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SYSTEMS THEORY LABORATORY

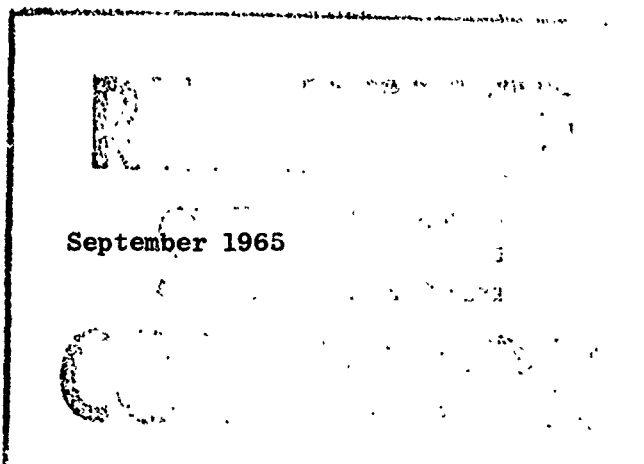
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J. Omura and T. Kailath



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Systems Theory Laboratory
Stanford Electronics Laboratories
Stanford University Stanford, California

ABSTRACT

We present a compilation of probability density functions, distribution functions, and characteristic functions for several arithmetic combinations of Gaussian random variables. These functions, the distribution functions in particular, need to be known for the evaluation of the error probabilities of many communication systems. We endeavour, as far as possible, to express these functions in "closed form" in terms of well known (and tabulated) transcendental functions. Our purpose is to have a convenient list of canonical forms which will be useful when working with arithmetic combinations of Gaussian random variables.

Since distribution functions are the most useful, they are listed first. In the second half the density functions, characteristic functions, and some moments are listed. Both halves follow the same outline so that one can easily locate corresponding functions relating to the same random variable. Since many distribution functions could not be found, in the list of distribution functions there are dashes, --, to indicate these yet unknown functions.

Some miscellaneous results are given in Appendix A. The transcendental functions are defined in Appendix B along with some useful relationships.

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DEFINITIONS AND NOTATION

Throughout this report capital letters will represent vectors or matrices with the exception of F , G , and transcendental functions. $F(\cdot)$ and $G(\cdot)$ are the only letters used to represent distribution functions of random variables with $f(\cdot)$ and $g(\cdot)$ representing the corresponding density functions. $\psi(\mu)$ will denote all characteristic functions. Letters x and y are used only for Gaussian random variables and $\underline{X}$ and $\underline{Y}$ are the only letters denoting Gaussian vectors.

$\underline{X}$ is always a column vector with independent equal variance Gaussian random variables as components. That is, for $\underline{X}$ n -dimensional

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $\{x_k\}_{k=1}^n$ are independent Gaussian random variables with equal variances, σ^2 . If $E\underline{X} = \underline{A}$ then this class of Gaussian vectors will be denoted as $N_n(\underline{A}, \sigma^2)$. Hence $\underline{X} \in N_n(\underline{A}, \sigma^2)$ means that $\underline{X}$ is an n -dimensional Gaussian vector with independent components and with common variance σ^2 and mean $E\underline{X} = \underline{A}$.

Only Gaussian vectors of the same order are allowed to be statistically dependent. If $\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2)$ and $\underline{X}^{(2)} \in N_n(\underline{B}, \sigma_2^2)$ are dependent then we mean dependence in the following manner.

$$\underline{X}^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}, \quad \underline{X}^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{bmatrix}; \quad \underline{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and

$$E(x_j^{(1)} - a_j)(x_k^{(2)} - b_k) = \begin{cases} 0 & j \neq k \\ \rho\sigma_1\sigma_2 & j = k \end{cases} \quad j, k = 1, 2, \dots, n$$

That is, only components of $X^{(1)}$ and $X^{(2)}$ having identical subscripts can be correlated. We shall summarize this dependence by the matrix.

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \quad |\rho| < 1$$

$\underline{M}$ is the common covariance matrix of the vector-pair components. We shall also define

$$\underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_1^2(1-\rho^2)} & -\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)} \\ -\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)} & \frac{1}{\sigma_2^2(1-\rho^2)} \end{bmatrix}$$

Clearly if $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are independent then $\rho = 0$.

If $\underline{X} \in N_n(\underline{0}, \sigma^2)$, we shall define

$$r = \|\underline{X}\| = \left(\sum_{k=1}^n x_k^2 \right)^{1/2}$$

as the Rayleigh variable of order n . r^2 is essentially the central chi square variable of order n .

If $\underline{X} \in N_n(\underline{A}, \sigma^2)$ we shall define

$$a = \|\underline{A}\| = \left(\sum_{k=1}^n a_k^2 \right)^{1/2}$$

as the norm of the mean and

$$v = \|\underline{X}\| = \left(\sum_{k=1}^n x_k^2 \right)^{1/2}$$

as the Rice variable of order n . v^2 is essentially the noncentral chi square variable of order n . Throughout this paper r and v will denote only these random variables.

We shall define an inner product for $\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2)$ and $\underline{X}^{(2)} \in N_n(\underline{B}, \sigma_2^2)$ as

$$(\underline{X}^{(1)}, \underline{X}^{(2)}) = \sum_{k=1}^n x_k^{(1)} x_k^{(2)}$$

For any positive real number, c , we define $[c]$ as

$$[c] = \min_n \{n; n \geq c; n \text{ is an integer}\}$$

If z is a random variable and $f(\cdot)$ its density function, we shall write its (cumulative) distribution function as

$$F(z) = \int_{-\infty}^z f(t) dt$$

and its characteristic function as

$$\psi_z(\mu) = \int_{-\infty}^{\infty} e^{i\mu r} f(r) dr, \quad i = \sqrt{-1}$$

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I. FUNDAMENTAL VARIABLES

A. GAUSSIAN

$$x \in N_1(a, \sigma^2)$$

$$F(x) = \begin{cases} \frac{1}{2} - \operatorname{erf}\left(\frac{a-x}{\sqrt{2\sigma^2}}\right) & x \leq a \\ \frac{1}{2} + \operatorname{erf}\left(\frac{x-a}{\sqrt{2\sigma^2}}\right) & x > a \end{cases}$$

[Ref. 5, p. 136]

B. RAYLEIGH

$$\underline{X} \in N_n(\underline{0}, \sigma^2); \quad r = \|\underline{X}\|$$

1. n = 1

$$F(r) = \operatorname{erf}\left(\frac{r}{\sqrt{2\sigma^2}}\right) \quad r \geq 0$$

[Ref. 5, p. 136]

2. n = 2

$$F(r) = 1 - e^{-r^2/2\sigma^2} \quad r \geq 0$$

3. n = 2k (even)

$$F(r) = 1 - e^{-r^2/2\sigma^2} \sum_{j=0}^{k-1} \frac{1}{j!} \left(\frac{r^2}{2\sigma^2}\right)^j \quad r \geq 0$$

[Ref. 5, p. 134]

4. n

C. RICE

$$\underline{X} \in N_n(\underline{A}, \sigma^2), \quad a = \|\underline{A}\|, \quad v = \|\underline{X}\|$$

1. n = 1

$$F(v) = \begin{cases} \frac{1}{2} \operatorname{erf} \left(\frac{v+|a|}{\sqrt{2\sigma^2}} \right) - \frac{1}{2} \operatorname{erf} \left(\frac{|a|-v}{\sqrt{2\sigma^2}} \right) & 0 \leq v \leq |a| \\ \frac{1}{2} \operatorname{erf} \left(\frac{v+|a|}{\sqrt{2\sigma^2}} \right) + \frac{1}{2} \operatorname{erf} \left(\frac{v-|a|}{\sqrt{2\sigma^2}} \right) & v > |a| \end{cases}$$

[Ref. 5, p. 136]

2. n = 2

$$F(v) = 1 - Q \left(\frac{a}{\sigma}, \frac{v}{\sigma} \right) \quad v \geq 0 \quad [\text{Ref. 19, p. 159}]$$

3. n = 2k (even)

$$F(v) = 1 - Q_k \left(\frac{a}{\sigma}, \frac{v}{\sigma} \right) \quad v \geq 0$$

4. n = 2k+1 (odd)

5. n

II. RATIO

A. GAUSSIAN/GAUSSIAN

$$x \in N_1(a, \sigma_1^2), \quad y \in N_1(b, \sigma_2^2)$$

Let
$$z = \frac{y}{x}, \quad c = \frac{\sigma_2}{\sigma_1}$$

1. Independent

a. $a = b = 0$

$$F(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left(\frac{z}{c} \right) \quad [\text{Ref. 5, p. 30}]$$

b. $a \neq 0, \quad b = 0, \quad \sigma_1^2 = \sigma_2^2 = 1$

$$F(z) = \begin{cases} \frac{1}{\pi} \tan^{-1} \left(\frac{1}{|z|} \right) - 2V \left(\frac{a|z|}{\sqrt{1+z^2}}; \frac{a}{\sqrt{1+z^2}} \right) & z < 0 \\ \frac{1}{2} & z = 0 \\ 1 - \frac{1}{\pi} \tan^{-1} \left(\frac{1}{z} \right) + 2V \left(\frac{az}{\sqrt{1+z^2}}; \frac{a}{\sqrt{1+z^2}} \right) & z > 0 \end{cases}$$

[Ref. 29, p. 118]

c. $a \neq 0, \quad b \neq 0$

2. Dependent

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

a. $a = b = 0$

$$F(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left(\frac{\frac{z}{c} - \rho}{(1-\rho^2)^{1/2}} \right) \quad [\text{Ref. 14}]$$

b. $a \neq 0, \quad b = 0$

c. $a \neq 0, \quad b \neq 0$

B. GAUSSIAN/RAYLEIGH (INDEPENDENT)

$$\underline{x} \in N_1(0, \sigma_1^2), \quad \underline{X} \in N_n(\underline{0}, \sigma_2^2); \quad r = \|\underline{X}\|$$

Let $z = \frac{x}{r}, \quad c = \frac{\sigma_2}{\sigma_1}$

1. $n = 1$

$$F(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} (cz) \quad [\text{Ref. 5, p. 30}]$$

2. n = 2

$$F(z) = \frac{1}{2} + \frac{z}{2(c^2 + z^2)^{1/2}} \quad [\text{Ref. 5, p. 50}]$$

3. n = 3

$$F(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(cz) + \frac{cz}{\pi(1+c^2 z^2)} \quad [\text{Ref. 5, p. 30}]$$

4. n = 2k (even); $\sigma_1^2 = \sigma_2^2 = 1$

$$F(z) = \begin{cases} \frac{1}{2} - \frac{1}{2} \left(\frac{z^2}{1+z^2} \right)^{1/2} \sum_{j=1}^k \binom{2j-2}{j-1} \left(\frac{1}{4+4z^2} \right)^{j-1} & z < 0 \\ \frac{1}{2} + \frac{1}{2} \left(\frac{z^2}{1+z^2} \right)^{1/2} \sum_{j=1}^k \binom{2j-2}{j-1} \left(\frac{1}{4+4z^2} \right)^{j-1} & z \geq 0 \end{cases}$$

[Ref. 16, p. 345]

5. n

C. GAUSSIAN/RICE (INDEPENDENT)

$$x \in N_1(0, \sigma_1^2), \quad \underline{X} \in N_n(\underline{A}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad v = \|\underline{X}\|$$

Let

$$q = \frac{x}{v}, \quad c = \frac{\sigma_2}{\sigma_1}$$

$$1. \quad \underline{n = 1; \quad \sigma_1^2 = \sigma_2^2 = 1}$$

$$F(q) = \begin{cases} \frac{1}{\pi} \tan^{-1} \left(\frac{1}{|q|} \right) - 2V \left(\frac{a|q|}{\sqrt{1+q^2}}, \frac{a}{\sqrt{1+q^2}} \right) & q < 0 \\ \frac{1}{2} & q = 0 \\ 1 - \frac{1}{\pi} \tan^{-1} \left(\frac{1}{q} \right) + 2V \left(\frac{aq}{\sqrt{1+q^2}}, \frac{a}{\sqrt{1+q^2}} \right) & q > 0 \end{cases}$$

[Ref. 29, p. 118]

$$2. \quad \underline{n = 2}$$

$$F(q) = \begin{cases} Q(\beta, \alpha) - \left(\frac{\sigma_2 \alpha}{a} \right) \exp \left(- \frac{\alpha^2 + \beta^2}{2} \right) I_0(\alpha\beta) & q < 0 \\ 1 - Q(\alpha, \beta) + \left(\frac{\sigma_2 \beta}{a} \right) \exp \left(- \frac{\alpha^2 + \beta^2}{2} \right) I_0(\alpha\beta) & q \geq 0 \end{cases}$$

where

$$\alpha = \frac{a}{2\sigma_2} \left[1 - \frac{cq}{(1+c^2 q^2)^{1/2}} \right]; \quad \beta = \frac{a}{2\sigma_2} \left[1 + \frac{cq}{(1+c^2 q^2)^{1/2}} \right]$$

[Ref. 29, p. 119]

3. $n = 2k$ (even); $\sigma_1^2 = \sigma_2^2 = 1$

$$F(q) = \begin{cases} q(\alpha, \beta) - \left(\frac{\beta}{a}\right) \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) I_0(\alpha\beta) - \frac{1}{2} \left(\frac{q^2}{1+q^2}\right)^{1/2} \\ \quad \cdot \sum_{j=2}^k \binom{2j-2}{j-1} \left(\frac{1}{4+4q^2}\right)^{j-1} {}_1F_1\left(\frac{1}{2}, j, -\frac{a^2}{2(1+q^2)}\right) & q < 0 \\ \\ 1 - q(\alpha, \beta) + \left(\frac{\beta}{a}\right) \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) I_0(\alpha\beta) + \frac{1}{2} \left(\frac{q^2}{1+q^2}\right)^{1/2} \\ \quad \cdot \sum_{j=2}^k \binom{2j-2}{j-1} \left(\frac{1}{4+4q^2}\right)^{j-1} {}_1F_1\left(\frac{1}{2}, j, -\frac{a^2}{2(1+q^2)}\right) & q \geq 0 \end{cases}$$

where

$$\alpha = \frac{a}{2} \left[1 - \left(\frac{q^2}{1+q^2}\right)^{1/2} \right], \quad \beta = \frac{a}{2} \left[1 + \left(\frac{q^2}{1+q^2}\right)^{1/2} \right]$$

[Ref. 16, p. 343]

4. n

D. RAYLEIGH/RAYLEIGH

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$\text{Let} \quad z = \frac{r_2}{r_1}, \quad c = \frac{\sigma_2}{\sigma_1}$$

$$\text{a. } n = 1, \quad m = 1$$

$$F(z) = \frac{2}{\pi} \tan^{-1} \left(\frac{z}{c} \right) \quad z \geq 0 \quad [\text{Ref. 5, p. 30}]$$

$$\text{b. } n = 1, \quad m = 2$$

$$F(z) = 1 - \left(\frac{c^2}{c^2 + z^2} \right)^{1/2} \quad z \geq 0 \quad [\text{Ref. 5, p. 50}]$$

$$\text{c. } n = 1, \quad m = 2k \quad (\text{even}); \quad \sigma_1^2 = \sigma_2^2 = 1$$

$$F(z) = 1 - \left(\frac{1}{1 + z^2} \right) \sum_{j=1}^k \binom{2j-2}{j-1} \left(\frac{z^2}{4 + 4z^2} \right)^{j-1} \quad z \geq 0$$

[Ref. 16, p. 343]

$$\text{d. } n = 2, \quad m = 2$$

$$F(z) = 1 - \frac{c^2}{1 + z^2} \quad z \geq 0 \quad [\text{Ref. 5, p. 31}]$$

e. $n = m$; $\sigma_1^2 = \sigma_2^2 = 1$ (Recursive Form)

$$F_n(z) = F_{n-2}(z) - \frac{h \Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi} (n-2) \Gamma\left(\frac{n-2}{2}\right) (1+h^2)^{(n-1)/2}} \quad z \geq 0$$

where

$$h = \frac{1-z^2}{2z} \quad [\text{Ref. 14}]$$

f. n even, m even

$$F(z) = 1 - \frac{1}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \sum_{j=0}^{(m/2)-1} \binom{(m/2)-1}{j} \frac{(-1)^j}{\left(\frac{n}{2} + j\right)} \left(\frac{c^2}{c^2 + z^2}\right)^{(n/2)+j} \quad z \geq 0$$

[Ref. 29, p. 112]

g. n odd, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$F(z) = 1 - \left(\frac{1}{1+z^2}\right)^{1/2} - \left(\frac{1}{1+z^2}\right)^{n/2} \left\{ \sum_{k=0}^{(m-4)/2} \left(\frac{z^2}{1+z^2}\right)^{(m/2)-k-1} \right. \\ \left. + \frac{\Gamma\left(\frac{m+n}{2} - k - 1\right)}{\left(\frac{m}{2} - k - 1\right)! \Gamma\left(\frac{n}{2}\right)} - z^2 \sum_{k=0}^{(n-3)/2} (1+z^2)^k \right\} \quad z \geq 0$$

[Ref. 29, p. 119]

h. n odd, m odd

See Ref. 29, p. 118.

i. n, m

See Ref. 29.

2. Dependent

$$\underline{x}^{(1)} \in N_n(\underline{0}, 1), \quad \underline{x}^{(2)} \in N_n(\underline{0}, 1)$$

$$\underline{M} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad r_1 = \|\underline{x}^{(1)}\|, \quad r_2 = \|\underline{x}^{(2)}\|$$

Let
$$z = \frac{r_2}{r_1}$$

a. n = 1

$$F(z) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left(\frac{h}{\sqrt{1-\rho^2}} \right) \quad z \geq 0$$

where

$$h = \frac{1 - z^2}{2z}$$

b. n = 2

$$F(z) = \frac{1}{2} - \frac{h}{2\sqrt{1-\rho^2+h^2}} \quad z \geq 0$$

where

$$h = \frac{1 - z^2}{2z}$$

c. $n = 2k$ (even) (Recursive Form)

$$F_{2k}(z) = F_{2k-2}(z) - \frac{h(1-\rho^2)^{k-1} \Gamma\left(k - \frac{1}{2}\right)}{\sqrt{\pi} (2k-2)(k-2)! (1-\rho^2+h^2)^{k-(1/2)}} \quad z \geq 0$$

where

$$h = \frac{1 - z^2}{2z^2}$$

d. n (Recursive Form)

$$F_n(z) = F_{n-2}(z) - \frac{h(1-\rho^2)^{(n-2)/2} \Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi} (n-2) \Gamma\left(\frac{n-2}{2}\right) (1-\rho^2+h^2)^{(n-1)/2}} \quad z \geq 0$$

where

$$h = \frac{1 - z^2}{2z} \quad [\text{Ref. 14}]$$

E. RICE/RAYLEIGH (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{C}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{B}, \sigma_2^2)$$

$$r = \|\underline{X}^{(1)}\|, \quad v = \|\underline{X}^{(2)}\|, \quad b = \|\underline{B}\|$$

Let

$$u = \frac{v}{r}, \quad c = \frac{\sigma_2}{\sigma_1}$$

$$1. \quad \underline{n = 1, \quad m = 1; \quad \sigma_1^2 = \sigma_2^2 = 1}$$

$$F(u) = \frac{2}{\pi} \tan^{-1} u - 4V \left(\frac{b}{\sqrt{1+u^2}}, \frac{bu}{\sqrt{1+u^2}} \right) \quad u \geq 0$$

[Ref. 29, p. 118]

$$2. \quad \underline{n = 1, \quad m = 2}$$

$$F(u) = 2 \left[Q(\alpha, \beta) - \left(\frac{\sigma_2 \beta}{b} \right) \exp \left(- \frac{\alpha^2 + \beta^2}{2} \right) I_0(\alpha\beta) \right] \quad u \geq 0$$

where

$$\alpha = \frac{b}{2\sigma_2} \left[1 - \left(\frac{c^2}{c^2 + u^2} \right)^{1/2} \right], \quad \beta = \frac{b}{2\sigma_2} \left[1 + \left(\frac{c^2}{c^2 + u^2} \right)^{1/2} \right]$$

[Ref. 29, p. 119]

$$3. \quad \underline{n = 1, \quad m = 2k \quad (\text{even}); \quad \sigma_1^2 = \sigma_2^2 = 1}$$

$$F(u) = 2 \left[Q(\alpha, \beta) - \left(\frac{\beta}{b} \right) \exp \left(- \frac{\alpha^2 + \beta^2}{2} \right) I_0(\alpha\beta) \right] - \left(\frac{1}{1+u^2} \right)^{1/2} \exp \left(- \frac{b^2}{2(1+u^2)} \right) \\ \cdot \sum_{j=2}^k \binom{2j-2}{j-1} \left(\frac{u^2}{1+u^2} \right)^{j-1} {}_1F_1 \left(\frac{1}{2}; j; - \frac{b^2 u^2}{2(1+u^2)} \right) \quad u \geq 0$$

where

$$\alpha = \frac{b}{2} \left[1 - \left(\frac{1}{1+u^2} \right)^{1/2} \right], \quad \beta = \frac{b}{2} \left[1 + \left(\frac{1}{1+u^2} \right)^{1/2} \right]$$

[Ref. 16, p. 343]

4. $n = 2, m = 2$

$$F(u) = \frac{\sigma_1^2 u^2}{\sigma_2^2 + \sigma_1^2 u^2} \exp \left(- \frac{b^2}{2(\sigma_2^2 + \sigma_1^2 u^2)} \right) \quad u \geq 0$$

[Ref. 29, p. 110
and Appendix]

5. $n = m$

See Ref. 29.

