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Technical Report

388

R. Price

Output Signal-to-Noise Ratio
as a Criterion
in Spread-Channel Signaling

13 May 1965

Prepared under Electronic Systems Division Contract AF 19(62B)-500 by

Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts



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OUTPUT SIGNAL-TO-NOISE RATIO AS A CRITERION
IN SPREAD-CHANNEL SIGNALING

R. PRICE

Group 64

TECHNICAL REPORT 388

13 MAY 1965

LEXINGTON

MASSACHUSETTS

ABSTRACT

In transmission and receiver design for radar or communication systems whose noisy channels contain Gaussianly fluctuating multipath, it is convenient to adopt a receiver output signal-to-noise ratio (SNR) criterion even though best error performance is actually sought. We investigate the loss (expressed as an equivalent transmitter output reduction) attending the use of this criterion. It is shown that when a Karhunen-Laève analysis of the signaling system yields a largest eigenvalue that is suitably small, this loss is minor or negligible at all levels of error probability. Furthermore, it is easily possible to have a channel-perturbed transmission that is sufficiently weak and incoherent for this eigenvalue to guarantee low loss, yet not so weak that high output SNR (good error performance) is precluded.

Accepted for the Air Force
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OUTPUT SIGNAL-TO-NOISE RATIO AS A CRITERION IN SPREAD-CHANNEL SIGNALING

I. INTRODUCTION

A. Background and Summary of Main Results

It has been a basic tenet in certain radar astronomy detection studies^{1,2} that system performance is adequately measured[†] by a suitably defined receiver output signal-to-noise ratio (SNR), where the receiver is assumed to perform a quadratic operation on the incoming signal that reduces it to a single number.[‡] It is further assumed that the cascaded multiplicative (Gaussianly fluctuating multipath such as exists with a radar-astronomical target and which produces time- and frequency-spreading) and additive (white noise such as thermal or shot-effect noise) channel disturbances are independent, zero-mean Gaussian processes of known[§] correlation functions. One then readily finds the explicit quadratic-form receiver whose output SNR is the greatest and which has a straightforward engineering realization as a kind of radiometer.¹⁻³ With the receiver thus determined, the functional dependence of the system performance on the transmitted waveshape and energy can be exhibited as measured by output SNR. The studies conclude with the problem of optimizing the transmitted waveform for a given available energy, still according to the SNR criterion and also assuming receiver observation over all time. A difficult nonlinear integral equation is encountered, which has been solved only in the particular case where the shapes of the time- and frequency-spreading profiles are Gaussian curves. (The best fixed-energy transmission in these circumstances is found to be a Gaussianly shaped pulse having a nominal duration that is the geometric mean of the time spread and the reciprocal of the frequency spread, and performance is shown to become worse as the product of the time spread by the frequency spread increases.)

What progress has been possible, as outlined above, is due largely to the mathematical nicety of the quite ad hoc SNR criterion, the assumption of a quadratic-form receiver, and the Gaussianity of the channel disturbances. A more conventional and useful practice in radar and binary communications, where decisions have to be made, is to rate system performance in terms of the pair of probabilities that relate to the two types of error that can be committed

[†] That output SNR is a satisfactory performance measure is widely held in radio astronomy.³ In this older field the term "deflection," defined as the ratio of the increase in mean output produced by the signal sought to the output standard deviation in the absence of signal, is equivalent to the square root of our output SNR.

[‡] The radar astronomy detection system is equivalent to a binary communication system that employs on-off signaling over a fluctuating multipath channel.

[§] For this discussion, the absolute intensities of the channel disturbances need not be known.

in deciding between "target present" and "target absent" in the radar situation, or between the two possible transmitted symbols in binary communications. By adopting such an error-probability criterion while removing all restrictions on the form of the decision-making receiver, it is found for the above channel that the optimum receiver can be realized as a quadratic-form processor followed by a decision operation which is triggered by the value of the processor output^{1,2} (see Sec. II-A and Appendix A). Moreover, the particular error-probability criterion that is chosen has no effect whatsoever on the processor and determines only the decision level. Thus there is a clear degree of correspondence between the optimum receivers obtained under the SNR and error-probability criteria.

In general, however, the detailed specification of the quadratic-form processor in the optimum decision-making receiver differs from that of the SNR-maximizing quadratic-form receiver, being implicit in the sense of involving the solution of an integral equation and possibly being relatively hard to implement as well. Likewise, the best fixed-energy transmissions under the two kinds of criteria ordinarily will differ, the one that optimizes performance under an error-probability criterion being at present exceptionally difficult to determine.

The purpose of the present study is to provide proof, based on quite general yet exact error-probability analysis, that radar and binary communication systems can safely be designed for spread channels according to the SNR criterion, even though best error performance is the actual goal, as long as "low energy-coherence" (LEC) conditions[†] prevail in the channel. This is significant in that, under these conditions, one can now have full confidence in using the relatively tractable mathematics of the SNR criterion, without having to face the worry so frequently met in trying to relate output SNR to error performance — that of the lack of knowledge of the output probability distributions.

Specifically, the first main result of the present study of a dual,[‡] spread-channel signaling system is that if, under LEC conditions, the quadratic-form, error-optimum signal processors that appear in the receiver are replaced by the SNR-maximizing generalized radiometers, very little increase in the transmission amplitude (keeping its waveshape unchanged) is needed to overcome the ensuing degradation in error performance.[§] It will be shown in Sec. III-A that the necessary increase is upper-bounded by a measure of the degree to which LEC conditions prevail, and that this upper bound does not depend at all on the values of the error probabilities or on whether a radar or a binary communication system is considered.

[†] Our LEC condition [made quantitative through (2.5)] involves, on the one hand, the incoherence produced by the time- and frequency-spreading of the fluctuating multipath, and on the other hand, the temporal behavior of the ratio of the average received signaling power to the noise spectral density. The more time- and/or frequency-spreading that there is, the more energy can be transmitted while LEC conditions still prevail; in fact, signals having infinite duration and energy can be sent over the channel without violating LEC, as shown by the bound (2.7). Consequently, LEC does not per se preclude high output SNR or low error probabilities. [See also the remarks preceding (2.19).]

[‡] This is a twofold iteration of the single-channel system considered in Refs. 1 and 2 and Appendix A, and is introduced in order to model either a radar system or equivalent on-off binary communication system, or a binary communication system of the more modern kind where energy is emitted with either of the symbols to be conveyed.

[§] To give an example in the context of radio astronomy, let us suppose that according to some error criterion, and with a white-noise background of 100°K, a spectral line of an arbitrary width and a peak density of 1°K is just detectable using a SNR-maximizing radiometer. Then by (2.8) and the receiving loss bound established in Sec. III-A, 0.99°K is a lower bound to the peak density of an identically shaped line that would be detectable with the optimum receiver, assuming the same observation interval for reception. The game hardly appears worth the candle, and there would be even less to be gained were the background noise to exceed 100°K.

That there is such a small effective difference in performance under LEC conditions is not surprising, considering that the error-optimum processing approaches the SNR-maximizing processing as the channel noise intensity becomes infinite (see Sec. II-B). In fact, Bello⁴ has already demonstrated this small performance difference, making the now unnecessary approximation that the processor outputs are Gaussianly distributed.

The second main result proceeds directly from Pierce's finding⁵ that the error probability for optimum reception of binary symmetric signaling over spread or diversity channels can be expressed as a real integral involving the system eigenvalues (see Sec. IV). For this situation only, it will be shown that the error probability of the error-optimum receiver can be bounded above and below by expressions involving the maximized output SNR that is attained when generalized radiometers are substituted for the quadratic-form processors appearing in the optimum receiver. Under LEC conditions these bounds are close in terms of decibels of transmission amplitude (in fact they are close in an absolute sense under extreme LEC conditions). By taking this result together with the low receiving loss established as the first main result, we conclude that one can safely proceed to design both transmission and reception on the basis of the SNR criterion, in a LEC-spread-channel, binary symmetric communication system. That is, we can be sure that the overall "design loss," defined in terms of an equivalent reduction in transmission amplitude attending the adoption of the SNR criterion, will be small.

B. General System Description

The system to be analyzed consists of a duplicate pair of the radar or on-off communication systems treated in Refs. 1 and 2, operating in reciprocal fashion over a pair of noisy, fluctuating multipath channels that are identical but statistically independent. Specifically, a single bit of information is sent by transmitting either a known narrow-band waveform $\text{Re} \{ \mathfrak{X}(t) e^{i\omega_0 t} \}$ over one channel and nothing over the other, or vice versa.[†] The transmitted waveform is converted by the fluctuating multipath of its associated channel into a (generally nonstationary) zero-mean, narrow-band Gaussian process $z(t)$ having the correlation function^{1,2}

$$\begin{aligned} \varphi_z(t, \tau) = \frac{1}{2} \text{Re} \left\{ \iint_{-\infty}^{\infty} \mathfrak{X}(t - \lambda) \mathfrak{X}^*(\tau - \lambda) \Phi_y(\omega, \lambda) \right. \\ \left. \times \exp [i(\omega + \omega_0)(t - \tau)] d\omega d\lambda \right\} \end{aligned} \quad (1.1)$$

where $\Phi_y(\omega, \lambda)$ is the scattering function of the fluctuating multipath. This (real, non-negative) function describes the power spectra (in ω) of the Gaussian fluctuations ("y" processes) that produce the frequency spreading and that occur with mutual independence at the various time-spreading multipath delays (in λ). Setting $\tau = t$ and integrating over all t in $(-\infty, \infty)$ to obtain the total average received signaling energy, we find

$$\overline{E_z} = \left[\frac{1}{2} \int_{-\infty}^{\infty} |\mathfrak{X}(t)|^2 dt \right] \left[\iint_{-\infty}^{\infty} \Phi_y(\omega, \lambda) d\omega d\lambda \right] \quad (1.2)$$

[†] The two channels might be carried by a single propagation medium if wide-deviation frequency-shift or time-shift keying were employed.⁶

so that the double integral of the scattering function is equal to the ratio of the total average received signaling energy to that transmitted.

