})^o$ on some interval containing $[\alpha^o, \alpha_J']$. This solution satisfies (KT-3) and

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(KT-4) and equals $(\underline{x}*(\alpha), \underline{u}*(\alpha))$ on $[\alpha^O, \alpha_J']$ by Corollary 2.1.
If $\alpha_J' = 1$, the solution of (Pα) on $[0,1]$ is complete. If
$\alpha_J' < 1$, however, $(\underline{x}^{S^O}(\alpha), \underline{u}^{S^O}(\alpha))$ does not satisfy (KT-3) and
(KT-4) for any $\alpha \in (\alpha_J', \alpha_{J+1}')$, for otherwise by Corollary 2.1
applied at α_J', S^O would be valid on $[\alpha_J', \alpha_{J+1}']$, which would
violate the definition of J. Clearly the scalar α' defined in
Step 2 is precisely α_J', and (i) and (ii) hold.

Next we prove (iii). Any set S which satisfies (8.1) is valid
at α', by Corollary 1.2 and the fact that $(\underline{x}^{S^O}(\alpha'), \underline{u}^{S^O}(\alpha')) =$
$(\underline{x}*(\alpha'), \underline{u}*(\alpha'))$. Applying Theorem 2 at α', we see that if S
satisfies (8.1) then (=S)α has a solution as stated on $[\alpha', \alpha'+\epsilon_1]$
for some $\epsilon_1 > 0$. By Corollary 1.1 we know that at least one such
S, say S', is valid on $[\alpha', \alpha'+\epsilon_2]$ for some $0 < \epsilon_2 \leq \epsilon_1$; by
Corollary 2.1 applied at α', $(\underline{x}^{S'}(\alpha), \underline{u}^{S'}(\alpha))$ satisfies (KT-3)
and (KT-4) on $[\alpha', \alpha'+\epsilon]$ for some $0 < \epsilon \leq \epsilon_2$. Since there is but
a finite number of sets satisfying (8.1), S' will be found after a
finite number of trials.

Finally, we prove (iv). It was established in the proof of (i)
that Step 3 is entered each time a point of change α' is encountered
at Step 2 such that the current set S^O being used at Step 2 is not
valid immediately above α'. It was established in the proof of
(iii) that Step 3 finds a set which is valid immediately above α'
in a finite number of trails, and control is returned to Step 2 along
with the new valid set. By convention we have taken α increasing,
and by Theorem 3 there is but a finite number of points of change on

[0,1]; it follows that Step 3 will only have to be executed a finite number of times before termination obtains.

Remark: At Step 2, α' need not be the <u>next</u> point of change above α^o, for S^o may remain valid on an interval spanning several points of change. The algorithm could be modified to require $S^o = B\alpha$ at Step 2, so that α' would assume, in turn, the values of each point of change of $B\alpha$; or one could require $S^o = A\alpha$ at Step 2, so that α' would assume, in turn, the values of each point of change of $A\alpha$. The minimum requirement (the one adopted here) is $A\alpha \subseteq S^o \subseteq B\alpha$ at Step 2, and seems more symmetrical and less arbitrary than either of the extreme requirements just mentioned.

From the proof of the Basic Theorem, it is clear that the Basic Conceptual Algorithm can be paraphrased as follows.

Step 1: By any convenient method, find the optimal solution $(\underline{x}*(0), \underline{u}*(0))$ of (Po). Set $\alpha^o = 0$, S^o equal to any set valid at $\alpha = 0$, and $(\underline{x},\underline{u})^o = (\underline{x}*(0), \underline{u}*(0))$.

Step 2: Solve $(=S^o)\alpha$ as α increases above α^o for its unique continuous solution satisfying the left end-point condition $(\underline{x}^{S^o}(\alpha^o), \underline{u}^{S^o}(\alpha^o)) = (\underline{x},\underline{u})^o$, namely $(\underline{x}*(\alpha), \underline{u}*(\alpha))$, until either $\alpha = 1$ or a point of change α' of $A\alpha$ or $B\alpha$ is encountered to the right of which S^o is no longer valid. In the first case,

terminate; in the second case, set $(\underline{x},\underline{u})^\circ = (\underline{x}^*(\alpha'),$ $\underline{u}^*(\alpha'))$ and go to Step 3.

Step 3: Among all sets valid at α', find one which is valid to the right of α'. Call it S'. Set $\alpha^\circ = \alpha'$, $S^\circ = S'$, and return to Step 2.

See Appendix B for graphical illustrations of this algorithm.

Now that the Basic Conceptual Algorithm has been theoretically justified, we take up computational considerations.

3. A Basic Computational Algorithm

In order to implement the Basic Conceptual Algorithm, it is necessary to have a method of actually solving $(=S)\alpha$ as α changes parametrically. Only in certain simple cases is it possible or economical to solve these equations analytically, and so usually numerical methods must be used. We recommend Newton's method, or a variation thereof, as an efficient means of solving $(=S)\alpha$ on a digital computer as α changes by small discrete jumps.

After proving the applicability of Newton's method, we state and prove a Basic Computational Algorithm. Some necessary computational refinements are then briefly indicated, with further details being added in Appendix C.

3.1 Newton's Method

Newton's method is briefly reviewed in Appendix C. Under Conditions 1 through 4, it is easily seen from Theorem C.1 of Appendix C and the proof of Theorem 2 that for each $\alpha_o \in [0,1]$, Newton's method applied

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to $(=S)\alpha_o$ is well-defined and quadratically convergent to $(\underline{x}*(\alpha_o)$, $\underline{u}*(\alpha_o))$ if S is valid at α_o and if the starting point $(\underline{x},\underline{u})^o$ is in a sufficiently small neighborhood of $(\underline{x}*(\alpha_o)$, $\underline{u}*(\alpha_o))$. Since $(\underline{x}*(\alpha)$, $\underline{u}*(\alpha))$ is continuous, by taking $\Delta\alpha$ small enough $(\underline{x}*(\alpha_o-\Delta\alpha)$, $\underline{u}*(\alpha_o-\Delta\alpha))$ is such a starting point. In other words, Newton's method is applicable point by point. Does there exist $\Delta\alpha > 0$ such that a computational algorithm can be designed using Newton's method to solve $(=S)\alpha$ with $\Delta\alpha$ as a fixed step size throughout? The answer is affirmative, and requires a proof that the size of the neighborhoods mentioned above may be taken to be bounded away from zero.

Theorem 4.1:

Let Conditions 1 through 4 hold, let $\alpha_o \in [0,1]$ not a point of change be fixed, and let S be valid at α_o.

Then there exists a scalar $r' > 0$, which does not depend on α_o or on S, such that Newton's method applied to equations $(=S)\alpha_o$ is well-defined and quadratically convergent to $(\underline{x}*(\alpha_o)$, $\underline{u}*(\alpha_o))$ if the starting point $(\underline{x},\underline{u})^o$ is in the ($n+m$ dimensional) neighborhood $N_{r'}(\underline{x}*(\alpha_o)$, $\underline{u}*(\alpha_o))$.

Proof:

1. We shall use the notation and observations immediately following the proof of Theorem 3. To prove this theorem it is sufficient to show that for each j $(j = 0,\ldots,N)$ there exists a scalar $r(j) > 0$ such that the following assertions hold on $N_{r(j)}$ $(\underline{x}*(\alpha_o)$, $\underline{u}*(\alpha_o))$ for any fixed $\alpha_o \in [\alpha_j', \alpha_{j+1}']$ and any S valid on $[\alpha_j', \alpha_{j+1}']$:

(a) The left-hand side of each equation of $(=S)\alpha_0$ is twice continuously differentiable with respect to $(\underline{x},\underline{u})$.

(b) The Jacobian $\dfrac{\partial((=S)\alpha_0)}{\partial(\underline{x},\underline{u})} \neq 0$.

(c) $\Delta(\underline{x},\underline{u}; \alpha_0, S) \leq L < 1$, where $\Delta(\underline{x},\underline{u};\alpha_0,S)$ is a certain upper estimate of the norm of the Jacobian matrix of the iteration function derived by applying Newton's method to $(=S)\alpha_0$ (see section 1 of Appendix C).

To see why this plan is sufficient, let $r' = \text{Min}\{r(0),\ldots,r(N)\}$, let $\alpha_0 \in [0,1]$ not a point of change be fixed, and let S be valid at α_0. Then for some j between 0 and N we have that S is valid on $[\alpha'_j, \alpha'_{j+1}]$ and $\alpha_0 \in [\alpha'_j, \alpha'_{j+1}]$. Applying Theorem C.2 of Appendix C, we see that Newton's method applied to $(=S)\alpha_0$ is well-defined and quadratically convergent to $(\underline{x}^*(\alpha_0), \underline{u}^*(\alpha_0))$ if the starting point $(\underline{x},\underline{u})^0 \in N_{r'}(\underline{x}^*(\alpha_0), \underline{u}^*(\alpha_0))$.

2. Let j be fixed, $0 \leq j \leq N$, let $\alpha_0 \in [\alpha'_j, \alpha'_{j+1}]$, and let S be any set which is valid on $[\alpha'_j, \alpha'_{j+1}]$.

By Condition 1, the left-hand side of each equation of $(=S)\alpha_0$ is twice continuously differentiable with respect to $(\underline{x},\underline{u})$ on some open neighborhood of $(\underline{x}^*(\alpha_0), \underline{u}^*(\alpha_0))$.

The Jacobian $\dfrac{\partial((=S)\alpha_0)}{\partial(\underline{x},\underline{u})} \neq 0$ at $(\underline{x}^*(\alpha_0), \underline{u}^*(\alpha_0))$ by the proof of Theorem 2. As a consequence of Condition 1, this Jacobian is continuous with respect to $(\underline{x},\underline{u})$ on some open neighborhood of $(\underline{x}^*(\alpha_0), \underline{u}^*(\alpha_0))$. One concludes that the Jacobian does not vanish in some open neighborhood of $(\underline{x}^*(\alpha_0), \underline{u}^*(\alpha_0))$.

It can be shown in a straightforward manner (see Henrici, 1964, p. 106) that $\Delta(\underline{x},\underline{u}; \alpha_o, S)$ vanishes at $(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$. By Condition 1 this function is continuous with respect to $(\underline{x},\underline{u})$ on some open neighborhood of $(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$. One concludes that $\Delta(\underline{x},\underline{u}; \alpha_o, S) \leq L$, where $0 \leq L < 1$, on some open neighborhood about $(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$.

Summarizing this part of the proof, we assert that (a), (b), and (c) hold on some open neighborhood of $(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$ when $\alpha_o \in [\alpha_j', \alpha_{j+1}']$ and S is any set which is valid on $[\alpha_j', \alpha_{j+1}']$.

3. Since $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ is continuous on the compact set $[\alpha_j', \alpha_{j+1}']$, the image set

$$\Gamma \overset{\triangle}{=} \{(\underline{x},\underline{u}): \ (\underline{x},\underline{u}) = (\underline{x}^*(\alpha),\underline{u}^*(\alpha)) \ \text{ for some } \ \alpha, \alpha_j' \leq \alpha \leq \alpha_{j+1}'\}$$

is compact. It follows from the compactness of Γ and the result of part 2 of this proof that there exists a scalar $r(j) > 0$ such that (a), (b), and (c) hold on $N_{r(j)}(\underline{x}^*(\alpha_o), \underline{u}^*(\alpha_o))$ when $\alpha_o \in [\alpha_j', \alpha_{j+1}']$ and S is any set which is valid on $[\alpha_j', \alpha_{j+1}']$.

When Conditions 1 through 4 hold, we define ℓ_1 to be the minimum distance between any two points of change on $[0,1]$, and ℓ_2 to be the length of the shortest of all the intervals $I\alpha_o$ defined in Theorem 2 applied at every point of change on $[0,1]$ with each set which is valid at each point of change. Define $\overline{\ell} = \frac{1}{3} \text{Min}\{\ell_1, \ell_2\}$. Note that $(\underline{x}^S(\alpha), \underline{u}^S(\alpha))$ is uniquely defined, by Theorem 2 applied at α_j', on $\overline{I}\alpha_j' \overset{\triangle}{=} [\alpha_j' - \overline{\ell}, \alpha_j' + \overline{\ell}]$, for any $1 \leq j \leq N$ and any S valid at α_j'.

<u>Theorem 4.2</u>:

Let Conditions 1 through 4 hold, let $\alpha' \in [0,1]$ be a particular point of change, and let S be valid at α'.

Then there exist scalars $r'' > 0$ and $0 < \ell'' \le \bar{\ell}$, which do not depend on α' or on S, such that Newton's method applied to $(=S)\alpha_o$ is well-defined and quadratically convergent to $(\underline{x}^S(\alpha_o), \underline{u}^S(\alpha_o))$ if $\alpha_o \in [\alpha'-\ell'', \alpha'+\ell'']$ and if the starting point $(\underline{x},\underline{u})^o \in N_{r''}(\underline{x}^S(\alpha_o), \underline{u}^S(\alpha_o))$.

<u>Proof</u>:

1. Since there is a finite number of points of change on $[0,1]$ and a finite number of valid sets at each, it is sufficient to show that the theorem holds with r and ℓ possibly depending on α' and S. This will be done by applying Theorem C.2 of Appendix C.

2. Let $\alpha' \in [0,1]$ be a particular point of change, and let S be valid at α'. It remains to demonstrate the existence of scalars $r > 0$ and $0 < \ell \le \bar{\ell}$ such that the following three assertions hold on $N_r(\underline{x}^S(\alpha_o), \underline{u}^S(\alpha_o))$ when $\alpha_o \in [\alpha'-\ell, \alpha'+\ell]$:

   (a) The left-hand side of each equation of $(=S)\alpha_o$ is twice differentiable with respect to $(\underline{x},\underline{u})$.

   (b) The Jacobian $\dfrac{\partial((=S)\alpha_o)}{\partial(\underline{x},\underline{u})} \ne 0$.

   (c) $\triangle(\underline{x},\underline{u}; \alpha_o,S) \le L < 1$.

3. In view of the fact that $(\underline{x}^S(\alpha'), \underline{u}^S(\alpha')) = (\underline{x}*(\alpha'), \underline{u}*(\alpha'))$, we may argue as in part 2 of the proof of Theorem 4.1 that (a), (b),

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and (c) hold for $\alpha_0 = \alpha'$ on some open neighborhood of $(\underline{x}^S(\alpha')$, $\underline{u}^S(\alpha'))$.