6. n even, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$F(u) = \left(\frac{u^2}{1 + u^2} \right)^{m/2} \exp \left(- \frac{b^2}{2(1 + u^2)} \right)$$

$$\cdot \sum_{j=0}^{(n/2)-1} \sum_{\ell=j}^{(n/2)-1} \frac{1}{j! 2^j} \binom{(m/2)+\ell-1}{\ell-j} \left(\frac{b^2 u^2}{1 + u^2} \right)^j \left(\frac{1}{1 + u^2} \right)^\ell \quad u \geq 0$$

[Ref. 29, p. 112]

7. n odd, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$F(u) = 2 \left[Q(\alpha, \beta) - \frac{\beta}{2} \exp \left(- \frac{\alpha^2 + \beta^2}{2} \right) I_0(\alpha\beta) \right] - \left(\frac{1}{1 + u^2} \right)^{n/2} \exp \left(- \frac{b^2}{2} \right)$$

$$\cdot \left\{ \sum_{j=0}^{(m-4)/2} \left(\frac{u^2}{1 + u^2} \right)^{(m-2j-2)/2} \frac{\Gamma \left(\frac{n + m - 2j - 2}{2} \right)}{\left(\frac{m - 2j - 2}{2} \right)! \Gamma \left(\frac{n}{2} \right)} \right.$$

$$\cdot {}_1F_1 \left(\frac{n + m - 2j - 2}{2}; \frac{m - 2j}{2}; 2\alpha\beta \right)$$

$$- u^{n-3} \sum_{j=0}^{(n-3)/2} (1+u^2)^j {}_1F_1\left(\frac{n-2j}{2}; 1; 2\alpha\beta\right) \left\} \quad u \geq 0$$

where

$$\alpha = \frac{b}{2} \left[1 - \left(\frac{1}{1+u^2} \right)^{1/2} \right], \quad \beta = \frac{b}{2} \left[1 + \left(\frac{1}{1+u^2} \right)^{1/2} \right]$$

[Ref. 29, p. 119]

8. n, m

See Ref. 29.

F. RICE/RICE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{B}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad b = \|\underline{B}\|, \quad \nu_1 = \|\underline{X}^{(1)}\|, \quad \nu_2 = \|\underline{X}^{(2)}\|$$

Let

$$q = \frac{\nu_2}{\nu_1}$$

$$1. \quad \underline{n = 1, \quad m = 1; \quad \sigma_1^2 = \sigma_2^2 = 1}$$

$$F(q) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left(\frac{q^2 - 1}{2q^2} \right) + 2V \left(\frac{aq + b}{\sqrt{1+q^2}}, \frac{a - bq}{\sqrt{1+q^2}} \right) \\ + 2V \left(\frac{aq - b}{\sqrt{1+q^2}}, \frac{a + bq}{\sqrt{1+q^2}} \right) \quad q \geq 0$$

[Ref. 29, p. 115]

2. $n = 2, m = 2$

$$F(q) = Q(\alpha, \beta) - \frac{\sigma_2^2 \alpha^2}{a^2} \exp \left(- \frac{\alpha^2 + \beta^2}{2} \right) I_0(\alpha\beta) \quad q \geq 0$$

where

$$\alpha = \left(\frac{a^2 q^2}{\sigma_2^2 + \sigma_1^2 q^2} \right)^{1/2}, \quad \beta = \left(\frac{b^2}{\sigma_2^2 + \sigma_1^2 q^2} \right)^{1/2}$$

[Ref. 29, p. 110]

For a generalization to dependent vectors see Ref. 35.

3. $n = m = 2k; \sigma_1^2 = \sigma_2^2 = 1$

$$F(q) = Q \left(\left(\frac{a^2 q^2}{1 + q^2} \right)^{1/2}, \left(\frac{b^2}{1 + q^2} \right)^{1/2} \right) - \frac{1}{1 + q^2} \exp \left(- \frac{a^2 q^2 + b^2}{2(1 + q^2)} \right) I_0 \left(\frac{abq}{1 + q^2} \right) \\ + \exp \left(- \frac{a^2 q^2 + b^2}{2(1 + q^2)} \right) \sum_{j=1-k}^{k-1} C_j(k-1, k-1; q) \left(\frac{bq}{a} \right)^j I_j \left(\frac{abq}{1 + q^2} \right) \quad q \geq 0$$

where

$$C_j(k-1, k-1; q) = \begin{cases} \sum_{l=j}^{k-1} \binom{k+l-1}{l-j} \left(\frac{q^2}{1+q^2} \right)^k \left(\frac{1}{1+q^2} \right)^l - \delta_{0m} \left(\frac{q^2}{1+q^2} \right) & j \geq 0 \\ -C_{-j}(k-1, k-1; \frac{1}{q}) & j < 0 \end{cases}$$

[Ref. 29, p. 112]

Note special case:

$$F(1) = Q\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) - \frac{1}{2} \exp\left(-\frac{a^2 + b^2}{4}\right) I_0\left(\frac{ab}{2}\right) \\ + \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{j=1}^{k-1} D_j \left[\left(\frac{b}{a}\right)^j - \left(\frac{a}{b}\right)^j \right] I_j\left(\frac{ab}{2}\right)$$

where

$$D_j = \sum_{\ell=j}^{k-1} \frac{(k-1+\ell)!}{(k-1+j)! (\ell-j)!} 2^{-\ell-k}$$

[Ref. 28, p. 17]

4. $n = m$

See Ref. 32.

5. n even, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$F(q) = Q\left(\left(\frac{a^2 q^2}{1+q^2}\right)^{1/2}, \left(\frac{b^2}{1+q^2}\right)^{1/2}\right) - \frac{1}{1+q^2} \exp\left(-\frac{a^2 q^2 + b^2}{2(1+q^2)}\right) I_0\left(\frac{abq}{1+q^2}\right) \\ + \exp\left(-\frac{a^2 q^2 + b^2}{2(1+q^2)}\right) \sum_{j=1-(m/2)}^{(n/2)-1} C_j\left(\frac{n}{2} - 1, \frac{m}{2} - 1; q\right) \\ \cdot \left(\frac{bq}{a}\right)^j I_j\left(\frac{abq}{1+q^2}\right) \quad q \geq 0$$

where

$$c_j\left(\frac{n}{2}-1, \frac{m}{2}-1; q\right) = \begin{cases} \sum_{\ell=j}^{(n-2)/2} \binom{(m/2)+\ell-1}{\ell-j} \left(\frac{q^2}{1+q^2}\right)^{n/2} \left(\frac{1}{1+q^2}\right)^\ell - \delta_{m0} \left(\frac{q^2}{1+q^2}\right) & j \geq 0 \\ -c_{-j}\left(\frac{m}{2}-1, \frac{n}{2}-1; \frac{1}{q}\right) & j < 0 \end{cases}$$

[Ref. 29, p. 112]

6. n, m

See Ref. 29.

III. SUM

A. CENTRAL CHI SQUARE (+) CENTRAL CHI SQUARE

1. Independent $\sigma_1^2 \neq \sigma_2^2$

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$\text{Let} \quad s = r_1^2 + r_2^2, \quad c = \frac{\sigma_2^2}{\sigma_1^2}$$

$$\text{a. } n = 1, \quad m = 1$$

$$G(s) = \frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2} \Lambda\left(\left|\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right|; \frac{s(\sigma_1^2 + \sigma_2^2)}{4\sigma_1^2\sigma_2^2}\right) \quad s \geq 0$$

$$\text{b. } n = 2, \quad m = 2$$

$$G(s) = \frac{1}{2|\sigma_1^2 - \sigma_2^2|} \left\{ \frac{1}{\alpha} (1 - e^{-\alpha s}) - \frac{1}{\beta} (1 - e^{-\beta s}) \right\} \quad s \geq 0$$

where

$$\alpha = \frac{\sigma_1^2 + \sigma_2^2 - |\sigma_1^2 - \sigma_2^2|}{4\sigma_1^2\sigma_2^2}, \quad \beta = \frac{\sigma_1^2 + \sigma_2^2 + |\sigma_1^2 - \sigma_2^2|}{4\sigma_1^2\sigma_2^2}$$

c. $n = m = 2k$

$$G(s) = \frac{1}{(k-1)!} \left(\frac{1}{2|\sigma_1^2 - \sigma_2^2|} \right)^k \sum_{j=0}^{k-1} \frac{(-1)^j (k+j-1)!}{j!} \left(\frac{2\sigma_1^2 \sigma_2^2}{|\sigma_1^2 - \sigma_2^2|} \right)^j$$

$$\cdot \left\{ \left[\frac{1}{\alpha^{k-j}} - e^{-\alpha s} \sum_{\ell=0}^{k-j-1} \frac{s^{k-j-\ell-1}}{(k-j-\ell-1)! \alpha^{\ell+1}} \right] \right.$$

$$\left. + (-1)^{k-j} \left[\frac{1}{\beta^{k-j}} - e^{-\beta s} \sum_{\ell=0}^{k-j-1} \frac{s^{k-j-\ell-1}}{(k-j-\ell-1)! \beta^{\ell+1}} \right] \right\} \quad s \geq 0$$

where

$$\alpha = \frac{\sigma_1^2 + \sigma_2^2 - |\sigma_1^2 - \sigma_2^2|}{4\sigma_1^2 \sigma_2^2}, \quad \beta = \frac{\sigma_1^2 + \sigma_2^2 + |\sigma_1^2 - \sigma_2^2|}{4\sigma_1^2 \sigma_2^2}$$

[Ref. 5, p. 134]

d. $n = m$

e. $n = 2, \quad m = 2k$

$$G(s) = 1 - \left(\frac{\sigma_1^2}{\sigma_1^2 - \sigma_2^2} \right)^k \left\{ \exp \left(-\frac{s}{2\sigma_1^2} \right) + \exp \left(-\frac{s}{2\sigma_2^2} \right) \left(\frac{\sigma_2^2}{\sigma_1^2} \right)^{k-1} \sum_{j=0}^{k-1} \sum_{\ell=0}^j \frac{1}{(j-\ell)!} \right.$$

$$\left. \cdot \left(\frac{\sigma_1^2 - \sigma_2^2}{2\sigma_1^2 \sigma_2^2} \right)^j (2\sigma_2^2)^\ell s^{j-\ell} \right\} \quad s \geq 0$$

[Ref. 5, p. 134]

f. n, m

2. Dependent

$$\underline{x}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{x}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{x}^{(1)}\|, \quad r_2 = \|\underline{x}^{(2)}\|$$

Let $s = r_1^2 + r_2^2$

a. n = 1

$$G(s) = \frac{2|\underline{W}|^{1/2}}{w_{11} + w_{22}} \Lambda \left(\frac{[(w_{11} - w_{22})^2 + 4w_{12}^2]^{1/2}}{w_{11} + w_{22}}, \frac{(w_{11} + w_{22})s}{4} \right) \quad s \geq 0$$

b. n = 2

$$G(s) = \frac{|\underline{W}|^{1/2}}{2[(w_{11} - w_{22})^2 + 4w_{12}^2]^{1/2}} \left\{ \frac{1}{\alpha} (1 - e^{-\alpha s}) - \frac{1}{\beta} (1 - e^{-\beta s}) \right\} \quad s \geq 0$$

where

$$\alpha = \frac{1}{4} \left\{ w_{11} + w_{22} - \left[(w_{11} - w_{22})^2 + 4w_{12}^2 \right]^{1/2} \right\}$$

$$\beta = \frac{1}{4} \left\{ w_{11} + w_{22} + \left[(w_{11} - w_{22})^2 + 4w_{12}^2 \right]^{1/2} \right\}$$

c. $n = 2k$ (even)

$$G(s) = \frac{|w|^k}{2^k (k-1)! \gamma^k} \sum_{j=0}^{k-1} \frac{(-1)^j (k+j-1)!}{j!} \left(\frac{2}{\gamma} \right)^j$$

$$\cdot \left\{ \left[\frac{1}{\alpha^{k-j}} - e^{-\alpha s} \sum_{\ell=0}^{k-j-1} \frac{s^{k-j-\ell-1}}{(k-j-\ell-1)! \alpha^{\ell+1}} \right] \right.$$

$$\left. + (-1)^{k-j} \left[\frac{1}{\beta^{k-j}} - e^{-\beta s} \sum_{\ell=0}^{k-j-1} \frac{s^{k-j-\ell-1}}{(k-j-\ell-1)! \beta^{\ell+1}} \right] \right\} \quad s \geq 0$$

where

$$\gamma = \left[(w_{11} - w_{22})^2 + 4w_{12}^2 \right]^{1/2}$$

$$\alpha = \frac{1}{4} \{ w_{11} + w_{22} - \gamma \}$$

$$\beta = \frac{1}{4} \{ w_{11} + w_{22} + \gamma \}$$

[Ref. 5, p. 134]

d. n

B. CENTRAL CHI SQUARE (+) NON CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad v = \|\underline{X}^{(1)}\|, \quad r = \|\underline{X}^{(2)}\|$$

Let $w = v^2 + r^2$

1. n = m

$$G(w) = \left(\frac{\sigma_1}{\sigma_2}\right)^n \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + j\right)}{j! \Gamma\left(\frac{n}{2}\right)} \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2}\right)^j \left[1 - Q_{n+j}\left(\frac{a}{\sigma_1}, \frac{\sqrt{w}}{\sigma_1}\right) \right] \quad w \geq 0$$

2. n even, m even

$$G(w) = \left(\frac{\sigma_1}{\sigma_2}\right)^m \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{m}{2} + j\right)}{j! \Gamma\left(\frac{m}{2}\right)} \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2}\right)^j \left[1 - Q_{(m+n)/2+j}\left(\frac{a}{\sigma_1}, \frac{\sqrt{w}}{\sigma_1}\right) \right] \quad w \geq 0$$

3. n, m

C. NON CENTRAL CHI-SQUARE (+) NON CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{B}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad b = \|\underline{B}\|, \quad v_1 = \|\underline{X}^{(1)}\|, \quad v_2 = \|\underline{X}^{(2)}\|$$

Let $t = v_1^2 + v_2^2$

1. n = m

$$G(t) = \left(\frac{\sigma_1}{\sigma_2}\right)^n \exp\left(-\frac{b^2}{2\sigma_2^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + \ell + j\right)}{\ell! \Gamma\left(\frac{n}{2} + j\right) j!} \cdot \left(\frac{b^2 \sigma_1^2}{2\sigma_2^2}\right)^j \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2}\right)^\ell \left[1 - Q_{n+j+\ell}\left(\frac{a}{\sigma_1}, \frac{\sqrt{t}}{\sigma_1}\right)\right] \quad t \geq 0$$

2. n even, m even

$$G(t) = \left(\frac{\sigma_1}{\sigma_2}\right)^m \exp\left(-\frac{b^2}{2\sigma_1^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{m}{2} + j + \ell\right)}{j! \ell! \Gamma\left(\frac{m}{2} + j\right)} \left(\frac{b^2 \sigma_1^2}{2\sigma_2^2}\right)^j \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2}\right)^\ell \cdot \left[1 - Q_{(m+n)/2+j+\ell}\left(\frac{a}{\sigma_1}, \frac{\sqrt{t}}{\sigma_1}\right)\right] \quad t \geq 0$$

3. n, m

IV. DIFFERENCE

A. CENTRAL CHI SQUARE (-) CENTRAL CHI SQUARE

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$\text{Let} \quad s = r_1^2 - r_2^2$$

$$\text{a. } n = 1, \quad m = 1$$

$$\text{b. } n = 2, \quad m = 2$$

$$G(s) = \begin{cases} \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \exp\left(-\frac{s}{2\sigma_2^2}\right) & s < 0 \\ 1 - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \exp\left(-\frac{s}{2\sigma_1^2}\right) & s \geq 0 \end{cases}$$

$$c. \quad n = m = 2k$$

$$G(s) = \begin{cases} a_k \exp\left(\frac{s}{2\sigma_2^2}\right) \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j} \frac{(k+j-1)!}{j! (k-j-\ell-1)!} \left(\frac{2\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^j \\ \quad \cdot (2\sigma_2^2)^{\ell+1} (-s)^{k-j-\ell} & s < 0 \\ \\ 1 - a_k \exp\left(-\frac{s}{2\sigma_1^2}\right) \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j} \frac{(k+j-1)!}{j! (k-j-\ell-1)!} \left(\frac{2\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^j \\ \quad \cdot (2\sigma_1^2)^{\ell+1} s^{k-j-\ell} & s \geq 0 \end{cases}$$

where

$$a_k = \frac{1}{(k-1)!} \left(\frac{1}{2(\sigma_1^2 + \sigma_2^2)} \right)^{k+1}$$

[Ref. 5, p. 134]

$$d. \quad n = m$$

$$e. \quad n = 2k, \quad m = 2$$

$$G(s) = \begin{cases} \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^k \exp \left(\frac{s}{2\sigma_2^2} \right) & s < 0 \\ \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^k \left\{ 1 - 2\sigma_2^2 + 2\sigma_2^2 \left(\frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2} \right)^k \left[1 - Q_k \left(0, \frac{\sqrt{s}}{\sigma_1} \right) \right] \right. \\ \quad \left. + 2\sigma_2^2 \exp \left(\frac{s}{2\sigma_2^2} \right) Q_k \left(0, \left(\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 \sigma_2^2} s \right)^{1/2} \right) \right\} & s \geq 0 \end{cases}$$

where

$$Q_k(0, b) = \exp \left(-\frac{b^2}{2} \right) \sum_{j=0}^{k-1} \frac{1}{j!} \left(\frac{b^2}{2} \right)^j$$

[Ref. 5, p. 134]

$$f. \quad n \text{ even}, \quad m \text{ even}; \quad \sigma_1^2 = \sigma_2^2 = 1$$

$$G(s) = \begin{cases} \frac{1}{2^{n/2}} \exp \left(\frac{s}{2} \right) \sum_{k=0}^{(m-2)/2} \sum_{j=0}^{m/2-k-1} \frac{\left(\frac{n}{2} + k - 1 \right)}{2^k k! \left(\frac{n}{2} - 1 \right)! j!} \left(\frac{-s}{2} \right)^j & s < 0 \\ \text{---} & s \geq 0 \end{cases}$$

[Ref. 13]

g. n, m

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

Let

$$s = r_1^2 - r_2^2$$

a. n = 1

b. n = 2

$$G(s) = \begin{cases} \frac{2|\underline{W}|}{\gamma\alpha} \exp\left(\frac{\alpha}{4}s\right) & s < 0 \\ 1 - \frac{2|\underline{W}|}{\gamma\beta} \exp\left(-\frac{\beta}{4}s\right) & s \geq 0 \end{cases}$$

where

$$\gamma = \left[(w_{11} + w_{22})^2 - 4w_{12}^2 \right]^{1/2}$$

$$\alpha = \gamma - (w_{11} - w_{22}), \quad \beta = \gamma + (w_{11} - w_{22})$$

c. $n = 2k$ (even)

$$G(s) = \begin{cases} a_k \exp\left(\frac{\alpha}{4}s\right) \sum_{j=0}^{k-1} \sum_{\ell=0}^{k+j-1} \frac{(k+j-1)!}{j! (k-j-\ell-1)!} \left(\frac{2}{\gamma}\right)^j \left(\frac{4}{\alpha}\right)^{\ell+1} (-s)^{k-j-\ell-1} & s < 0 \\ 1 - a_k \exp\left(-\frac{\beta}{4}s\right) \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{j! (k-j-\ell-1)!} \left(\frac{2}{\gamma}\right)^j \left(\frac{4}{\beta}\right)^{\ell+1} s^{k-j-\ell-1} & s \geq 0 \end{cases}$$

where

$$a_k = \frac{|\underline{w}|^k}{2^{k(k-1)!} \gamma^k}$$

$$\gamma = \left[(w_{11} + w_{22})^2 - 4w_{12}^2 \right]^{1/2}$$

$$\alpha = \gamma - (w_{11} - w_{22}), \quad \beta = \gamma + (w_{11} - w_{22})$$