Additive white Gaussian noise is injected into each channel, following the fluctuating multipath disturbance. The added noises are independent, and each is of spectral density N_0 , specified on the basis of a physical, single-sided spectrum measured in cycles/second.

In order to have a system model that will serve equally well either for communications employing balanced signaling of the type just described, or for radar or on-off signaling, we introduce the respective options of either making both channel outputs available to the receiver, or of allowing it to observe either channel output but not both. In the former or "communication" option, the receiver observes, at one channel output, a sample of the signal-plus-noise Gaussian process having correlation function $\varphi_z(t, \tau) + N_0 \delta(t - \tau)/2$ and, at the other channel output, a sample of a white-noise-only process whose correlation function is $N_0 \delta(t - \tau)/2$; the receiver is called upon to decide which channel output is which and hence to decide over which channel the transmission has been sent. In the radar mode the receiver decides to which of the above two Gaussian processes its observation belongs, and hence judges whether or not there has been a transmission over (or target in) the channel whose output it observes. This receiver description applies both to the optimum receiver now to be discussed and to the suboptimum receiver next considered.

II. OPTIMUM AND SUBOPTIMUM RECEPTION

A. Error-Optimum Reception and a Pair of Tests for Low, Largest Eigenvalue

Whatever observations may be available to it, the binary-choice receiver that achieves the best error performance, regardless of the details of the particular error-probability criterion adopted, is one that bases its decisions on the value of the likelihood ratio taken over all available observations, or on any monotonic function of this ratio such as the natural logarithm.^{7,8} For the radar option the logarithm of the likelihood ratio is that of the ratio of the probability measure of the single observation under the "transmitter on" hypothesis to its probability measure under the noise-only alternative. This logarithm can be the processing output of the optimum radar receiver, its value then determining the decision.

For the communication option the likelihood ratio is a similar ratio of probability measures — this time taken on the dual channel-output observation. By virtue of the assumed channel independence, this ratio factors into the product of a pair of likelihood ratios taken on the individual channel outputs. For either observed output, the hypothesis that there has been a transmission over its associated channel but not over the other, and the converse hypothesis, are equivalent to the radar hypotheses of transmitter "on" and "off," respectively. We therefore conclude, recognizing that "on" in one channel necessarily corresponds to "off" in the other and vice versa, that the optimum processor output for the communication receiver can be formed as the difference of the logarithm-likelihood outputs from a pair of optimum radar processors that operate individually on the channel outputs.

As a matter of mathematical convenience, and because, as earlier mentioned, we are free to choose any monotonic function of the likelihood ratio for the optimum processing output, we shall usually add an arbitrary constant to the logarithm of the likelihood ratio and shall multiply it by another (positive) constant as well, specifying the result to be the optimum processor output.

In a radar situation such shifting and scaling merely requires a compensatory resetting of the decision level, a detail which need not concern us for the purposes of the present study. Such modification is likewise permissible in the pair of optimum radar processors whose output difference has just been demonstrated to be the optimum communication processing output, as long as both radar processors undergo the same modification. In binary symmetric communication systems, in fact, where the decision level for the logarithm of the likelihood ratio would be set at zero, the decision level is obviously left unchanged by such addition or scaling.

Drawing on the Karhunen-Loève exposition of Refs. 9 and 10, it is found in Appendix A that we can write for the output of the optimum radar processor (modified as above) that operates on the signal $w_k(t)$ received over the k^{th} channel:

$$d_k^0 = \sum_{j=0}^{\infty} \frac{\tilde{\lambda}_j (w_{jk1}^2 + w_{jk2}^2)}{1 + \beta_j} \quad ; \quad k = 1, 2 \quad . \quad (2.1)$$

For radar, d_1^0 or d_2^0 is the optimum decision quantity; in communications it is $d_2^0 - d_1^0$. In (2.1),

$$\left. \begin{aligned} w_{jk1} &= \sqrt{2} \int_{-T/2}^{T/2} w_k(t) \operatorname{Re} \left\{ \tilde{\psi}_j(t) e^{i\omega_o t} \right\} dt \\ w_{jk2} &= \sqrt{2} \int_{-T/2}^{T/2} w_k(t) \operatorname{Re} \left\{ \tilde{\psi}_j(t) \exp[i\omega_o t + \frac{i\pi}{2}] \right\} dt \end{aligned} \right\} \quad (2.2)$$

where we assume without loss of generality that the observation interval is $(-T/2, T/2)$. Also,

$$\beta_j = \frac{2\tilde{\lambda}_j}{N_o} \quad (2.3)$$

and the $\tilde{\lambda}_j$ are equal to one-quarter the bounded, countable, non-negative eigenvalues associated with the orthonormal eigenfunctions $\tilde{\psi}_j(t)$ of the homogeneous linear integral equation

$$\begin{aligned} \int_{-T/2}^{T/2} \left[\iint_{-\infty}^{\infty} \tilde{\chi}(t-\lambda) \tilde{\chi}^*(\tau-\lambda) \Phi_y(\omega, \lambda) \exp[i\omega(t-\tau)] d\omega d\lambda \right] \\ \times \tilde{\psi}_j(\tau) d\tau = 4\tilde{\lambda}_j \tilde{\psi}_j(t) \quad ; \quad -\frac{T}{2} \leq t \leq \frac{T}{2} \quad . \end{aligned} \quad (2.4)$$

In (2.4) each eigenvalue is counted by the number of linearly independent eigenfunctions associated with it – frequently there will be just one.

The largest eigenvalue (making the quite unessential assumption that there is only one eigenfunction having this eigenvalue) of (2.4) is given the index $j = 0$. The noise-scaled largest eigenvalue $\beta_0 = 2\tilde{\lambda}_0/N_o$ is of prime interest to this study, for it will be shown in Secs. III and IV to set limits on the design loss associated with the adoption of the output-SNR criterion.

Although it is generally difficult to determine β_0 exactly, all that is really required for low design loss is that an upper bound to β_0 be small compared to unity. Such upper bounds are given in Refs. 2 and 4 in terms of $\varphi_z(t, \tau)$, the correlation function (1.1) of the channel-perturbed transmission (less noise). A key bound is

$$\beta_0 \leq \frac{2}{N_0} \max_{-T/2 \leq t \leq T/2} \int_{-T/2}^{T/2} \varphi_z(\tau, \tau) |\rho_z(t, \tau)| d\tau \quad (2.5)$$

Here $\rho_z(t, \tau) = \varphi_z(t, \tau) / \sqrt{\varphi_z(t, t) \varphi_z(\tau, \tau)}$ is a correlation coefficient whose magnitude, an index of the coherence of the (noiseless) channel-perturbed transmission, cannot exceed unity.

Observing that the right side of (2.5) involves a time-integrated interplay between the rate of arrival $\varphi_z(\tau, \tau)$ of average signaling energy and the signal coherence as measured by $|\rho_z(t, \tau)|$, we now define "low energy-coherence" (LEC) conditions to exist if and only if the right side of (2.5) is small compared to unity.[†] [Note: In Ref. 2, LEC is said to exist if either of a pair of bounds, one of which is that of (2.5), is small.]

Clearly, (2.5) involves detailed knowledge of $\mathfrak{X}(t)$ and $\Phi_y(\omega, \lambda)$ through (1.1); we now present another bound[‡] that requires much less information about the transmission and the channel. To obtain this new bound we multiply (2.4) on both sides by $\tilde{\psi}_j^*(t)$, set $j = 0$, and integrate on t from $-T/2$ to $T/2$. Using (A-13) and regrouping, we find

$$\begin{aligned} 4\tilde{\lambda}_0 &= \iint_{-\infty}^{\infty} \Phi_y(\omega, \lambda) d\omega d\lambda \left| \int_{-T/2}^{T/2} \tilde{\psi}_0(\tau) \mathfrak{X}^*(\tau - \lambda) e^{-i\omega\tau} d\tau \right|^2 \\ &\leq \int_{-\infty}^{\infty} d\lambda \max_{\omega} [\Phi_y(\omega, \lambda)] \int_{-T/2}^{T/2} \tilde{\psi}_0(\tau) \tilde{\psi}_0^*(t) \mathfrak{X}^*(\tau - \lambda) \\ &\quad \times \mathfrak{X}(t - \lambda) dt d\tau \int_{-\infty}^{\infty} \exp[i\omega(t - \tau)] d\omega \\ &= 2\pi \int_{-\infty}^{\infty} d\lambda \left[\max_{\omega} \Phi_y(\omega, \lambda) \right] \int_{-T/2}^{T/2} |\tilde{\psi}_0(\tau)|^2 |\mathfrak{X}(\tau - \lambda)|^2 d\tau \\ &\leq 2\pi \left[\max_t |\mathfrak{X}(t)|^2 \right] \int_{-\infty}^{\infty} \left[\max_{\omega} \Phi_y(\omega, \lambda) \right] d\lambda \int_{-T/2}^{T/2} |\tilde{\psi}_0(\tau)|^2 d\tau \end{aligned} \quad (2.6)$$

and again by (A-13), the second integral in (2.6) is unity. Therefore,

$$\beta_0 \frac{N_0}{2} = \tilde{\lambda}_0 \leq \frac{\pi}{2} \left[\max_t |\mathfrak{X}(t)|^2 \right] \int_{-\infty}^{\infty} \left[\max_{\omega} \Phi_y(\omega, \lambda) \right] d\lambda \quad (2.7)$$

and we see that if the channel frequency spreading is accompanied by a peak-power limitation on the transmitter, all the β_j may well remain below unity no matter how much transmitter energy is expended. When the transmission modulation $\mathfrak{X}(t)$ is a constant, we may precede the steps in (2.6) by integrating over all λ ; a tighter bound is then obtained in which $\max_{\omega}$ is moved outside the integral in (2.7). This special bound can be expressed simply in terms of the (physical, single-sided, cyclic frequency) spectral density $S(f)$ of the signal received in the absence of noise:

[†] Just as low energy-coherence guarantees $\beta_0 \ll 1$ through the inequality (2.5), one can in turn bound the right side of (2.5) by 4ν , if it is known that, for any t , νN_0 exceeds that fraction of the total average received signaling energy $\bar{E}_z$ lying at those τ -times for which $|\rho_z(t, \tau)| \geq \nu N_0 / \bar{E}_z$.