4. Since $(\underline{x}^S(\alpha), \underline{u}^S(\alpha))$ is continuous on the closed interval $\bar{I}\alpha'$, and therefore uniformly continuous, one may assert the existence of scalars $r > 0$ and $0 < \ell \leq \bar{\ell}$ such that (a), (b), (c) hold on $N_r(\underline{x}^S(\alpha_0), \underline{u}^S(\alpha_0))$ when $\alpha_0 \in [\alpha'-\ell, \alpha'+\ell]$.

By specializing Theorem 4.2 to $\alpha_0 = \alpha'$, and recalling that $(\underline{x}^S(\alpha'), \underline{u}^S(\alpha')) = (\underline{x}*(\alpha'), \underline{u}*(\alpha'))$ when S is valid at α', it is evident that Theorem 4.1 is still true if α_0 is permitted to be a point of change. Since $(\underline{x}*(\alpha), \underline{u}*(\alpha))$ is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$, and one immediately obtains the following corollary of Theorem 4.1.

Corollary 4.1:

Let Conditions 1 through 4 hold, let $\alpha_0 \in [0,1]$, and let S be valid at α_0.

Then there exists a scalar $\delta' > 0$, which does not depend on α_0 or on S, such that Newton's method applied to $(=S)\alpha_0$ is well-defined and quadratically convergent to $(\underline{x}*(\alpha_0), \underline{u}*(\alpha_0))$ if the starting point is $(\underline{x}*(\alpha_0-\delta), \underline{u}*(\alpha_0-\delta))$ and $|\delta| \leq \delta'$, $0 \leq \alpha_0-\delta \leq 1$.

A similar argument shows that Theorem 4.2 yields the following corollary.

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Corollary 4.2:

Let Conditions 1 through 4 hold, let $\alpha' \in [0,1]$ be a particular point of change, and let S be valid at α'.

Then there exist scalars $\delta'' > 0$ and $0 < \ell'' \leq \overline{\ell}$, which do not depend on α' or on S, such that Newton's method applied to $(=S)\alpha_0$ is well-defined and quadratically convergent to $(\underline{x}^S(\alpha_0), \underline{u}^S(\alpha_0))$ if $\alpha_0 \in [\alpha'-\ell'', \alpha'+\ell'']$ and if $(\underline{x}*(\alpha_0-\delta), \underline{u}*(\alpha_0-\delta))$ is the starting point and $|\delta| \leq \delta''$, $0 \leq \alpha_0-\delta \leq 1$.

3.2 The Basic Computational Algorithm

Using the results of the previous subsection, we can design a computational counterpart of the Basic Conceptual Algorithm by using Newton's method to solve $(=S)\alpha$ as α increases by steps of size $\Delta\alpha$. A useful idealization is obtained by assuming that there is no computational error. In view of the quadratic nature of the convergence of Newton's method, it is no less plausible to assume that Newton's method converges to an exact solution of $(=S)\alpha$ when it theoretically should converge.[8] An annotated flow chart of the Basic Computational Algorithm is given in Figure 1.

Theorem 5:

Assume that Conditions 1 through 4 hold, that there is no computational error, and that Newton's method converges to an exact solution of $(=S)\alpha$ when it theoretically should converge.

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[8] This assumption is strictly true only when f_1 and f_2 are quadratic polynomials and all constraints are linear, in which case $(=S)\alpha$ is a set of <u>linear</u> equations in $(\underline{x},\underline{u})$ and Newton's method therefore leads to an exact solution in a single iteration.

Then there exist $\epsilon > 0$ and $\Delta\alpha > 0$ such that the Basic Computational Algorithm is well-defined and will terminate with $J\Delta\alpha = 1$ in a finite number of computational steps.

Proof: Put

$$\epsilon_1 = \frac{1}{3} \underset{\substack{1 \leq j \leq N \\ i \in A\alpha'_j}}{Min} \{u_i^*(\alpha'_j)\} .$$

By construction, $\epsilon_1 > 0$. By the uniform continuity of $u_i^*(\alpha)$ $(i = 1,\ldots,m)$ on $[0,1]$, there exists a scalar $\delta_1 > 0$ such that $|\alpha-\alpha'_j| \leq \delta_1$ implies $|u_i^*(\alpha)-u_i^*(\alpha'_j)| < \epsilon_1$ $(i = 1,\ldots,m)$ for any j $(j = 1,\ldots,N)$. Put

$$\epsilon_2 = \frac{1}{3} \underset{\substack{1 \leq j \leq N \\ i \notin B\alpha'_j}}{Min} g_i(\underline{x}^*(\alpha'_j)) .$$

By construction, $\epsilon_1 > 0$. By the uniform continuity of $g_i(\underline{x}^*(\alpha))$ $(i = 1,\ldots,m)$ on $[0,1]$, there exists a scalar $\delta_2 > 0$ such that $|\alpha-\alpha'_j| \leq \delta_2$ implies $|g_i(\underline{x}^*(\alpha)) - g_i(\underline{x}^*(\alpha'_j))| < \epsilon_2$ $(i = 1,\ldots,m)$ for any j $(j = 1,\ldots,N)$.

Put $\epsilon^* = Min\{\epsilon_1,\epsilon_2\}$ and $\Delta\alpha^* = 1/K$, where K is the smallest integer satisfying $K \geq 2/Min\{\delta_1,\delta_2,\delta',\delta'',\ell''\}$. In view of the Basic Theorem, to prove this theorem it is sufficient to show that for these choices of ϵ and $\Delta\alpha$ Newton's method is well-defined and sure to be convergent as stated in Steps 2 and 3, and that the trials at Step 3 must lead to a success.

At each application of Newton's method during Step 2,
$(\underline{x}^{J-1}, \underline{u}^{J-1}) = (\underline{x}*((J-1)\Delta\alpha), \underline{u}*((J-1)\Delta\alpha))$ and S^O is valid at
$\alpha = (J-1)\Delta\alpha$. If S^O is valid at $J\Delta\alpha$, then since $\Delta\alpha* \leq \delta'$ we
have by Corollary 4.1 that Newton's method is well-defined and con-
vergent to $(\underline{x}*(J\Delta\alpha), \underline{u}*(J\Delta\alpha))$. If S^O is not valid at $J\Delta\alpha$,
then since $\Delta\alpha* \leq \ell" \leq \overline{\ell}$ there must be exactly one point of change
$\alpha' \leq 1$ on $[(J-1)\Delta\alpha, J\Delta\alpha]$; but S^O is valid at α', $\Delta\alpha* \leq \ell"$,
and $\Delta\alpha* \leq \delta'$, so by Corollary 4.2 Newton's method is well-defined
and convergent to $(\underline{x}^{S^O}(J\Delta\alpha), \underline{u}^{S^O}(J\Delta\alpha))$, and Step 3 is entered. By
the choice of $\epsilon*$, $A = A\alpha'$ and $B = B\alpha'$. Corollary 4.2 again applies,
and ensures that Newton's method is well-defined and convergent to
$(\underline{x}^S(J\Delta\alpha), \underline{u}^S(J\Delta\alpha))$ when $A \subseteq S \subseteq B$. The trials are sure to lead to
a success because some set which is valid at α' must also be valid
at $J\Delta\alpha$, since α' is the only point of change on $[(J-1)\Delta\alpha, J\Delta\alpha]$.

A word is in order about the consequences of taking ϵ and $\Delta\alpha$
different from $\epsilon*$ and $\Delta\alpha*$. This is of considerable practical
importance, since $\epsilon*$ and $\Delta\alpha*$ cannot be calculated beforehand.
It is possible to give a detailed discussion of the difficulties
caused in the Basic Computational Algorithm by "poor" choices of ϵ
and $\Delta\alpha$, but we shall limit the present discussion to a few general
remarks.

It is clear from the proof of the theorem that when $\epsilon = \epsilon*$, any
$\Delta\alpha \leq \Delta\alpha*$ will do; in fact, to every ϵ, $0 < \epsilon < \epsilon*$, there exists
$\Delta\alpha*(\epsilon)$, $0 < \Delta\alpha*(\epsilon) \leq \Delta\alpha*$, such that the Basic Computational Algorithm
is well-defined and computationally finite when ϵ and $\Delta\alpha$ are used

Step 1

Solve (Po). Put (x^0, u^0, S^0) equal to a solution of (KT-1) through (KT-4) at $\alpha=0$

Choose step size $\Delta\alpha > 0$
Choose $\epsilon > 0$
Put $J = 0$

Put $(x^{J-1}, u^{J-1}) = (x^J, u^J)$
Put $J = J+1$

No

Is $J\Delta\alpha \geq 1$?

Yes

Terminate

Step 2

Iterate from (x^{J-1}, u^{J-1}) to (x^J, u^J), the solution$^+$ of $(=S^0)J\Delta\alpha$, by Newton's method

Yes

(S^0 IS STILL VALID AT $J\Delta\alpha$)

Write $(x^*(J\Delta\alpha), u^*(J\Delta\alpha)) = (x^J, u^J)$

Is $g_i(x^J) \geq 0$, $\forall i \notin S^0$, and $u_i^J \geq 0$, $\forall i \in S^0$?

No

(S^0 IS NOT VALID AT $J\Delta\alpha$)

Put $A = \{i: u_i^{J-1} > \epsilon\}$ and $B = \{i: g_i(x^{J-1}) < \epsilon\}$

Choose S such that $A \subseteq S \subseteq B$ and S not tried before at the current value of J

Step 3

Put $S^0 = S$

Iterate from (x^{J-1}, u^{J-1}) to (x^J, u^J), the solution$^+$ of $(=S)J\Delta\alpha$, by Newton's method

Yes

(THIS TRIAL WAS SUCCESSFUL: S IS VALID AT $J\Delta\alpha$)

Is $g_i(x^J) \geq 0$, $\forall i \notin S$, and $u_i^J \geq 0$, $\forall i \in S$?

No

(THIS TRIAL WAS NOT SUCCESSFUL: S IS NOT VALID AT $J\Delta\alpha$)

Figure 1

Flow Chart of the Basic Computational Algorithm

$^+$ The notation used here is contradictory of that used elsewhere in this work: (x^J, u^J) actually means $(x^{S^0}(J\Delta\alpha), u^{S^0}(J\Delta\alpha))$ at Step 2, for example.

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and $\Delta\alpha \leq \Delta\alpha^*(\epsilon)$. Thus ϵ and $\Delta\alpha$ need not be exactly ϵ^* and $\Delta\alpha^*$ in order for the algorithm to be applicable. In general, however, the following qualitative assertions hold: (a) when ϵ is too small, there may be too few candidate sets at Step 3, i.e., there may be no set satisfying $A \subseteq S \subseteq B$ which is valid at $J\Delta\alpha$, so that Step 3 cannot be successfully completed; (b) when ϵ is too large, there may be too many candidate sets at Step 3, resulting in an excessive number of trials before Step 3 is successfully completed and possibly in the break-down of Newton's method (lack of convergence or lack of existence of the required inverse matrix) for the trial sets which are not valid at $J\Delta\alpha$ and do not satisfy the hypotheses of Corollary 4.2 applied at the point of change just before $J\Delta\alpha$; (c) when $\Delta\alpha$ is too small, the algorithm is applicable but requires more executions of Step 2 increments in α, thereby reducing the efficiency of the algorithm for a user who would be satisfied with knowing $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ for a coarser grid of values; and (d) when $\Delta\alpha$ is too large, Newton's method is apt to be ill-defined, or divergent, or convergent to the wrong solution of $(=S)J\Delta\alpha$, and it could happen that there is no set satisfying $A \subseteq S \subseteq B$ which is valid at $J\Delta\alpha$, so that Step 3 cannot be successfully completed.

It is evident that ϵ and $\Delta\alpha$ must be selected by trial and error. A more powerful approach would be to modify ϵ and $\Delta\alpha$ adaptively as the computations proceed: one would provide for monitoring the number of iterations used by Newton's method each time it is employed and also the number of candidate sets at Step 3, and the basic strategy would be to increase $\Delta\alpha$ and/or decrease ϵ when the algorithm is making

good progress and to decrease $\Delta\alpha$ and/or increase ϵ when the algorithm encounters difficulty. Such an approach was applied successfully in the design of the machine code used to solve the parametric problem of Chapter IV.

In addition to the possibility of increasing computational efficiency by adaptive selection of ϵ and $\Delta\alpha$, it is possible to greatly improve computational efficiency by using refinement, bordering, and partitioning methods for the inverse matrix required by Newton's method. A discussion of some of these devices is given in Appendix C. These devices, or others like them, should be incorporated into any machine code for implementing the present algorithm , or the number of matrix inversions required would probably preclude the use of Newton's method.

4.   Further Study of Step 3

Step 3 of the Basic Conceptual Algorithm involves a certain amount of trial and error: at the point of change α', try different sets S which are valid at α' (i.e., $A\alpha' \subseteq S \subseteq B\alpha'$) until one is found which is valid to the right of α'. When $B\alpha'-A\alpha'$ is a singleton, then no erroneous trials will be made at Step 3; for there are only two eligible sets, one of which was found at Step 2 not to be valid to the right of α'. When $B\alpha'-A\alpha'$ contains many constraint indices, however, many unsuccessful trials may have to be made before a set which is valid to the right of α' is found. It is therefore of interest to appraise how serious a difficulty the trial and error nature of Step 3 is likely to be, and to consider some ways of ameliorating this potential stumbling block.