[Ref. 5, p. 134]

d. n

B. NON CENTRAL CHI SQUARE (-) CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad v = \|\underline{X}^{(1)}\|, \quad r = \|\underline{X}^{(2)}\|$$

Let

$$w = v^2 - r^2$$

1. $n = 1, m = 1$

2. $n = 2, m = 2$

$$G(w) = \begin{cases} d \exp\left(\frac{w}{2\sigma_2^2}\right) & w < 0 \\ d \left\{ 1 - 2\sigma_2^2 + 2(\sigma_1^2 + \sigma_2^2) \exp\left(\frac{a^2}{2(\sigma_1^2 + \sigma_2^2)}\right) \left[1 - Q\left(\frac{a}{\sigma_1}, \frac{\sqrt{w}}{\sigma_1}\right) \right] \right. \\ \quad \left. + 2\sigma_2^2 \exp\left(\frac{w}{2\sigma_2^2}\right) Q\left(\left(\frac{a^2 \sigma_2^2}{\sigma_1^2(\sigma_1^2 + \sigma_2^2)}\right)^{1/2}, \left(\frac{\sigma_2^2 + \sigma_2^2}{\sigma_1^2 \sigma_2^2} w\right)^{1/2}\right) \right\} & w \geq 0 \end{cases}$$

where

$$d = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \exp\left(-\frac{a^2(\sigma_2^2 + 2\sigma_1^2)}{4\sigma_1^2(\sigma_1^2 + \sigma_2^2)}\right)$$

[Ref. 23]

3. $n = m$

$$4. \quad \underline{n = 2, \quad m = 2k; \quad \sigma_1^2 = \sigma_2^2 = 1}$$

$$G(w) = \begin{cases} \frac{1}{2} \exp\left(\frac{2w - a^2}{4}\right) \sum_{j=0}^{k-1} \frac{1}{2^j} L_j\left(-\frac{a^2}{2}\right) \sum_{\ell=0}^{k-j-1} \frac{1}{\ell!} \left(-\frac{w}{2}\right)^\ell & w < 0 \\ \text{---} & w \geq 0 \end{cases}$$

[Ref. 13]

$$5. \quad \underline{n = 2k, \quad m = 2}$$

$$G(w) = \begin{cases} d \exp\left(\frac{w}{2\sigma_2^2}\right) & w < 0 \\ d \left\{ 1 - 2\sigma_2^2 + 2\sigma_2^2 \left(\frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2}\right)^k \exp\left(\frac{a^2}{2(\sigma_1^2 + \sigma_2^2)}\right) \left[1 - Q_k\left(\frac{a}{\sigma_1}, \frac{\sqrt{w}}{\sigma_1}\right) \right] \right. \\ \quad \left. + 2\sigma_2^2 \exp\left(\frac{w}{2\sigma_2^2}\right) Q_k\left(\left(\frac{a^2 \sigma_2^2}{2\sigma_1^2(\sigma_1^2 + \sigma_2^2)}\right)^{1/2}, \left(\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 \sigma_2^2} w\right)^{1/2}\right) \right\} & w \geq 0 \end{cases}$$

where

$$d = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^k \exp\left(-\frac{a^2(\sigma_2^2 + 2\sigma_1^2)}{4\sigma_1^2(\sigma_1^2 + \sigma_2^2)}\right)$$

6. n even, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$G(w) = \begin{cases} \frac{1}{2^{n/2}} \exp\left(\frac{2w - a^2}{4}\right) \sum_{k=0}^{(m/2)-1} \frac{1}{2^k} L_k^{(n/2)-1}\left(-\frac{a^2}{2}\right) \\ \quad \cdot \sum_{j=0}^{(m/2)-k-1} \frac{1}{j!} \left(-\frac{w}{2}\right)^j & w < 0 \\ \text{---} & w \geq 0 \end{cases}$$

[Ref. 13]

7. n, m

C. NON CENTRAL CHI SQUARE (-) NON CENTRAL CHI SQUARE

V. PRODUCT

A. GAUSSIAN WITH ZERO MEAN

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

Let $z = \underline{X}^{(1)'} \underline{X}^{(2)} = (\underline{X}^{(1)}, \underline{X}^{(2)})$

a. $n = 1$

b. $n = 2, \quad c = \sigma_1 \sigma_2$

$$F(z) = \begin{cases} \frac{1}{2} \exp\left(\frac{z}{c}\right) & z < 0 \\ 1 - \frac{1}{2} \exp\left(-\frac{z}{c}\right) & z \geq 0 \end{cases}$$

c. $n = 2k, \quad c = \sigma_1 \sigma_2$

$$F(z) = \begin{cases} \frac{\exp\left(\frac{z}{c}\right)}{2^{k-2} c^k (k-1)!} \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{2^j j! (k-j-\ell-1)!} c^{j+\ell+1} (-z)^{k-j-\ell-1} & z < 0 \\ 1 - \frac{\exp\left(-\frac{z}{c}\right)}{2^{k-2} c^k (k-1)!} \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{2^j j! (k-j-\ell-1)!} c^{j+\ell+1} z^{k-j-\ell-1} & z \geq 0 \end{cases}$$

[Ref. 5, p. 134]

d. n

e. Angle

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Let $z = (\underline{X}^{(1)}, \underline{X}^{(2)})$

a. n = 1

b. n = 2

$$F(z) = \begin{cases} \frac{1-\rho}{2} \exp\left(\frac{z}{\sigma_1 \sigma_2 (1-\rho)}\right) & z < 0 \\ 1 - \frac{1+\rho}{2} \exp\left(-\frac{z}{\sigma_1 \sigma_2 (1+\rho)}\right) & z \geq 0 \end{cases}$$

$$c. \quad n = 2k, \quad c = \sigma_1 \sigma_2, \quad d = \sqrt{w_{11} w_{22}}$$

$$F(z) = \begin{cases} \frac{e^{\alpha z}}{2^k (k-1)! c^k} \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{j! (k-j-\ell-1)!} \left(\frac{1}{2d}\right)^j \\ \quad \cdot \left(\frac{1}{\alpha}\right)^{\ell+1} (-z)^{k-j-\ell-1} & z < 0 \\ \\ 1 - \frac{e^{-\beta z}}{2^k (k-1)! c^k} \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{j! (k-j-\ell-1)!} \left(\frac{1}{2d}\right)^j \\ \quad \cdot \left(\frac{1}{\beta}\right)^{\ell+1} z^{k-j-\ell-1} & z \geq 0 \end{cases}$$

where

$$\alpha = d - w_{12} = \sqrt{w_{11} w_{22}} - w_{12}$$

$$\beta = d + w_{12} = \sqrt{w_{11} w_{22}} + w_{12}$$

[Ref. 5, p. 134]

d. n

e. Angle

B. GAUSSIAN WITH ONE NON ZERO MEAN (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma^2), \quad a = \|\underline{A}\|$$

Let

$$z = (\underline{x}^{(1)}, \underline{x}^{(2)})$$

1. $n = 1$

2. $n = 2$

$$F(z) = \begin{cases} \frac{1}{2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(\frac{z}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{l=0}^j \sum_{r=0}^{j-l} C_{jlr} \frac{a^{2j} (-z)^{j-l-r}}{2^{2j+l} (\sigma^2)^{2j-l-r}} & z < 0 \\ 1 - \frac{1}{2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(-\frac{z}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{l=0}^j \sum_{r=0}^{j-l} C_{jlr} \frac{a^{2j} z^{j-l-r}}{2^{2j+l} (\sigma^2)^{2j-l-r}} & z \geq 0 \end{cases}$$

where

$$C_{jlr} = \frac{(j+l)!}{j! l! (j-l-r)!}$$

3. $n = 2k$ (even)

$$F(z) = \begin{cases} \left(\frac{1}{2\sigma^2}\right)^k \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(\frac{z}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{l=0}^{k+j-1} \sum_{r=0}^{k+j-l-1} d_{jlr} \cdot \frac{a^{2j} (-z)^{k+j-l-r-1}}{2^{l+2j} (\sigma^2)^{2j-l-r-1}} & z < 0 \\ 1 - \left(\frac{1}{2\sigma^2}\right)^k \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(-\frac{z}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{l=0}^{k+j-1} \sum_{r=0}^{k+j-l-1} d_{jlr} \cdot \frac{a^{2j} z^{k+j-l-r-1}}{2^{l+2j} (\sigma^2)^{2j-l-r-1}} & z \geq 0 \end{cases}$$

where

$$d_{j\ell r} = \frac{(k+j+\ell-1)!}{j! (k+j-1)! \ell! (k+j-\ell-r-1)!}$$

[Ref. 5, p. 134]

4. n

C. GAUSSIAN WITH NON ZERO MEANS

D. RAYLEIGH (x) RAYLEIGH

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$\text{Let} \quad z = r_1 r_2, \quad c = \sigma_1 \sigma_2$$

a. $n = 1, \quad m = 1$

b. $n = 2, \quad m = 2$

$$F(z) = 1 - \frac{z}{c} K_1\left(\frac{z}{c}\right) \quad z \geq 0 \quad [\text{Ref. 6}]$$

c. $n = m = 2k$

d. $n, \quad m$

2. Dependent

E. RAYLEIGH (×) RICE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad v = \|\underline{X}^{(1)}\|, \quad r = \|\underline{X}^{(2)}\|$$

Let

$$u = vr$$

$$1. \quad \underline{n = 1, \quad m = 1}$$

$$2. \quad \underline{n = 2, \quad m = 2}$$

$$F(u) = 1 - \exp \left(- \frac{a^2}{2\sigma_1^2} \right) \sum_{j=0}^{\infty} \left(\frac{1}{j!} \right)^2 \left(\frac{a}{2\sigma_1} \right)^{2j} \left(\frac{u}{\sigma_1 \sigma_2} \right)^{j+1} k_{j+1} \left(\frac{u}{\sigma_1 \sigma_2} \right) \quad u \geq 0$$

$$3. \quad \underline{n, \quad m}$$

F. RICE (×) RICE

I. FUNDAMENTAL VARIABLES

A. GAUSSIAN

$$x \in N_1(a, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{(x-a)^2}{2\sigma^2} \right)$$

$$\psi_x(\mu) = \exp \left(i\mu a - \frac{1}{2} \mu^2 \sigma^2 \right)$$

For $a = 0$

$$E x^{2k} = \sum_{j=0}^{[(k-1)/2]} \left(\frac{k!}{j! 2^j} \right)^2 \frac{\sigma^{2k}}{(k-2j)!} \quad k \geq 0$$

B. RAYLEIGH

$$\underline{X} \in N_n(\underline{0}, \sigma^2), \quad r = \|\underline{X}\|, \quad s = r^2$$

1. $n = 1$

$$f(r) = \frac{2}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{r^2}{2\sigma^2} \right) \quad r \geq 0$$

$$E r^k = \frac{(2\sigma^2)^{k/2}}{\sqrt{\pi}} \Gamma \left(\frac{k+1}{2} \right) \quad k \geq 0$$

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2} \right)^{1/2}$$

2. n = 2

$$f(r) = \frac{r}{\sigma^2} \exp \left(- \frac{r^2}{2\sigma^2} \right) \quad r \geq 0$$

$$Er^k = (2\sigma^2)^{k/2} \Gamma \left(\frac{k}{2} + 1 \right) \quad k \geq 0$$

$$\psi_s(\mu) = \frac{1}{1 - 2i\mu\sigma^2}$$

3. n = 2k (even)

$$f(r) = \frac{2r^{2k-1}}{(2\sigma^2)^k (k-1)!} \exp \left(- \frac{r^2}{2\sigma^2} \right) \quad r \geq 0$$

$$Er^\ell = (2\sigma^2)^{\ell/2} \frac{\Gamma \left(k + \frac{\ell}{2} \right)}{(k-1)!} \quad \ell \geq 0$$

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2} \right)^k$$

4. n

$$f(r) = \frac{2r^{n-1}}{(2\sigma^2)^{n/2} \Gamma \left(\frac{n}{2} \right)} \exp \left(- \frac{r^2}{2\sigma^2} \right) \quad r \geq 0$$

[Ref. 24. p. 29]

$$Er^k = (2\sigma^2)^{k/2} \frac{\Gamma \left(\frac{n+k}{2} \right)}{\Gamma \left(\frac{n}{2} \right)} \quad k \geq 0$$

[Ref 24, p. 72]

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2} \right)^{n/2}$$

C. RICE

$$\underline{X} \in N_n(\underline{A}, \sigma^2), \quad a = \|\underline{A}\|, \quad v = \|\underline{X}\|, \quad t = v^2$$

1. n = 1

$$f(v) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{v^2 + a^2}{2\sigma^2}\right) \left[\exp\left(\frac{va}{\sigma^2}\right) + \exp\left(-\frac{va}{\sigma^2}\right) \right] \quad v \geq 0$$

$$E v^k = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}} {}_1F_1\left(\frac{k+1}{2}, \frac{1}{2}; \frac{a^2}{2\sigma^2}\right) \quad k \geq 0$$

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2} \right)^{1/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2}\right)$$

2. n = 2

$$f(v) = \frac{v}{\sigma^2} \exp\left(-\frac{v^2 + a^2}{2\sigma^2}\right) I_0\left(\frac{va}{\sigma^2}\right) \quad v \geq 0$$

$$E v^k = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \Gamma\left(\frac{1}{2} k + 1\right) {}_1F_1\left(\frac{1}{2} k + 1, 1; \frac{a^2}{2\sigma^2}\right)$$

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2} \right) \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2}\right)$$

3. $n = 2k$ (even)

$$f(v) = \frac{v^k}{\sigma^2 a^{k-1}} \exp \left(- \frac{v^2 + a^2}{2\sigma^2} \right) I_{k-1} \left(\frac{va}{\sigma^2} \right) \quad v \geq 0$$

$$E v^\ell = (2\sigma^2)^{\ell/2} \exp \left(- \frac{a^2}{2\sigma^2} \right) \frac{\Gamma(k + \frac{\ell}{2})}{(k-1)!} {}_1F_1 \left(k + \frac{\ell}{2}; k; \frac{a^2}{2\sigma^2} \right) \quad \ell \geq 0$$

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2} \right)^k \exp \left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2} \right)$$

4. $n = 2k + 1$ (odd)

$$f(v) = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{v}{a} \right)^k \exp \left(- \frac{v^2 + a^2}{2\sigma^2} \right) \left\{ \exp \left(\frac{va}{\sigma^2} \right) \sum_{j=0}^{k-1} \frac{(-1)^j (k+j-1)!}{j! (k-j-1)!} \left(\frac{\sigma^2}{2va} \right)^j \right. \\ \left. + (-1)^k \exp \left(- \frac{va}{\sigma^2} \right) \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j! (k-j-1)!} \left(\frac{\sigma^2}{2va} \right)^j \right\} \quad v \geq 0$$

$$E v^\ell = (2\sigma^2)^{\ell/2} \exp \left(- \frac{a^2}{2\sigma^2} \right) \frac{\Gamma(k + \frac{\ell+1}{2})}{\Gamma(k + \frac{1}{2})} {}_1F_1 \left(k + \frac{\ell+1}{2}, k + \frac{1}{2}; \frac{a^2}{2\sigma^2} \right)$$

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2} \right)^{k+(1/2)} \exp \left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2} \right)$$

5. n

$$f(v) = \frac{a}{\sigma^2} \left(\frac{v}{a}\right)^{n/2} \exp\left(-\frac{v^2 + a^2}{2\sigma^2}\right) I_{(n-2)/2}\left(\frac{va}{\sigma^2}\right) \quad v \geq 0$$

[Ref. 24, p. 28]

$$Ev^k = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(\frac{n+k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} {}_1F_1\left(\frac{n+k}{2}, \frac{n}{2}; \frac{a^2}{2\sigma^2}\right) \quad k \geq 0.$$

[Ref. 24, p. 72]

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right)^{n/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2}\right)$$

D. JOINT DENSITIES

1. Gaussian

a. Let

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

be a gaussian vector with $E\underline{Y} = \underline{A}$ and $\underline{K} = E(\underline{Y} - \underline{A})(\underline{Y} - \underline{A})'$.

Then

$$f(\underline{Y}) = \frac{1}{(2\pi)^{n/2} |\underline{K}|^{1/2}} \exp\left(-\frac{1}{2} (\underline{Y} - \underline{A})' \underline{K}^{-1} (\underline{Y} - \underline{A})\right)$$

b. Let

$$\underline{U} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

Then,

$$\psi_{\underline{z}}(\underline{U}) = \exp \left(i \underline{U}' \underline{A} - \frac{1}{2} \underline{U}' \underline{K} \underline{U} \right)$$

c. For $n = 2$, $\underline{A} = 0$,

$$\underline{K} = \begin{bmatrix} \sigma^2 & \rho \sigma^2 \\ \rho \sigma^2 & \sigma^2 \end{bmatrix}$$

$$E y_1^{k_1} y_2^{k_2} = \begin{cases} 0 & k_1 + k_2 \text{ odd} \\ \left(\frac{i}{2} \right)^{(k_2 - k_1)/2} \sigma^{(k_1 + k_2)} k_1! k_2! & \\ \sum_{j=0}^{k_1} \frac{(2j)! \rho^{k_1 - 2j}}{(j! 2^j)^2 (k_1 - 2j)! \left(2j + \frac{k_2 - k_1}{2} \right)!} & \begin{matrix} k_1 + k_2 \text{ even} \\ k_2 \geq k_1 \end{matrix} \end{cases}$$

2. Rayleigh

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}; \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$\text{Let} \quad r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$f(r_1, r_2) = \frac{(r_1 r_2)^{n/2}}{(2|w_{12}|)^{(n-2)/2} |\underline{M}|^{n/2} \Gamma\left(\frac{n}{2}\right)} \exp \left(-\frac{1}{2} (w_{11} r_1^2 + w_{22} r_2^2) \right)$$

$$\cdot I_{(n-2)/2}(r_1 r_2 |w_{12}|) \quad r_1, r_2 \geq 0$$

[Ref. 24, p. 34]

$$E r_1^{k_1} r_2^{k_2} = \frac{2^{(k_1+k_2)/2} |\underline{W}|^{n/2} \Gamma\left(\frac{n+k_1}{2}\right) \Gamma\left(\frac{n+k_2}{2}\right)}{w_{11}^{(n+k_1)/2} w_{22}^{(n+k_2)/2} \Gamma^2\left(\frac{n}{2}\right)}$$

$$\cdot {}_2F_1\left(\frac{n+k_1}{2}, \frac{n+k_2}{2}, \frac{n}{2}; \frac{w_{12}^2}{w_{11} w_{22}}\right) \quad k_1, k_2 > 0$$

[Ref. 24, p. 73]

3. Rice

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{A}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}; \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$\text{Let } a = \|\underline{A}\|, \quad \nu_1 = \|\underline{X}^{(1)}\|, \quad \nu_2 = \|\underline{X}^{(2)}\|$$

$$\begin{aligned} f(\nu_1, \nu_2) &= \frac{\nu_1 \nu_2 \Gamma\left(\frac{n-2}{2}\right)}{|\underline{M}|^{n/2}} \left(\frac{2}{a^2 w_{11} w_{12} w_{22}} \right)^{(1/2)(n-2)} \\ &\cdot \exp \left(-\frac{1}{2} (w_{11} \nu_1^2 + w_{22} \nu_2^2) - \frac{1}{2} (w_{11} + w_{22}) a^2 \right) \\ &\cdot \sum_{j=0}^{\infty} (-1)^j \left(\frac{1}{2} n + j - 1 \right) \binom{n+j-3}{n-3} I_{(1/2)n+j-1}(\nu_1 \nu_2 w_{12}) \\ &\cdot I_{(1/2)n+j-1}(\nu_1 a w_1) I_{(1/2)n+j-1}(\nu_2 a w_2) \end{aligned}$$

where

$$w_1 = w_{11} + w_{12}, \quad w_2 = w_{22} + w_{12}$$

[Ref. 24, p. 32]

II. RATIO

A. GAUSSIAN/GAUSSIAN

$$x \in N_1(a, \sigma_1^2), \quad y \in N_1(b, \sigma_2^2), \quad M = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$$

Let $z = \frac{y}{x}, \quad c = \frac{\sigma_2}{\sigma_1}$

1. Independent

$$\rho = 0$$

a. $a = b = 0$

$$f(z) = \frac{c}{\pi(c^2 + z^2)}$$

$$\psi_z(\mu) = e^{-|\mu|c}$$

b. $a \neq 0, \quad b = 0$

$$f(z) = \frac{\sigma_1 \sigma_2}{\pi(\sigma_2^2 + \sigma_1^2 z^2)} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) + \frac{a\sigma_2^2}{\sqrt{2\pi}(\sigma_2^2 + \sigma_1^2 z^2)^{3/2}} \\ \cdot \exp\left(-\frac{a^2 z^2}{2(\sigma_2^2 + \sigma_1^2 z^2)}\right) \operatorname{erf}\left(\frac{a\sigma_2^2}{\sqrt{2} \sigma_1 \sigma_2 (\sigma_2^2 + \sigma_1^2 z^2)^{1/2}}\right)$$