[‡] This bound has already been found by Bella,⁴ who assumes discrete rather than continuous multipath.

$$\tilde{\lambda}_0 \leq \max_f \frac{S(f)}{2} \quad (2.8)$$

This bound, which is well known, actually becomes an equality if the observation length T grows infinite.

If the observation interval is $(-\infty, \infty)$, a companion bound to (2.7) may be found by expressing $\tilde{\psi}_0(\tau)$ and $X^*(\tau - \lambda)$ in the first line of (2.6) in terms of their respective Fourier transforms $\Psi_0(\omega)$ and $X(\omega)$:

$$\begin{aligned} 4\tilde{\lambda}_0 &= \iint_{-\infty}^{\infty} \Phi_y(\omega, \lambda) d\omega d\lambda \left| (2\pi)^{-2} \iint_{-\infty}^{\infty} \Psi_0(\omega') X(\omega'') \right. \\ &\quad \times \exp[-i\omega\tau + i\omega'\tau + i\omega''(\tau - \lambda)] d\tau d\omega' d\omega'' \left. \right|^2 \\ &= \iint_{-\infty}^{\infty} \Phi_y(\omega, \lambda) d\omega d\lambda \left| (2\pi)^{-1} \int_{-\infty}^{\infty} \Psi_0(\omega - \omega'') X(\omega'') e^{-i\omega''\lambda} d\omega'' \right|^2. \end{aligned}$$

Then following a development paralleling that of (2.6) and using Parseval's relation:

$$\int_{-\infty}^{\infty} |\Psi_0(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |\tilde{\psi}_0(\tau)|^2 d\tau = 2\pi$$

we find

$$\tilde{\lambda}_0 \leq \frac{1}{4} \left[\max_{\omega} |X(\omega)|^2 \right] \int_{-\infty}^{\infty} \left[\max_{\lambda} \Phi_y(\omega, \lambda) \right] d\omega \quad (2.9)$$

Thus channel delay spreading together with a limit on the energy spectral density of the transmission can also act to prevent the β_j from exceeding unity.[†]

Also for infinite observation interval, we can obtain by proceeding from the first line of (2.6),

$$\frac{\beta_0}{2\pi} = \frac{\tilde{\lambda}_0}{\pi N_0} \leq \frac{\overline{E_z}}{N_0} \frac{\max_{\omega, \lambda} \Phi_y(\omega, \lambda)}{\iint_{-\infty}^{\infty} \Phi_y(\omega, \lambda) d\omega d\lambda} \quad (2.10)$$

where $\overline{E_z}$ is the total average received signaling energy. This bound has been given in Ref. 2 and shows an interplay between the average energy-to-noise-density ratio and the time- and frequency-spreading.

[†]The bound (2.9) is actually valid for any observation interval $(-T/2, T/2)$, since a Sturmian Separation Theorem¹¹ (called to the author's attention by the late Dr. M. J. Levin of Lincoln Laboratory) implies that the largest eigenvalue of (2.4) is a nondecreasing function of T . Viewed in terms of Bello's duality theory,¹² (2.9) is merely the Fourier dual of (2.7).

B. The Generally Suboptimum, SNR-Maximizing Processor

As pointed out in Appendix A, practical implementation of the processing (2.1)-(2.2) requires solving the generally difficult integral equation (A-10). We note, however, that as $N_0 \rightarrow \infty$ and therefore $\beta_0 \rightarrow 0$ in (2.1) (all the other $\beta_j \rightarrow 0$ as well, since β_0 is the largest), d_k^0 approaches the explicit limit:

$$d_k^S = \sum_{j=0}^{\infty} \tilde{\lambda}_j (w_{jk1}^2 + w_{jk2}^2) = \iint_{-T/2}^{T/2} w_k(t) w_k(\tau) \varphi_z(t, \tau) dt d\tau \quad ; \quad k = 1, 2 \quad (2.11)$$

This is obtained from (A-9)-(A-10) and the remark immediately following (A-10). Because the processor (2.11) involves the correlation function $\varphi_z(t, \tau)$ of the fluctuating-multipath-perturbed transmission quite explicitly, and in fact has been realized specifically as a generalized radiometer in Refs. 1 and 2, one is tempted to substitute it in lieu of the optimum radar processor wherever the latter is employed in the receiver. A loss in error performance is generally[†] the cost of this convenience, but one may expect the loss not to be severe when β_0 is small. A major purpose of this study is to confirm this expectation by placing an upper limit on this loss in terms of an equivalent transmitter output reduction, a limit that involves β_0 and approaches zero with β_0 .

The explicit processor (2.11) has the significant property that it achieves the maximum output SNR among all processors of the broad linear-quadratic class:

$$P[w(t)] = \iint_{-T/2}^{T/2} [aw(t) + f(t)] [bw(\tau) + g(\tau)] K(t, \tau) dt d\tau \quad (2.12)$$

where a , b , $f(t)$, $g(t)$, and $K(t, \tau)$ are quite arbitrary but not random. (This is so whether or not the signal received in the absence of noise is narrow band.) To show this, we make the expansions in the eigenfunctions of (A-2):

$$\left. \begin{aligned} K(t, \tau) &= \sum_{m,n=0}^{\infty} a_{mn} \psi_m(t) \psi_n(\tau) \\ f(t) &= \sum_{m=0}^{\infty} f_m \psi_m(t) \\ g(t) &= \sum_{m=0}^{\infty} g_m \psi_m(t) \end{aligned} \right\} -\frac{T}{2} \leq t, \tau \leq \frac{T}{2} \quad (2.13)$$

whereupon we have, using (A-1),

$$P[w(t)] = \sum_{m,n=0}^{\infty} a_{mn} (aw_m + f_m) (bw_n + g_n) \quad (2.14)$$

[†] Should there be just a finite number of β_j , all of equal value and hence equal to β_0 , comparison of (2.11) and (2.1) reveals that the processor (2.11) is in fact optimum. In general, however, (2.11) is suboptimum.

with the w_m independent, zero-mean Gaussian variates derived from $w(t)$. The mean value of $P[w(t)]$ is seen to be $ab \sum_{m=0}^{\infty} a_{mm} \overline{w_m^2} + \sum_{m,n=0}^{\infty} a_{mn} f_m g_n$, so that by (A-6) the change in the mean output that occurs when a transmission is sent over the channel is equal to $ab \sum_{m=0}^{\infty} a_{mm} \lambda_m$, where the λ_m are the eigenvalues of (A-2), which are shown in Appendix A to occur in equal pairs for narrow-band situations. It is straightforward to find the variance of $P[w(t)]$, using the fact that the fourth moment of a zero-mean Gaussian variate is three times the square of the second moment. Again with the help of (A-6), we find that, when the received signal $w(t)$ is noise alone,

$$\begin{aligned}
\overline{P[w(t)]^2} - (\overline{P[w(t)]})^2 &= \left(\frac{abN_o}{2}\right)^2 \sum_{m,n=0}^{\infty} a_{mn}(a_{mn} + a_{nm}) \\
&\quad + \frac{N_o}{2} \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} aa_{mn}g_n + ba_{nm}f_n \right]^2 \\
&\leq \left(\frac{abN_o}{2}\right)^2 \sum_{m,n=0}^{\infty} a_{mn}(a_{mn} + a_{nm}) \\
&= \frac{(abN_o)^2}{8} \sum_{m,n=0}^{\infty} (a_{mn} + a_{nm})^2 \\
&\leq \frac{(abN_o)^2}{2} \sum_{m=0}^{\infty} a_{mm}^2. \tag{2.15}
\end{aligned}$$

Thus the output signal-to-noise ratio, defined as the square of the mean output change divided by the variance for noise alone, is found to have the upper bound:

$$R_P \leq \frac{2 \left[\sum_{m=0}^{\infty} a_{mm} \lambda_m \right]^2}{N_o^2 \sum_{m=0}^{\infty} a_{mm}^2}. \tag{2.16}$$

By the Schwarz inequality the upper bound of (2.16) is in turn upper-bounded by $(2/N_o^2) \sum_{m=0}^{\infty} \lambda_m^2$, but reference to (2.15) shows that this bound can actually be attained by R_P if we choose $a_{mm} = \lambda_m$, $a_{mn} = 0$ for $m \neq n$, $f_m = 0 = g_m$ for all m , and $a = 1 = b$. Therefore, by (2.14), maximum output SNR is achieved with the processor that forms $\sum_{m=0}^{\infty} \lambda_m w_m^2$, which is recognized as the same sum as in (2.11) when $w_k(t)$ is substituted for $w(t)$ and narrow-band conditions are invoked to give pairing of the eigenvalues.

Having established the maximum-SNR property of the processor (2.11), we note yet a further upper bound to its output SNR:

$$R = \frac{2}{N_o^2} \sum_{m=0}^{\infty} \lambda_m^2 \leq \frac{2}{N_o^2} \left[\sum_{m=0}^{\infty} \lambda_m \right]^2 = \frac{2}{N_o^2} \left[\int_{-T/2}^{T/2} \varphi_z(t, t) dt \right]^2 \leq 2 \left(\frac{\overline{E_z}}{N_o} \right)^2. \tag{2.17}$$

Here we have used the Mercer expansion:⁹

$$\varphi_z(t, \tau) = \sum_{m=0}^{\infty} \lambda_m \psi_m(t) \psi_m(\tau) \quad ; \quad -\frac{T}{2} \leq t, \tau \leq \frac{T}{2} \quad (2.18)$$

and, setting $t = \tau$, have integrated both sides of (2.18) over $(-T/2, T/2)$ with the help of (A-3). For the narrow-band systems actually under consideration, the fact that the eigenvalues appear in pairs means that the quantity on the right of the first inequality in (2.17) can be halved, giving a tighter bound.