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It is possible to argue heuristically that $B\alpha'-A\alpha'$, which may be referred to as the set of <u>degenerate</u> constraints at α', will ordinarily consist of only one constraint. Let $\alpha_o \in [0,1]$ be fixed, and assume that Conditions 1 through 4 hold. From the sufficiency of the Kuhn-Tucker Theorem, it follows that $\underline{x}^*(\alpha_o)$ also is the optimum solution to the problem

$$\text{Maximize } f(\underline{x};\alpha_o) \text{ subject to } g_i(\underline{x}) \geq 0, \quad \forall\, i \in A\alpha_o .$$
$$\underline{x}$$

In other words, all constraints except those of $A\alpha_o$ are redundant. The fact that some of them, namely those of $B\alpha_o-A\alpha_o$, happen to be <u>exactly</u> satisfied at $\underline{x}^*(\alpha_o)$ can be viewed as an "accident." It seems more likely that a redundant constraint will be <u>slack</u> at $\underline{x}^*(\alpha_o)$, as those of $M-B\alpha_o$ are. If α_o is not a point of change, we conclude that $B\alpha_o-A\alpha_o$ is likely to be empty ($B\alpha_o-A\alpha_o = \emptyset$ implies that there is exactly one valid set at α_o). The set $B\alpha_o-A\alpha_o$ is sure to contain at least one constraint, however, when α_o is a point of change, for as α traverses the unit interval continuity dictates that the only way a constraint can make the transition from slack to active or conversely is to pass through $B\alpha-A\alpha$. Unless there is strong interdependence between different constraints, not more than one or two constraints are likely to be involved in such a transition at any given point of change.

Remark: The last observation brings up an interesting point regarding the testing of new mathematical programming algorithms. Often a new algorithm is applied to a number of problems whose data were generated "randomly" in an effort to gain computational

experience quickly and to judge the efficiency of the algorithm. In our case this procedure would very likely lead to results biased in favor of our algorithm. The reason, of course, is that "interdependence" between constraints is less likely to occur when problem data are generated randomly than when problem data derive from real applications; the result is that Step 3 will rarely require any erroneous trials for problems with randomized data.

The above heuristic argument, although somewhat comforting, does not preclude the possibility of $B\alpha'-A\alpha'$ being quite numerous (by Condition 4, $B\alpha$ can be composed of at most n constraint indices, and so $B\alpha'-A\alpha'$ could have up to n constraints). Faced with this possibility, one may follow two main courses of inquiry. One may attempt to construct methods of perturbing $(P\alpha)$ so as to ensure that $B\alpha-A\alpha$ consists of only one or two constraints at each point of change (see Markowitz, 1956, p. 125, and Zahl, 1964, p. 156). Alternatively, one may attempt to devise rules for deciding in what order the trials should be made at Step 3 (the Basic Conceptual Algorithm is ambiguous in this respect) so as to tend to keep the number of erroneous trials small. We choose to follow the second course of inquiry, because (a) this type of investigation is conspicuously lacking at present (for a notable exception in the context of a related problem see Theil and Van de Panne, 1960), and (b) the second course of inquiry must be undertaken before the need for perturbation can be established.

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## 4.1 Preliminary Remarks on Determining the Order of Trials at Step 3

We begin by establishing some terminology. Suppose that Step 2
has ended with the point of change $\alpha' < 1$. Let $\alpha'+$ be a point between
α' and the next largest point of change. If S is valid at α' but
not at $\alpha'+$, the unique continuous solution of $(=S)\alpha$ satisfying the
left end-point value $(\underline{x}^*(\alpha'), \underline{u}^*(\alpha'))$ violates either (KT-3) or
(KT-4), or possibly both, as α increases above α'. In other words,
S "causes an <u>alarm</u>" as α increases[9] above α'. A violation of
(KT-3) is called a <u>feasibility alarm</u>, while a violation of (KT-4) is
called an <u>optimality alarm</u>. By continuity, the set of feasibility alarms
must be contained in $B\alpha'-S$, and the set of optimality alarms must be
contained in the set $S-A\alpha'$; hence all alarms are from $B\alpha'-A\alpha'$. Since
S is not valid at $\alpha'+$, by Corollary 1.2 either $\{S-B\alpha'+\} \neq \emptyset$ or
$\{A\alpha'+ - S\} \neq \emptyset$. The set $S-B\alpha'+$ will be called the <u>excess</u> of S at
$\alpha'+$, and $A\alpha'+ - S$ will be called the <u>deficiency</u> of S at $\alpha'+$.
Clearly the smallest change in S which will result in a set which
is valid at $\alpha'+$ is to delete its excess and add its deficiency. The
number of constraint indices of $\{A\alpha'+ - S\} \cup \{S-B\alpha'+\}$ is therefore
a measure of the minimum <u>distance</u>,[10] which we denote by $d(S)$, between
S and the collection of all sets which are valid at $\alpha'+$.

---

[9] Since $x_i^S(\alpha)$ and $u_i^S(\alpha)$ are <u>analytic</u> functions, there is an $\epsilon > 0$
such that each component of $(\underline{g}(\underline{x}^S(\alpha), \underline{u}^S(\alpha))$ has constant sign on
$(\alpha', \alpha'+\epsilon)$. It is in this sense that we define the alarms caused by
S "as α increases above α'."

[10] The distance between a set C and a set D, where C and D are
both subsets of M, can be defined as the number of elements in the set
$\{C-D\} \cup \{D-C\}$. It is readily verified that this definition meets all of
the usual requirements of a distance metric and hence makes a metric
space out of the set of all subsets of M.

Figure 2 is designed to help the reader visualize the various sets mentioned above for a hypothetical case, and it will be convenient to refer to it occasionally during the rest of this section. Each dot represents a constraint--fifteen in all. The constraints in S are circled to distinguish them from the others. Constraints 6, 8, and 10 are labelled "g" to signify that they are potential feasibility alarms ($B\alpha'$-S), and constraints 7, 9, and 11 are labelled "u" to signify that they are potential optimality alarms (S-$A\alpha'$). The deficiency of S at α'+ is precisely constraint 6, and the excess is constraint 11.

Can one guess, by observing which feasibility and optimality alarms S causes as α increases above α', what changes can be made in S in order for it to be valid at α'+? It is tempting to conjecture that any constraint (in S) which yields an optimality alarm should be deleted from S, for it is well-known (e.g., see Wilde, 1962) that a dual variable may be interpreted as giving the marginal decrease of the value of the objective function with respect to an increase in the "right-hand side" of the corresponding constraint. Similarly, it is tempting to conjecture that any constraint (not in S) which yields a feasibility alarm should be added to S in order that it remain satisfied as α increases above α'. If this line of reasoning were correct, then by deleting the constraints which yield optimality alarms and adding those which yield feasibility alarms, one could obtain from S a set which is valid at α'+; for the optimality alarms would coincide with the excess of S and the feasibility alarms would

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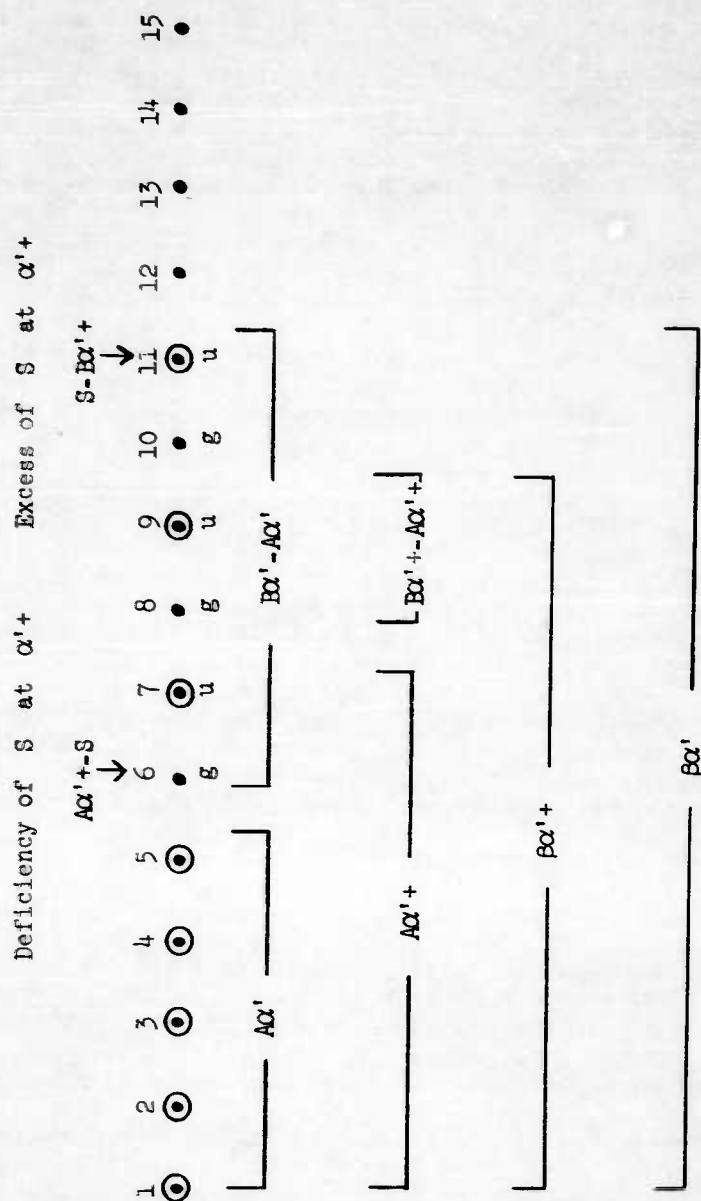
Figure 2

Note: Constraints in S are circled.

coincide with the deficiency of S.   Unfortunately this is not the case, because the interactions between constraints which are degenerate at α' have been ignored. It is therefore possible to construct simple examples (see Appendix B) for which there are _false_ and _silent_ alarms. By a false alarm we mean a feasibility alarm which is not from the deficiency of S at $\alpha'+$ and not from the set of degenerate constraints at $\alpha'+$, or an optimality alarm which is not from the excess of S at $\alpha'+$ and not from the set of degenerate constraints at $\alpha'+$. By a silent feasibility alarm we mean the absence of a feasibility alarm from a constraint in the deficiency of S at $\alpha'+$, and by a silent optimality alarm we refer to the absence of an optimality alarm from a constraint in the excess of S at $\alpha'+$.   In terms of Figure 2, a false feasibility alarm would be an alarm from constraint number 10, a false optimality alarm would be an alarm from 7, a silent feasibility alarm would be the absence of an alarm from 6, and a silent optimality alarm would be the absence of an alarm from 11. Note that the alarms from the set of constraints which are degenerate at $\alpha'+(B\alpha'+ - A\alpha'+)$, if any, are immaterial— for the presence or absence of these constraints (numbers 8 and 9 in Figure 2) for a trial set does not affect its validity at $\alpha'+$.

The above remarks indicate that not very much information about what constitutes a valid set at $\alpha'+$ can be gleaned from a trial which fails at Step 3. Evidently the statement of Corollary 1.1 that $A\alpha' \subseteq A\alpha'+ \subseteq B\alpha'+ \subseteq B\alpha'$ is about as strong a statement as can be made. As has already been pointed out, this is already a very strong statement

in the likely event that there are only a few degenerate constraints at α'. Yet when a trial set fails at Step 3 there is one clue to the identity of a set which is valid at $\alpha'+$ that can be salvaged: at least one of the alarms given during a failure is from the deficiency or excess at $\alpha'+$ of the trial set. In the next subsection we shall prove this fact. The result will then be used to devise an ordering of trials at Step 3.

4.2 Sharpening Corollary 2.1

Lemma 6.1:

Let $\alpha' \in [0,1]$ be a point of change, let S be valid at α', and assume that Conditions 1 through 4 hold.

Then there exists a convex set $X' \supset X$ and an open interval containing and symmetric about α' and contained in $\overline{I}\alpha'$ such that, for each fixed value of α in this interval, $\underline{x}^S(\alpha)$ is the optimal solution of

$$\text{Maximize} \quad f(\underline{x};\alpha)$$
$$\underline{x} \in X'$$

$$\text{subject to} \quad g_i(\underline{x}) = 0, \quad \forall i \in \{S - S^+\alpha\}$$
$$g_i(\underline{x}) \geq 0, \quad \forall i \in S^+\alpha \,,$$

where $S^+\alpha \subseteq \{i \in S: \ u_i^S(\alpha) \geq 0\}$.

Proof: Arguing as in Proposition 3 and using the continuity of $\underline{u}^S(\alpha)$ and the fact that $\underline{u}^S(\alpha') = \underline{u}*(\alpha') \geq \underline{0}$, one obtains (here we employ the notations of Proposition 3) that $\underset{\mu}{\text{Max}} \ \xi_\mu(\Delta_x^2(f(\underline{x};\alpha) + \sum_1^m u_i^S(\alpha)g_i(\underline{x})))$

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is negative on $X \times \alpha'$ and continuous on some open region containing this direct product set. By the compactness and convexity of $X \times \alpha'$, it follows that the hessian of the Lagrangian function $f(\underline{x};\alpha) + \sum_{i}^{m} u_i^S(\alpha) g_i(\underline{x})$ is negative definite on some open convex region $X' \times I'\alpha'$ containing $X \times \alpha'$. In view of A.3, the Lagrangian function must be strictly concave with respect to $\underline{x}$ on the open convex set X' for each fixed value of $\alpha \in I'\alpha'$.

Now $\underline{x}^S(\alpha') = \underline{x}^*(\alpha') \in X \subset X'$, X' open; since $\underline{x}^S(\alpha)$ is continuous on $\overline{I}\alpha'$, one obtains that $\underline{x}^S(\alpha) \in X'$ for all α sufficiently near α'. Since the gradient with respect to $\underline{x}$ of the Lagrangian function vanishes at $\underline{x}^S(\alpha)$, we conclude by A.6 that $\underline{x}^S(\alpha)$ is the global maximum of that function on the convex set X' for any fixed α sufficiently near α'. Using the fact that $u_i^S(\alpha) = 0$, $\forall\, i \notin S$, and $g_i(\underline{x}^S(\alpha)) = 0$, $\forall\, i \in S$, one obtains, for any fixed α sufficiently near α', that

$$(14) \qquad f(\underline{x}^S(\alpha);\,\alpha) \geq f(\underline{x};\alpha) + \sum_{1}^{m} u_i^S(\alpha)\, g_i(\underline{x}), \quad \forall\, \underline{x} \in X'\,.$$

Since $\sum_{1}^{m} u_i^S(\alpha)\, g_i(\underline{x}) \geq 0$ for all $\underline{x}$ such that $g_i(\underline{x}) = 0$, $\forall\, i \in \{S - S^+\alpha\}$, and $g_i(\underline{x}) \geq 0$, $\forall\, i \in S^+\alpha$, where $S^+\alpha \subseteq \{i \in S: u_i^S(\alpha) \geq 0\}$, the conclusion of the lemma follows from (14).