[Ref. 4, p. 60]

c. $a \neq 0, b \neq 0$

$$f(z) = \frac{\sigma_1 \sigma_2}{\pi(\sigma_2^2 + \sigma_1^2 z^2)} \exp \left(-\frac{1}{2} \left(\frac{b^2}{\sigma_2^2} + \frac{a^2}{\sigma_1^2} \right) \right) + \frac{a\sigma_2^2 + b\sigma_1^2 z}{\sqrt{2\pi} (\sigma_2^2 + \sigma_1^2 z^2)^{3/2}} \\ \cdot \exp \left(-\frac{(b - az)^2}{2(\sigma_2^2 + \sigma_1^2 z^2)} \right) \operatorname{erf} \left(\frac{a\sigma_2^2 + b\sigma_1^2 z}{\sqrt{2} \sigma_1 \sigma_2 (\sigma_2^2 + \sigma_1^2 z^2)^{1/2}} \right)$$

[Ref. 4, p. 60]

2. Dependent

a. $a = b = 0$

$$f(z) = \frac{c(1 - \rho^2)^{1/2}}{\pi(c^2 - 2\rho cz + z^2)}$$

$$\psi_z(\mu) = \exp \left(i\mu\rho c - |\mu|(1 - \rho^2)^{1/2} c \right)$$

b. $a \neq 0, b = 0$

$$f(z) = \frac{\sigma_1 c_2 (1 - \rho^2)^{1/2}}{\pi(\sigma_2^2 - 2\rho z \sigma_1 \sigma_2 + \sigma_1^2 z^2)} \exp \left(-\frac{a^2}{2\sigma_1^2 (1 - \rho^2)} \right) + \frac{a\rho\sigma_1 z - a\sigma_2^2}{\sqrt{2\pi} (\sigma_2^2 - 2\rho\sigma_1\sigma_2 z + \sigma_1^2 z^2)^{3/2}} \\ \cdot \exp \left(-\frac{a^2 z^2}{2(\sigma_2^2 - 2\rho\sigma_1\sigma_2 z + \sigma_1^2 z^2)} \right) \\ \cdot \operatorname{erf} \left(\frac{a\sigma_1\sigma_2\rho z - a\sigma_2^2}{\sqrt{2} \sigma_1 \sigma_2 (1 - \rho^2) (\sigma_2^2 - 2\sigma_1\sigma_2\rho z + \sigma_1^2 z^2)^{1/2}} \right)$$

c. $a \neq 0, b \neq 0$

$$f(z) = \frac{\sigma_1 \sigma_2 (1 - \rho^2)^{1/2}}{\pi (\sigma_2^2 - 2\sigma_1 \sigma_2 \rho z + \sigma_1^2 z^2)} \exp \left(- \frac{1}{2(1-\rho^2)} \left(\frac{b^2}{\sigma_2^2} - \frac{2\rho ab}{\sigma_1 \sigma_2} + \frac{a^2}{\sigma_1^2} \right) \right) \\ + \frac{\sigma_2 (b\sigma_1 \rho - a\sigma_2) + z\sigma_1 (a\rho - b\sigma_1)}{\sqrt{2\pi} (\sigma_2^2 - 2\rho\sigma_1 \sigma_2 z + \sigma_1^2 z^2)^{3/2}} \exp \left(- \frac{1}{2} \left(\frac{(b - za)^2}{\sigma_2^2 - 2\sigma_1 \sigma_2 \rho + \sigma_1^2 z^2} \right) \right) \\ \cdot \operatorname{erf} \left(\frac{\sigma_2 (\rho b \sigma_1 - a \sigma_2) + z \sigma_1 (\rho a \sigma_2 - b \sigma_1)}{\sqrt{2} \sigma_1 \sigma_2 (1 - \rho^2) (\sigma_2^2 - 2\rho\sigma_1 \sigma_2 z + \sigma_1^2 z^2)^{1/2}} \right)$$

[Ref. 4, p. 60]

B. GAUSSIAN/RAYLEIGH (INDEPENDENT)

$$x \in N_1(0, \sigma_1^2); \quad \underline{X} \in N_n(\underline{0}, \sigma_2^2); \quad r = \|\underline{X}\|$$

$$\text{Let} \quad z = \frac{x}{r}, \quad c = \frac{\sigma_2}{\sigma_1}$$

1. $n = 1$

$$f(z) = \frac{c}{\pi(1 + c^2 z^2)}$$

$$\psi_z(\mu) = e^{-|\mu|/c}$$

2. $n = 2$

$$f(z) = \frac{c}{2(1 + c^2 z^2)^{3/2}}$$

3. $n = 3$

$$f(z) = \frac{2c}{\pi(1 + c^2 z^2)^2}$$

$$\psi_z(\mu) = \left(1 + \frac{|\mu|}{c}\right) e^{-|\mu|/c}$$

4. $n = 2k$ (even)

$$\psi(z) = \frac{c\Gamma\left(k + \frac{1}{2}\right)}{\sqrt{\pi} (k-1)! (1 + c^2 z^2)^{k+(1/2)}}$$

5. n

$$f(z) = \frac{c\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right) (1 + c^2 z^2)^{(n+1)/2}} = \frac{c}{B\left(\frac{1}{2}n, \frac{1}{2}\right) (1 + c^2 z^2)^{(n+1)/2}}$$

[Ref. 24, p. 57]

C. GAUSSIAN/RICE (INDEPENDENT)

$$\mathbf{x} \in N_1(0, \sigma_1^2), \quad \underline{\mathbf{X}} \in N_n(\underline{\mathbf{A}}, \sigma_2^2), \quad a = \|\underline{\mathbf{A}}\|, \quad v = \|\underline{\mathbf{X}}\|$$

$$\text{Let} \quad q = \frac{\mathbf{x}}{v}, \quad c = \frac{\sigma_2}{\sigma_1}$$

1. $n = 1$

$$f(q) = \frac{\sigma_1 \sigma_2}{\pi(\sigma_1^2 + \sigma_2^2 q^2)} \exp\left(-\frac{a^2}{2\sigma_2^2}\right) {}_1F_1\left(1, \frac{1}{2}, \frac{a^2 \sigma_1^2}{2\sigma_2^2(\sigma_1^2 + \sigma_2^2 q^2)}\right)$$

2. $n = 2$

$$f(q) = \frac{\sigma_1^2 \sigma_2^2}{2(\sigma_1^2 + \sigma_2^2 q^2)^{3/2}} \exp \left(- \frac{a^2}{4\sigma_2^2} \left(\frac{\sigma_1^2 + 2\sigma_2^2 q^2}{\sigma_1^2 + \sigma_2^2 q^2} \right) \right) \\ \cdot \left[\left(1 + \frac{a^2 \sigma_1^2}{2\sigma_2^2(\sigma_1^2 + \sigma_2^2 q^2)} \right) I_0 \left(\frac{a^2 \sigma_1^2}{4\sigma_2^2(\sigma_1^2 + \sigma_2^2 q^2)} \right) \right. \\ \left. + \frac{a^2 \sigma_1^2}{2\sigma_2^2(\sigma_1^2 + \sigma_2^2 q^2)} I_1 \left(\frac{a^2 \sigma_1^2}{4\sigma_2^2(\sigma_1^2 + \sigma_2^2 q^2)} \right) \right]$$

3. $n = 2k$ (even)

$$f(q) = \frac{\sigma_2^2}{\sigma_1^2 B\left(k, \frac{1}{2}\right) \left(1 + \frac{\sigma_2^2}{\sigma_1^2} z^2 \right)^{k+(1/2)}} \\ \cdot \exp \left(- \frac{a^2}{2\sigma_2^2} \right) {}_1F_1 \left(k + \frac{1}{2}, k, \frac{a^2 \sigma_1^2}{2\sigma_2^2(\sigma_1^2 + \sigma_2^2 q^2)} \right)$$

4. n

$$f(q) = \frac{\sigma_2}{\sigma_1 B\left(\frac{1}{2}n, \frac{1}{2}\right) \left(1 + \frac{\sigma_2^2}{\sigma_1^2} q^2\right)^{(n+1)/2}} \\ \cdot \exp\left(-\frac{a^2}{2\sigma_2^2}\right) {}_1F_1\left(\frac{n+1}{2}, \frac{n}{2}; \frac{a^2 \sigma_1^2}{2\sigma_2^2(\sigma_1^2 + \sigma_2^2 q^2)}\right)$$

[Ref. 24, p. 56]

D. RAYLEIGH/RAYLEIGH

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$\text{Let} \quad z = \frac{r_2}{r_1}, \quad c = \frac{\sigma_2}{\sigma_1}$$

a. $n = m = 1$

$$f(z) = \frac{2c}{\pi(c^2 + z^2)} \quad z \geq 0$$

$$\psi_z(\mu) = e^{-|\mu|c}$$

b. $n = 1, m = 2$

$$f(z) = \frac{zc^2}{(c^2 + z^2)^{3/2}} \quad z \geq 0$$

c. $n = 1, m = 2k$

$$f(z) = \frac{2z^{2k-1} c^{2k}}{B\left(\frac{1}{2}, k\right) (c^2 + z^2)^{k+(1/2)}} \quad z \geq 0$$

d. $n = m = 2$

$$f(z) = \frac{2zc^2}{(c^2 + z^2)^2} \quad z \geq 0$$

$$Ez = \frac{\pi}{2c} \quad [\text{Ref. 24, p. 73}]$$

e. $n = m$

$$f(z) = \frac{2z^{n-1} c^n}{B\left(\frac{n}{2}, \frac{n}{2}\right) (c^2 + z^2)^n} \quad z \geq 0$$

$$Ez^k = \frac{\Gamma\left(\frac{n+k}{2}\right) \Gamma\left(\frac{n-k}{2}\right)}{c^k \left[\Gamma\left(\frac{n}{2}\right)\right]^{1/2}} \quad 0 \leq k \leq n$$

f. n even, m even

$$f(z) = \frac{2z^{m-1} c^m}{B\left(\frac{n}{2}, \frac{m}{2}\right) (c^2 + z^2)^{(m+n)/2}} \quad z \geq 0$$

[Ref. 24, p. 52]

g. n odd, m even

Same as f.

h. n odd, m odd

Same as f.

i. n, m

Same as f.

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

Let
$$z = \frac{r_2}{r_1}$$

a. n = 1

$$f(z) = \frac{2|\underline{W}|^{1/2}(w_{11} + w_{22}z^2)}{\pi \left[(w_{11} + w_{22}z^2)^2 - 4w_{12}^2z^2 \right]} \quad z \geq 0$$

b. $n = 2$

$$f(z) = \frac{2|\underline{w}| (w_{11} + w_{22}z^2)}{\left[(w_{11} + w_{22}z^2)^2 - 4w_{12}^2z^2 \right]^{3/2}} \quad z \geq 0$$

$$Ez = \frac{|\underline{w}| \pi}{2w_{11}^{3/2} w_{22}} {}_2F_1\left(\frac{3}{2}, \frac{1}{2}, 1; \frac{w_{12}^2}{w_{11}w_{22}}\right)$$

c. $n = 2k$

$$f(z) = \frac{2|\underline{w}|^k z^{2k-1} (w_{11} + w_{22}z^2)}{B(k, k) \left[(w_{11} + w_{22}z^2)^2 - 4w_{12}^2z^2 \right]^{k+(1/2)}} \quad z \geq 0$$

$$Ez^\ell = \frac{|\underline{w}|^k \Gamma\left(k + \frac{\ell}{2}\right) \Gamma\left(k - \frac{\ell}{2}\right)}{w_{11}^{k+(\ell/2)} w_{22}^{k-(\ell/2)} [(k-1)!]^2} {}_2F_1\left(k + \frac{\ell}{2}, k - \frac{\ell}{2}, k; \frac{w_{12}^2}{w_{11}w_{22}}\right) \quad 0 \leq \ell < 2k$$

d. n

$$f(z) = \frac{2|\underline{w}|^{n/2} z^{n-1} (w_{11} + w_{22}z^2)}{B\left(\frac{1}{2}n, \frac{1}{2}n\right) \left[(w_{11} + w_{22}z^2)^2 - 4w_{12}^2z^2 \right]^{(n+1)/2}} \quad z \geq 0$$

[Ref. 24, p. 50]

$$Ez^k = \frac{|\underline{w}|^{n/2} \Gamma\left(\frac{n+k}{2}\right) \Gamma\left(\frac{n-k}{2}\right)}{w_{11}^{(n+k)/2} w_{22}^{(n-k)/2} \left[\Gamma\left(\frac{n}{2}\right)\right]^2} {}_2F_1\left(\frac{n+k}{2}, \frac{n-k}{2}, \frac{n}{2}; \frac{w_{12}^2}{w_{11}w_{22}}\right) \quad 0 \leq k < n$$

[Ref. 24, p. 73]

E. RICE/RAYLEIGH (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{B}, \sigma_2^2)$$

$$b = \|\underline{B}\|, \quad r = \|\underline{X}^{(1)}\|, \quad v = \|\underline{X}^{(2)}\|, \quad c = \frac{\sigma_2}{\sigma_1}$$

$$\text{Let} \quad u = \frac{v}{r}$$

$$1. \quad \underline{n = 1, \quad m = 1}$$

$$f(u) = \frac{2\sigma_1\sigma_2}{\pi(\sigma_2^2 + \sigma_1^2u^2)} \exp\left(-\frac{b^2}{2\sigma_2^2}\right) {}_1F_1\left(1, \frac{1}{2}; \frac{u^2b^2\sigma_1^2}{2\sigma_2^2(\sigma_2^2 + \sigma_1^2u^2)}\right) \quad u \geq 0$$

$$2. \quad \underline{n = 1, \quad m = 2}$$

$$f(u) = \frac{u\sigma_1^2\sigma_2}{(\sigma_2^2 + \sigma_1^2u^2)^{3/2}} \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) \{(1 + 2\alpha\beta)I_0(\alpha\beta) + 2\alpha\beta I_1(\alpha\beta)\} \quad u \geq 0$$

where

$$\alpha = \frac{b}{2\sigma_2} \left[1 - \left(\frac{\sigma_2^2}{\sigma_2^2 + \sigma_1^2u^2} \right)^{1/2} \right], \quad \beta = \frac{b}{2\sigma_2} \left[1 + \left(\frac{\sigma_2^2}{\sigma_2^2 + \sigma_1^2u^2} \right)^{1/2} \right]$$

3. $n = 1, m = 2k \text{ (even)}$

$$f(u) = \frac{2u^{2k-1} \sigma_1^{2k} \sigma_2}{B\left(\frac{1}{2}, k\right) (\sigma_2^2 + \sigma_1^2 u^2)^{k+(1/2)}} \exp\left(-\frac{b^2}{2\sigma_2^2}\right) {}_1F_1\left(k + \frac{1}{2}, k; \frac{u^2 b^2 \sigma_1^2}{2\sigma_2^2 (\sigma_2^2 + \sigma_1^2 u^2)}\right) \quad u \geq 0$$

4. $n = 2, m = 2$

$$f(u) = \frac{2u \sigma_1^2 \sigma_2^2}{(\sigma_2^2 + \sigma_1^2 u^2)^2} \exp\left(-\frac{b^2}{2(\sigma_2^2 + \sigma_1^2 u^2)}\right) \left\{1 + \frac{u^2 b^2 \sigma_1^2}{2\sigma_2^2 (\sigma_2^2 + \sigma_1^2 u^2)}\right\} \quad u \geq 0$$

5. $n = m$

$$f(u) = \frac{2u^{n-1} \sigma_1^n \sigma_2^n}{B\left(\frac{n}{2}, \frac{n}{2}\right) (\sigma_2^2 + \sigma_1^2 u^2)^n} \exp\left(-\frac{b^2}{2\sigma_2^2}\right) {}_1F_1\left(n, \frac{n}{2}; \frac{u^2 b^2 \sigma_1^2}{2\sigma_2^2 (\sigma_2^2 + \sigma_1^2 u^2)}\right) \quad u \geq 0$$

6. $n \text{ even}, m \text{ even}$

$$f(u) = \frac{2u^{m-1} \sigma_1^m \sigma_2^n}{B\left(\frac{n}{2}, \frac{m}{2}\right) (\sigma_2^2 + \sigma_1^2 u^2)^{(m+n)/2}} \cdot \exp\left(-\frac{b^2}{2\sigma_2^2}\right) {}_1F_1\left(\frac{m+n}{2}, \frac{m}{2}; \frac{u^2 b^2 \sigma_1^2}{2\sigma_2^2 (\sigma_2^2 + \sigma_1^2 u^2)}\right) \quad u \geq 0$$

[Ref. 24, p. 52]

7. n odd, m even

Same as 6.

8. n, m

Same as 6.

F. RICE/RICE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{B}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad b = \|\underline{B}\|, \quad \nu_1 = \|\underline{X}^{(1)}\|, \quad \nu_2 = \|\underline{X}^{(2)}\|$$

Let

$$q = \frac{\nu_2}{\nu_1}$$

1. n = 1, m = 1

$$f(q) = \frac{2\sigma_1\sigma_2}{\sigma_2^2 + \sigma_1^2 q^2} \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{k+j}{j} \frac{1}{\Gamma(k + \frac{1}{2}) \Gamma(j + \frac{1}{2})}$$

$$\cdot \left(\frac{a^2\sigma_2^2}{2\sigma_1^2}\right)^k \left(\frac{q^2 b^2 \sigma_1^2}{2\sigma_2^2}\right)^j \frac{1}{(\sigma_2^2 + \sigma_1^2 q^2)^{k+j}} \quad q \geq 0$$

(or Differentiate Distribution)

$$2. \quad \underline{n = 2, \quad m = 2}$$

$$f(q) = \frac{2q\sigma_1^2\sigma_2^2}{(\sigma_2^2 + \sigma_1^2q^2)^2} \exp\left(-\frac{a^2q^2 + b^2}{2(\sigma_2^2 + \sigma_1^2q^2)}\right) \\ \cdot \left[\left(1 + \frac{a^2\sigma_2^4 + q^2b^2\sigma_1^4}{2\sigma_1^2\sigma_2^2(\sigma_2^2 + \sigma_1^2q^2)}\right) I_0\left(\frac{qab}{\sigma_2^2 + \sigma_1^2q^2}\right) + \frac{qab}{\sigma_2^2 + \sigma_1^2q^2} \right. \\ \left. \cdot I_1\left(\frac{qab}{\sigma_2^2 + \sigma_1^2q^2}\right) \right] \quad q \geq 0$$

$$3. \quad \underline{n = m = 2k}$$

$$f(q) = (-1)^k \frac{ab}{\sigma_1^2\sigma_2^2} \left(\frac{q}{ab}\right)^k \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right) \\ \cdot \frac{d^k}{dh^k} \left[\frac{1}{2h} \exp\left(\frac{1}{4h}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}q^2\right)\right) I_{k-1}\left(\frac{qab}{2h\sigma_1^2\sigma_2^2}\right) \right] \quad q \geq 0$$

$$\text{evaluated at} \quad h = \frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{q^2}{\sigma_2^2}\right) \quad [\text{Ref. 24, p. 52}]$$

4. n = m

$$f(q) = \frac{2q^{n-1} \sigma_1^n \sigma_2^n}{(\sigma_2^2 + \sigma_1^2 q^2)^n} \exp \left(-\frac{1}{2} \left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2} \right) \right) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(n+k+j-1)!}{k! j! \Gamma\left(\frac{n}{2} + k\right) \Gamma\left(\frac{n}{2} + j\right)} \\ \cdot \left(\frac{a^2 \sigma_2^2}{2\sigma_1^2} \right)^k \left(\frac{q^2 b^2 \sigma_1^2}{2\sigma_2^2} \right)^j \frac{1}{(\sigma_2^2 + \sigma_1^2 q^2)^{k+j}} \quad q \geq 0$$

5. n even, m even

$$f(q) = \frac{2a^{m-1} \sigma_1^m \sigma_2^n}{(\sigma_2^2 + \sigma_1^2 q^2)^{(m+n)/2}} \exp \left(-\frac{1}{2} \left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2} \right) \right) \\ \cdot \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{m+n}{2} + k + j\right)}{k! j! \Gamma\left(\frac{n}{2} + k\right) \Gamma\left(\frac{m}{2} + j\right)} \left(\frac{a^2 \sigma_2^2}{2\sigma_1^2} \right)^k \left(\frac{q^2 b^2 \sigma_1^2}{2\sigma_2^2} \right)^j \\ \cdot \frac{1}{(\sigma_2^2 + \sigma_1^2 q^2)^{k+j}} \quad q \geq 0$$

[Ref. 24, p. 51]

(or Differentiate Distribution)

6. n, m

Same as 5.