Thus we see from (2.17) that a good output SNR requires, not unexpectedly, a good ratio of total average received signaling energy to noise density. However, we have also seen by (2.10) how the latter condition by no means need imply that (in narrow-band situations) $\beta_0 = 2\tilde{\lambda}_0/N_0 = 2\lambda_0/N_0$ be large. Hence, as Bello⁴ has observed, we can certainly have weak-signal situations, as for example are practically the rule in radio astronomy, where β_0 is quite small while the output SNR for a radiometer detector is high. To test situations in which it is suspected that β_0 may in fact be large, one can use the simple lower bound $\lambda_0 \sum_{m=0}^{\infty} \lambda_m \geq \sum_{m=0}^{\infty} \lambda_m^2$, leading with the aid of (2.17) to

$$\beta_0 = \frac{2\lambda_0}{N_0} \geq \frac{R}{(\overline{E_z}/N_0)} \quad . \quad (2.19)$$

(This can be tightened, if desired, by replacing $\overline{E_z}$ by the average signaling energy received just in the observation interval, rather than over all time.)

Aside from its use in (2.19), why are we interested in the value of the maximized output SNR? — especially since some doubt is cast on its usefulness as a criterion by the fact[†] that the error-optimum detector (2.1) or (A-9) must generally have a lower SNR than the suboptimum detector (2.11). The answer is that for binary symmetric communication over the dual channel and error-optimum reception, it is possible to obtain (as will be shown in Sec. IV) bounds on the error probability in terms of this maximum SNR R — bounds that are close in terms of transmission amplitude when β_0 is small. Furthermore, through the bound established in Sec. III on the effective transmission-amplitude reduction associated with the use of SNR-maximizing processors rather than ones that are error-optimum, we can also bound the error probability of the suboptimum SNR-maximizing detector in terms of its output SNR. Whereas exact error probabilities are difficult to determine for either type of reception, the maximum attainable SNR is relatively easy to evaluate; squaring (2.18), integrating t and τ over $(-T/2, T/2)$, and using (A-3), we find (remembering that in narrow-band situations there are a pair of equal λ_m for each β_m)

$$R = \frac{2}{N_0} \sum_{m=0}^{\infty} \lambda_m^2 = \sum_{m=0}^{\infty} \beta_m^2 = \frac{2}{N_0} \iint_{-T/2}^{T/2} \varphi_z^2(t, \tau) dt d\tau \quad . \quad (2.20)$$

[†] As a point of interest, Rudnick¹³ has found that with a definition of output SNR that differs from the one given above, maximum SNR is achieved among a broad class of processors (not just mixed linear-quadratic) by one whose output is a fairly simple function of (2.1), the error-optimum output. (Sebestyen¹⁴ actually noted this result earlier, attributing it to R. Hines.)

C. Derivation of Error Probabilities for Optimum Reception

In order to determine error probabilities for reception that uses the optimum processor (2.1), we need to know the statistics of the observables w_{jk1} and w_{jk2} that appear in (2.1) and (2.2). It is established in Appendix A that the w_{jk1} , w_{jk2} , $j = 0, 1, 2, \dots$, $k = 1$ or 2 , are mutually independent, zero-mean Gaussian variates having variance $a_k \tilde{\chi}_j + (N_o/2)$, where a_k is unity or zero depending on whether or not there has been a transmission over the k^{th} channel. Those variates for $k = 1$ are also independent of those for $k = 2$ by virtue of the assumed independence between the channel disturbances.

Multiplying by $4/N_o^2$, we obtain from (2.1),

$$\frac{4}{N_o^2} d_k^o = \sum_{j=0}^{\infty} \frac{\beta_j (1 + a_k \beta_j) v_{jk}}{(1 + \beta_j)} ; \quad k = 1, 2 \quad (2.21)$$

where the v_{jk} are mutually independent for all j, k , and each is the sum of a pair of independent, squared Gaussian variates of zero mean and unit variance. Each v_{jk} is thus distributed as a central chi-square variate of two degrees of freedom, and they therefore all have the probability density

$$p(v) = \begin{cases} (\frac{1}{2}) e^{-v/2} & ; \quad v \geq 0 \\ 0 & ; \quad v < 0 \end{cases} \quad (2.22)$$

and the characteristic function

$$\overline{e^{i\mu v}} = (1 - 2i\mu)^{-1} ; \quad i = \sqrt{-1} \quad (2.23)$$

In radar we deal only with d_k^o , $k = 1$ or 2 , and announce that there has been a transmission over the k^{th} channel if d_k^o exceeds some decision level $DN_o^2/4$; otherwise, we decide that there has been no transmission in that channel. Here we are interested in the probabilities of the two types of decision error that can occur — P_M^o , the probability that the optimum detector misses the transmission because d_k^o chances not to exceed $DN_o^2/4$ even though $a_k = 1$, and P_F^o , the probability that there is a false alarm because d_k^o happens to exceed $DN_o^2/4$ even though $a_k = 0$.

In dual-channel communications as earlier described, an error is committed if $(d_2^o - d_1^o)$ fails to exceed the decision level $DN_o^2/4$ when $a_2 = 1$, $a_1 = 0$, or if $(d_2^o - d_1^o)$ exceeds $DN_o^2/4$ when $a_2 = 0$, $a_1 = 1$. Since the v_{jk} of (2.21) are statistically identical as well as mutually independent for all j and k , a simple argument involving an interchange of the $k = 1$ and $k = 2$ indices shows that the probability for the latter type of communication error is the same as that for $(d_2^o - d_1^o)$ being less than $-DN_o^2/4$ with $a_2 = 1$, $a_1 = 0$. In particular, the two communication error probabilities are equal when $D = 0$, which is the setting used for binary symmetric communications. [There is zero probability that $(d_2^o - d_1^o)$ is exactly equal to the decision level.] Therefore, in the communication situation we need consider only the $a_2 = 1$, $a_1 = 0$ case, but must deal with all real values of D .

In terms of its characteristic function $\overline{e^{i\mu d}}$, the probability that the decision quantity d [which is d_k^o for radar, $(d_2^o - d_1^o)$ for communications] fails to exceed $DN_o^2/4$ is the same as the probability that it is less than $DN_o^2/4$, and is given by the contour integral

$$P\left(d \leq \frac{DN_o^2}{4}\right) = P\left(d < \frac{DN_o^2}{4}\right) = (2\pi)^{-1} \int_C (-i\mu)^{-1} e^{-i\mu D} \overline{\exp\left[\frac{i4\mu d}{N_o^2}\right]} d\mu \quad (2.24)$$

Here C is a line parallel to the real axis and displaced from it by any positive imaginary amount that is less than the smallest positive imaginary coordinate of any singularity of $\exp[i4\mu d/N_o^2]$, integration being in the direction of increasing real coordinate. In all cases, $4d/N_o^2$ is, by (2.21), a weighted sum of the independent variates v_{jk} , so that its characteristic function is the product of the characteristic functions of the individual terms of the sum. The individual characteristic functions are given by (2.23) with $c_{jk}\mu$ substituted for μ , where c_{jk} is the weight associated with v_{jk} .

Referring to (2.21) with $a_k = 1$ and using (2.24), we find for the radar situation,

$$P_M^o = (2\pi)^{-1} \int_{C_1} \frac{e^{-i\mu D} d\mu}{(-i\mu) \prod_{j=0}^{\infty} (1 - 2i\beta_j \mu)} \quad (2.25)$$

Similarly, with $a_k = 0$,

$$1 - P_F^o = (2\pi)^{-1} \int_{C_2} \frac{e^{-i\mu D} d\mu}{(-i\mu) \prod_{j=0}^{\infty} \left[1 - \frac{2i\beta_j \mu}{1 + \beta_j}\right]} \quad (2.26)$$

The contours C_1 and C_2 can be any lines paralleling the real axis and having a positive imaginary coordinate.

A closer relationship between (2.25) and (2.26) can be established by choosing C_1 to have an imaginary coordinate that is larger by $i/2$ than that of C_2 . Then making the substitution $\mu = \mu' - i/2$ and dropping the prime, we have

$$1 - P_F^o = (2\pi)^{-1} e^{-D/2} \left[\prod_{j=0}^{\infty} (1 + \beta_j) \right] \int_{C_1} \frac{e^{-i\mu D} d\mu}{(-i\mu - 1/2) \prod_{j=0}^{\infty} (1 - 2i\beta_j \mu)} \quad (2.27)$$

The similarity of the contour integrals in (2.27) and (2.25) is a feature peculiar to optimum processing. By differentiating (2.25) and (2.27) with respect to D , the ratio of the probability density of d_k^o for $a_k = 0$ to that for $a_k = 1$, evaluated at $d_k^o = DN_o^2/4$, is found to be just $e^{-D/2} \prod_{j=0}^{\infty} (1 + \beta_j)$. This is a result of d_k^o being monotonically related to the likelihood ratio and hence being a sufficient statistic, so that it is a measure of the likelihood ratio taken on its own probability distributions under the two hypotheses.^{15,16} As we shall see in Appendix B, this feature of optimum processing can be quite useful.

For the communication situation the output from the optimum processing is $d = (d_2^o - d_1^o)$, and from (2.21) and (2.23) we find for $a_2 = 1$, $a_1 = 0$ that the characteristic function of $4d/N_o^2$ is:

$$\overline{\exp\left[\frac{i4\mu d}{N_o^2}\right]} = \prod_{j=0}^{\infty} (1 - 2i\beta_j \mu)^{-1} \left[1 + \frac{2i\beta_j \mu}{(1 + \beta_j)}\right]^{-1} \quad (2.28)$$

Substituting (2.28) in (2.24), letting $\mu = (i + 2\mu')/4$ as suggested by Pierce,⁵ with μ' running along the real axis as a permissible contour [the smallest positive imaginary coordinate of a singularity

being $i(1 + \beta_0)/(2\beta_0)$, and then dropping the prime, we have for the probability of error at the decision-level setting $DN_0^2/4$,

$$P_e^O(D) = (2\pi)^{-1} e^{D/4} \left[\prod_{j=0}^{\infty} (1 + \beta_j) \right] \int_{-\infty}^{\infty} \frac{\exp[-i\mu D/2] d\mu}{(\frac{1}{2} - i\mu) \prod_{j=0}^{\infty} [(1 + \beta_j/2)^2 + (\beta_j\mu)^2]} \quad (2.29)$$

When $D = 0$, the integral in (2.29) becomes purely real with limits $(0, \infty)$; this is Pierce's result,⁵ which again is a consequence of the processing optimality, and which will receive further attention in Sec. IV and Appendix B.