Remark: An easy proof of this lemma can be constructed from the Kuhn-Tucker Theorem when all constraints are linear; in this case X' may be taken to be E^n. When all constraints are linear, specialization of the Kuhn-Tucker Theorem reveals that $(=S)\alpha$ are necessary and sufficient conditions for a maximum of $f(\underline{x};\alpha)$ subject to $g_i(\underline{x}) = 0$, $\forall\, i \in S$.

Remark: The region X' may be taken to be contained in the open region
mentioned in Condition 1.

Theorem 6:

Let $\alpha' \in [0,1]$ be a point of change, let S be valid at α',
and assume that Conditions 1 through 4 hold.

Then there exists an open interval containing and symmetric about
α' and contained in $\overline{I}\alpha'$ such that, for each fixed value of α
in this interval, the following three assertions are equivalent:

(i) S is valid at α.

(ii) $(\underline{x}^S(\alpha), \underline{u}^S(\alpha)) = (\underline{x}*(\alpha), \underline{u}*(\alpha))$.

(iii) $g_i(\underline{x}^S(\alpha)) \geq 0, \quad \forall\ i \in \{A\alpha\text{-}S\}$
$u_i^S(\alpha) \geq 0, \quad \forall\ i \in \{S\text{-}B\alpha\}.$

Proof: The equivalence of (i) and (ii) and the fact that (ii)
implies (iii) are known from Corollary 2.1. To complete the proof of
the theorem, it is sufficient to show that (iii) implies (ii) on the
interval mentioned in Lemma 6.1.

Assume that (iii) holds for some fixed value of α in the interval
mentioned in Lemma 6.1. Using the assumption that $u_i^S(\alpha) \geq 0$,
$\forall\ i \in \{S\text{-}B\alpha\}$, and applying Lemma 6.1 with $S^+\alpha = \{S\text{-}B\alpha\}$, one may
assert the existence of a convex set $X' \supset X$ such that $\underline{x}^S(\alpha)$ is an
optimal solution of

(15)   Maximize $f(\underline{x};\alpha)$ subject to $g_i(\underline{x}) = 0, \quad \forall\ i \in \{B\alpha \cap S\}$
$\underline{x} \in X'$
$g_i(\underline{x}) \geq 0, \quad \forall\ i \in \{S\text{-}B\alpha\}$.

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Using the assumption that $g_i(\underline{x}^S(\alpha)) \geq 0$, $\forall\ i \in \{A\alpha{-}S\}$, we have that $\underline{x}^S(\alpha)$ is feasible in

$$
\begin{aligned}
&\text{Maximize}\quad f(\underline{x};\alpha)\\
&\underline{x} \in X'\\
&\qquad\qquad \text{subject to}\quad g_i(\underline{x}) = 0, \quad \forall\ i \in \{B\alpha \cap S\}\\
&\qquad\qquad\qquad\qquad\qquad g_i(\underline{x}) \geq 0, \quad \forall\ i \in \{S{-}B\alpha\} \cup \{A\alpha{-}S\} \ .
\end{aligned}
$$
(16)

Since the feasible region of (16) is included in that of (15), $\underline{x}^S(\alpha)$ must be an optimal solution of (16).

It follows from A.4 and A.6 and the fact that $(\underline{x}^*(\alpha), \underline{u}^*(\alpha))$ satisfies $(=A\alpha)\alpha$ that $\underline{x}^*(\alpha)$ is optimal in

(17)   Maximize $f(\underline{x};\alpha)$ subject to $g_i(\underline{x}) \geq 0$, $\forall\ i \in A\alpha$.
         $\underline{x} \in X'$

Since the feasible region of (16) is included in that of (17), and since $\underline{x}^*(\alpha)$ is feasible in (16), $\underline{x}^*(\alpha)$ must be optimal in (16). That is, both $\underline{x}^*(\alpha)$ and $\underline{x}^S(\alpha)$ are optimal in (16); thus $f(\underline{x}^*(\alpha);\alpha) = f(\underline{x}^S(\alpha);\alpha)$. Because $\underline{x}^S(\alpha)$ is feasible in (17), therefore, we finally have that $\underline{x}^S(\alpha)$ is optimal in (17). Since (17) must have a <u>unique</u> optimal solution by A.2, $\underline{x}^S(\alpha) = \underline{x}^*(\alpha)$. This implies, by Condition 4, that $\underline{u}^S(\alpha) = \underline{u}^*(\alpha)$. Thus (ii) holds.

The significance of this sharpening of Corollary 2.1 is that it rules out the possibility that all alarms are either false or from the set of degenerate constraints at $\alpha'+$ when S is not valid at $\alpha'+$. That is, at least one alarm is from the deficiency or excess of S at $\alpha'+$.

4.3 Modification of Step 3--Determining the Order of Trials

Suppose that Step 2 has ended with the point of change $\alpha' < 1$. Designate the set of alarms which are given by S^O (the set used during Step 2) as α increases above α' by T. Applying Theorem 6 at α', we know that at least one of the alarms is from the excess or deficiency of S^O at $\alpha'+$. Unfortunately, we do not know which one. A logical way of proceeding at Step 3 is to modify S^O by one constraint at a time for each constraint in T, i.e., try the sets $S^O \pm i$ for each $i \in T$, where the symbol $S^O \pm i$ means $S^O \cup i$ if $i \notin S^O$ and $S^O - i$ if $i \in S^O$. This notation is designed to avoid having to distinguish between feasibility and optimality alarms. In other words, add the constraints which were feasibility alarms to S^O and delete constraints which were optimality alarms from S^O one at a time until each alarm has been heeded individually. Note that $S^O \pm i$, $i \in T$, is valid at α' since all alarms caused by a set which is valid at α' must be from $B\alpha' - A\alpha'$. Hence $S^O \pm i$, $i \in T$, satisfies (8.1).

When T has been exhausted by this first generation of trials, at least one trial set, say $S^O \pm i_o$, is one unit of distance closer to a valid set at $\alpha'+$. If $d(S^O) = 1$ then $S^O \pm i_o$ is valid at $\alpha'+$ and Step 3 has been successfully completed. If $d(S^O) > 1$ then $d(S^O \pm i_o) = d(S^O) - 1 > 0$, and a second generation of trials is necessary. At each first generation trial, let T_i denote the alarms due to $S^O \pm i$, $i \in T$. At the second generation one should try $S^O \pm i \pm j$ for all $i \in T$ and all $j \in T_i$. The symbol $S^O \pm i \pm j$

means $\{S^O \pm i\} \cup j$ if $j \not\in S^O \pm i$ and $\{S^O \pm i\} - j$ if $j \in S^O \pm i$. Applying Theorem 6 at α' with $S = S^O \pm i_o$, we see that at least one of the alarms due to $S^O \pm i_o$ is from the excess or deficiency of $S^O \pm i_o$ at $\alpha'+$, but we do not know which one. Hence at least one of the sets $S^O \pm i_o \pm j$, $j \in T_{i_o}$, is one unit of distance closer to a set which is valid at $\alpha'+$. Designate one such set by $S^O \pm i_o \pm j_o$. If $d(S^O) = 2$ then $S^O \pm i_o \pm j_o$ is valid at $\alpha'+$, and Step 3 has been successfully completed. If $d(S^O) > 2$, then $d(S^O \pm i_o \pm j_o) = d(S^O)-2 > 0$, and a <u>third generation</u> of trials is necessary.

The third generation of trials is constructed in a manner analogous to the preceding generations, and so on for the higher order generations. If at any trial a set is encountered which has been tried before, it may, of course, be discarded.

At each generation the distance from <u>some</u> trial set, and perhaps several, to the collection of all sets which are valid at $\alpha'+$ is decreased by one unit. Since $d(S^O)$ is finite (in fact it is bounded by the number of constraints in $B\alpha'-A\alpha'$ minus the number of constraints in $B\alpha'+ - A\alpha'+$), after a finite number of generations of trials a set which is valid at $\alpha'+$ will be obtained--after exactly $d(S^O)$ generations, in fact. The nearest valid set is, it will be recalled, S^O plus its deficiency at $\alpha'+$ minus its excess at $\alpha'+$. These rules are summarized below.

Order of Trials at Step 3

1. Let T denote the alarms which are given by S^O as α increases above α'. At the first generation of trials,

try $S^o \pm i$ for each $i \in T$. Let T_i denote the set of alarms which are given by $S^o \pm i$, $i \in T$, as α increases above α'. If $T_i = \emptyset$ for some $i* \in T$, then $S^o \pm i*$ is valid at $\alpha'+$, and Step 3 has been completed; otherwise, go on to the second generation of trials.

2. At the second generation of trials, try $S^o \pm i \pm j$ for each $i \in T$ and all $j \in T_i$. Let T_{ij} be the set of alarms which are given by $S^o \pm i \pm j$, $i \in T$ and $j \in T_i$, as α increases above $\alpha'+$. If $T_{ij} = \emptyset$ for some $i* \in T$ and $j* \in T_{i*}$, then $S^o \pm i* \pm j*$ is valid at $\alpha'+$, and Step 3 has been completed; otherwise, go on to a third generation of trials.

Etc.   (Omit any sets which have been tried previously.)

Since the only modification of Step 3 being suggested here is a more complete specification of the order in which the trial sets are to be considered, and since this order has been shown to lead to a successful completion of Step 3, the assertions of the Basic Theorem still apply to the Basic Conceptual Algorithm with Step 3 modified as above.

If these rules are to be incorporated into the Basic Computational Algorithm, then in order to ensure that Theorem 6 — and hence the above rules — applies, it is necessary to take $\Delta\alpha$ less than one-half the length of the smallest of the intervals of Theorem 6 applied at each point of change.

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We do not hold that the order of trials suggested here is the most efficient order which can be devised. However, the following advantages are to be noted:

(1) Each unsuccessful trial helps to determine the order of successive trials.

(2) The suggested order of trials always leads to the (unique) valid set nearest S^o.

(3) A valid set is found after exactly $d(S^o)$ generations of trials. In this sense search termination is predictable, although not a priori so.

(4) S^o is deformed one constraint at a time from trial to trial, so that the computational machinery is upset the least amount possible.

5. Some Extensions

5.1 Linear Equality Constraints

Let the constraints of $(P\alpha)$ include some linear equality constraints. It is clear that if each such constraint is written as a pair of inequality constraints (i.e., if the pair $g_i(\underline{x}) \geq 0$, $-g_i(\underline{x}) \geq 0$ is written in place of $g_i(\underline{x}) = 0$), then Condition 4 never holds. Fortunately, it can be shown that a simple modification of the Basic Conceptual Algorithm obviates this difficulty: always include the linear equality constraints in S^o at Step 2 and in the trial sets at Step 3 (ignore any optimality alarms that such constraints may give). If all of the constraints happen to be linear equalities, in fact, Step 3 would disappear entirely.

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5.2 More General Parametric Problems

With appropriate modifications of the four conditions, it can be shown (Geoffrion, 1965) that many of the results of this chapter apply to any one-dimensional perturbation of

(P$\underline{p}$)   Maximize $f(\underline{x},\underline{p})$ subject to $\underline{g}(\underline{x},\underline{p}) \geqq \underline{0}$,
              $\underline{x}$

where the parameter $\underline{p} = (p_1,\ldots,p_k)$ varies over a convex set P in E^k, $f(\underline{x},\underline{p})$ is continuous in $(\underline{x},\underline{p})$ and strictly concave in $\underline{x}$ for each $\underline{p} \in P$, and $g_i(\underline{x},\underline{p})$ $(i = 1,\ldots,m)$ is concave in $(\underline{x},\underline{p})$. By a one-dimensional perturbation of (P$\underline{p}$) we mean a parametric problem of the form

$$\text{Maximize} \quad f(\underline{x},\underline{p}' + \alpha(\underline{p}'' - \underline{p}'))$$
$$\underline{x}$$

$$\text{subject to} \quad \underline{g}(\underline{x},\underline{p}' + \alpha(\underline{p}'' - \underline{p}')) \geqq \underline{0}$$

for each value of $\alpha \in [0,1]$, where $\underline{p}'$, $\underline{p}'' \in P$.

It is evident that (P$\underline{p}$) is general enough to include many of the parametric problems of interest to those who wish to perform sensitivity analysis on concave programming problems.

CHAPTER IV

An Illustrative Example

A simple model of a firm will be used to illustrate the manipulation and solution of a decision problem under uncertainty by means of the techniques presented in the preceding three chapters.

1. A Decision Problem Under Uncertainty

Consider a hypothetical firm which produces and sells n products in an imperfectly competitive market. Assume that the cost of producing and selling each unit of product i is c_i dollars per unit, and that the total dollar revenue accruing from the sale of x_i units of product i is $r_i(x_i) = (a_i + b_i\beta - d_i)x_i + (d_i/k_i)\ell n(k_i x_i + 1)$, where a_i, d_i, k_i are positive scalars, $\ell n(\cdot)$ denotes the natural log, and β is a price index. The interpretation of $r_i(x_i)$ becomes clearer if one examines $dr_i(x_i)/dx_i = a_i + b_i\beta - d_i + d_i/(k_i x_i + 1)$. Since $dr_i(0)/dx_i = a_i + b_i\beta$ and $dr_i(\infty)/dx_i = a_i + b_i\beta - d_i$, we see that price gradually decreases from $a_i + b_i\beta$ (notice the linear dependence on the price index) to $a_i + b_i\beta - d_i$ dollars per unit as production increases without bound. The value of k_i determines the rapidity of the price decrease, and it is easily shown that a proportion $0 < t < 1$ of the total possible price decrease d_i is achieved at $x_i = t/(1-t)k_i$.

If we denote the (short-run) resource and other constraints (including $\underline{x} \geq \underline{0}$) by $\underline{g}(\underline{x}) \geq \underline{0}$, then assuming that the firm can sell all it produces the profit maximization problem is

$$(1) \quad \underset{\underline{x}}{\text{Maximize}} \quad \sum_{i=1}^{n} (a_i + b_i\beta - d_i)x_i + (d_i/k_i)\ell n(k_i x_i + 1) - c_i x_i$$

$$\text{subject to } \underline{g}(\underline{x}) \geq \underline{0} .$$

We shall assume that all functions and coefficients are known except the price index β, which will be regarded as a random variable with a known cumulative distribution function $\Phi(\beta)$.