G. JOINT DENSITY

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Also

$$\underline{X}^{(3)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(4)} \in N_n(\underline{0}, \sigma_2^2), \quad \underline{M}$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|, \quad r_3 = \|\underline{X}^{(3)}\|, \quad r_4 = \|\underline{X}^{(4)}\|$$

$$\text{Let} \quad z_1 = \frac{r_3}{r_1} \quad \text{and} \quad z_2 = \frac{r_4}{r_2}$$

Here

- r_1 and r_3 are independent
- r_2 and r_4 are independent
- r_1 and r_2 are related by $\underline{M}$
- r_3 and r_4 are related by $\underline{M}$

$$f(z_1, z_2) = \frac{4|\underline{W}|^n (z_1 z_2)^{n-1}}{\Gamma^2\left(\frac{n}{2}\right) [w_{11} w_{22} (1+z_1)^2 (1+z_2)^2]^n} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{[(n+k+j-1)!]^2}{k! j! \Gamma\left(\frac{n}{2} + k\right) \Gamma\left(\frac{n}{2} + j\right)} \\ \cdot \left(\frac{w_{12}^2}{w_{11} w_{22} (1+z_1)^2 (1+z_2)^2} \right)^{k+j} (z_1 z_2)^{2j} \quad z_1, z_2 \geq 0$$

[Ref. 24, p. 54]

III. SUM

A. CENTRAL CHI SQUARE (+) CENTRAL CHI SQUARE

1. Independent $\sigma_1^2 \neq \sigma_2^2$

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2); \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$\text{Let} \quad s = r_1^2 + r_2^2, \quad c = \frac{\sigma_2^2}{\sigma_1^2}$$

$$a. \quad n = m = 1$$

$$g(s) = \frac{1}{2\sigma_1^2\sigma_2^2} \exp\left(-\frac{(\sigma_1^2 + \sigma_2^2)s}{4\sigma_1^2\sigma_2^2}\right) I_0\left(\frac{|\sigma_1^2 - \sigma_2^2|s}{4\sigma_1^2\sigma_2^2}\right) \quad s \geq 0$$

$$Es^k = \frac{2^{2k+1} k! \sigma_1^{2k+1} \sigma_2^{2k+1}}{(\sigma_1^2 + \sigma_2^2)^{k+1}} {}_2F_1\left(\frac{k+1}{2}, \frac{k}{2} + 1, 1; \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right) \quad k \geq 0$$

$$\psi_s(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 - 2i\mu\sigma_2^2)}\right)^{1/2}$$

$$b. \quad n = m = 2$$

$$g(s) = \frac{1}{2|\sigma_1^2 - \sigma_2^2|} \exp\left(-\frac{(\sigma_1^2 + \sigma_2^2)s}{4\sigma_1^2\sigma_2^2}\right) \left\{ \exp\left(\frac{|\sigma_1^2 - \sigma_2^2|s}{4\sigma_1^2\sigma_2^2}\right) - \exp\left(-\frac{|\sigma_1^2 - \sigma_2^2|s}{4\sigma_1^2\sigma_2^2}\right) \right\}$$

$s \geq 0$

$$Es^k = \frac{2^{2k+2} (k+1)! \sigma_1^{2k+2} \sigma_2^{2k+2}}{(\sigma_1^2 + \sigma_2^2)^{k+2}} {}_2F_1\left(\frac{k}{2} + 1, \frac{k+3}{2}, \frac{3}{2}; \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right) \quad k \geq 0$$

$$\psi_s(\mu) = \frac{1}{(1 - 2i\mu\sigma_1^2)(1 - 2i\mu\sigma_2^2)}$$

$$c. \quad n = m = 2k$$

$$g(s) = \frac{1}{s^{(k-1)!}} \left(\frac{s}{2|\sigma_1^2 - \sigma_2^2|}\right)^k \exp\left(-\frac{(\sigma_1^2 + \sigma_2^2)s}{4\sigma_1^2\sigma_2^2}\right) \left\{ \exp\left(\frac{s|\sigma_1^2 - \sigma_2^2|}{4\sigma_1^2\sigma_2^2}\right) \right. \\ \cdot \sum_{j=0}^{k-1} \frac{(-1)^j (k+j-1)!}{j! (k+j-1)!} \left(\frac{2\sigma_1^2\sigma_2^2}{(\sigma_1^2 - \sigma_2^2)s}\right)^j + (-1)^k \exp\left(-\frac{s|\sigma_1^2 - \sigma_2^2|}{4\sigma_1^2\sigma_2^2}\right) \\ \cdot \left. \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j! (k-j-1)!} \left(\frac{2\sigma_1^2\sigma_2^2}{s|\sigma_1^2 - \sigma_2^2|}\right)^j \right\} \quad s \geq 0$$

$$Es^\ell = \frac{2^{2k+2\ell} (2k+\ell-1)! \sigma_1^{2k+2\ell} \sigma_2^{2k+2\ell}}{(2k-1)! (\sigma_1^2 + \sigma_2^2)^{2k+\ell}} \\ \cdot {}_2F_1\left(k + \frac{\ell}{2}, k + \frac{\ell+1}{2}, k + \frac{1}{2}; \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right) \quad \ell \geq 0$$

$$\psi_s(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 - 2i\mu\sigma_2^2)}\right)^k$$

d. $n = m$

$$g(s) = \frac{\sqrt{\pi}}{2\sigma_1\sigma_2\Gamma(\frac{n}{2})} \left(\frac{s}{2|\sigma_1^2 - \sigma_2^2|} \right)^{(n-1)/2} \exp \left(- \frac{(\sigma_1^2 + \sigma_2^2)s}{4\sigma_1^2\sigma_2^2} \right) \cdot I_{(n-1)/2} \left(\frac{|\sigma_1^2 - \sigma_2^2|s}{4\sigma_1^2\sigma_2^2} \right) \quad s \geq 0$$

$$Es^k = \frac{2^{n+2k} (n+k-1)! \sigma_1^{n+2k} \sigma_2^{n+2k}}{(n-1)! (\sigma_1^2 + \sigma_2^2)^{k+n}}$$

$$\cdot {}_2F_1 \left(\frac{n+k}{2}, \frac{n+k+1}{2}, \frac{n}{2}; \left(\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^2 \right) \quad k \geq 0$$

[Ref. 24, p. 74]

$$\psi_s(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 - 2i\mu\sigma_2^2)} \right)^{n/2}$$

e. $n = 2, \quad m = 2k$

$$g(s) = \frac{\sigma_2^{2k-2}}{2(\sigma_2^2 - \sigma_1^2)^k} \exp \left(- \frac{s}{2\sigma_1^2} \right)$$

$$\cdot \left\{ \exp \left(\frac{s(\sigma_2^2 - \sigma_1^2)}{2\sigma_1^2\sigma_2^2} \right) - \sum_{j=0}^{k-1} \frac{1}{j!} \left(\frac{s(\sigma_2^2 - \sigma_1^2)}{2\sigma_1^2\sigma_2^2} \right)^j \right\} \quad s \geq 0$$

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2} \right) \left(\frac{1}{1 - 2i\mu\sigma_2^2} \right)^k$$

f. n, m

$$g(s) = \frac{s^{(n+m-2)/2}}{2^{(n+m)/2} \sigma_1^n \sigma_2^m \Gamma\left(\frac{n+m}{2}\right)} \exp\left(-\frac{s}{2\sigma_1^2}\right) {}_1F_1\left(\frac{m}{2}, \frac{n+m}{2}, \frac{(\sigma_2^2 - \sigma_1^2)s}{2\sigma_1^2\sigma_2^2}\right) \quad s \geq 0$$

[Ref. 24, p. 61]

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2} \right)^{n/2} \left(\frac{1}{1 - 2i\mu\sigma_2^2} \right)^{m/2}$$

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

Let

$$s = r_1^2 + r_2^2$$

a. $n = 1$

$$g(s) = \frac{\exp\left(-\frac{(w_{11} + w_{22})s}{4}\right)}{2|\underline{M}|^{1/2}} I_0\left(\frac{1}{4}s \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}\right) \quad s \geq 0$$

$$E_s^k = \frac{2^{2k+1} k!}{|\underline{M}|^{1/2} (w_{11} + w_{22})^{k+1}} {}_2F_1\left(\frac{k+1}{2}, \frac{k}{2} + 1, 1; \frac{(w_{11} - w_{22})^2 + 4w_{12}^2}{(w_{11} + w_{22})^2}\right) \quad k \geq 0$$

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{(\sigma_2^2 - 2i\mu|\underline{M}|)(\sigma_1^2 - 2i\mu|\underline{M}|) - \rho^2 \sigma_1^2 \sigma_2^2} \right)^{1/2}$$

b. $n = 2$

$$g(s) = \frac{1}{2|\underline{M}| \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}} \{e^{-\alpha s} - e^{-\beta s}\} \quad s \geq 0$$

$$\alpha = \frac{1}{4} \left\{ w_{11} + w_{22} - \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2} \right\}$$

$$\beta = \frac{1}{4} \left\{ w_{11} + w_{22} + \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2} \right\}$$

$$E_s^k = \frac{2^{2k+2} (k+1)!}{|\underline{M}| (w_{11} + w_{22})^{k+2}} {}_2F_1\left(\frac{k}{2} + 1, \frac{k+3}{2}, \frac{3}{2}; \frac{(w_{11} - w_{22})^2 + 4w_{12}^2}{(w_{11} + w_{22})^2}\right) \quad k \geq 0$$

$$\psi_s(\mu) = \frac{|\underline{M}|}{(\sigma_2^2 - 2i\mu|\underline{M}|)(\sigma_1^2 - 2i\mu|\underline{M}|) - \rho^2 \sigma_1^2 \sigma_2^2}$$

$$c. \quad n = 2k$$

$$g(s) = \frac{s^{k-1}}{(2|\underline{M}|)^k (k-1)! \gamma^k} \left\{ e^{-\alpha s} \sum_{j=0}^{k-1} \frac{(-1)^j (k+j-1)!}{j! (k-j-1)!} \left(\frac{2}{s\gamma} \right)^j \right. \\ \left. + (-1)^k e^{-\beta s} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j! (k-j-1)!} \left(\frac{2}{s\gamma} \right)^j \right\} \quad s \geq 0$$

$$\gamma = \left[(w_{11} - w_{22})^2 + 4w_{12}^2 \right]^{1/2}$$

$$\alpha = \frac{1}{4} [w_{11} + w_{22} - \gamma]$$

$$\beta = \frac{1}{4} [w_{11} + w_{22} + \gamma]$$

$$Es^\ell = \frac{2^{2k+2\ell} (2k+\ell-1)!}{|\underline{M}|^k (2k-1)! (w_{11} + w_{22})^{2k+\ell}}$$

$$\cdot {}_2F_1 \left(k + \frac{\ell}{2}, k + \frac{\ell+1}{2}, k + \frac{1}{2}; \frac{(w_{11} - w_{22})^2 + 4w_{12}^2}{(w_{11} + w_{22})^2} \right) \quad \ell \geq 0$$

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}| \right) \left(\sigma_1^2 - 2i\mu|\underline{M}| \right) - \rho^2 \sigma_1^2 \sigma_2^2} \right)^k$$

d. n

$$g(s) = \frac{\left(\frac{\pi}{2}\right)^{1/2} s^{(n-1)/2} \exp\left(-\frac{(w_{11} + w_{22})s}{4}\right)}{(2|\underline{M}|)^{n/2} \Gamma\left(\frac{n}{2}\right) \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{(n-1)/2}}$$

$$\cdot I_{(n-1)/2} \left(\frac{1}{4} s \left[(w_{11} - w_{22})^2 + 4w_{12}^2 \right]^{1/2} \right) \quad s \geq 0$$

[Ref. 24, p. 58]

$$Es^k = \frac{2^{n+2k} (n+k-1)!}{|\underline{M}|^{n/2} (n-1)! (w_{11} + w_{22})^{n+k}}$$

$$\cdot {}_2F_1\left(\frac{n+k}{2}, \frac{n+k+1}{2}, \frac{n+1}{2}; \frac{(w_{11} - w_{22})^2 + 4w_{12}^2}{(w_{11} + w_{22})^2}\right) \quad k \geq 0$$

[Ref. 24, p. 74]

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{(\sigma_2^2 - 2i\mu|\underline{M}|)(\sigma_1^2 - 2i\mu|\underline{M}|) - \rho^2 \sigma_1^2 \sigma_2^2} \right)^{n/2}$$

B. CENTRAL CHI SQUARE (+) NON CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad v = \|\underline{X}^{(1)}\|, \quad r = \|\underline{X}^{(2)}\|$$

Let

$$w = v^2 + r^2$$

1. $n = m$

$$g(w) = \frac{1}{2\sigma_1^2} \left(\frac{\sigma_1}{\sigma_2} \right)^n \left(\frac{w}{a^2} \right)^{(n-1)/2} \exp \left(- \frac{w + a^2}{2\sigma_1^2} \right) \\ \cdot \sum_{j=0}^{\infty} \frac{\Gamma(\frac{n}{2} + j)}{j! \Gamma(\frac{n}{2})} \left(\frac{\sqrt{w} (\sigma_2^2 - \sigma_1^2)}{a\sigma_2^2} \right)^j I_{n+j-1} \left(\frac{\sqrt{w} a}{\sigma_1^2} \right) \quad w \geq 0$$

$$\psi_w(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_2^2)(1 - 2i\mu\sigma_1^2)} \right)^{n/2} \exp \left(- \frac{i\mu a^2}{1 - 2i\mu\sigma_1^2} \right)$$

2. n even, m even

$$g(w) = \frac{1}{2\sigma_1^2} \left(\frac{\sigma_1}{\sigma_2} \right)^m \left(\frac{w}{a^2} \right)^{(m+n-2)/4} \exp \left(- \frac{w + a^2}{2\sigma_1^2} \right) \\ \cdot \sum_{j=0}^{\infty} \frac{\Gamma(\frac{m}{2} + j)}{j! \Gamma(\frac{m}{2})} \left(\frac{\sqrt{w} (\sigma_2^2 - \sigma_1^2)}{a\sigma_2^2} \right)^j I_{[(m+n)/2] + j - 1} \left(\frac{\sqrt{w} a}{\sigma_1^2} \right)$$

[Ref. 24, p. 60]

$$\psi_w(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_2^2} \right)^{m/2} \left(\frac{1}{1 - 2i\mu\sigma_1^2} \right)^{n/2} \exp \left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2} \right)$$

3. n, m

Same as 2.

C. NON CENTRAL CHI SQUARE (+) NON CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{B}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad b = \|\underline{B}\|, \quad \nu_1 = \|\underline{X}^{(1)}\|, \quad \nu_2 = \|\underline{X}^{(2)}\|$$

Let $t = \nu_1^2 + \nu_2^2$

1. $n = m$

$$g(t) = \frac{1}{2\sigma_1^2} \left(\frac{t}{a^2}\right)^{(n-1)/2} \left(\frac{\sigma_1}{\sigma_2}\right)^n \exp\left(-\frac{t}{2\sigma_1^2}\right) \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right)$$

$$\cdot \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + j + \ell\right)}{k! j! \Gamma\left(\frac{n}{2} + j\right)} \left(\frac{\sqrt{t} b^2 \sigma_1^2}{2a\sigma_2^4}\right)^j \left(\frac{\sqrt{t} (\sigma_2^2 - \sigma_1^2)}{a\sigma_2^2}\right)^{\ell}$$

$$\cdot I_{n+j+\ell-1}\left(\frac{\sqrt{t} a}{\sigma_1^2}\right) \quad t \geq 0$$

$$\psi_t(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 - 2i\mu\sigma_2^2)}\right)^{n/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right) \exp\left(\frac{i\mu b^2}{1 - 2i\mu\sigma_2^2}\right)$$

2. n even, m even

$$g(t) = \frac{1}{2\sigma_1^2} \left(\frac{t}{a} \right)^{(n+m-2)/4} \left(\frac{\sigma_1}{\sigma_2} \right)^m \exp \left(- \frac{t}{2\sigma_1^2} \right) \exp \left(- \frac{1}{2} \left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2} \right) \right)$$

$$\cdot \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\Gamma(\frac{m}{2} + j + \ell)}{j! \ell! \Gamma(\frac{m}{2} + j)} \left(\frac{\sqrt{t} b^2 \sigma_1^2}{2a\sigma_2^2} \right)^j \left(\frac{\sqrt{t} (\sigma_2^2 - \sigma_1^2)}{a\sigma_2^2} \right)^\ell$$

$$\cdot I_{[(n+m)/2] + j + \ell - 1} \left(\frac{\sqrt{t} a}{\sigma_1^2} \right)$$

$t \geq 0$

[Ref. 24, p. 59]

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2} \right)^{n/2} \left(\frac{1}{1 - 2i\mu\sigma_2^2} \right)^{m/2} \exp \left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2} \right) \exp \left(\frac{i\mu b^2}{1 - 2i\mu\sigma_2^2} \right)$$

3. n, m

Same as 2.

IV. DIFFERENCE

A. CENTRAL CHI SQUARE (-) CENTRAL CHI SQUARE

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

Let

$$s = r_1^2 - r_2^2$$

a. $n = m = 1$

$$g(s) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)s}{4\sigma_1^2\sigma_2^2}\right) K_0\left(\frac{(\sigma_1^2 + \sigma_2^2)s}{4\sigma_1^2\sigma_2^2}\right)$$

$$\psi_s(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 + 2i\mu\sigma_2^2)}\right)^{1/2}$$

b. $n = m = 2$

$$g(s) = \frac{1}{2(\sigma_1^2 + \sigma_2^2)} \exp\left(-\frac{s(\sigma_2^2 - \sigma_1^2) + |s|(\sigma_1^2 + \sigma_2^2)}{4\sigma_1^2\sigma_2^2}\right)$$

$$\psi_s(\mu) = \frac{1}{(1 - 2i\mu\sigma_1^2)(1 + 2i\mu\sigma_2^2)}$$

$$c. \quad n = m = 2k$$

$$g(s) = \frac{1}{2(\sigma_1^2 + \sigma_2^2)(k-1)!} \left(\frac{|s|}{2(\sigma_1^2 + \sigma_2^2)} \right)^{k-1} \exp \left(- \frac{s(\sigma_2^2 - \sigma_1^2) + |s|(\sigma_1^2 + \sigma_2^2)}{4\sigma_1^2\sigma_2^2} \right)$$

$$\cdot \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!(k-j-1)!} \left(\frac{2\sigma_1^2\sigma_2^2}{|s|(\sigma_1^2 + \sigma_2^2)} \right)^j$$

$$\psi_s(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 + 2i\mu\sigma_2^2)} \right)^k$$

$$d. \quad n = m$$

$$g(s) = \frac{1}{2\sqrt{\pi\sigma_1^2\sigma_2^2} \Gamma(\frac{n}{2})} \left(\frac{|s|}{2(\sigma_1^2 + \sigma_2^2)} \right)^{(n-1)/2} \exp \left(- \frac{(\sigma_2^2 - \sigma_1^2)s}{4\sigma_1^2\sigma_2^2} \right)$$

$$\cdot K_{(n-1)/2} \left(\frac{(\sigma_1^2 + \sigma_2^2)|s|}{4\sigma_1^2\sigma_2^2} \right)$$

$$\psi_s(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 + 2i\mu\sigma_2^2)} \right)^{n/2}$$

$$e. \quad n = 2k, \quad m = 2$$

$$g(s) = \begin{cases} \frac{(-s)^{k-(1/2)} \exp\left(\frac{s}{2\sigma_2^2}\right)}{2^{k+(1/2)} \sigma_1^{2k-1} \sigma_2 \left(\sigma_1^2 + \sigma_2^2\right)^{1/2} [(k-1)!]^2} \sum_{j=0}^k \frac{k!}{(k-j)!} \cdot \left(-\frac{2\sigma_1^2 \sigma_2^2}{s(\sigma_1^2 + \sigma_2^2)}\right)^j & s \leq 0 \\ \frac{\sigma_1 \sigma_2^{2k-1} \exp\left(-\frac{s}{2\sigma_1^2}\right)}{(2s)^{1/2} (\sigma_1^2 + \sigma_2^2)^{k+(1/2)}} & s \geq 0 \end{cases}$$

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right) \left(\frac{1}{1 + 2i\mu\sigma_2^2}\right)^k$$

f. n even, m even

$$g(s) = \left\{ \begin{array}{l} \frac{(-s)^{(m+n-4)/4} \exp \left(- \frac{(\sigma_2^2 - \sigma_1^2)s}{4\sigma_1^2\sigma_2^2} \right)}{2^{(m+n)/4} \sigma_1^n \sigma_2^m \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{(n+m)/4} \Gamma\left(\frac{m}{2}\right)} \\ \quad \cdot W_{(m-n)/4, (n+m-2)/4} \left(- \frac{s(\sigma_1^2 + \sigma_2^2)}{2\sigma_1^2\sigma_2^2} \right) \quad s < 0 \\ \\ \frac{s^{(m+n-4)/4} \exp \left(- \frac{(\sigma_2^2 - \sigma_1^2)s}{4\sigma_1^2\sigma_2^2} \right)}{2^{(m+n)/4} \sigma_1^n \sigma_2^m \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{(m+n)/4} \Gamma\left(\frac{n}{2}\right)} \\ \quad \cdot W_{(n-m)/4, (n+m-2)/4} \left(\frac{s(\sigma_1^2 + \sigma_2^2)}{2\sigma_1^2\sigma_2^2} \right) \quad s \geq 0 \end{array} \right.$$

[Ref. 24, p. 65]

$$s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2} \right)^{n/2} \left(\frac{1}{1 + 2i\mu\sigma_2^2} \right)^{m/2}$$

g. n, m

Same as f.