III. AN UPPER BOUND TO EQUIVALENT LOSS IN TRANSMITTED ENERGY ATTRIBUTABLE TO THE USE OF SUBOPTIMUM, SNR-MAXIMIZING PROCESSORS

A. Development of the Bound

In Sec. II-B we introduced the SNR-maximizing processor (2.11). This processor has the merit of being quite explicit in its specification, but suffers from generally being suboptimal. When it is used in a radar receiver, for example, the false-alarm probability P_F^S for a given miss probability P_M^S will in general be higher (and certainly will never be lower) than that for the optimum detector operating with the same miss probability ($P_M^S = P_M^O$). (It follows that the same statement must be true with the false alarm and miss probabilities interchanged.)

Similarly, in the communication situation, let us suppose that with either the optimum or the suboptimum processing there is the same probability of mistaking "off" in the first channel and "on" in the second ($a_1 = 0, a_2 = 1$) for the reverse transmission ($a_1 = 1, a_2 = 0$). This is accomplished by adjusting the respective decision levels $D_O N_0^2/4$ and $D_S N_0^2/4$ so that the error probabilities $P_e^O(D_O)$ and $P_e^S(D_S)$ are equal. Then with the suboptimum processing the probability of mistaking "on" in the first channel and "off" in the second for its converse is generally greater (and is never less) than that with optimum processing [i.e., $P_e^S(-D_S) \geq P_e^O(-D_O)$, the k -interchange argument used earlier on $(d_2^O - d_1^O)$ also being valid for the suboptimum detection]. It follows that in binary symmetric signaling with $D_O = 0 = D_S$, the (now single) error probability for the suboptimum processing generally exceeds that for the optimum processing [$P_e^S(0) \geq P_e^O(0)$].

The statements made in the preceding two paragraphs are a direct consequence of what is meant by error-optimum processing, to which statistical detection (or decision) theory is addressed,^{7,8} and will not be proven here.

In assessing the loss in system performance that occurs in favoring the suboptimum, SNR-maximizing processing for its relative engineering convenience, a measure that is reasonable from an engineering standpoint is the decrease in transmission amplitude that will degrade the error performance for optimum reception to that obtained with the suboptimum processing and the original amplitude. We restrict our attention solely to changes in transmission amplitude because, in general, the $\{\beta_j\}$ that specify the error probabilities through (2.25), (2.27), and (2.29) depend in complicated ways [see (2.3)-(2.4)] on the transmission and channel scattering function. In shrinking the amplitude, we at least know by (2.4) that the $\{\beta_j\}$ will all be reduced by the same factor as is the square of the average amplitude, or the transmitted energy. This is essentially a conservative policy; moreover, there may well be restrictions on the bandwidth of the transmission or other waveshape limitations that must be observed – it is likely that such constraints will still be satisfied in the amplitude reduction.

From the earlier observation, made in connection with (2.11), that the suboptimum processing approaches optimality as β_0 , the largest of the β_j , approaches zero, we may expect the loss in system performance caused by suboptimum processing to decrease with β_0 . In agreement with this expectation, we shall now prove that the loss never exceeds $10 \log_{10} (1 + \beta_0)$ decibels in equivalent transmission reduction, either for radar or binary communications. Because of the complicated way in which the error probabilities depend on the $\{\beta_j\}$, one cannot in general hope for more than an upper bound to the loss; even if the loss could be assessed precisely, it would depend in detail on all the $\{\beta_j\}$, and would no doubt be a very unwieldy expression.

To obtain the above bound, let us imagine a new, "clairvoyant" receiver containing a pair of processors that, like (2.1) and (2.11), have for their outputs sums involving the squared observables w_{jk1} , w_{jk2} ,

$$d_k^c = \sum_{j=0}^{\infty} \frac{\lambda_j (1 + \beta_j) (w_{jk1}^2 + w_{jk2}^2)}{1 + \beta_j [1 + \beta_j + a_k (\beta_0 - \beta_j)] / (1 + \beta_0)} ; \quad k = 1, 2 \quad (3.1)$$

where these observables are obtained from the received signals $w_1(t)$, $w_2(t)$ through (2.2). This receiver is termed clairvoyant in that, as indicated in (3.1), it is assumed to know the values of a_1 and a_2 and hence to know what transmission has taken place. However, this information is not used directly in making the decision; rather, decision is based on the values of d_1^c and d_2^c in the same way that it is for the optimum and the suboptimum, SNR-maximizing processor outputs.

Referring to (2.11), we see that when $a_k = 1$, $d_k^c = d_k^s$, and that when $a_k = 0$, $d_k^c \geq d_k^s$, for any received signal $w_k(t)$ that is simultaneously supplied to the clairvoyant and suboptimum receivers. In radar, therefore, both receivers will have the same miss probability for the same decision-level setting, whereas the clairvoyant receiver will have a false-alarm probability that is no less than that of the suboptimum receiver.

By the same token, when $a_2 = 1$, $a_1 = 0$ in binary communications, the deciding difference ($d_2^c - d_1^c$) for clairvoyant reception is never larger, for given $w_1(t)$, $w_2(t)$, than that for the suboptimum processing ($d_2^s - d_1^s$). Thus for any given common setting of the decision level, positive or negative, the suboptimum receiver has no higher an error probability than the clairvoyant. We conclude that both for radar and for binary communications, the suboptimum receiver cannot be outperformed by the clairvoyant receiver, and we turn our attention to the performance of the latter.

Recalling that the sum $(w_{jk1}^2 + w_{jk2}^2)$ can be written $(a_k \lambda_j + N_o/2) v_{jk}$, with the $\{v_{jk}\}$ as in (2.21), and multiplying (3.1) by $4/N_o^2$, we find upon examining the $a_k = 1$ and $a_k = 0$ cases separately that (3.1) is identical to (2.21), except for having $\beta_j(1 + \beta_j)/(1 + \beta_0)$ substituted for β_j in (2.21) and the result multiplied by $(1 + \beta_0)$. Thus the error probabilities for the clairvoyant receiver are identical to those obtained with optimum reception in a new system having, in lieu of the noise-scaled eigenvalues $\{\beta_j\}$, the reduced eigenvalues $\{\beta_j(1 + \beta_j)/(1 + \beta_0)\}$, and having the decision level set at $(1 + \beta_0)$ times that of the clairvoyant receiver in the original system.

We have thus shown, by means of the clairvoyant receiver as a "bridge," that for given transmission and channels the suboptimum receiver is no worse in error performance than is optimum reception in a new system whose eigenvalues (signaling energies in the observables $\{w_{jk1}\}$, $\{w_{jk2}\}$) are reduced.

Two gaps remain in the development of the loss bound. First, the eigenvalues are in general not uniformly reduced in the new system relative to the original, since the ratio $(1 + \beta_j)/(1 + \beta_0)$ varies unless all the $\{\beta_j\}$ are equal. As explained earlier, in seeking a new, optimum-processing system that is outperformed by the suboptimum processing in the original system, we want the eigenvalues of the new system to be uniformly reduced so that the new transmission will simply be the original one reduced in amplitude. We therefore introduce yet another new system in which the $\{\beta_j\}$ of the original system are even further reduced,[†] but now uniformly, to $\{\beta_j/(1 + \beta_0)\}$, and next need to prove that the error performance of optimum reception in this latest system is no better than that of optimum reception in the system having the $\{\beta_j(1 + \beta_j)/(1 + \beta_0)\}$. The proof is left to Appendix B, where the sensible result[‡] is obtained that for optimum reception the error performance is never improved by decreasing any one of the β_j when the other $\{\beta_j\}$ are arbitrary but held fixed. This proof establishes that in a given spread-channel radar or binary communication system of the kind considered in this report, the receiver error performance for the suboptimum, SNR-maximizing processing is at least as good as that for optimum reception, provided that the waveform transmitted to the optimum receiver is $(1 + \beta_0)^{-1/2}$ times that sent to the suboptimum receiver.

With optimum processing and the original amplitude, the error performance can be no worse than with suboptimum processing; thus the error performance of the SNR-maximizing receiver is bracketed between that for optimum processing with the original, full amplitude and that for optimum processing after the $(1 + \beta_0)^{-1/2}$ amplitude reduction. Rather than bounding the suboptimum error performance, however, we want to set limits on the effective transmission loss that attends SNR-maximizing reception at a given level of error performance. This second of the two gaps previously mentioned is closed again with the help of Appendix B, where it is shown that just as optimum-reception error performance is never improved by decreasing the $\{\beta_j\}$, it is likewise never worsened by increasing them. Remembering that the $\{\beta_j\}$ are proportional to the square of transmission amplitude, and assuming that there is some ϵ for which the optimum receiver, operating with the transmission amplitude reduced by the factor $\epsilon^{-1/2}$, has the same error performance as that for the suboptimum processing with the full amplitude, we can thereby demonstrate that ϵ must lie in the range $[1, (1 + \beta_0)^{-1}]$. The loss attending SNR-maximizing processing therefore lies between zero and $10 \log_{10} (1 + \beta_0)$ decibels.

B. Test of the Bound in Two Binary Communication Systems

The argument just concluded has established, for any spread-channel radar or binary communication system of the type considered, that $10 \log_{10} (1 + \beta_0)$ is an upper bound to the maximum decibel loss in transmitted amplitude that is suffered in effect when the suboptimum processing (2.11) is used in lieu of the generally more difficult optimum processing (2.1). A key part of the argument involved the introduction of a clairvoyant receiver whose response to any observable

[†] Another choice is to reduce the $\{\beta_j(1 + \beta_j)/(1 + \beta_0)\}$ to $\{\beta_j(\beta_{\min}/\beta_0)\}$, where $\beta_{\min}$ is the minimum β_j . This leads to $10 \log_{10}(\beta_0/\beta_{\min})$ decibels as an additional upper bound on the maximum equivalent transmission reduction associated with suboptimum, SNR-maximizing processing. For spread channels the greatest lower bound of the $\{\beta_j\}$ is generally zero, so that this result is of limited utility.