2. Circumventing Uncertainty by a Vector Maximum Reformulation

In order to circumvent the uncertainty attending the objective function of (1), we elect to employ one of the approaches considered at some length in Chapter I: a vector maximum reformulation using the expected value criterion and the maximum .05-fractile criterion (some fractile other than the .05-fractile could be used if desired). Assume that $\Phi(\beta)$ is continuous, strictly increasing on the entire real line, and that its mean is zero (if the mean is not zero, it can be incorporated into the a_i). One derives that the mean and .05-fractile of the objective function for fixed $\underline{x}$ are, respectively,

$$f_1(\underline{x}) = \sum_{i=1}^{n} (a_i - d_i - c_i)x_i + (d_i/k_i)\ell n(k_i x_i + 1)$$

$$f_2(\underline{x}) = \begin{cases} f_1(\underline{x}) + \Phi^{-1}(.05) \sum_{i=1}^{n} b_i x_i & \text{if } \sum b_i x_i \geq 0 \\[3ex] f_1(\underline{x}) + \Phi^{-1}(.95) \sum_{i=1}^{n} b_i x_i & \text{if } \sum b_i x_i \leq 0 \,. \end{cases}$$

In place of (1) we consider the vector maximum problem

(2)
$$\underset{\underline{x}}{\text{"Maximize"}} \quad f_1(\underline{x}), \quad f_2(\underline{x})$$
$$\text{subject to } \underline{g}(\underline{x}) \geq \underline{0} \,.$$

The efficient outcomes of (2) are to be computed and plotted (as in Figure 3 below) so as to present a "tradeoff curve" between the two

criteria. A decision-maker then subjectively determines a point on the tradeoff curve, and implements the corresponding optimal production schedule.

3. An Equivalent Parametric Programming Reformulation

The hessian of $f_1(\underline{x})$ is a diagonal matrix, with $-k_i d_i/(k_i x_i + 1)^2$ on the diagonal. When $\underline{x} \geq \underline{0}$, the assumed positivity of k_i and d_i implies that this hessian is negative definite. By A.3, therefore, $f_1(\underline{x})$ is seen to be strictly concave on the non-negative orthant. An enumeration of cases shows that $f_2(\underline{x})$ is also strictly concave on the non-negative orthant when $\Phi^{-1}(.05) \leq 0$ and $\Phi^{-1}(.95) \geq 0$. In view of our assumption that the mean is 0, it is reasonable to assume that this last condition holds. Assuming further that each constraint function is concave, we conclude that Proposition 6 of Chapter II applies.[1] Hence to find all efficient solutions of (2) it is equivalent to find the optimal solutions of

$$\text{(3)} \qquad \begin{array}{c} \text{Maximize} \quad (1-\alpha)f_1(\underline{x}) + \alpha f_2(\underline{x}) \\ \underline{x} \\[1em] \text{subject to } \underline{g}(\underline{x}) \geq \underline{0} \end{array}$$

for each value of α in the unit interval.

Consider (3) with α fixed. The presence of the logical condition in the definition of f_2 makes the solution of (3) somewhat

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[1] It is easy to see that Proposition 6 still holds if the f_i are assumed to be strictly concave on X, and not necessarily on E^n.

awkward.   One approach is to solve the pair of problems

$$\text{Maximize} \quad (1-\alpha)f_1(\underline{x}) + \alpha[f_1(\underline{x}) + \Phi^{-1}(.05) \sum b_i x_i]$$
$$\underline{x}$$

(4)
$$\text{subject to} \quad \underline{g}(\underline{x}) \geq \underline{0}$$

$$\sum b_i x_i \geq 0$$

$$\text{Maximize} \quad (1-\alpha)f_1(\underline{x}) + \alpha[f_1(\underline{x}) + \Phi^{-1}(.95) \sum b_i x_i]$$
$$\underline{x}$$

(5)
$$\text{subject to} \quad \underline{g}(\underline{x}) \geq \underline{0}$$

$$\sum b_i x_i \leq 0 .$$

The optimal value of (3) equals the larger of the optimal values
of (4) and (5), since the feasible regions of (4) and (5) are merely
a dichotomy of that of (3). We shall avoid this complication, however,
by requiring of our numerical example that $b_i \geq 0$ $(i = 1,\ldots,n)$;
since $\underline{x} \geq \underline{0}$, this condition implies that $\sum b_i x_i \geq 0$, and therefore
(3) may be rewritten as

$$\text{Maximize} \quad (1-\alpha)f_1(\underline{x}) + \alpha[f_1(\underline{x}) + \Phi^{-1}(.05) \sum b_i x_i]$$
(6)
$$\underline{x}$$

$$\text{subject to} \quad \underline{g}(\underline{x}) \geq \underline{0} .$$

4.   Solving the Parametric Problem

We shall solve a numerical example based on (6) with $n = 4$
and $m = 7$. Table 1 gives the numerical data for the objective

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function, and the constraints[2] are:

$$x_i \geq 0, \quad i = 1, \ldots, 4$$

$$-.01x_1 \; -.01x_2 \; -.04x_3 \; -.04x_4 + 2 \geq 0$$

$$-.4\, x_1 \; -.4\, x_2 \; -.1\, x_3 \; -.1\, x_4 + 20 \geq 0$$

$$-.01x_1^2 \; -.01x_2^2 \; -.01x_3^2 \; -.01x_4^2 + 15 \geq 0 \; .$$

	i = 1	i = 2	i = 3	i = 4
a_i	10.0	12.0	10.5	11.0
b_i	0.0634	0.0950	0.6740	0.7540
c_i	8.0	10.0	8.5	9.0
d_i	2.50	2.55	2.20	2.25
k_i	0.12	0.13	0.045	0.050

Table 1[3]

It is further assumed that β is normally distributed with zero mean and unit variance. Hence $\Phi^{-1}(.05) = -1.64$.

It is clear, since $k_i, b_i \geq 0$ $(i = 1, \ldots, 4)$, that f_1, f_2 and g_i $(i = 1, \ldots, 7)$ are analytic on some open region containing the non-negative orthant. Because the constraints are concave, therefore,

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[2] Each x_i represents hundreds of units of product i. The last three constraints are to be interpreted as constraints on three resources, which we refer to as resources A, B, and C respectively. Resources are measured in thousands of units.

[3] The units of the coefficients are such that f_1 and f_2 are in thousands of dollars.

Condition 1 of Chapter III is satisfied. Since resources are limited in all real problems of this type, Condition 2 is not restrictive, and in fact holds for the feasible region of our numerical example. It was observed above that the hessian of f_1 is negative definite on the non-negative orthant, and the same is true for f_2, so that Condition 3 holds. We shall not bother to verify whether Condition 4 is satisfied by our numerical example.

A version of the Basic Computational Algorithm for solving (6) was coded for the Burroughs B5000 computer. No attempt was made to optimize program efficiency beyond the incorporation of a simple variable step size feature (see the last two paragraphs of section 3, Chapter III). The results of the computation are presented in Figures 1, 2, and 3. Figure 1 is a graph of the optimal production schedule, $\underline{x}*(\alpha)$, as a function of α. Note the markers at the following values for α, each of which is a point of change marking an execution of Step 3: 0.6024, 0.7819, 0.8338. Since no false or silent alarms are encountered at any of these points, Step 3 is executed in each case with no erroneous trials. Figure 2 presents graphs of $u_i^*(\alpha)$ and $g_i(\underline{x}*(\alpha))$ $(i = 5,6,7)$. Note that the dual variables (or "shadow prices") $u_i^*(\alpha)$ $(i = 1,...,4)$ are not graphed, since they are identically zero on $[0,1]$, and that it is not necessary to graph the non-negativity constraints. Figure 3 is a plot of the efficient outcomes associated with the two criterion functions--a tradeoff curve. It shows, for example, that production plan $\underline{x}*(0.807)$ guarantees a profit of at least \$32,700 with probability .95 and an expected profit of \$79,100.
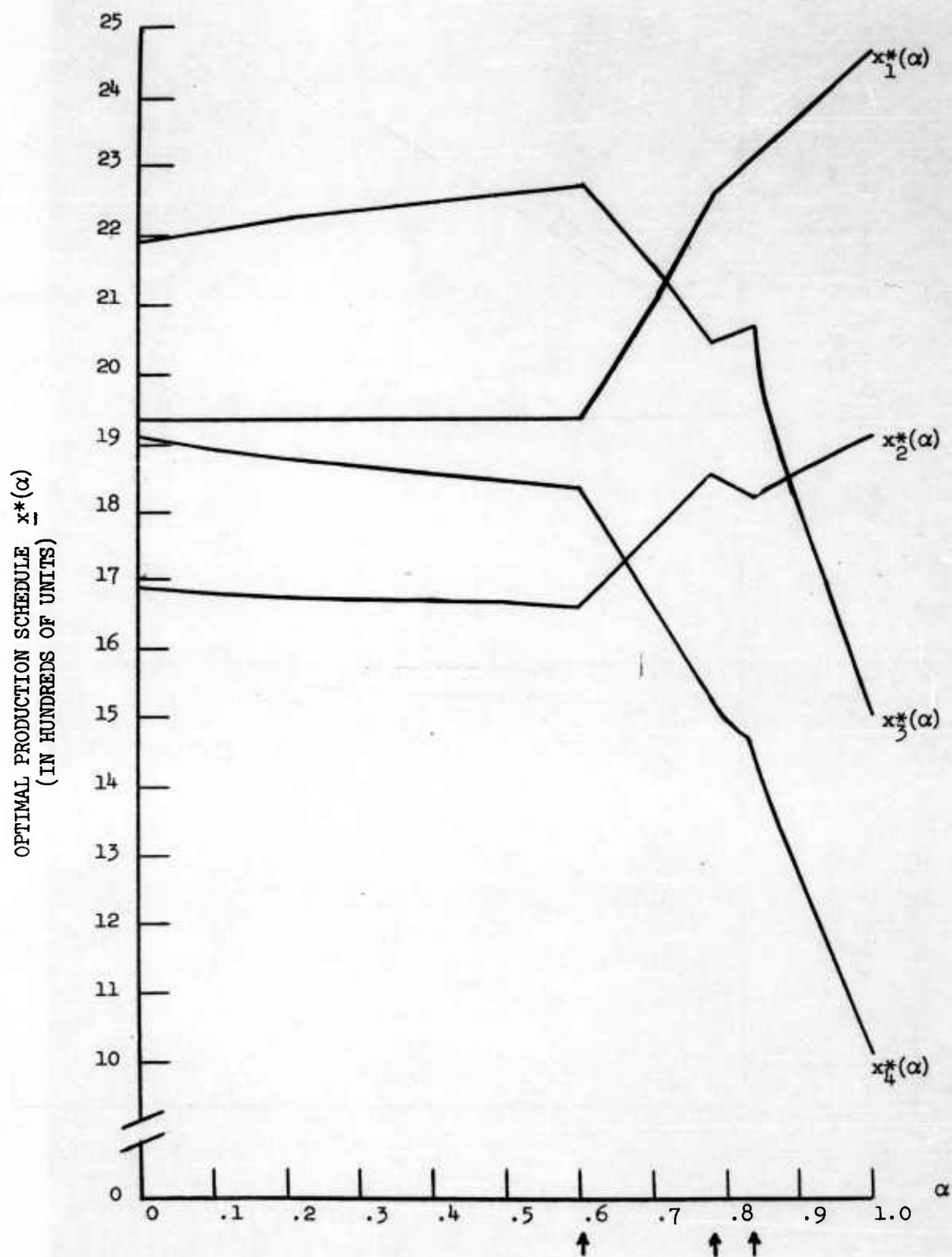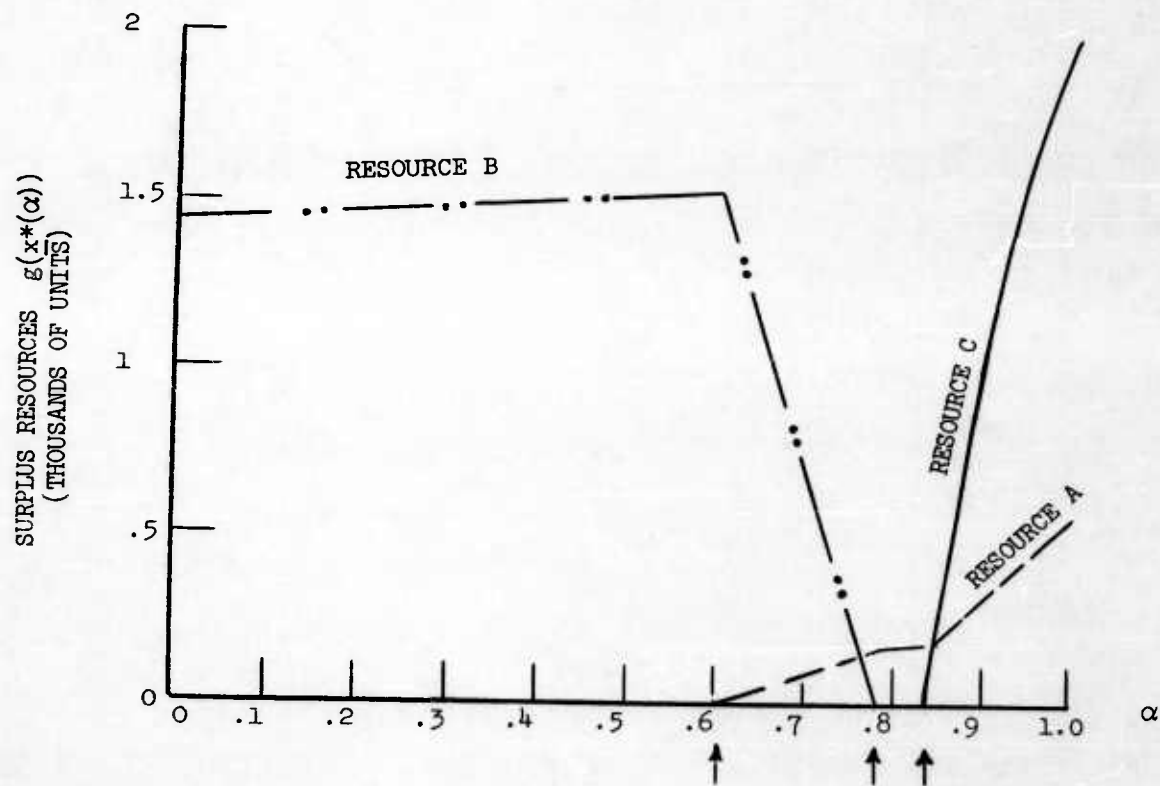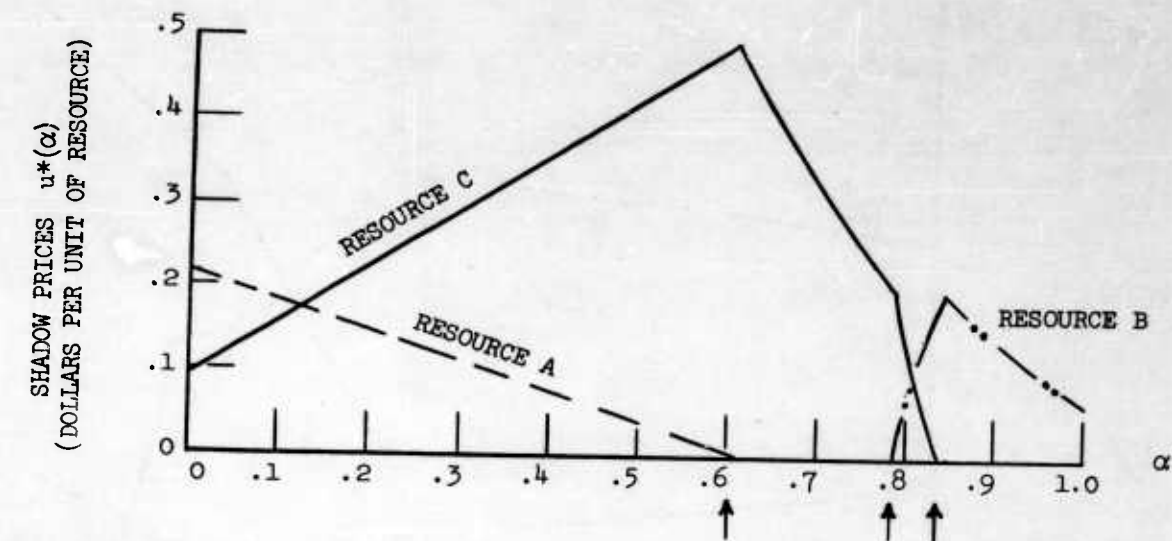
114