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$\text{Let } s = r_1^2 - r_2^2, \quad \gamma = \left[(w_{11} + w_{22})^2 - 4w_{12}^2 \right]^{1/2}$$

a. $n = 1$

$$g(s) = \frac{\exp\left(-\frac{(w_{11} - w_{22})s}{4}\right)}{2\pi|\underline{M}|^{1/2}} K_0\left(\frac{|s|\gamma}{4}\right)$$

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{(\sigma_2^2 - 2i\mu|\underline{M}|)(\sigma_1^2 + 2i\mu|\underline{M}|) - \rho^2\sigma_1^2\sigma_2^2} \right)^{1/2}$$

b. $n = 2$

$$g(s) = \begin{cases} \frac{2 \exp\left(\frac{\alpha s}{4}\right)}{|\underline{M}| \gamma \alpha} & s < 0 \\ 1 - \frac{2 \exp\left(\frac{\beta s}{4}\right)}{|\underline{M}| \gamma \beta} & s > 0 \end{cases}$$

$$\alpha = \gamma - (w_{11} - w_{22}), \quad \beta = \gamma + (w_{11} - w_{22})$$

$$\psi_s(\mu) = \frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 + 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}$$

c. $n = 2k$

$$g(s) = \frac{|s|^{k-1} \exp\left(-\frac{(w_{11} - w_{22})s}{4}\right)}{(k-1)! 2^k |\underline{M}|^k \gamma^k} \cdot \exp\left(-\frac{\gamma|s|}{4}\right) \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j! (k-j-1)!} \left(\frac{2}{|s|\gamma}\right)^j$$

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 + 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2} \right)^k$$

d. n

$$g(s) = \frac{|s|^{(n-1)/2} \exp\left(-\frac{(w_{11} - w_{22})s}{4}\right)}{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right) 2^{(n+1)/2} |\underline{M}|^{n/2} \gamma^{(n-1)/2}} K_{(n-1)/2}\left(\frac{\gamma|s|}{4}\right)$$

[Ref. 24, p. 61]

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 + 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2} \right)^{n/2}$$

B. NON CENTRAL CHI SQUARE (-) CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad v = \|\underline{X}^{(1)}\|, \quad r = \|\underline{X}^{(2)}\|$$

Let $w = v^2 - r^2$

1. $n = 1, m = 1$

$$g(w) = \left\{ \begin{array}{l} \frac{\exp\left(-\frac{a^2}{2\sigma_1^2}\right) \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)w}{4\sigma_1^2\sigma_2^2}\right)}{\pi \left[-2w(\sigma_1^2 + \sigma_2^2)\right]^{1/2}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{-wa^4\sigma_2^2}{8\sigma_1^6(\sigma_2^2 + \sigma_1^2)}\right)^{j/2} \\ \quad \cdot W_{-k/2, k/2} \left(-\frac{(\sigma_1^2 + \sigma_2^2)w}{2\sigma_1^2\sigma_2^2}\right) \quad w < 0 \\ \\ \frac{\exp\left(-\frac{a^2}{2\sigma_1^2}\right) \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)w}{4\sigma_1^2\sigma_2^2}\right)}{\left[2w(\sigma_1^2 + \sigma_2^2)\right]^{1/2}} \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(j + \frac{1}{2})} \left(\frac{wa^4\sigma_2^2}{8\sigma_1^6(\sigma_1^2 + \sigma_2^2)}\right)^{j/2} \\ \quad \cdot W_{k/2, k/2} \left(\frac{(\sigma_1^2 + \sigma_2^2)w}{2\sigma_1^2\sigma_2^2}\right) \quad w \geq 0 \end{array} \right.$$

$$\psi_w(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 + 2i\mu\sigma_2^2)}\right)^{1/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

2. $n = 2, m = 2$

$$g(w) = \begin{cases} \frac{c}{2\sigma_2^2} \exp\left(\frac{w}{2\sigma_2^2}\right) & w \leq 0 \\ \frac{c}{2\sigma_2^2} \exp\left(\frac{w}{2\sigma_2^2}\right) Q\left(\left(\frac{a^2 \sigma_2^2}{\sigma_1^2 (\sigma_1^2 + \sigma_2^2)}\right)^{1/2}; \left(\frac{w(\sigma_1^2 + \sigma_2^2)}{\sigma_1^2 \sigma_2^2}\right)^{1/2}\right) & w > 0 \end{cases}$$

[Ref. 23]

where

$$c = \frac{\sigma_2^2}{2(\sigma_1^2 + \sigma_2^2)} \exp\left(-\frac{a^2(\sigma_2^2 + 2\sigma_1^2)}{4\sigma_1^2(\sigma_1^2 + \sigma_2^2)}\right)$$

$$\psi_w(\mu) = \frac{1}{(1 - 2i\mu\sigma_1^2)(1 + 2i\mu\sigma_2^2)} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

3. $n = m$

$$g(w) = \begin{cases} \frac{(-w)^{(n-2)/2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{n/2} (\sigma_1^2 + \sigma_2^2)^{n/2} \Gamma\left(\frac{n}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} \\ \cdot \left(\frac{-wa^4\sigma_2^2}{8\sigma_1^6(\sigma_1^2 + \sigma_2^2)}\right)^{j/2} w^{-j/2, (n+j-2)/2} \left(-\frac{(\sigma_1^2 + \sigma_2^2)w}{2\sigma_1^2\sigma_2^2}\right) & w < 0 \\ \\ \frac{w^{(n-2)/2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{n/2} (\sigma_1^2 + \sigma_2^2)^{n/2}} \sum_{j=0}^{\infty} \frac{1}{j! \Gamma\left(\frac{n}{2} + j\right)} \\ \cdot \left(\frac{wa^4\sigma_2^2}{8\sigma_1^6(\sigma_1^2 + \sigma_2^2)}\right)^{j/2} w^{j/2, (n+j-2)/2} \left(\frac{(\sigma_1^2 + \sigma_2^2)w}{2\sigma_1^2\sigma_2^2}\right) & w \geq 0 \end{cases}$$

$$\psi_w(\mu) = \left(\frac{1}{(1 - 2i\mu\sigma_1^2)(1 + 2i\mu\sigma_2^2)}\right)^{n/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

$$4. \quad n = 2, \quad m = 2k$$

$$g(w) = \left\{ \begin{array}{l} \frac{(-w)^{(k-1)/2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{(k+1)/2} \sigma_1^2 \sigma_2^{2k} \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{(k+1)/2}} \sum_{j=0}^{\infty} \frac{1}{j! (k-1)!} \\ \cdot \left(\frac{-wa^4 \sigma_2^2}{8\sigma_1^6 (\sigma_1^2 + \sigma_2^2)}\right)^{j/2} w^{(k-j-1)/2, (k+j)/2} \exp\left(-\frac{(\sigma_1^2 + \sigma_2^2)w}{2\sigma_1^2 \sigma_2^2}\right) \quad w < 0 \\ \\ \frac{w^{(k-1)/2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{(k+1)/2} \sigma_1^2 \sigma_2^{2k} \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{(k+1)/2}} \sum_{j=0}^{\infty} \frac{1}{j! j!} \\ \cdot \left(\frac{wa^4 \sigma_2^2}{8\sigma_1^6 (\sigma_1^2 + \sigma_2^2)}\right)^{j/2} w^{(1+j-k)/2, (k+j)/2} \exp\left(-\frac{(\sigma_1^2 + \sigma_2^2)w}{2\sigma_1^2 \sigma_2^2}\right) \quad w \geq 0 \end{array} \right.$$

$$\psi_w(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right) \left(\frac{1}{1 + 2i\mu\sigma_2^2}\right)^k \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

$$5. \quad \underline{n = 2k, \quad m = 2}$$

$$g(w) = \begin{cases} \frac{d}{2\sigma_2^2} \exp\left(\frac{w}{2\sigma_2^2}\right) & w \leq 0 \\ \frac{d}{2\sigma_2^2} \exp\left(\frac{w}{2\sigma_2^2}\right) Q_k \left(\left(\frac{a^2 \sigma_2^2}{2\sigma_1^2 (\sigma_1^2 + \sigma_2^2)} \right)^{1/2} ; \left(\frac{w (\sigma_1^2 + \sigma_2^2)}{\sigma_1^2 \sigma_2^2} \right)^{1/2} \right) & w \geq 0 \end{cases}$$

$$\psi(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2} \right)^k \left(\frac{1}{1 + 2i\mu\sigma_2^2} \right) \exp \left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2} \right)$$

where

$$d = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^k \exp \left(- \frac{a^2 (\sigma_2^2 + 2\sigma_1^2)}{4\sigma_1^2 (\sigma_1^2 + \sigma_2^2)} \right)$$

6. n even, m even

$$g(w) = \left\{ \begin{array}{l} \frac{(-w)^{(n+m-4)/4} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{(m+n)/4} \sigma_1^n \sigma_2^m \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{(m+n)/4}} \sum_{j=0}^{\infty} \frac{1}{j! \Gamma\left(\frac{m}{2}\right)} \\ \cdot \left(\frac{-wa^4\sigma_2^2}{8\sigma_1^6(\sigma_1^2 + \sigma_2^2)}\right)^{j/2} w^{(m-n-2k)/4, (n+m+2k-2)/4} \left(-\frac{(\sigma_1^2 + \sigma_2^2)w}{2\sigma_1^2\sigma_2^2}\right) \quad w < 0 \\ \\ \frac{w^{(n+m-4)/2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \exp\left(-\frac{(\sigma_2^2 - \sigma_1^2)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{(m+n)/4} \sigma_1^n \sigma_2^m \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{(m+n)/4}} \sum_{j=0}^{\infty} \frac{1}{j! \Gamma\left(\frac{n}{2} + j\right)} \\ \cdot \left(\frac{wa^4\sigma_2^2}{8\sigma_1^6(\sigma_1^2 + \sigma_2^2)}\right)^{j/2} w^{(n-m+2k)/4, (n+m+2k-2)/4} \left(\frac{(\sigma_1^2 + \sigma_2^2)w}{2\sigma_1^2\sigma_2^2}\right) \quad w \geq 0 \end{array} \right.$$

[Ref. 24, p. 64]

$$\psi_w(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)^{n/2} \left(\frac{1}{1 + 2i\mu\sigma_2^2}\right)^{m/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

7. n, m

Same as 6.

C. NON CENTRAL CHI SQUARE (-) NON CENTRAL CHI SQUARE

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{B}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad b = \|\underline{B}\|, \quad \nu_1 = \|\underline{X}^{(1)}\|, \quad \nu_2 = \|\underline{X}^{(2)}\|$$

Let

$$t = \nu_1^2 - \nu_2^2$$

$$g(t) = \left\{ \begin{array}{l} \frac{(-t)^{(m+n-4)/4} \exp \left[-\frac{1}{2} \left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2} \right) \right] \exp \left(-\frac{(\sigma_2^2 - \sigma_1^2)t}{4\sigma_1^2\sigma_2^2} \right)}{2^{(m+n)/4} \sigma_1^n \sigma_2^m \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{(n+m)/4}} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \\ \cdot \frac{1}{j! \ell! \Gamma\left(\frac{m}{2} + \ell\right)} \left(\frac{-ta^4\sigma_2^2}{8\sigma_1^6(\sigma_1^2 + \sigma_2^2)} \right)^{j/2} \left(\frac{-tb^4\sigma_1^2}{8\sigma_2^6(\sigma_1^2 + \sigma_2^2)} \right)^{\ell/2} \\ \cdot {}^N_{[(m-n)/4] + [(j-\ell)/2]; [(n+m-2)/4] + [(j+\ell)/2]} \left(-\frac{(\sigma_2^2 - \sigma_1^2)t}{2\sigma_1^2\sigma_2^2} \right) \quad t < 0 \\ \\ \frac{t^{(m+n-4)/4} \exp \left[-\frac{1}{2} \left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2} \right) \right] \exp \left(-\frac{(\sigma_2^2 - \sigma_1^2)t}{4\sigma_1^2\sigma_2^2} \right)}{2^{(m+n)/4} \sigma_1^n \sigma_2^m \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{(n+m)/4}} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \\ \cdot \frac{1}{j! \ell! \Gamma\left(\frac{n}{2} + j\right)} \left(\frac{ta^4\sigma_2^2}{8\sigma_1^6(\sigma_1^2 + \sigma_2^2)} \right)^{j/2} \left(\frac{tb^4\sigma_1^2}{8\sigma_2^6(\sigma_1^2 + \sigma_2^2)} \right)^{\ell/2} \\ \cdot {}^N_{[(n-m)/4] + [(j-\ell)/2]; [(m+n-2)/4] + [(j+\ell)/2]} \left(\frac{(\sigma_2^2 - \sigma_1^2)t}{2\sigma_1^2\sigma_2^2} \right) \quad t \geq 0 \end{array} \right.$$

[Ref. 24, p. 63]

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2} \right)^{n/2} \left(\frac{1}{1 + 2i\mu\sigma_2^2} \right)^{m/2} \exp \left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2} \right) \exp \left(- \frac{i\mu b^2}{1 + 2i\mu\sigma_2^2} \right)$$

V. PRODUCT

A. GAUSSIAN WITH ZERO MEANS

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\text{Let } z = (\underline{X}^{(1)}, \underline{X}^{(2)}) = \sum_{k=1}^n x_k^{(1)} x_k^{(2)}; \quad c = \sigma_1 \sigma_2$$

a. $n = 1$

$$f(z) = \frac{1}{\pi c} K_0 \left(\frac{|z|}{c} \right) \quad [\text{Ref. 24, p. 45}]$$

$$\psi_z(\mu) = \left(\frac{1}{1 + c^2 \mu^2} \right)^{1/2}$$

b. $n = 2$

$$f(z) = \frac{1}{2c} \exp \left(- \frac{|z|}{c} \right)$$

$$\psi_z(\mu) = \frac{1}{1 + c^2 \mu^2}$$

c. $n = 2k$

$$f(z) = \frac{|z|^{k-1} \exp \left(- \frac{|z|}{c} \right)}{2^{k-2} c^k (k-1)!} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j! (k-j-1)!} \left(\frac{c}{2|z|} \right)^j$$

$$\psi_z(\mu) = \left(\frac{1}{1 + c^2 \mu^2} \right)^k$$

d. n

$$f(z) = \frac{|z|^{(n-1)/2}}{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right) 2^{(n-1)/2} c^{(n+1)/2}} K_{(n-1)/2} \left(\frac{|z|}{c} \right)$$

[Ref. 24, p. 42]

$$\psi_z(\mu) = \left(\frac{1}{1 + c^2 \mu^2} \right)^{n/2}$$

e. Angle $z = (\underline{x}^{(1)}, \underline{x}^{(2)}) = \|\underline{x}^{(1)}\| \|\underline{x}^{(2)}\| \cos \phi$
 Phi (ϕ) is the angle between $\underline{x}^{(1)}$ and $\underline{x}^{(2)}$.

$$\pi(\phi) = \frac{(\sin \phi)^{n-2}}{B\left(\frac{1}{2}, \frac{n-1}{2}\right)} \quad 0 < \phi < 2\pi$$

[Ref. 24, p. 47]

2. Dependent

$$\underline{x}^{(1)} \in N_n(0, \sigma_1^2), \quad \underline{x}^{(2)} = N_n(0, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$z = (\underline{x}^{(1)}, \underline{x}^{(2)}) = \sum_{k=1}^n x_k^{(1)} x_k^{(2)}$$

a. $n = 1$

$$f(z) = \frac{1}{\pi} |\underline{w}|^{1/2} \exp(-w_{12}z) K_0(|z| \sqrt{w_{11}w_{22}})$$

[Ref. 24, p. 45]

$$\psi_z(\mu) = \left(\frac{1}{1 - 2i\mu\rho c + \mu^2 c^2 (1-\rho^2)} \right)^{1/2}$$

b. $n = 2$

$$f(z) = \frac{\exp(-w_{12}z)}{2c} \exp(-|z| \sqrt{w_{11}w_{22}})$$

$$\psi_z(\mu) = \frac{1}{1 - 2i\mu\rho c + \mu^2 c^2 (1-\rho^2)}$$

c. $n = 2k$

$$f(z) = \frac{|z|^{k-1} \exp(-w_{12}z) \exp(-|z| \sqrt{w_{11}w_{22}})}{(k-1)! 2^k c^k} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j! (k-j-1)!}$$

$$\cdot \left(\frac{1}{2|z| \sqrt{w_{11}w_{22}}} \right)^j$$

$$\psi_z(\mu) = \left(\frac{1}{1 - 2i\mu\rho c + \mu^2 c^2 (1-\rho^2)} \right)^k$$

d. n

$$f(z) = \frac{|z|^{(n-1)/2} \exp(-w_{12}z)}{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right) 2^{(n-1)/2} |\underline{M}|^{1/2} c^{(n-1)/2} K_{(n-1)/2}(|z| \sqrt{w_{11}w_{22}})}$$

[Ref. 24, p. 42]

$$\psi_z(\mu) = \left(\frac{1}{1 - 2i\mu\rho c + \mu^2 c^2 (1-\rho^2)} \right)^{n/2}$$

e. Angle $z = (\underline{X}^{(1)}, \underline{X}^{(2)}) = \|\underline{X}^{(1)}\| \|\underline{X}^{(2)}\| \cos \phi$
 $\Phi(\phi)$ is the angle between $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$.

$$\pi(\phi) = \frac{(n-1) \Gamma(n) (\sin \phi)^{n-2}}{|\underline{M}|^{n/2} \sqrt{\pi} \Gamma\left(n + \frac{1}{2}\right) (\sqrt{w_{11}w_{22}} + w_{12} \cos \phi)^n} \\ \cdot {}_2F_1\left(n, \frac{1}{2}, n + \frac{1}{2}; \frac{w_{12} \cos \phi - \sqrt{w_{11}w_{22}}}{w_{12} \cos \phi + \sqrt{w_{11}w_{22}}}\right)$$

[Ref. 24, p. 45]

B. GAUSSIAN WITH ONE NON ZERO MEAN (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma^2), \quad a = \|\underline{A}\|$$

Let

$$z = (\underline{X}^{(1)}, \underline{X}^{(2)}) = \sum_{k=1}^n x_k^{(1)} x_k^{(2)}$$

1. n = 1

$$f(z) = \frac{1}{\sqrt{\pi} \sigma^2} \exp \left(-\frac{a^2}{2\sigma^2} \right) \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(j + \frac{1}{2})} \left(\frac{a^2 |z|}{4\sigma^4} \right)^j K_j \left(\frac{|z|}{\sigma^2} \right)$$

$$\psi_z(\mu) = \left(\frac{1}{1 + \sigma^4 \mu^2} \right)^{1/2} \exp \left(-\frac{\mu^2 \sigma^2 a^2}{2(1 + \sigma^4 \mu^2)} \right)$$

2. n = 2

$$f(z) = \frac{1}{2\sigma^2} \exp \left(-\frac{a^2}{2\sigma^2} \right) \exp \left(-\frac{|z|}{\sigma^2} \right) \sum_{j=0}^{\infty} \sum_{\ell=0}^j \frac{(j+\ell)!}{2^\ell j! j! \ell! (j-\ell)!} \left(\frac{a^2}{4\sigma^2} \right)^j \left(\frac{|z|}{\sigma^2} \right)^{j-\ell}$$

$$\psi_z(\mu) = \frac{1}{1 + \sigma^4 \mu^2} \exp \left(-\frac{\mu^2 \sigma^2 a^2}{2(1 + \sigma^4 \mu^2)} \right)$$