[‡] With the suboptimum, SNR-maximizing processing, however, decreasing an eigenvalue can actually improve the receiver error performance — see the discussion following (3.11).

w_{jkl} of the received signals is individually compared with that of the suboptimum receiver. Such logic completely ignores the relative importance of the various observables in contributing to overall receiver performance; we may therefore expect the cited upper bound to be rather conservative.

In order to see how loose this bound may be in particular cases, and hopefully to aid future efforts to obtain a tighter but still universal bound that involves only the largest noise-scaled eigenvalue β_0 , we have analyzed two particular communication systems. Both systems are binary symmetric, the decision level being set at zero. System 1 has just two pairs of noise-scaled eigenvalues, of values β_0 and $\beta_1 = \gamma\beta_0$, $0 \leq \gamma \leq 1$, and is more typical of discrete diversity communication^{17,18} than of communication over spread channels, since the latter generally involves a countably infinite set of eigenvalues.

System 2 better fits a spread-channel situation, but is somewhat nonphysical in that it is the limit, as $N \rightarrow \infty$, of a system having a finite number N of pairs of noise-scaled eigenvalues, the largest pair having value β_0 and the remainder all being equal to $\sqrt{\rho/N}$. In the limit, the remainder eigenvalues are all properly smaller than β_0 , no matter what the system parameters β_0 and ρ may be. On the other hand, the total average signaling energy becomes infinite, being proportional to $\beta_0 + (N-1)\sqrt{\rho/N}$; this does not upset the validity of the present analysis, however.[†]

In either of these communication systems, the receiver bases its decision on the sign of the quantity

$$d_\alpha = \left[\tilde{\lambda}_0(w_{021}^2 + w_{022}^2) + \alpha \tilde{\lambda}_1 \sum_{j=1}^{N-1} (w_{j21}^2 + w_{j22}^2) \right] - \left[\tilde{\lambda}_0(w_{011}^2 + w_{012}^2) + \alpha \tilde{\lambda}_1 \sum_{j=1}^{N-1} (w_{j11}^2 + w_{j12}^2) \right] \quad (3.2)$$

where $\tilde{\lambda}_0 = \beta_0 N_0/2$ in both systems, $\tilde{\lambda}_1 = \gamma\beta_0 N_0/2$ and $N = 2$ in System 1, and $\tilde{\lambda}_1 = (N_0/2)\sqrt{\rho/N}$ while $N \rightarrow \infty$ in System 2. The first bracketed term in (3.2) is supplied by the processor operating from the second channel, and the second is generated by the other processor. Adjustment of the processing parameter α permits us to obtain the suboptimum processing (2.11) when $\alpha = 1$, and (except for an irrelevant gain factor) the optimum processing when $\alpha = (1 + \beta_0)/(1 + 2\tilde{\lambda}_1/N_0)$. In the same fashion as (2.29) was obtained from (2.24), infinite integrals can be obtained for the respective error probabilities ${}_Y P_e(\alpha, \beta_0)$ and ${}_P P_e(\alpha, \beta_0)$ for Systems 1 and 2, with reception as in (3.2). Evaluation of the integrals, where a Gaussian limit is involved in the second system for $N \rightarrow \infty$, yields the rather complicated expressions

[†] System 2 falls in a class considered by Hájek^{19,20} and Middleton,²¹ where the expression (2.1) for the optimum processing fails to converge, and yet the likelihood ratio (A-8) is finite positive. Here an optimum receiver exists in principle, but not in the form (2.1), and it has nonzero probability of error [see (3.4)]. Like (2.1), the expression (2.11) for the suboptimum processing also fails to converge; so does (3.2), which includes both (2.1) and (2.11). Therefore the results to be obtained for System 2 should not be viewed as actually obtainable, but can be approached in a physical system as closely as one may desire.

$$\gamma P_e(\alpha, \beta_0) = \frac{(1 + \beta_0)(2 + \gamma\beta_0) + (\alpha\gamma)^2(1 + \gamma\beta_0)(2 + \beta_0) + (\alpha\gamma)(2 + \beta_0)(2 + \gamma\beta_0)}{(2 + \beta_0)(2 + \gamma\beta_0)(1 + \alpha\gamma + \beta_0)(1 + \alpha\gamma + \alpha\gamma^2\beta_0)} \quad (3.3)$$

and

$$\begin{aligned} \rho P_e(\alpha, \beta_0) &= \frac{1}{2} [1 - \operatorname{erf}(\sqrt{\rho/4})] - \frac{(1 + \beta_0)}{2(2 + \beta_0)} \exp \left[\rho\alpha \frac{(\alpha + \beta_0 + \beta_0^2)}{(\beta_0 + \beta_0^2)^2} \right] \\ &\times \{1 - \operatorname{erf}[\sqrt{\rho/4} (2\alpha + \beta_0 + \beta_0^2) (\beta_0 + \beta_0^2)^{-1}]\} + \frac{\exp[\rho\alpha\beta_0^{-2}(\alpha - \beta_0)]}{2(2 + \beta_0)} \\ &\times \{1 - \operatorname{erf}[\sqrt{\rho/4} \beta_0^{-1}(2\alpha - \beta_0)]\} \end{aligned} \quad (3.4)$$

where

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad . \quad (3.5)$$

For the suboptimum processing, $\alpha = 1$, and for optimum processing, yielding the minimum $\gamma P_e(\alpha, \beta_0)$ or $\rho P_e(\alpha, \beta_0)$, $\alpha = (1 + \beta_0)/(1 + \gamma\beta_0)$ in System 1 and $\alpha = (1 + \beta_0)$ in System 2.

To learn the loss due to suboptimum processing in System 1 at given values of β_0 and γ , one sets $\alpha = 1$ in (3.3), notes the error probability, and then sets $\alpha = (1 + \beta_0)/(1 + \gamma\beta_0)$, substitutes $\epsilon_r\beta_0$ for β_0 , and finds the value of ϵ_r that yields the same error probability. The effective transmission amplitude loss attending the suboptimum processing is then $10 \log_{10} \epsilon_r^{-1}$. Likewise, one determines the loss in System 2 at given β_0 and ρ by determining $\rho P_e(1, \beta_0)$ from (3.4), substituting $\epsilon_r\beta_0$ and $\epsilon_r^2\rho$ for β_0 and ρ , respectively, in (3.4) and finding for what ϵ_r the error probability $\rho P_e(1 + \beta_0, \beta_0)$ is the same. (The substitution of $\epsilon_r^2\rho$ for ρ is based on the fact that ρ is proportional to the squared value of the remainder eigenvalues; also, $\alpha = 1 + \beta_0$ for optimum reception in System 2, since these eigenvalues, while not zero, are individually negligible.)

Although exact solution for ϵ_r is a matter of trial and error, some insight can be had by examining the behavior of the error probability for β_0 near zero and for β_0 asymptotically large. After expanding (3.3) about $\beta_0 = 0$ to terms of order β_0^3 , we find that the loss for System 1 behaves as

$$\epsilon_r^{-1} \xrightarrow{\beta_0 \rightarrow 0} 1 + \beta_0^2 \left(\frac{1 - \gamma}{1 + \gamma} \right)^2 (1 + \gamma^{-1} + \gamma^{-2})^{-1} \quad . \quad (3.6)$$

The coefficient multiplying β_0^2 has maximum value (in the range $0 \leq \gamma \leq 1$) equal to 0.0193 at $\gamma = 0.35$. At the other extreme, if we let β_0 increase so that both $\beta_0 \gg 1$ and $\gamma\beta_0 \gg 1$, still with $0 \leq \gamma \leq 1$, we find

$$\epsilon_r^{-1} \xrightarrow{\beta_0 \rightarrow \infty} \sqrt{(\gamma/3) (1 + \gamma + \gamma^2) (\gamma^2 + \beta_0^{-1})^{-1}} \quad (3.7)$$

and the value of γ that maximizes ϵ_r^{-1} for a given large β_0 is, asymptotically, $\gamma = \beta_0^{-1/2}$. With γ thus set, we find

$$\epsilon_r^{-1} \xrightarrow{\beta_0 \rightarrow \infty} \left(\frac{\sqrt{\beta_0}}{6} \right)^{1/2} . \quad (3.8)$$

Equations (3.6) and (3.8) show that the bound $(1 + \beta_0)$ that has been established for ϵ_r^{-1} grows progressively more conservative as $\beta_0 \rightarrow 0$ or $\beta_0 \rightarrow \infty$, at least for this particular communication system. The ratio $\beta_0/(\epsilon_r^{-1} - 1)$, which is a measure of the tightness of the bound, has a minimum value of about 68, obtained for β_0 around 5 to 6 and γ about 0.2.

An incidental result is

$$\epsilon_r^{-1} \xrightarrow{\gamma \rightarrow 1} 1 + \beta_0^2 \left[\frac{(\gamma - 1)^2}{6(2 + \beta_0)(1 + \beta_0)} \right] . \quad (3.9)$$

Equations (3.6) and (3.9) suggest that $(1 + \beta_0^2)$ may bound ϵ_r^{-1} when $\beta_0 \leq 1$ and $0 \leq \gamma \leq 1$. Numerical checks made at selected points in this (β_0, γ) region support this conjecture although no proof is yet available for System 1, let alone for all radar and communication systems.

For System 2, expansion of (3.4) about $\beta_0 = 0$ to terms of order β_0^3 and $\rho^{3/2}$ shows that

$$\epsilon_r^{-1} \xrightarrow{\beta_0 \rightarrow 0} 1 + \beta_0^2 \eta^4 \left[\frac{(2 + \eta^{-2})(1 - \operatorname{erf} \eta) - \eta^{-1}(2/\sqrt{\pi}) e^{-\eta^2}}{(1 - \operatorname{erf} \eta) + \eta(2/\sqrt{\pi}) e^{-\eta^2}} \right] \quad (3.10)$$

where $\eta = \sqrt{\rho}/\beta_0$. The coefficient multiplying β_0^2 in (3.10) has a maximum value of 0.0995, which is achieved at $\eta = 0.94$. This compares with a maximum coefficient of 0.0193 for System 1.