Figure 1
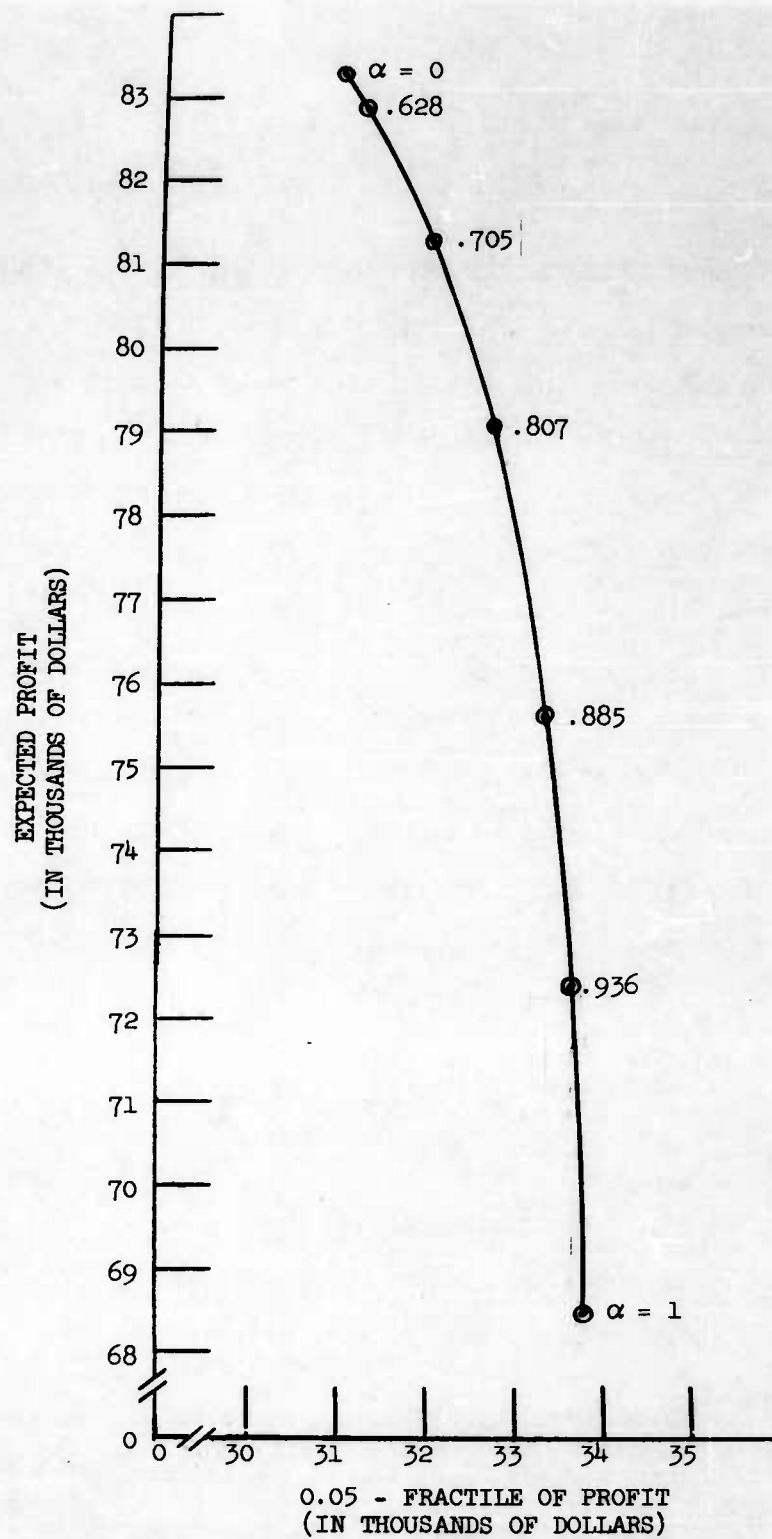
115

Figure 2

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Figure 3

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APPENDIX A

Some Properties of Convex Sets and Concave Functions

A set S in E^n is said to be <u>convex</u> if $(\lambda \underline{x}' + (1-\lambda)\underline{x}'') \in S$ whenever $\underline{x}', \underline{x}'' \in S$ and $0 \leq \lambda \leq 1$.

A function $f(\underline{x})$ which is defined on a convex set S is said to be <u>concave</u> if $f(\lambda \underline{x}' + (1-\lambda)\underline{x}'') \geq \lambda f(\underline{x}') + (1-\lambda) f(\underline{x}'')$ whenever $\underline{x}', \underline{x}'' \in S$ and $0 \leq \lambda \leq 1$. If the first inequality holds strictly whenever $\underline{x}' \neq \underline{x}''$ and $0 < \lambda < 1$, $f(\underline{x})$ is said to be <u>strictly concave</u>. The function $-f(\underline{x})$ is said to be <u>convex</u> or <u>strictly convex</u> according as $f(\underline{x})$ is concave or strictly concave. When the convex set S is not specified explicitly, it is implicitly taken to be the entire space.

The following properties of convex sets and concave functions are used in the text. The proofs, most of which follow easily from the definitions, may be found in Fenchel (1953) or Zoutendijk (1960).

A.1 If $g_i(\underline{x})$ $(i = 1,\ldots,m)$ are concave functions on E^n, then $\{\underline{x}: g_i(\underline{x}) \geq 0, \; i = 1,\ldots,m\}$ is a closed and convex set.

A.2 Any local maximum of a concave function on a convex set is also a global maximum over that set; a strictly concave function can have at most one local maximum.

A.3 A twice-differentiable function defined on a convex set S is concave if and only if its hessian matrix is negative

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semidefinite at each $\underline{x} \in S$. If the hessian is negative definite at each $\underline{x} \in S$, then the function is strictly concave (the converse is <u>not</u> true in general, but does hold when the function is a quadratic polynomial and $S = E^n$).

A.4 If $f_i(\underline{x})$ $(i = 1,\ldots,k)$ are concave functions on a convex set S, and $u_i \geq 0$ $(i = 1,\ldots,k)$, at least one $u_i > 0$, then $\sum_1^k u_i f_i(\underline{x})$ is concave on S; if $f_i(\underline{x})$ is strictly concave for some i such that $u_i > 0$, then $\sum_1^k u_i f_i(\underline{x})$ is strictly concave.

A.5 A concave or convex function on a convex set S is continuous at every relative interior point of S.

A.6 If $f(\underline{x})$ is differentiable and concave on a convex set S and $\nabla_x f(\underline{x}^o) = \underline{0}$, $\underline{x}^o \in S$, then $f(\underline{x}^o) \geq f(\underline{x})$ for all $\underline{x} \in S$.

A.7 The Theorem of the Separating Hyperplane asserts that if S and T are two convex sets in E^n with no interior point in common, then there exist an n-vector $\underline{v} \neq \underline{0}$ and a scalar c such that $\sum v_i s_i \leq c \leq \sum v_i t_i$ for all $\underline{s} \in S$, $\underline{t} \in T$ (see Karlin, 1959, p. 398 for a proof).

APPENDIX B

Graphical Examples

We shall illustrate the Basic Conceptual Algorithm by considering three examples of the form

$$\text{(B.1)} \quad \underset{\underline{x}}{\text{Maximize}} \quad \alpha \sum_{1}^{n} -(x_i - c_i')^2 + (1-\alpha) \sum_{1}^{n} -(x_i - c_i)^2$$

$$\text{subject to} \quad \underline{a}_i^t \underline{x} + b_i \geq 0, \quad i = 1, \ldots, m.$$

The first example is well-behaved in the sense that there are no false or silent alarms (see section 4.1 for definitions of "false" and "silent" alarms), whereas in the second and third examples such troubles do occur.

Problems of the form (B.1) are among the simplest which can be subsumed under the present theory: both objective functions are quadratic and linearly separable, and the constraints are linear. The fact that false and silent alarms can occur for such problems seems to render unlikely the existence of a special class of $(P\alpha)$ for which false and silent alarms cannot occur.

The examples to be given are presented and analyzed graphically rather than numerically because (B.1) is readily amenable to graphical interpretation when $n = 2$ (the case considered here). Let α be fixed. When S is a consistent set, i.e., when $X_S \overset{\triangle}{=} \{\underline{x}: \underline{a}_i^t \underline{x}_i + b_i = 0, \forall i \in S\} \neq \emptyset$, it follows from the Kuhn-Tucker Theorem that $(=S)\alpha$ is necessary and sufficient for a maximum of the objective function subject to $\underline{x} \in X_S$. From the circularity of the level

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curves of this particular objective function it is evident that this constrained maximum is just the point of X_S nearest to the unconstrained maximum $\hat{\underline{x}}(\alpha) \equiv \alpha\underline{c}' + (1-\alpha)\underline{c}$.

Each figure is drawn in $\underline{x}$-space $(n = 2)$ with two constraints $(m = 2)$. The loci of $g_1(\underline{x}) = 0$, $g_2(\underline{x}) = 0$, the unconstrained maximum $\hat{\underline{x}}(\alpha)$, and the constrained maximum $\underline{x}*(\alpha)$ (the heavy line) are drawn, as well as certain features pertaining to the points of change. Light lines representing the projection of $\hat{\underline{x}}(\alpha)$ onto the feasible region are also drawn; in view of the circularity of the level curves of the objective function for fixed α, these lines are in the direction of the gradient of the objective function at $\underline{x}*(\alpha)$. The gradients of the constraints point into the feasible region.

From $(=S)\alpha$ we see that the dual variables express minus the gradient of the objective function at $\underline{x}^S(\alpha)$ as a linear combination of the gradients of the constraints in S. The signs of $u_i^S(\alpha)$ $(i \in S)$ are easily determined by visual inspection of the figures.

The first example is presented graphically in Figure B.1. At $\alpha = 0$ the unconstrained maximum $\hat{\underline{x}}(0)$ is interior to the feasible region. Thus the constrained maximum $\underline{x}*(0)$ equals $\hat{\underline{x}}(0)$ and $Bo = \emptyset$, which implies that $Ao = \emptyset$ since $A\alpha \subseteq B\alpha$ for all α. We are obliged to let $S^o = \emptyset$, for the empty set is the only valid set at $\alpha = 0$ (recall that S is valid at α if and only if $A\alpha \subseteq S \subseteq B\alpha$). Step 1 is complete. Step 2 demands that we solve $(=\emptyset)\alpha$ as α increases above 0 until an alarm is given, i.e., until $\underline{x}^\emptyset(\alpha)$ leaves the feasible region or $u_i^\emptyset(\alpha)$ becomes negative

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for some i. The last alternative (an optimality alarm) cannot happen for $S^o = \emptyset$, for $(=\emptyset)\alpha$ requires $\underline{u}^{\emptyset}(\alpha) = \underline{0}$. Only the first alternative (a feasibility alarm) can occur. Equations $(=\emptyset)\alpha$ are easily seen to be the conditions for an unconstrained maximum. Since $\hat{\underline{x}}(0)$ is interior to the feasible region for $0 \leq \alpha < \alpha_1$, no alarms are given on $[0,\alpha_1)$; $(\underline{x}^{\emptyset}(\alpha), \underline{u}^{\emptyset}(\alpha)) = (\underline{x}*(\alpha), \underline{u}*(\alpha)) = (\hat{\underline{x}}(\alpha),\underline{0})$ and $A\alpha = B\alpha = \emptyset$ on $[0,\alpha_1)$. At α_1 the unconstrained maximum happens to be on the boundary of the feasible region, but beyond α_1 it violates the first constraint, i.e. $(=\emptyset)\alpha$ leads to a feasibility alarm for g_1 just above α_1. Thus α_1 is the point of change which completes Step 2, and $(\underline{x}^{\emptyset}(\alpha_1), \underline{u}^{\emptyset}(\alpha_1)) = (\underline{x}*(\alpha_1), \underline{u}*(\alpha_1)) = (\hat{\underline{x}}(\alpha_1),\underline{0})$, $A\alpha_1 = \emptyset$, $B\alpha_1 = \{1\}$. Since $\alpha_1 < 1$, we go to Step 3. Two sets are valid at α_1: $\emptyset$ and $\{1\}$. The former was seen at Step 2 not to be valid above α_1, and so the latter must be. Control is now returned to Step 2 with $S^o = \{1\}$.