3. n = 2k (even)

$$f(z) = \frac{1}{2\sigma^2} \left(\frac{|z|}{2\sigma^2} \right)^{k-1} \exp \left(-\frac{a^2}{2\sigma^2} \right) \exp \left(-\frac{|z|}{\sigma^2} \right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{k+j-1} \frac{(k+j+\ell-1)!}{2^\ell j! (k+j-1)! \ell! (k+j-\ell-1)!} \left(\frac{a^2}{4\sigma^2} \right)^j \left(\frac{|z|}{\sigma^2} \right)^{j-\ell}$$

$$\psi_z(\mu) = \left(\frac{1}{1 + \sigma^4 \mu^2} \right)^k \exp \left(- \frac{\mu^2 \sigma^2 a^2}{2(1 + \sigma^4 \mu^2)} \right)$$

4. n

$$f(z) = \frac{1}{\sqrt{\pi} \sigma^2} \left(\frac{z}{2\sigma^2} \right)^{(n-1)/2} \exp \left(- \frac{a^2}{2\sigma^2} \right) \sum_{j=0}^{\infty} \frac{1}{j! \Gamma\left(\frac{n}{2} + j\right)}$$

$$\cdot \left(\frac{a^2 |z|}{2\sigma^2} \right)^{[(n-1)/2] + j} K_{[(n-1)/2] + j} \left(\frac{|z|}{\sigma^2} \right)$$

[Ref. 32]

$$\psi_z(\mu) = \left(\frac{1}{1 + \mu^2 \sigma^4} \right)^{n/2} \exp \left(- \frac{\mu^2 \sigma^2 a^2}{2(1 + \mu^2 \sigma^4)} \right)$$

C. GAUSSIAN WITH NON ZERO MEANS

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma^2), \quad \underline{X}^{(2)} \in N_n(\underline{B}, \sigma^2)$$

$$a = \|\underline{A}\|, \quad b = \|\underline{B}\|$$

$$\text{Let } z = (\underline{X}^{(1)}, \underline{X}^{(2)}) = \sum_{k=0}^n x_k^{(1)} x_k^{(2)}$$

$$\alpha^2 = \frac{1}{4} \left((\underline{A+B}), (\underline{A+B}) \right) = \frac{a^2 + b^2}{4} + \frac{1}{2} \sum_{k=1}^n a_k b_k$$

$$\beta^2 = \frac{1}{4} \left((\underline{A-B}), (\underline{A-B}) \right) = \frac{a^2 + b^2}{4} - \frac{1}{2} \sum_{k=1}^n a_k b_k$$

$$f(z) = \begin{cases} \frac{(-z)^{(n-2)/2} \exp\left(-\frac{a^2 + b^2}{2\sigma^2}\right)}{2^{n/2} \sigma^n} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{j! \ell! \Gamma\left(\frac{n}{2} + \ell\right)} \\ \cdot \left(\frac{-\alpha^4 z}{2\sigma^6}\right)^{j/2} \left(\frac{-\beta^4 z}{2\sigma^6}\right)^{\ell/2} W_{(\ell-j)/2, (n+\ell+j-1)/2} \left(\frac{-2z}{\sigma^2}\right) & z < 0 \\ \frac{z^{(n-2)/2} \exp\left(-\frac{a^2 + b^2}{2\sigma^2}\right)}{2^{n/2} \sigma^n} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{j! \ell! \Gamma\left(\frac{n}{2} + j\right)} \\ \cdot \left(\frac{\alpha^4 z}{2\sigma^6}\right)^{j/2} \left(\frac{\beta^4 z}{2\sigma^6}\right)^{\ell/2} W_{(j-\ell)/2, (n+j+\ell-1)/2} \left(\frac{2z}{\sigma^2}\right) & z \geq 0 \end{cases}$$

[Ref. 24, p. 63]

$$\psi_z(\mu) = \left(\frac{1}{1 + \sigma^2 \mu^2} \right)^{n/2} \exp \left(- \frac{(a^2 + b^2) \sigma^2 \mu^2 - 2i\mu \sum_{r=0}^n a_r b_r}{1 + \sigma^4 \mu^2} \right)$$

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma^2), \quad \underline{X}^{(2)} \in N_n(\underline{B}, \sigma^2)$$

$$a = \|\underline{A}\|, \quad b = \|\underline{B}\|, \quad M = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}; \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$\text{Let} \quad z = (\underline{X}^{(1)}, \underline{X}^{(2)}) = \sum_{k=1}^n x_k^{(1)} x_k^{(2)}$$

$$\alpha^2 = \frac{1}{4} \left((\underline{A} + \underline{B}), (\underline{A} + \underline{B}) \right) = \frac{a^2 + b^2}{4} + \frac{1}{2} \sum_{k=1}^n a_k b_k$$

$$\beta^2 = \frac{1}{4} \left((\underline{A} - \underline{B}), (\underline{A} - \underline{B}) \right) = \frac{a^2 + b^2}{4} - \frac{1}{2} \sum_{k=1}^n a_k b_k$$

$$f(z) = \begin{cases} \frac{(-z)^{(n-2)/2} \exp(-2\beta^2 \sqrt{w_{11} w_{22}})}{2^{n/2} \sigma^n} \exp(-w_{12} z) \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{j! \ell! \Gamma(\frac{n}{2} + \ell)} \\ \cdot \left(-\frac{\alpha^4(1-\rho)z}{2(1+\rho)^3 \sigma^6} \right)^{j/2} \left(-\frac{\beta^4(1+\rho)z}{2(1-\rho)^3 \sigma^6} \right)^{\ell/2} \\ \cdot w_{(\ell-j)/2, (n+j+\ell-1)/2} (-2z \sqrt{w_{11} w_{22}}) & z < 0 \\ \\ \frac{z^{(n-2)/2} \exp(-2\beta^2 \sqrt{w_{11} w_{22}})}{2^{n/2} \sigma^n} \exp(-w_{12} z) \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{j! \ell! \Gamma(\frac{n}{2} + j)} \\ \cdot \left(\frac{\alpha^4(1-\rho)z}{2(1+\rho)^3 \sigma^6} \right)^{j/2} \left(\frac{\beta^4(1+\rho)z}{2(1-\rho)^3 \sigma^6} \right)^{\ell/2} \\ \cdot w_{(j-\ell)/2, (n+j+\ell-1)/2} (2z \sqrt{w_{11} w_{22}}) & z \geq 0 \end{cases}$$

[Ref. 24, p. 63]

$$\psi_z(\mu) = \left(\frac{1}{\gamma}\right)^{n/2} \exp \left(- \frac{(a^2 + b^2)\sigma^2\mu^2 - 2(i\mu + \rho\sigma^2\mu^2) \sum_{r=0}^n a_r b_r}{\gamma} \right)$$

$$\gamma = 1 - 2i\mu\sigma^2\rho + \mu^2\sigma^4(1-\rho^2)$$

D. RAYLEIGH (X) RAYLEIGH

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

$$\text{Let} \quad z = r_1 r_2, \quad c = \sigma_1 \sigma_2$$

$$\text{a. } n = m = 1$$

$$f(z) = \frac{2}{\pi c} K_0\left(\frac{z}{c}\right) \quad z \geq 0$$

$$Ez^k = (2c)^k \frac{\Gamma^2\left(\frac{k+1}{2}\right)}{\pi} \quad k \geq 0$$

$$\text{b. } n = m = 2$$

$$f(z) = \frac{z}{c^2} K_0\left(\frac{z}{c}\right) \quad z \geq 0$$

$$Ez^k = (2c)^k \Gamma^2\left(\frac{k}{2} + 1\right) \quad k \geq 0$$

$$c. \quad n = m = 2k$$

$$f(z) = \frac{4}{z} \left(\frac{z}{2c} \right)^{2k} \frac{1}{[(k-1)!]^2} K_0 \left(\frac{z}{c} \right) \quad z \geq 0$$

$$Ez^\ell = (2c)^\ell \frac{\Gamma^2 \left(k + \frac{\ell}{2} \right)}{[(k-1)!]^2} \quad \ell \geq 0$$

[Ref. 24, p. 73]

$$d. \quad n, \quad m$$

$$f(z) = \frac{4}{z} \left(\frac{z}{2c} \right)^{(m+n)/2} \frac{1}{\Gamma \left(\frac{n}{2} \right) \Gamma \left(\frac{m}{2} \right)} K_{(n-m)/2} \left(\frac{z}{c} \right) \quad z \geq 0$$

[Ref. 24, p. 49]

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{0}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}; \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

Let

$$z = r_1 r_2$$

$$f(z) = \frac{|\underline{w}|^{n/2} z^{n/2}}{(2|w_{12}|)^{(n-2)/2} \Gamma(\frac{n}{2})} I_{(n/2)-1} (z|w_{12}|) K_0 (z\sqrt{w_{11}w_{22}}) \quad z \geq 0$$

[Ref. 24, p. 47]

$$Ez^k = \frac{2^k |\underline{w}|^{n/2} \Gamma^2(\frac{n+k}{2})}{(w_{11}w_{22})^{(n+k)/2} \Gamma^2(\frac{n}{2})} {}_2F_1\left(\frac{n+k}{2}, \frac{n+k}{2}, \frac{n}{2}; \frac{w_{12}^2}{w_{11}w_{22}}\right)$$

[Ref. 24, p. 73]

E. RAYLEIGH ($\times$) RICE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{0}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad v = \|\underline{X}^{(1)}\|, \quad r = \|\underline{X}^{(2)}\|$$

Let

$$u = vr$$

$$1. \quad \underline{n = 1, \quad m = 1}$$

$$f(u) = \frac{2}{\sqrt{\pi} \sigma_1 \sigma_2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(j + \frac{1}{2})} \left(\frac{a}{2\sigma_1^2}\right)^{2j} \left(\frac{u\sigma_1}{\sigma_2}\right)^j K_j\left(\frac{u}{\sigma_1 \sigma_2}\right)$$

$$u \geq 0$$

$$2. \quad \underline{n = 2, \quad m = 2}$$

$$f(u) = \frac{u}{\sigma_1^2 \sigma_2^2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \sum_{j=0}^{\infty} \frac{1}{[j!]^2} \left(\frac{a}{2\sigma_1^2}\right)^{2j} \left(\frac{u\sigma_1}{\sigma_2}\right)^j K_j\left(\frac{u}{\sigma_1 \sigma_2}\right) \quad u \geq 0$$

3. n, m

$$f(u) = \frac{4}{u} \left(\frac{u}{2\sigma_1\sigma_2} \right)^{(m+n)/2} \frac{\exp \left(-\frac{a^2}{2\sigma_1^2} \right)}{\Gamma\left(\frac{m}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j! \Gamma\left(\frac{n}{2} + j\right)} \left(\frac{a}{2\sigma_1^2} \right)^{2j} \left(\frac{u\sigma_1}{\sigma_2} \right)^j$$

$$\cdot K_{[(n-m)/2]+j} \left(\frac{u}{\sigma_1\sigma_2} \right) \quad u \geq 0$$

[Ref. 24, p. 48]

F. RICE (X) RICE

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma_1^2), \quad \underline{X}^{(2)} \in N_m(\underline{B}, \sigma_2^2)$$

$$a = \|\underline{A}\|, \quad b = \|\underline{B}\|, \quad \nu_1 = \|\underline{X}^{(1)}\|, \quad \nu_2 = \|\underline{X}^{(2)}\|$$

Let

$$q = \nu_1 \nu_2$$

$$f(q) = \frac{4}{q} \left(\frac{q}{2\sigma_1\sigma_2} \right)^{(n+m)/2} \exp \left(-\frac{1}{2} \left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2} \right) \right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{j! \ell! \Gamma\left(\frac{n}{2} + j\right) \Gamma\left(\frac{m}{2} + \ell\right)}$$

$$\cdot \left(\frac{a \sqrt{q}}{2\sigma_1^2} \right)^{2j} \left(\frac{b \sqrt{q}}{2\sigma_2^2} \right)^{2\ell} \left(\frac{\sigma_1}{\sigma_2} \right)^{j-\ell} K_{[(n-m)/2]+j-\ell} \left(\frac{q}{\sigma_1\sigma_2} \right) \quad q \geq 0$$

[Ref. 24, p. 48]

$$E_Q^p = (2\sigma_1\sigma_2)^p \frac{\Gamma\left(\frac{n+p}{2}\right) \Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right)} \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right) \\ \cdot {}_1F_1\left(\frac{n+p}{2}; \frac{n}{2}; \frac{a^2}{2\sigma_1^2}\right) {}_1F_1\left(\frac{m+p}{2}; \frac{m}{2}; \frac{b^2}{2\sigma_2^2}\right) \quad p > 0$$

[Ref. 24]

G. JOINT DENSITY

$$\underline{X}^{(1)} \in N_n(0, \sigma_1^2), \quad \underline{X}^{(2)} \in N_n(0, \sigma_2^2)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Also

$$\underline{X}^{(3)} \in N_n(0, \sigma_1^2), \quad \underline{X}^{(4)} \in N_n(0, \sigma_2^2), \quad \underline{M}$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|, \quad r_3 = \|\underline{X}^{(3)}\|, \quad r_4 = \|\underline{X}^{(4)}\|$$

$$\text{Let} \quad z_1 = r_1 r_3, \quad z_2 = r_2 r_4$$

Here r_1 and r_3 are independent

r_2 and r_4 are independent

r_1 and r_2 are related by $\underline{M}$

r_3 and r_4 are related by $\underline{M}$

$$f(z_1, z_2) = \frac{|\underline{w}|^n (z_1 z_2)^{n-1}}{2^{2n-4} \Gamma^2\left(\frac{n}{2}\right)} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{j! \ell! \Gamma\left(\frac{n}{2} + j\right) \Gamma\left(\frac{n}{2} + \ell\right)}$$

$$\cdot \left(\frac{z_1 z_2 w_{12}^2}{4} \right)^{j+\ell} K_{j-\ell}(z_1 w_{11}) K_{j-\ell}(z_2 w_{22}) \quad z_1, z_2 > 0$$

[Ref. 24, p. 49]

VI. GENERAL QUADRATIC FORMS [Ref. 31]

Let

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

be a Gaussian vector with

$$\underline{EY} = \underline{A} \quad \text{and} \quad \underline{K} = E(\underline{Y}-\underline{A})'(\underline{Y}-\underline{A})$$

A. $Q = \underline{Y}'\underline{W}\underline{Y}$

Let $Q = \underline{Y}'\underline{W}\underline{Y}$ be the quadratic form where $\underline{W}$ is an $n \times n$ matrix.

Let

$$\{\lambda_k\}_{k=1}^n$$

be the eigenvalues of $\underline{W}\underline{K}$.

1. $\underline{EY} = \underline{0}$ (zero mean)

$$\psi_Q(\mu) = |\underline{I} - 2i\mu \underline{W}\underline{K}|^{-1/2} = \left(\frac{1}{\prod_{k=1}^n (1 - 2i\mu\lambda_k)} \right)^{1/2}$$

$$EQ = \sum_{k=1}^n \lambda_k$$

$$\text{Var } Q = 2 \sum_{k=1}^n \lambda_k^2$$

$$2. \quad \underline{EY} = \underline{A}$$

$$\psi_Q(\mu) = |\underline{I} - 2i\mu \underline{W} \underline{K}|^{-1/2} \exp \left(-\frac{1}{2} \underline{A}' \underline{K}^{-1} [\underline{I} - (\underline{I} - 2i\mu \underline{W} \underline{K})^{-1}] \underline{A} \right)$$

$$B. \quad Q = \underline{Y}' \underline{W} \underline{Y} + \underline{B}' \underline{Y}$$

Let $Q = \underline{Y}' \underline{W} \underline{Y} + \underline{B}' \underline{Y}$; $\underline{B}' = n \times 1$ column vector. $\underline{W}$ is an $n \times n$ matrix.

$$1. \quad \underline{EY} = \underline{0} \quad (\text{zero mean})$$

$$\psi_Q(\mu) = |\underline{I} - 2i\mu \underline{W} \underline{K}|^{-1/2} \exp \left(-\frac{\mu^2}{2} \underline{B}' \underline{K} (\underline{I} - 2i\mu \underline{W} \underline{K})^{-1} \underline{B} \right)$$

$$2. \quad \underline{EY} = \underline{A}$$

$$\psi_Q(\mu) = |\underline{I} - 2i\mu \underline{W} \underline{K}|^{-1/2} \exp \left(i\mu (\underline{A}' \underline{K} \underline{A} + \underline{B}' \underline{A}) \right)$$

$$\cdot \exp \left(-\frac{\mu^2}{2} (2\underline{W} \underline{A} + \underline{B})' \underline{K} (\underline{I} - 2i\mu \underline{W} \underline{K})^{-1} (2\underline{W} \underline{A} + \underline{B}) \right)$$

APPENDIX A. MISCELLANEOUS FORMS

1. A Zero Mean Complex Quadratic Form [Ref. 2]

$\underline{X}$ and $\underline{Y}$ are complex zero mean Gaussian vectors. Hence $\underline{EX} = \underline{EY} = \underline{0}$. Components of the same vector are independent as usual.

$$\text{Given: } E x_k x_k^* = E |x_k|^2 = m_{xx} \quad k = 1, \dots, n$$

$$E y_k y_k^* = E |y_k|^2 = m_{yy} \quad k = 1, \dots, n$$

$$E x_k^* y_k = m_{xy}, \quad E x_k y_k^* = m_{xy}^* \quad k = 1, \dots, n$$

$$\text{Let } \underline{M} = \begin{bmatrix} m_{xx} & m_{xy} \\ m_{xy}^* & m_{yy} \end{bmatrix}$$

a, b be real and c complex.

Define

$$q_k = a |x_k|^2 + b |y_k|^2 + c x_k^* y_k + c^* x_k y_k^* \quad k = 1, \dots, n$$

$$\underline{P} = \underline{M} \begin{bmatrix} a & c \\ c^* & b \end{bmatrix} = \begin{bmatrix} p_{xx} & p_{xy} \\ p_{xy}^* & p_{yy} \end{bmatrix}$$

$$q = \sum_{k=1}^n q_k$$

$$\alpha = \left\{ \left[\frac{p_{xx} + p_{yy}}{2|\underline{p}|} \right]^2 - \frac{1}{|\underline{p}|} \right\}^{1/2} + \frac{p_{xx} + p_{yy}}{|\underline{p}|}$$

$$\beta = \left\{ \left[\frac{p_{xx} + p_{yy}}{2|\underline{p}|} \right]^2 - \frac{1}{|\underline{p}|} \right\}^{1/2} - \frac{p_{xx} + p_{yy}}{2|\underline{p}|}$$

a. $n = 1$

$$g(q) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} e^{\beta q} & q < 0 \\ \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha q} & q \geq 0 \end{cases}$$

$$G(q) = \begin{cases} \frac{\alpha}{\alpha+\beta} e^{\beta q} & q < 0 \\ 1 - \frac{\beta}{\alpha+\beta} e^{-\alpha q} & q \geq 0 \end{cases}$$

b. n

$$g(q) = \begin{cases} e^{\beta q} \sum_{k=0}^{n-1} \frac{(\alpha\beta)^n}{(\alpha+\beta)^{n+k}} \binom{n+k-1}{k} \frac{(-q)^{n-k-1}}{(n-k-1)!} & q < 0 \\ e^{-\alpha q} \sum_{k=0}^{n-1} \frac{(\alpha\beta)^n}{(\alpha+\beta)^{n+k}} \binom{n+k-1}{k} \frac{(-q)^{n-k-1}}{(n-k-1)!} & q \geq 0 \end{cases}$$

$$G(q) = \begin{cases} \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \frac{(\alpha\beta)^n}{(\alpha+\beta)^{n+k}} \binom{n+k-1}{k} \frac{(-q)^{n-k-j-1}}{(n-k-j-1)! \beta^{j+1}} e^{\beta q} & q < 0 \\ 1 - \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \frac{(\alpha\beta)^n}{(\alpha+\beta)^{n+k}} \binom{n+k-1}{k} \frac{(-1)^{n-k-1} q^{n-k-j-1}}{(n-k-j-1)! \alpha^{j+1}} e^{-\alpha q} & q \geq 0 \end{cases}$$

2. Rayleigh (+) Rayleigh (Independent) [Ref. 30]

$$\underline{X}^{(1)} \in N_2(\underline{0}, 1); \quad \underline{X}^{(2)} \in N_2(\underline{0}, 1)$$

$$r_1 = \|\underline{X}^{(1)}\|, \quad r_2 = \|\underline{X}^{(2)}\|$$

Let

$$z = r_1 + r_2$$

$$f(z) = \exp\left(-\frac{z^2}{4}\right) \left[\left(\frac{z^2}{4} - \frac{1}{2}\right) \sqrt{\pi} \operatorname{erf}\left(\frac{z}{2}\right) + \frac{z}{2} \exp\left(-\frac{z^2}{4}\right) \right] \quad z \geq 0$$

3. Gaussian (x) Rayleigh (Independent) [Ref. 30]

$$x \in N_1(0, \sigma_1^2), \quad \underline{X} \in N_2(\underline{0}, \sigma_2^2)$$

$$r = \|\underline{X}\|$$

Let

$$z = xr$$

$$f(z) = \frac{1}{2\sigma_1\sigma_2} \exp\left(-\frac{|z|}{\sigma_1\sigma_2}\right)$$

4. Gaussian (×) Rayleigh (+) Gaussian (Independent)

$$x \in N_1(0,1), \quad \underline{X} \in N_2(\underline{0},1), \quad y \in N_1(0,\sigma^2)$$

$$r = \|\underline{X}\|$$

Let

$$w = xr + y$$

$$f(w) = \frac{\exp\left(-w + \frac{\sigma^2}{2}\right)}{2} \left\{ 1 - \frac{\operatorname{erf}\left[\frac{\sigma}{\sqrt{2}} - \frac{w}{\sqrt{2}\sigma}\right] + \operatorname{erf}\left[\frac{\sigma}{\sqrt{2}} + \frac{w}{\sqrt{2}\sigma}\right]}{2} \right\} \quad [\text{Ref. 30}]$$

5. Gaussian (+) Rayleigh (Independent)

$$x \in N_1(0,\sigma_1^2), \quad \underline{X} \in N_2(\underline{0},\sigma_2^2)$$

$$r = \|\underline{X}\|$$

Let

$$z = x + r$$

$$f(z) = \frac{\sigma_1 \exp\left(-\frac{z^2}{2\sigma_1^2}\right)}{\sqrt{2\pi}(\sigma_2^2 + \sigma_1^2)} + \frac{z\sigma_2 \exp\left(-\frac{z^2\sigma_2^2}{2(\sigma_1^2 + \sigma_2^2)}\right)}{2(\sigma_1^2 + \sigma_2^2)^{3/2}} \left[1 + \operatorname{erf}\left(\frac{z\sigma_2}{\sigma_1 \sqrt{2(\sigma_1^2 + \sigma_2^2)}}\right) \right] \quad [\text{Ref. 30}]$$

6. Gaussian (+) Double Rayleigh (Independent)

$$x \in N_1(0,\sigma^2), \quad \underline{X} \in N_2(\underline{0},1)$$

$$r = \|\underline{X}\| \quad \text{and} \quad r^* = \begin{cases} r & \text{with probability } \frac{1}{2} \\ -r & \text{with probability } \frac{1}{2} \end{cases}$$

Let

$$w = x + r^*$$

$$f(w) = \frac{\sigma \exp\left(-\frac{w^2}{2\sigma^2}\right)}{\sqrt{2\pi} (1 + \sigma^2)} + \frac{|w| \exp\left(-\frac{w^2}{2(1 + \sigma^2)}\right)}{2(1 + \sigma^2)^{3/2}} \operatorname{erf}\left[\frac{|w|}{\sigma \sqrt{2(1 + \sigma^2)}}\right]$$

[Ref. 30]

7. General Products (Independent)

a. $x_k \in N_1(0,1)$, $y_k \in N_1(0,1)$, $k = 1, \dots, n$; all independent.