If $\eta = \sqrt{\rho}/\beta_0$ is held constant and β_0 is made large, we find, by studying the exponential error compression that takes place in (3.4), that

$$\epsilon_r^{-1} \xrightarrow{\beta_0 \rightarrow \infty} \frac{\sqrt{\beta_0}}{2} \quad (3.11)$$

regardless of the value of η . Comparing (3.11) with (3.8), we see that the System 2 is superior to System 1 in testing the bound at high β_0 as well as at low. In fact, the ratio $\beta_0/(\epsilon_r^{-1} - 1)$ reaches as low as 13.3 at $\beta_0 = 5$, $\rho = 16$, compared to the minimum value of about 68 for System 1. We conclude that any universal upper bound to the loss that is of the form $10 \log_{10}(1 + k\beta_0)$ cannot have k less than 0.075.

Finally, it may be noted by setting $\alpha = 1$ in (3.4) that when ρ is large and β_0 exceeds 2, increasing β_0 can actually increase the error probability. Thus, the reasonable notion proven in Appendix B, that increasing any one of the eigenvalues cannot cause degradation in the error performance for optimum reception, certainly is not true of reception using the suboptimum, SNR-maximizing processing.

IV. ERROR PROBABILITY RELATED TO MAXIMIZED OUTPUT SNR FOR BINARY SYMMETRIC COMMUNICATIONS

A. Preliminary Remarks

We have shown in Sec. III that if the largest noise-scaled eigenvalue β_0 is small compared to unity, little effective loss in transmission amplitude is suffered in a spread-channel radar or communication system when reception employs the explicit, but generally suboptimum, SNR-maximizing processor (2.11) in lieu of the implicit, error-optimum one (2.1). This result

confirms processor output SNR as a legitimate receiver design criterion when $\beta_0 \ll 1$, but does not make use of the actual value of output SNR attained by the SNR-maximizing processing. Since [as shown in (2.20)] this maximum value R is just as explicit as the receiver that yields it, one would hope that it could be related to error performance, so that output SNR could be used as a convenient overall system performance criterion and thereby guide transmission design as well as receiver synthesis.

The results of this section fulfill this hope within the context of binary symmetric communications. For such systems we develop bounds on the error probability for optimum reception, bounds that for any particular transmission-channel combination involve only β_0 and the output SNR R of the SNR-maximizing processing. Through the decibel-loss bound $\log_{10}(1 + \beta_0)$ already established for maximum-SNR reception, we can then bound the overall effective loss attending maximum-SNR design by a function involving only the maximum output SNR R_m available under the system constraints and β_{0m} , the associated maximum- R value of β_0 .

By using these results one can show, for example, that when $\beta_0 \leq 0.1$, the error probability for the optimum receiver is given by a simple monotonically decreasing function of $\epsilon^2 R$, where ϵ lies between 0.844 and 1.0, and that when $\beta_{0m} \leq 0.1$ the overall "design loss" associated with maximum-SNR transmission and reception is less than 1.2 db. If in addition to the condition $\beta_{0m} \leq 0.1$, interest is confined to situations in which the error probability for either receiver is less than 0.01, the design loss will be less than 0.9 db; even small design-loss bounds are met with yet lower β_{0m} and error probability.

With regard to the foregoing, let us define "design loss" more explicitly than was done at the end of Sec. 1-A, where the term was first introduced. Suppose that in a spread-channel, binary symmetric communication system one is somehow able to find, within specified constraints, the transmission that minimizes the error probability for optimum reception. Then this transmitter-receiver combination yields the minimum error probability that is attainable for the given channel under the specified constraints, and transmitter-receiver design for maximum output SNR can lead to no better error performance. If, however, the "design loss" is known to be less than M db, then one can be sure that the error performance obtained with maximum-SNR design is at least as good as that resulting from minimum-error design, if the latter is made to suffer the handicap of an M -db reduction in transmission amplitude.[†]

No design-loss or error-probability bounds have yet been obtained for binary communications in which the decision level is nonzero. For radar, we do have the rather weak but quite general bounds:[‡]

$$P_F P_M \leq P_e \quad ; \quad P_F + P_M \geq P_e \quad ; \quad \frac{(P_F^o + P_M^o)}{2} \geq P_e^o \quad (4.1)$$

[†] Equation (2.1) indicates that in order to maintain optimum reception, a readjustment of the processing is generally required after such an amplitude reduction [unlike the maximum-SNR processing (2.11)]; in the absence of such readjustment the error performance of the handicapped design can be even worse.

[‡] These bounds resulted from discussions with Drs. R. S. Kennedy and B. Reiffen of Lincoln Laboratory.

which loosely relate radar system performance to that of a binary symmetric communication system.[†] Here P_e is the error probability for any binary symmetric communication system that uses dual, statistically independent and identical channels and identical channel processors, where the sign of the processor output difference determines the binary decision. The quantities P_F and P_M are the false-alarm and miss probabilities, respectively, for any given decision level D_r in a radar system that is identical to the given communication system except for the absence of one channel and its processor. We note that when attention is confined to error-optimum processing, as indicated by the "o" superscripts, an improved lower bound is available for the sum of the radar error probabilities. This third bound in (4.1) is the simplest to establish, for it merely affirms that $\min_{D_r} (P_F^o + P_M^o)/2$, the lowest error probability attainable when a binary symmetric communication system is hampered by the removal of one of the channel outputs from its receiver input, cannot be less than P_e^o , the lowest error probability attainable with this output restored.

The first bound in (4.1) is obtained by noting that $P_F P_M$ is the probability that the output of the processor that receives noise alone exceeds D_r , while at the same time the output of the other processor falls below D_r even though signaling energy reaches it. This joint event implies that the output difference has such a sign that an error is produced in the communication situation; hence the probability P_e of communication error is at least as great as $P_F P_M$, the probability of the joint event. Similar reasoning for the contrary joint event leads to $(1 - P_F)(1 - P_M)$, and hence $(1 - P_F - P_M)$, being upper-bounded by $(1 - P_e)$; this yields the second bound in (4.1).

We now develop the earlier-mentioned error bounds for spread-channel, binary symmetric communication, first obtaining from them the cited design-loss bounds. We then show in Sec. IV-C, as an incidental result, that in some circumstances tight error bounds can be achieved.

B. Analysis for Decibel-Loss Bounds

Equation (2.29), with the decision level $DN_o^2/4$ set at zero, gives the error probability for optimum reception in spread-channel, binary symmetric communications. Noting the even and odd symmetry of the real and imaginary parts, respectively, of the integrand in (2.29) after it has been rationalized, we have⁵

$$P_e^o(0) = F(\{\beta_j\}) G(\{\beta_j\}) \quad (4.2)$$

where

[†] From (4.1) and (4.11) we obtain the bound for optimum spread-channel radar reception and any decision level (and hence for any radar receiver as well):

$$P_F^a + P_M^a \geq 1 - \operatorname{erf} \sqrt{R/4} \quad .$$

It is believed that this bound remains valid when tightened by the replacement of $R/4$ with $R/8$. (There would be no question, were the receiver output Gaussianly distributed, with the same variance for echo presence as for its absence.) Appendix C outlines a derivation of the improved bound, which calls on a theorem (interesting in both a statistical and a circuit-theory context) that is as yet apparently unproven but that seems to be true.

$$F(\{\beta_j\}) = \prod_{j=0}^{\infty} (1 + \beta_j) \left(1 + \frac{\beta_j}{2}\right)^{-2} \quad (4.3)$$

$$G(\{\beta_j\}) = \frac{1}{\pi} \int_0^{\pi/2} d\Theta \left\{ \prod_{j=0}^{\infty} \left[1 + \left(\frac{\beta_j \tan \Theta}{2 + \beta_j} \right)^2 \right] \right\}^{-1} \quad (4.4)$$

and where the transformation $2\mu = \tan \Theta$ has been made.

The terms in (4.3) have the bounds

$$\exp \left[-\frac{\beta_j^2}{4} \right] \leq (1 + \beta_j) \left(1 + \frac{\beta_j}{2}\right)^{-2} \leq \exp \left[-\frac{\beta_j^2}{4} \left(1 + \frac{\beta_j}{2}\right)^{-2} \right] \quad (4.5)$$

This can be verified by observing that (4.5) is true for $\beta_j = 0$, and then by finding that the logarithmic derivatives of the quantities in (4.5), taken with respect to β_j , rank as in (4.5) for $\beta_j \geq 0$.

Thus, since $\beta_j \leq \beta_0$,

$$\exp \left[-\frac{R}{4} \right] \leq F(\{\beta_j\}) \leq \exp \left[-\frac{R}{4} \left(1 + \frac{\beta_0}{2}\right)^{-2} \right] \quad (4.6)$$

where

$$R = \sum_{j=0}^{\infty} \beta_j^2 \quad (4.7)$$

is, by (2.20), the maximized processor output SNR.