To execute Step 2 for the second time we must solve $(=\{1\})\alpha$ as α increases above α_1 until an alarm obtains. These equations are the conditions for a maximum of the objective function subject to the first constraint being exactly satisfied. As α increases above α_1, $\underline{x}^1(\alpha)$ moves along the portion of the boundary determined by the first constraint; since minus the gradient of the objective function at $\underline{x}^1(\alpha)$ is expressed as $u_1^1(\alpha)$ times the gradient of g_1, it is geometrically clear that $u_1^1(\alpha)$ grows increasingly positive as α increases. Hence no alarms are given until α_2 is passed, when the second constraint begins to be violated. We have

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$\underline{x}^1(\alpha) = \underline{x}^*(\alpha)$, $u_1^1(\alpha) = u_1^*(\alpha) > 0$, $u_2^1(\alpha) = u_2^*(\alpha) = 0$, $A\alpha = B\alpha = \{1\}$ on (α_1, α_2). Since $\alpha_2 < 1$ is the point of change at which Step 2 is completed, we go to Step 3. Now $A\alpha_2 = \{1\}$ and $B\alpha_2 = \{1,2\}$, so that $\{1\}$ and $\{1,2\}$ are valid at α_2; since the former was seen not to be valid just above α_2, the latter must be. Control is returned to Step 2 again, this time with $S^o = \{1,2\}$.

Step 2 now requires that $(=\{1,2\})\alpha$ be solved as α increases above α_2 until an alarm occurs. These equations are the conditions for a maximum of the objective function subject to both constraints being satisfied exactly. Since the intersection of the two equality constraints determines a unique point, $\underline{x}^{1,2}(\alpha)$ is constant for all α. The projection lines of $\hat{\underline{x}}(\alpha)$ onto the feasible region and the interpretation of the dual variables make it clear that $\underline{u}^{1,2}(\alpha) > \underline{0}$ on (α_2, α_3), $u_1^{1,2}(\alpha_3) = 0$, $u_2^{1,2}(\alpha_3) > 0$, and $u_1^{1,2}(\alpha) < 0$, $u_2^{1,2}(\alpha) > 0$ for $\alpha > \alpha_3$. In other words, an optimality alarm occurs for the first constraint just above α_3, so that Step 2 is complete at that point of change. Going to Step 3, we see that $A\alpha_3 = \{2\}$, $B\alpha_3 = \{1,2\}$; since the latter is not valid just above α_3 the former must be. Control is returned to Step 2 with $S^o = \{2\}$.

At Step 2, $(=\{2\})\alpha$ must be solved as α increases above α_3. Reasoning as before, we see that $\{2\}$ remains valid on $[\alpha_3, 1]$. Hence $\underline{x}^2(\alpha) = \underline{x}^*(\alpha)$, $A\alpha = B\alpha = \{2\}$, $u_1^2(\alpha) = 0$, and $u_2^2(\alpha) > 0$ on $(\alpha_3, 1]$.

This completes the solution of the first example. A summary appears in Table B.1. Note that there were no false or silent alarms, and no erroneous trials at any Step 3.

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The second and third examples are presented graphically in Figures B.2 and B.3. The summaries which appear in the corresponding Tables B.2 and B.3 can be constructed by following the lines of reasoning illustrated in the above discussion of the first example. Nevertheless, certain of the entries are reasoned out below. The second example is designed to show that false feasibility and silent optimality alarms can occur, the third to show that false optimality and silent feasibility alarms can occur.

The second example is very much like the first, except that the unconstrained maximum happens to pass through the vertex of the feasible region. At $\alpha = \alpha_1$: $\underline{x}^*(\alpha_1) = \hat{\underline{x}}(\alpha_1)$, $A\alpha_1 = \emptyset$, and $B\alpha_1 = \{1,2\}$. At Step 3 one must solve $(=S)\alpha$ for α just above α_1, S valid at α_1, until a set which is valid just above α_1 is found. The four sets $\emptyset$, $\{1\}$, $\{2\}$, and $\{1,2\}$ are valid at α_1. If one tries $\emptyset$, it is clear that $\underline{x}^{\emptyset}(\alpha) \equiv \hat{\underline{x}}(\alpha)$ violates both constraints as α increases above α_1, and also that only $\{2\}$ is valid just above α_1. Hence there is a false feasibility alarm for g_1, for g_1 is not in the deficiency of $\emptyset$ and is not degenerate just above α_1. See the second line of Table B.2. If one tries $\{1\}$, $(=\{1\})\alpha$ are the conditions for a maximum of the objective function subject to the first constraint being exactly satisfied. It is evident that $\underline{x}^1(\alpha)$ violates the second constraint above α_1, i.e. a feasibility alarm for g_2 obtains. Since minus the gradient of the objective function at $\underline{x}^1(\alpha)$ is expressed as $u_1^1(\alpha)$ times the gradient of g_1, $u_1^1(\alpha)$ is seen to be positive above α_1. Thus no optimality

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alarm obtains for g_1, which means, in view of the unique validity of $\{2\}$ above α_1 and the fact that g_1 is in the excess of $\{1\}$ just above α_1, that $(=\{1\})\alpha$ leads to a silent optimality alarm. See the third line of Table B.2.

In the third and last example, the unconstrained maximum again happens to pass through the vertex of the feasible region. At $\alpha = \alpha_1$, we have $\underline{x}^*(\alpha_1) = \hat{\underline{x}}(\alpha_1)$, $A\alpha_1 = \emptyset$, and $B\alpha_1 = \{1,2\}$. The valid sets at α_1 are $\emptyset$, $\{1\}$, $\{2\}$, and $\{1,2\}$. The only set which is valid just above α_1 is $\{2\}$. If one tries $\{1\}$ at Step 3, $\underline{x}^1(\alpha)$ evidently remains feasible. Since g_2 is in the deficiency of $\{1\}$ just above α_1, we see that $(=\{1\})\alpha$ leads to a silent feasibility alarm, as recorded in the third line of Table B.3. If one tries $\{1,2\}$, $\underline{x}^{1,2}(\alpha)$ must remain at the intersection of the two equality constraints. It is graphically clear that minus the gradient of the objective function at $\underline{x}^{1,2}(\alpha) = \underline{x}^*(\alpha_1)$ is represented by a negative linear combination of the gradients of the constraints as α increases above α_1, so that optimality alarms occur for both constraints. Since g_2 is not in the excess of $\{1,2\}$ and is not degenerate just above α_1, a false optimality alarm registers for the second constraint. See the fifth line of Table B.3.

Figure B.1

α	Valid Sets at α: S	Feasibility and Optimality Alarms Due to S Just Above α		Deficiency and Excess of S Just Above α	
		Feasibility	Optimality	Deficiency	Excess
$[0, \alpha_1)$	$\emptyset$	----	----	----	----
α_1	$\emptyset$	{1}	None	{1}	None
	{1}	None	None	None	None
(α_1, α_2)	{1}	----	----	----	----
α_2	{1}	{2}	None	{2}	None
	{1,2}	None	None	None	None
(α_2, α_3)	{1,2}	----	----	----	----
α_3	{1,2}	None	{1}	None	{1}
	{2}	None	None	None	None
$(\alpha_3, 1]$	{2}	----	----	----	----

Table B.1

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$\underline{c}'$ $\underline{\hat{x}}(1)$

$\underline{x}^*(1)$

$\nabla_x g_1(\underline{x}^*(\alpha_1))$

$\nabla_x g_2(\underline{x}^*(\alpha_1))$

$\underline{\hat{x}}(\alpha_1) \equiv \underline{x}^*(\alpha_1)$

Figure B.2

$\underline{x}^*(0)$ $\underline{\hat{x}}(0)$

$\underline{c}$

$g_1(\underline{x}) = 0$ $g_2(\underline{x}) = 0$

α	Valid Sets at α: S	Feasibility and Optimality Alarms Due to S Just Above α		Deficiency and Excess of S Just Above α	
		Feasibility	Optimality	Deficiency	Excess
$[0,\alpha_1)$	$\emptyset$	----	----	----	----
α_1	$\emptyset$	{1,2}[1]	None	{2}	None
	{1}	{2}	None[2]	{2}	{1}
	{2}	None	None	None	None
	{1,2}	None	{1}	None	{1}
$(\alpha_1,1]$	{2}	----	----	----	----

Table B.2

[1] False feasibility alarm for g_1
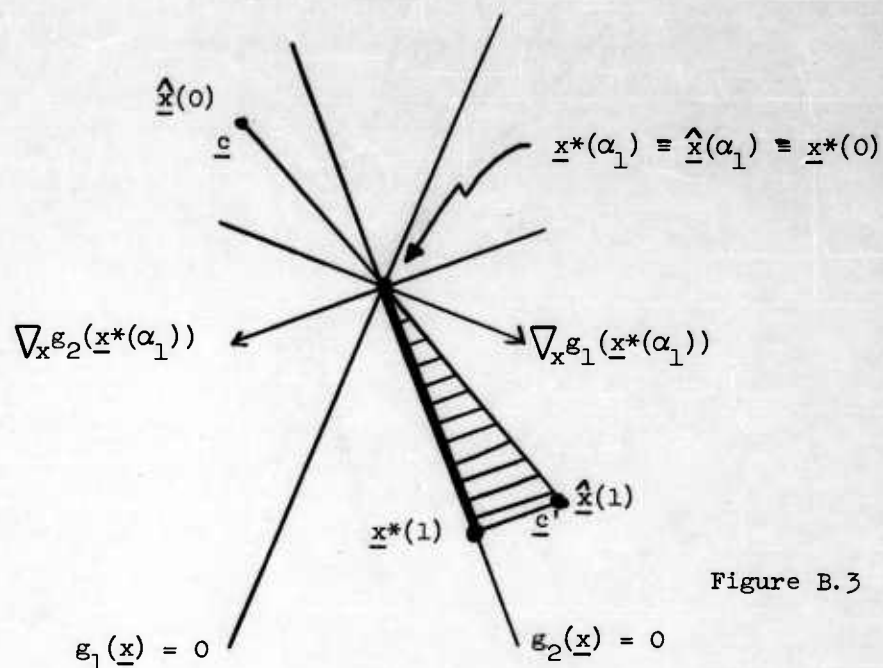
[2] Silent optimality alarm; no optimality alarm for u_1.

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Figure B.3

α	Valid Sets at α: S	Feasibility and Optimality Alarms Due to S Just Above α		Deficiency and Excess of S Just Above α	
		Feasibility	Optimality	Deficiency	Excess
$[0,\alpha_1)$	$\{1,2\}$	----	----	----	----
α_1	$\emptyset$	$\{2\}$	None	$\{2\}$	None
	$\{1\}$	None[1]	$\{1\}$	$\{2\}$	$\{1\}$
	$\{2\}$	None	None	None	None
	$\{1,2\}$	None	$\{1,2\}$[2]	None	$\{1\}$
$(\alpha_1,1]$	$\{2\}$	----	----	----	----

Table B.3

[1] Silent feasibility alarm: no feasibility alarm for g_2.

[2] False optimality alarm for u_2.

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APPENDIX C

Computational Devices

1. ### Newton's Method

Newton's method is based on using successive linear approximations for solving systems of nonlinear equations when good first approximations of the solutions are available. In order to solve the system

(C.1)
$$
\begin{aligned}
f_1(\underline{x}) &= 0 \\
f_2(\underline{x}) &= 0 \\
&\vdots \\
f_n(\underline{x}) &= 0 \ ,
\end{aligned}
$$

where $\underline{x}$ is an n-vector, Newton's method is the recursion

(C.2)
$$
\underline{x}^{k+1} = \underline{x}^k - \left[\frac{\partial f_i(\underline{x}^k)}{\partial x_j} \right]^{-1} \underline{f}(\underline{x}^k), \qquad k = 0, 1, 2, \ldots
$$

where $\underline{x}^o$ is a given starting point. The stated inverse must exist in order for (C.2) to be well-defined, of course. We denote the right-hand side of (C.2) by $\underline{F}(\underline{x}^k)$. $\underline{F}(\underline{x})$ is the iteration function obtained by applying Newton's method to (C.1).

There are numerous versions of conditions under which Newton's method can be guaranteed to converge. The following theorem is typical.

Theorem C.1 :

Assume that $f_i(\underline{x})$ $(i = 1,\ldots,n)$ is continuously differentiable on some neighborhood of $\underline{x}^*$, and that the Jacobian $\dfrac{\partial(\underline{f}(\underline{x}))}{\partial(\underline{x})}$ does not

vanish at $\underline{x}*$, where $\underline{f}(\underline{x}*) = \underline{0}$. Then Newton's method is well-defined and quadratically convergent to $\underline{x}*$ if the starting point $\underline{x}^o$ is in a sufficiently small neighborhood of $\underline{x}*$.

See Householder (1953, p. 136) for a proof.

Quadratic convergence of the sequence $\langle \underline{x}^k \rangle$ $(k = 0,1,\ldots.)$ to $\underline{x}*$ means that (here $\|\cdot\|$ denotes the Euclidean norm)

$$\lim_{k \to \infty} \frac{\|\underline{x}^k - \underline{x}*\|}{\|\underline{x}^{k-1} - \underline{x}*\|^2} = \text{a constant} \neq 0 \;.$$

By way of contrast, linear convergence would mean that

$$\lim_{k \to \infty} \frac{\|\underline{x}^k - \underline{x}*\|}{\|\underline{x}^{k-1} - \underline{x}*\|} = \text{a constant} \neq 0 \;.$$

Evidently the quadratic convergence of Newton's method is a highly desirable feature. The price one pays for it is the necessity of evaluating an inverse matrix at each iteration, and having to have a good starting point. To ameliorate the first disadvantage, at some expense of speed of convergence, approximate inverses can be used. Often one can achieve a substantial net gain in computational efficiency by judicious application of this idea (see, for example, Ostrowski, 1960, and Householder, 1953, p. 136).