Let

$$z = \prod_{k=1}^n \frac{x_k}{y_k}$$

1) $n = 1$

$$f(z) = \frac{1}{\pi(1 + z^2)}$$

2) $n = 2$

$$f(z) = \frac{1}{\pi^2(z^2 - 1)} \ln(z^2)$$

3) $n = 3$

$$f(z) = \frac{1}{2! \pi^3(z^2 + 1)} \{[\ln z^2]^2 + 1\}$$

4) $n = 4$

$$f(z) = \frac{1}{3! \pi^4 (z^2 - 1)} \{ [\ln z^2]^3 + 4[\ln z^2] \}$$

5) $n = 5$

$$f(z) = \frac{1}{4! \pi^5 (z^2 + 1)} \{ [\ln z^2]^4 + 10[\ln z^2] + 9 \}$$

6) n

$$f(z) = \frac{1}{\pi^n [1 - (-1)^n z^{-1}]} \sum_{k=0}^{n-1} \frac{1}{(n-1-k)! k!} (\ln z)^{n-1-k} \frac{d^k}{ds^k} [s \csc s]^n \Big|_{s=0}$$

[Ref. 34]

b. Other Products and Discussion of the Mellin Transformation

See Ref. 34.

8. Instantaneous Frequency of Narrow Band Gaussian Noise

See Ref. 35.

APPENDIX B. TRANSCENDENTAL FUNCTIONS

1. Definitions and Special Forms of Transcendental Functions

a. Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt; \quad x \geq 0$$

$$\operatorname{erfc}(x) = \frac{1}{2} [1 - \operatorname{erf}(x)] = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt; \quad x \geq 0$$

b. Modified Bessel Functions

First Kind

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{\alpha+2k}}{k! \Gamma(\alpha+k+1)} \quad x \geq 0$$

$$I_n(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k}}{k! (n+k)!} \quad n \text{ integer}$$

$$I_{-1/2}(x) = \left(\frac{1}{2\pi x}\right)^{1/2} (e^x + e^{-x})$$

$$I_{n+(1/2)}(x) = \frac{1}{\sqrt{2\pi x}} \left[e^x \sum_{k=0}^n \frac{(-1)^k (n+k)!}{k! (n-k)! (2x)^k} + (-1)^{n+1} \right.$$

$$\left. \cdot e^{-x} \sum_{k=0}^n \frac{(n+k)!}{k! (n-k)! (2x)^k} \right]$$

Second Kind

$$k_{\alpha}(x) = \int_0^{\infty} \exp(-x \cosh t) \cosh \alpha t \, dt \quad x \geq 0$$

$$k_{n+(1/2)}(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \sum_{k=0}^n \frac{(n+k)!}{k! (n-k)! (2x)^k} \quad \begin{array}{l} n \text{ integer} \\ x \geq 0 \end{array}$$

c. Marcum Q Functions

$$Q_m(\alpha, \beta) = \int_{\beta}^{\infty} t \left(\frac{t}{2}\right)^{m-1} \exp\left(-\frac{t^2 + \alpha^2}{2}\right) I_{m-1}(\alpha t) \, dt \quad \begin{array}{l} \alpha \geq 0 \\ \beta \geq 0 \end{array}$$

$$Q_m(0, \beta) = \exp\left(-\frac{\beta^2}{2}\right) \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{\beta^2}{2}\right)^k$$

$$Q_m(\alpha, 0) = 1$$

d. Bivariate Normal Function

$$v(\alpha, \beta) = \frac{1}{2\pi} \int_0^{\alpha} e^{-t^2/2} \, dt \int_0^{(\beta/\alpha)t} e^{-x^2/2} \, dx \quad \begin{array}{l} \alpha \geq 0 \\ \beta \geq 0 \end{array}$$

e. Elliptically Normal Probability Function

$$\Lambda(\alpha, \beta) = \int_0^{\beta} e^{-t} I_0(\alpha t) \, dt \quad \begin{array}{l} \alpha \geq 0 \\ \beta \geq 0 \end{array}$$

$$\Lambda(0, \beta) = 1 - e^{-\beta}$$

$$\Lambda(\alpha, \infty) = \frac{1}{\sqrt{1 - \alpha^2}} ; \quad \Lambda(1, \beta) = \beta e^{-\beta} (I_0(\beta) + I_1(\beta))$$

f. Gamma and Beta Functions

Gamma Functions

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt \quad \alpha > 0$$

$$\Gamma(n) = (n-1)! = (n-1)(n-2)(n-3) \dots (3)(2)(1) \quad n \text{ integer}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}; \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Beta Functions

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt; \quad \begin{matrix} \alpha > 0 \\ \beta > 0 \end{matrix}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \begin{matrix} \alpha > 0 \\ \beta > 0 \end{matrix}$$

$$B(m, n) = \frac{(m-1)! (n-1)!}{(m+n-1)!} \quad n, m \text{ integers}$$

g. Hypergeometric and Confluent Hypergeometric Functions

Confluent Hypergeometric Functions

$${}_1F_1(\alpha, \beta; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{x^k}{k!} \quad (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \quad \beta \neq 0, -1, -2, \dots$$

$${}_1F_1(\alpha, \beta; x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 e^{xt} t^{\alpha-1} (1-t)^{\beta-\alpha-1} dt \quad \beta > \alpha$$

$${}_1F_1(\alpha, \alpha; x) = e^x$$

$${}_1F_1(m, n; x) = \frac{(n-1)!}{(m-1)!} \sum_{k=0}^{\infty} \frac{(m+k-1)!}{k! (n+k-1)!} x^k \quad m, n \text{ integers}$$

$${}_1F_1(n+1, n; x) = \left(1 + \frac{x}{n}\right) e^x$$

$${}_1F_1(1, n+1; x) = \frac{n!}{x^n} \left[e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!} \right]$$

Hypergeometric Function

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tx)^{-\alpha} dt \quad \alpha > \beta > 0$$

h. Generalized Laguerre Polynomial

$$L_n^\alpha(z) = \frac{e^z z^{-\alpha}}{n!} \frac{d^n}{dz^n} (e^z z^{n+\alpha}) = \frac{\Gamma(\alpha+n+1)}{n! \Gamma(\alpha+1)} {}_1F_1(-n, \alpha+1, z)$$

[Ref. 21, p. 189]

$$L_n^\alpha(z) = \sum_{m=0}^n \binom{n+\alpha}{n-m} \frac{(-z)^m}{m!}$$

i. Whittaker Functions

$$W_{\alpha, \beta}(x) = \frac{x^{\beta+(1/2)} e^{-x/2}}{\Gamma(\beta - \alpha + \frac{1}{2})} \int_0^\infty e^{-xt} t^{\beta-\alpha-(1/2)} (1+t)^{3+\alpha-(1/2)} dt$$

$$\beta - \alpha + \frac{1}{2} > 0$$

For n, m integers

$$W_{(1-m-n)/2, (n+m)/2}(x) = \frac{e^{-x/2}}{x^{(n+m-1)/2}} \quad x > 0$$

$$W_{(m+n-1)/2, (n+m)/2}(x) = x^{(m+n+1)/2} e^{-x/2} \sum_{k=0}^{n+m-1} \frac{(n+m-1)!}{(n+m-k-1)! x^{k+1}} \quad x \geq 0$$

$$W_{-n/2, m+[(n-1)/2]}(x) = \frac{x^{m+(n/2)} e^{-x/2}}{(m+n-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(m+n+k-1)!}{x^{m+n+k}} \quad x > 0$$

$$W_{n/2, m+[(n-1)/2]}(x) = \frac{x^{m+(n/2)} e^{-x/2}}{(m-1)!} \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{(m+k-1)!}{x^{m+k}} \quad x > 0$$

$$W_{n/2, m-[(n-1)/2]}(x) = \frac{x^{m-[(n-1)/2]} e^{-x/2}}{(m-n)!} \sum_{k=0}^m \binom{m}{k} \frac{(m-n+k)!}{x^{m-n+k}} \quad \begin{matrix} x > 0 \\ m \geq n \end{matrix}$$

$$W_{-n/2, m-[(n-1)/2]}(x) = \frac{x^{m-[(n-1)/2]} e^{-x/2}}{m!} \sum_{k=0}^{m-n} \binom{m-n}{k} \frac{(m+k)!}{x^{m+k+1}} \quad \begin{matrix} x > 0 \\ m \geq n \end{matrix}$$

2. Some Useful Relationships between Transcendental Functions

a. Marcum Q Function

$$1) \quad n = 1, \quad Q_1(\alpha, \beta) = Q(\alpha, \beta)$$

$$Q(\alpha, \beta) = \exp \left(-\frac{\alpha^2 + \beta^2}{2} \right) \sum_{k=0}^{\infty} \left(\frac{\alpha}{\beta} \right)^k I_k(\alpha\beta) \quad \beta > \alpha > 0$$

$$Q(\alpha, \beta) = 1 - \exp \left(-\frac{\alpha^2 + \beta^2}{2} \right) \sum_{k=0}^{\infty} \left(\frac{\beta}{\alpha} \right)^k I_k(\alpha\beta) \quad \alpha > \beta$$

$$Q(\alpha, \alpha) = \frac{1}{2} \left[1 + e^{-\alpha^2} I_0(\alpha^2) \right]$$

$$Q(\alpha, \beta) + Q(\beta, \alpha) = 1 + \exp \left(-\frac{\alpha^2 + \beta^2}{2} \right) I_0(\alpha\beta)$$

For $\alpha \gg \beta \gg 1$

$$Q(\alpha, \beta) \sim 1 - \frac{1}{\alpha - \beta} \sqrt{\frac{\beta}{2\pi\alpha}} \exp \left(-\frac{(\alpha - \beta)^2}{2} \right)$$

For $\beta \gg \alpha \gg 1$

$$Q(\alpha, \beta) \sim \frac{1}{\beta - \alpha} \sqrt{\frac{\alpha}{2\pi\beta}} \exp \left(-\frac{(\beta - \alpha)^2}{2} \right)$$

$$\int_0^\gamma e^{x/c} Q(\sqrt{2a}; \sqrt{2bx}) dx = -c + \frac{bc^2}{bc-1} \exp\left(\frac{a}{bc-1}\right) \left\{ 1 - Q\left(\sqrt{\frac{2abc}{bc-1}}, \sqrt{\frac{2(bc-1)\gamma}{c}}\right) \right\}$$

$$+ ce^{\gamma/c} Q(\sqrt{2a}, \sqrt{2b\gamma}) \quad \text{For } a, b, c > 0$$

$$\frac{d}{dx} Q(\alpha, x) = -x \exp\left(-\frac{x^2 + \alpha^2}{2}\right) I_0(\alpha x) \quad x \geq 0$$

$$\frac{d}{dx} Q(x, \beta) = \beta \exp\left(-\frac{x^2 + \beta^2}{2}\right) I_1(\beta x) \quad x \geq 0$$

$$\exp\left(-\frac{x}{c} - c\right) \sum_{k=0}^{\infty} \frac{(c^2 x)^{k/2}}{(k!)^2} w_{k/2, (k+1)/2}(x) = Q(\sqrt{2c}, \sqrt{2x}) \quad x \geq 0$$

2) m

$$Q_m(\alpha, \beta) = Q(\alpha, \beta) + \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) \sum_{k=1}^{m-1} \left(\frac{\beta}{\alpha}\right)^k I_k(\alpha\beta)$$

$$\int_0^\gamma e^{x/c} Q_m(\sqrt{2a}; \sqrt{2bx}) dx = -c + c \left(\frac{bc}{bc-1}\right)^m \exp\left(\frac{a}{bc-1}\right)$$

$$\cdot \left\{ 1 - Q_m\left(\sqrt{\frac{2abc}{bc-1}}, \sqrt{\frac{2(bc-1)\gamma}{c}}\right) \right\}$$

$$+ ce^{\gamma/c} Q_m(\sqrt{2a}, \sqrt{2b\gamma}) \quad a, b, c > 0$$

$$x^{(m-1)/2} \exp\left(-\frac{x}{c} - c\right) \sum_{k=0}^{\infty} \frac{(c^2 x)^{k/2}}{k! (m+k-1)!} w_{(m+k-1)/2, (m+k)/2}(x) = Q_m(\sqrt{2c}; \sqrt{2x})$$

$$x \geq 0$$

b. Modified Bessel Functions

First Kind

$$\frac{d}{dx} I_{\alpha}(x) = \frac{I_{\alpha-1}(x) + I_{\alpha+1}(x)}{2}$$

$$\int x^n I_{n-1}(x) dx = x^n I_n(x)$$

$$\int x^{-n} I_{n+1}(x) dx = x^{-n} I_n(x)$$

$$I_n(x) = I_{-n}(x)$$

$$\int e^x I_0(x) dx = x e^x [I_0(x) - I_1(x)]$$

$$\int e^{-x} I_0(x) dx = x e^{-x} [I_0(x) + I_1(x)]$$

$$\int e^x I_1(x) dx = e^x [(1-x)I_0(x) + xI_1(x)]$$

$$\int e^{-x} I_1(x) dx = e^{-x} [(1+x)I_0(x) + xI_1(x)]$$

$$\int_0^{\infty} x e^{-\gamma x^2} I_{\mu}(\alpha x) I_{\mu}(\beta x) dx = \frac{1}{2\gamma} \exp\left(\frac{\alpha^2 + \beta^2}{4\gamma}\right) I_{\mu}\left(\frac{\alpha\beta}{2\gamma}\right) \quad \begin{matrix} \mu > -1 \\ \gamma > 0 \end{matrix}$$

Second Kind

$$\int x^n K_{n-1}(x) dx = -x^n K_n(x)$$

$$\int x^{-n} K_{n+1}(x) dx = -x^{-n} K_n(x)$$

$$\int_0^\infty e^{-ax} K_0(bx) dx = \frac{1}{\sqrt{b^2 - a^2}} \arccos \frac{a}{b} \quad b > |a|$$

$$\int_0^\infty x^{\mu-1} K_\nu(\alpha x) dx = \frac{2^{\mu-2}}{\alpha^\mu} \Gamma\left(\frac{\mu+\nu}{2}\right) \Gamma\left(\frac{\mu-\nu}{2}\right) \quad \begin{matrix} \mu+\nu > 0 \\ \alpha > 0 \end{matrix}$$

c. Whittaker Function and Gamma Function

$$w_{0,\beta}(x) = \left(\frac{x}{\pi}\right)^{1/2} K_\beta\left(\frac{x}{2}\right) \quad x \geq 0$$

$$2^{\alpha-2} \Gamma\left(\frac{\alpha-1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right) = \Gamma\left(\frac{1}{2}\right) \Gamma(\alpha-1)$$

d. Confluent Hypergeometric Function

$${}_1F_1(\alpha, \beta; x) = e^x {}_1F_1(\beta-\alpha, \beta; -x)$$

$${}_1F_1(\alpha, \alpha+1, -x) = \alpha x^{-\alpha} \int_0^x e^{-t} t^{\alpha-1} dt = z^{-\alpha} \Gamma(\alpha+1) I\left(\frac{x}{\sqrt{\alpha}}, \frac{1}{\alpha}\right) \quad \alpha > 0$$

where $I(\cdot, \cdot)$ is the incomplete gamma function.

$${}_1F_1(1, a+1; x) = e^{-x} x^a \Gamma(a+1) I\left(\frac{x}{\sqrt{a}}, a-1\right) \quad a > 0$$

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) = \frac{\sqrt{\pi}}{2x} \operatorname{erf} x \quad x > 0$$

$${}_1F_1(-n, 1; x) = L_n(x) \quad \text{original Laguerre polynomial}$$

$${}_1F_1(-n, \alpha+1, x) = \frac{n! \Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} L_n^\alpha(x) \quad \text{general Laguerre polynomial}$$

$${}_1F_1\left(\alpha + \frac{1}{2}, 2\alpha+1, x\right) = \frac{2^{2\alpha} \Gamma(\alpha+1) e^{x/2}}{x^\alpha} I_\alpha\left(\frac{x}{2}\right)$$

$${}_1F_1\left(\frac{1}{2}, 1; -x\right) = e^{-x/2} I_0\left(\frac{x}{2}\right)$$

$${}_1F_1\left(\frac{1}{2}, 2; -x\right) = e^{-x/2} \left[I_0\left(\frac{x}{2}\right) + I_1\left(\frac{x}{2}\right) \right]$$

$${}_1F_1\left(-\frac{1}{2}, 1, -x\right) = e^{-x/2} \left[(1+x) I_0\left(\frac{x}{2}\right) + x I_1\left(\frac{x}{2}\right) \right]$$

$${}_1F_1\left(\frac{3}{2}, 1, -x\right) = e^{-x/2} \left[(1-x) I_0\left(\frac{x}{2}\right) + x I_1\left(\frac{x}{2}\right) \right]$$

$${}_1F_1\left(\frac{3}{2}, 2, -x\right) = e^{-x/2} \left[I_0\left(\frac{x}{2}\right) - I_1\left(\frac{x}{2}\right) \right]$$

For recursive relationships see Ref. 19.

$$\int_0^\infty x^\beta e^{-ax^2} I_\mu(\gamma x) dx = \frac{1}{2a^{(\beta+\mu+1)/2}} \left(\frac{\gamma}{2}\right)^\mu \frac{\Gamma\left(\frac{\beta+\mu+1}{2}\right)}{\Gamma(\mu+1)} {}_1F_1\left(\frac{\beta+\mu+1}{2}, \mu+1, \frac{\gamma^2}{4a}\right)$$

$$a > 0, \quad \beta + \mu > -1$$

e. Generalized Laguerre Polynomial

$$\frac{n!}{\Gamma(\alpha+n+1)} \int_0^\infty e^{-z} z^\alpha L_n^\alpha(z) L_k^\alpha(z) dz = \delta_{nk}$$

f. Special

$$\sum_{k=m}^n \binom{\ell+k}{k-m} (1+x)^{n-k} = \sum_{k=m}^n \binom{n+\ell+1}{k-m} x^{n-k}$$

where m , n , and ℓ are integers and $0 \leq m \leq n$.

[Ref. 29]

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