For the purpose of proving Theorems 4.1 and 4.2, we find it more convenient to employ

<u>Theorem C.2</u>:

Let $\underline{x}^*$ satisfy $\underline{f}(\underline{x}^*) = \underline{0}$. Assume that there exists a neighborhood $N_r(\underline{x}^*)$ on which the following three assertions hold:

(a) The functions $f_i(\underline{x})$ $(i = 1,\ldots,n)$ are twice continuously differentiable.

(b) The Jacobian $\dfrac{\partial(\underline{f}(\underline{x}))}{\partial(\underline{x})} \neq 0$.

(c) $\left[\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial F_i(\underline{x})}{\partial x_j}\right)^2\right]^{1/2} \leq L < 1.$

Then Newton's method (C.2) is well-defined and quadratically convergent to $\underline{x}^*$ if the starting point $\underline{x}^o$ is in $N_r(\underline{x}^*)$.

This theorem follows from results given in Householder (1953, p. 135) and Henrici (1964, p. 101).

<u>Remark</u>: The square-root expression in (c) is an upper estimate of the Euclidean norm of the Jacobian matrix of $\underline{F}(\underline{x})$ (see Faddeeva, 1959, p. 121).

For reference we record the recursion equation of Newton's method applied to $(=S)\alpha_o$. We have, for $k = 0,1,2,\ldots$

$$(C.3)\quad \left(\frac{\underline{x}}{\underline{u}_S}\right)^{k+1} = \left(\frac{\underline{x}}{\underline{u}_S}\right) - \left[\begin{array}{c|c} H & D^t \\ \hline D & 0 \end{array}\right]^{-1}\left(\frac{\nabla_x\left(f(\underline{x};\alpha) + \sum_S u_i\, g_i(\underline{x})\right)}{g_S(\underline{x})}\right)$$

where $H = \nabla_x^2\left(f(\underline{x};\alpha) + \sum_S u_i\, g_i(\underline{x})\right)$, D is the matrix whose rows are $\nabla_x g_i(\underline{x})$ $(i \in S)$, and $\underline{u}_S$ and $\underline{g}_S$ are the vectors obtained

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by deleting from $\underline{u}$ and $\underline{g}(\underline{x})$ the components not in S; all quantities on the right-hand side are evaluated at α_o and $(\underline{x},\underline{u}_S)^k$. Note that the equations $u_i = 0$ $(i \notin S)$, which are a part of $(=S)\alpha_o$, have been omitted from the recursion because they are already solved.

In order to have a compact notation for the square-root expression in (c) of Theorem C.2 specialized to $(=S)\alpha_o$, we denote by $\Delta(\underline{x},\underline{u}: \alpha_o,S)$ the square-root of the sum of the squares of all the elements of the Jacobian matrix of the iteration function appearing in (C.3) (i.e., of the Jacobian matrix of the right-hand side of (C.3) considered as a vector-valued function of $(\underline{x},\underline{u}_S))$.

2. Convenient Partitions of the Inverse Matrix Required by Newton's Method

Let

$$\begin{bmatrix} H & D^t \\ \hline D & 0 \end{bmatrix}^{-1}$$

be defined as in (C.3). Under our conditions, it is easily verified that

$$(C.4) \quad \begin{bmatrix} H & D^t \\ \hline D & 0 \end{bmatrix}^{-1} = \begin{bmatrix} H^{-1} - H^{-1}D^t(DH^{-1}D^t)^{-1}DH^{-1} & H^{-1}D^t(DH^{-1}D^t)^{-1} \\ \hline (DH^{-1}D^t)^{-1}DH^{-1} & -(DH^{-1}D^t)^{-1} \end{bmatrix}.$$

Let there be s elements in S (by Condition 4, $s \leq n$). The inversion of the $n+s$ by $n+s$ matrix has been reduced to the inversion of two matrices, one n by n (H) and the other s by s $(DH^{-1}D^t)$, and to several matrix multiplications.

Whereas the size of H remains constant no matter what S is, the dimension of $DH^{-1}D^t$ does vary with S, for during Step 3 rows are added to and deleted from D as S changes. It is advantageous to use bordering methods to pass from an available $(DH^{-1}D^t)^{-1}$ to the next when S is changed at Step 3. We shall consider the case in which one row is added to "the bottom" of D, and also the case in which the last row of D is deleted. Results similar to the following can be derived to cover the addition or deletion of an arbitrary row, and also multiple additions and/or deletions.

If one row d is to be added to D, then

$$\left\{ \begin{bmatrix} D \\ \hline d \end{bmatrix} H^{-1}[D^t \vdots d^t] \right\}^{-1} = \begin{bmatrix} DH^{-1}D^t & DH^{-1}d^t \\ \hline dH^{-1}D^t & dH^{-1}d^t \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} (DH^{-1}D^t)^{-1} + \dfrac{Qd^t dQ^t}{dRd^t} & \dfrac{Qd^t}{dRd^t} \\ \hline \dfrac{dQ^t}{dRd^t} & 1/dRd^t \end{bmatrix} ,$$

where $Q = -(DH^{-1}D^t)^{-1}DH^{-1}$ and $R = H^{-1} - H^{-1}D^t(DH^{-1}D^t)^{-1}DH^{-1}$. Note that Q and R are immediately available from (C.4)

Let D be written

$$\begin{bmatrix} D_1 \\ \hline d \end{bmatrix} ,$$

and let $(DH^{-1}D^t)^{-1}$ be written

$$\begin{bmatrix} T_1 & T_2 \\ \hline T_2^t & T_3 \end{bmatrix} ,$$

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where T_3 is 1 by 1 (i.e., T_3 is a scalar). If row d is deleted from D, then $(D_1 H^{-1} D_1^t)^{-1} = T_1 - T_2 \, T_2^t / T_3$.

3.   A Refinement Method for Approximate Matrix Inverses

Suppose that A is a square matrix whose inverse exists and is desired to be found, and that an approximate inverse B_o is available. The error inherent in B_o causes the matrix $I - AB_o$ not to vanish. If[1] $\|I - AB_o\| \leq L < 1$, then the recursion

$$B_k = B_{k-1} + B_{k-1}(I - AB_{k-1}), \qquad k = 1, 2, \ldots$$

converges to A^{-1}, and the considerable rapidity of the convergence is apparent from the estimate

$$\|B_k - A^{-1}\| \leq \|B_o\| \, L^{2k} / (1-L) \ .$$

See Faddeeva (1959, pp. 99-102) for further details on this method, which is due to H. Hotelling.

It is clear that this device can be used to great advantage in maintaining an arbitrarily accurate approximation to H^{-1} as α increases (for the elements of H, and therefore of H^{-1}, are continuous functions of α on the unit interval), and also to $(DH^{-1}D^t)^{-1}$ so long as S stays the same.

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[1] We define the norm $\|A\|$ of any n by n matrix A as $\displaystyle \operatorname*{Max}_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$. Other norms could be used, but this one (the so-called "$p = 1$ norm") is particularly convenient for computational purposes.

4.   Formulae for $\ d(\underline{x}^S(\alpha), \underline{u}^S(\alpha))/d\alpha$

It may be shown by implicitly differentiating $(=S)\alpha$ that the following additional conclusions can be added to Theorem 2:   for $\alpha \in I\alpha_o$,

$$d(\underline{x}^S(\alpha))/d\alpha = - R(\nabla_x f_1(\underline{x}) - \nabla_x f_2(\underline{x}))$$

$$d(\underline{u}_S^S(\alpha))/d\alpha = Q(\nabla_x f_1(\underline{x}) - \nabla_x f_2(\underline{x})) \ ,$$

where R and Q are as in section 2 above and all quantities are evaluated at $(\underline{x}^S(\alpha), \underline{u}_S^S(\alpha))$.

These formulae are of possible interest for the purpose of facilitating the convergence of Newton's method, when fairly large step sizes are being used, by extrapolating to better starting points. Note that R and Q are immediately available from (C.4).

BIBLIOGRAPHY

Apostol, T., <u>Mathematical Analysis</u>, Addison-Wesley, Reading, Massachusetts, 1957.

Arrow, K. J., L. Hurwicz, and H. Uzawa, "Constraint Qualifications in Maximization Problems," <u>Naval Research Logistics Quarterly</u>, Vol. 8, pp. 175-190 (1961).

Bochner, S., and W. Martin, <u>Several Complex Variables</u>, Princeton University Press, Princeton, New Jersey, 1948.

Boot, J. C. G., "Notes on Quadratic Programming: The Kuhn-Tucker and Theil-Van de Panne Conditions, Degeneracy, and Equality Constraints," <u>Management Science</u>, Vol. 8, pp. 85-98 (1961).

Boot, J. C. G., "Binding Constraint Procedures of Quadratic Programming," <u>Econometrica</u>, Vol. 31, pp. 464-498 (1963a).

Boot, J. C. G., "On Sensitivity Analysis in Convex Quadratic Programming Problems," <u>Operations Research</u>, Vol. 11, pp. 771-786 (1963b).

Charnes, A., and W. Cooper, "Chance-Constrained Programming," <u>Management Science</u>, Vol. 6, pp. 73-79 (1959).

Charnes, A., and W. Cooper, "Deterministic Equivalents for Optimizing and Satisficing Under Chance Constraints," <u>Operations Research</u>, Vol. 11, pp. 18-39 (1963).

Dantzig, G., "Linear Programming Under Uncertainty," <u>Management Science</u>, Vol. 1, pp. 197-206 (1955).

Dorn, W., "Non-Linear Programming--A Survey," <u>Management Science</u>, Vol. 9, pp. 171-208 (1963).

Dvoretzsky, A., J. Kiefer, and J. Wolfowitz, "The Inventory Problem," Econometrica, Vol. 20, pp. 187-222 (1952).

Faddeeva, V., Computational Methods of Linear Algebra, (C. D. Benster, translator), Dover Publications, New York, 1959.

Fenchel, W., Convex Cones, Sets, and Functions, Lecture notes, Department of Mathematics, Princeton University, 1953.

Gass, S., and T. Saaty, "The Computational Algorithm for the Parametric Objective Function," Naval Research Logistics Quarterly, Vol. 2, pp. 39-45 (1955).

Geoffrion, A., "Continuity in Parametric Concave Programming Problems," Western Management Science Institute Working Paper (to appear, 1965).

Graves, R., and P. Wolfe, Recent Advances in Mathematical Programming, McGraw-Hill, New York, 1963.

Hadley, G., Nonlinear and Dynamic Programming, Addison-Wesley, Reading, Massachusetts, 1964.

Henrici, P., Elements of Numerical Analysis, Wiley, New York, 1964.

Hodges, J., and E. Lehmann, "The Use of Previous Experience in Reaching Statistical Decisions," Annals of Mathematical Statistics, Vol. 23, pp. 369-407 (1952).

Householder, A., Principles of Numerical Analysis, McGraw-Hill, New York, 1953.

Houthakker, H., "The Capacity Method of Quadratic Programming," Econometrica, Vol. 28, pp. 62-87 (1960).

Karlin, S., Mathematical Methods and Theory in Games, Programming, and Economics, Vol. I, Addison-Wesley, Reading, Massachusetts, 1959.

Kataoka, S., "A Stochastic Programming Model," _Econometrica_, Vol. 31, pp. 181-196 (1963).

Kuhn, H., and A. Tucker, "Nonlinear Programming," in _Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability_, (J. Neyman, ed.), University of California Press, Berkeley, 1951, pp. 481-492.

Luce, R., and H. Raiffa, _Games and Decisions_, Wiley, 1957.

Madansky, A., "Methods of Solution of Linear Programs Under Uncertainty," _Operations Research_, Vol. 10, pp. 463-471 (1962).

Madansky, A., "Linear Programming Under Uncertainty," in Graves and Wolfe (1963), pp. 103-110.

Mangasarian, O., and J. B. Rosen, "Inequalities for Stochastic Non-Linear Programming Problems," _Operations Research_, Vol. 12, pp. 143-154 (1964).

Markowitz, H., "The Optimization of a Quadratic Function Subject to Linear Constraints," _Naval Research Logistics Quarterly_, Vol. 3, pp. 111-133 (1956).

Neyman, J., and E. Pearson, "On the Problem of the Most Efficient Tests of Statistical Hypotheses," _Philosophical Transactions of the Royal Society_, A-231, pp. 289-337 (1933).

Ostrowski, A., _Solution of Equations and Systems of Equations_, Academic Press, New York, 1960.

Parzen, E., _Modern Probability Theory and Its Applications_, Wiley, 1960.

Reiter, S., "Surrogates for Uncertain Decision Problems: Minimal Information for Decision-Making," _Econometrica_, Vol. 25, pp. 339-345 (1957).

Saaty, T., and J. Bram, <u>Nonlinear Mathematics</u>, McGraw-Hill, New York, 1964.

Simon, H., "Dynamic Programming Under Uncertainty with a Quadratic Criterion Function," <u>Econometrica</u>, Vol. 24, pp. 74-81 (1956).

Theil, H., and C. Van De Panne, "Quadratic Programming as an Extension of Classical Quadratic Maximization," <u>Management Science</u>, Vol. 7, pp. 1-20 (1960).

Theil, H., <u>Optimal Decision Rules for Government and Industry</u>, North-Holland Publishing Company, Amsterdam, 1964.

Wilde, D., "Differential Calculus in Nonlinear Programming," <u>Operations Research</u>, Vol. 10, pp. 764-773 (1962).

Wolfe, P., "The Simplex Method for Quadratic Programming," <u>Econometrica</u>, Vol. 27, pp. 382-398 (1959).

Wolfe, P., "Recent Developments in Nonlinear Programming," The RAND Corporation, Report R-401-PR, May 1962. An abridgement was published in Graves and Wolfe (1963), pp. 67-86.

Zahl, S., "A Deformation Method for Quadratic Programming," <u>Journal of the Royal Statistical Society</u>, Series B, Vol. 26, pp. 141-160 (1964).

Zoutendijk, G., <u>Methods of Feasible Directions</u>, Elsevier, Amsterdam, 1960.