Equation (4.4) likewise can be bounded in terms of β_0 and R . First we note that each term of the product in (4.4) can be upper-bounded using

$$1 + x \leq e^x$$

and that the product itself certainly exceeds unity by the sum of the ratios that appear in the terms. Thus⁵

$$\frac{1}{\pi} \int_0^{\pi/2} \exp[-S \tan^2 \Theta] d\Theta \leq G(\{\beta_j\}) \leq \frac{1}{\pi} \int_0^{\pi/2} d\Theta (1 + S \tan^2 \Theta)^{-1} \quad (4.8)$$

where

$$S = \sum_{j=0}^{\infty} \left(\frac{\beta_j}{2 + \beta_j} \right)^2 \quad (4.9)$$

An individual term in (4.9) has the bounds $\beta_j^2/4$, $[\beta_j/(2 + \beta_0)]^2$, so that by (4.7), S lies between $R/4$ and $(R/4) (1 + \beta_0/2)^{-2}$; evaluating⁵ the integrals in (4.8) and employing these bounds on S we have

$$\frac{1}{2} e^{R/4} (1 - \operatorname{erf} \sqrt{R/4}) \leq G(\{\beta_j\}) \leq \left[2 + 2\sqrt{\frac{R}{4} \left(1 + \frac{\beta_0}{2}\right)^{-2}} \right]^{-1} \quad (4.10)$$

where the error function is defined in (3.5). Multiplying (4.6) by (4.10), the error probability of the optimum receiver is found to be bounded as

$$\frac{1}{2} (1 - \operatorname{erf} \sqrt{\frac{R}{4}}) \leq P_e^O(0) \leq \frac{\exp[-(R/4)(1 + \beta_0/2)^{-2}]}{2 [1 + \sqrt{(R/4)(1 + \beta_0/2)^{-2}}]} \quad (4.11)$$

The right member of (4.11) can be written

$$\frac{e^{-G_1^2}}{2(1 + G_1)} = \frac{1}{2} \{1 - \operatorname{erf}[G_1/\kappa(G_1)]\} \quad (4.12)$$

where $G_1 = \sqrt{R/4}/(1 + \beta_0/2)$, and there is a unique solution for $\kappa(G_1)$. Some selected values of $\kappa(G_1)$ are given in Table I.[†]

TABLE I									
$G_1 = \frac{\sqrt{R/4}}{(1 + \beta_0/2)}$	0	0.2786	0.3691	0.5467	1.061	2.061	3.053	4.046	∞
$\kappa(G_1)$	$\lim_{G_1 \rightarrow 0} = \frac{2}{\sqrt{\pi}}$ $= 1.1284$	1.1145	1.1074	1.0934	1.0613	1.0303	1.0177	1.0115	1

Equations (4.11)-(4.12) effectively bound the error probability for optimum reception to within $10 \log_{10} \{(1 + \beta_0/2) \kappa[\sqrt{R/4}/(1 + \beta_0/2)]\}$ decibels. Applying (4.11)-(4.12) to obtain a bound on the loss associated with maximum-SNR design of a binary symmetric signaling system, we recall from Sec. III that the error probability $P_e^S(0)$ for SNR-maximizing reception will not exceed that for optimum reception if the latter is given the handicap of a $10 \log_{10} (1 + \beta_0)$ decibel reduction in transmission amplitude. Dividing both β_0 and $\sqrt{R}$ by $(1 + \beta_0)$ accordingly, it follows that $P_e^S(0)$ is upper-bounded by the right member of (4.12) with $(1 + 3\beta_0/2)$ replacing $(1 + \beta_0/2)$ in G_1 , where R is the maximum output SNR R_m attainable under the system constraints, and β_0 has the value β_{0m} attending the SNR maximization.

Turning now to minimum-error design, there is a transmission that yields an error probability $P_e^O(0)_{\min}$ that is the minimum attainable under the system constraints and that with SNR-maximizing reception would provide some output SNR $R_{\min P}$; $P_e^O(0)_{\min}$ cannot be less than the left member of (4.11) with $R = R_{\min P}$. Thus if this minimum-error transmission is reduced by $10 \log_{10} \{(1 + 3\beta_{0m}/2) \kappa[\sqrt{R_m/4}/(1 + 3\beta_{0m}/2)]\}$ decibels while the receiver is kept optimum, the resulting $P_e^O(0)$ will, by the argument of the preceding paragraph, be at least as great as the error probability $P_e^S(0)$ obtained with maximum-SNR design and no transmission-amplitude handicap. [Here we have used the monotonicity of the left member of (4.11) together with the fact that $R_m \geq R_{\min P}$.] This establishes that the design loss associated with the use of the SNR criterion cannot exceed $10 \log_{10} \{(1 + 3\beta_{0m}/2) \kappa[\sqrt{R_m/4}/(1 + 3\beta_{0m}/2)]\}$ decibels. The examples cited in the preliminaries to this section are obtained from this bound and the one given at the beginning of the preceding paragraph, together with reference to (4.11) and Table I.

[†] Note that R and β_0 in G_1 cannot in general be independently specified, since by (4.7) R cannot be less than β_0^2 .

C. Analysis for Tight Error-Probability Bounds

In certain situations it is possible to have error-probability bounds that are quite tight in the usual sense, rather than as measured indirectly, by an effective decibel difference in transmission amplitude. Such special situations exist when β_0 is small not only compared to unity but to R^{-1} , the reciprocal of the maximized output SNR, and where in addition this SNR is unity or higher.

Under these circumstances, we can return to $G(\{\beta_j\})$ of (4.4) and obtain for it a tighter upper bound than that of (4.10). The bound is established through a partial expansion of the product in (4.4), leading to the inequality

$$\begin{aligned} \prod_{j=0}^{\infty} \left[1 + \left(\frac{\beta_j x}{2 + \beta_j} \right)^2 \right] &\geq 1 + x^2 S + \left(\frac{x^4 S^2}{2} \right) \left[1 - \frac{\beta_0^2}{S(2 + \beta_0)^2} \right] + \frac{x^6 S^3}{6} \left[1 - \frac{3\beta_0^2}{S(2 + \beta_0)^2} \right] \\ &\geq \left[1 - \frac{3\beta_0^2}{R} \right] \left[1 + (xV)^2 + \frac{(xV)^4}{2} + \frac{(xV)^6}{6} \right] \end{aligned} \quad (4.13)$$

where R and S are given by (4.7) and (4.9), respectively, and $V^2 = (R/2) (1 + \beta_0/2)^{-2}$. (Bounds involving higher powers of V could be obtained if so desired.)

Substituting (4.13) in (4.4) and evaluating the integral, we find

$$\begin{aligned} G(\{\beta_j\}) &\leq \left(2 - \frac{6\beta_0^2}{R} \right)^{-1} \left[\frac{1 - 0.328058V - 0.259672V^2 + 0.156838V^3 + 0.124144V^4}{1 + 0.791542V - 0.373462V^2 - 0.295611V^3 + 0.266012V^4 + 0.210560V^5} \right] \\ &\leq 1.04503 \left(2 - \frac{6\beta_0^2}{R} \right)^{-1} \exp \left[\frac{R}{4} \left(1 + \frac{\beta_0}{2} \right)^{-2} \right] \left[1 - \operatorname{erf} \sqrt{\frac{R}{4} \left(1 + \frac{\beta_0}{2} \right)^{-2}} \right] ; \\ &\quad \text{for } \beta_0 \leq \sqrt{\frac{R}{3}} \end{aligned} \quad (4.14)$$

where the higher upper bound, which was established numerically, becomes an equality as $(R/4) (1 + \beta_0/2)^{-2}$ approaches infinity. Replacement of the upper bound in (4.10) by the right member of (4.14) leads to a similar change in the upper bound of (4.11), and we now have

$$\frac{1}{2} \left[1 - \operatorname{erf} \sqrt{\frac{R}{4}} \right] \leq P_e^0(0) \leq 0.523 \left(1 - \frac{3\beta_0^2}{R} \right)^{-1} \left[1 - \operatorname{erf} \sqrt{\frac{R}{4} \left(1 + \frac{\beta_0}{2} \right)^{-2}} \right] \quad (4.15)$$

The final step is concerned with the ratio of the two bracketed terms in (4.15). We develop the bounds, using (3.5),

$$\begin{aligned} 1 \leq \frac{1 - \operatorname{erf}(a-b)}{1 - \operatorname{erf} a} &= \frac{1 - \operatorname{erf} a + (2/\sqrt{\pi}) \int_a^{a-b} e^{-t^2} dt}{1 - \operatorname{erf} a} \leq 1 + \frac{(2/\sqrt{\pi}) b \exp[-(a-b)^2]}{1 - \operatorname{erf} a} \\ &\leq 1 + \left(\frac{2b}{\sqrt{\pi}} \right) (1 + a\sqrt{\pi}) \exp[2ab - b^2] ; \quad 0 \leq b \leq a \end{aligned} \quad (4.16)$$

Here Pierce's bound⁵

$$1 - \operatorname{erf} a \geq \frac{e^{-a^2}}{1 + a\sqrt{\pi}} ; \quad a \geq 0 \quad (4.17)$$

has been used. To apply (4.16) we set $a = \sqrt{R/4}$, $a - b = \sqrt{(R/4)(1 + \beta_0/2)^{-2}}$, and find that

$$0 \leq b \leq \frac{\beta_0}{2} \sqrt{\frac{R}{4}} \quad (4.18)$$

Since β_0 will certainly be less than 2 for weak-signal conditions, the condition $0 \leq b \leq a$ on (4.16) is satisfied, and we have

$$\begin{aligned} 1 \leq \frac{1 - \operatorname{erf} \sqrt{(R/4)(1 + \beta_0/2)^{-2}}}{1 - \operatorname{erf} \sqrt{R/4}} &\leq 1 + \frac{\beta_0}{2} \left(\frac{R}{2} + \sqrt{\frac{R}{\pi}} \right) \exp \left[\frac{R\beta_0(1 + \beta_0/4)}{4(1 + \beta_0/2)^2} \right] \\ &\leq 1 + \frac{1}{4} (3\beta_0 \frac{R}{2} + \beta_0) \exp \left[\frac{\beta_0 R}{4} \right] \end{aligned} \quad (4.19)$$

where the inequality $(R/2) + \sqrt{R/2} \leq [3(R/2) + 1]/2$ has been employed.

Taking the ratio r of the upper bound in (4.15) to the lower bound and using (4.19),

$$1 \leq r \leq 1.05 \left(1 - \frac{3\beta_0^2}{R} \right)^{-1} \left\{ 1 + 0.25(3\beta_0 \frac{R}{2} + \beta_0) \exp \left[\frac{\beta_0 R}{2} \right] \right\} \quad (4.20)$$

and we can therefore determine the optimum-reception error probability to within ± 2.5 percent if $\beta_0 R \ll 1$ and $R \geq 1$. This also proves to be true for the suboptimum, SNR-maximizing reception under the same conditions when one applies the receiving-loss bound $10 \log_{10} (1 + \beta_0)$ decibels.

V. CONCLUSIONS

This study has been primarily concerned with confirming that under frequently met spread-channel conditions, one can safely employ a receiver output-SNR system design criterion (specified in Sec. II-B), even though error-probability optimization is the actual goal. In contrast to error-probability criteria, which lead to having to solve an integral equation (A-10) for the optimum receiver and to even more difficult problems in finding the best transmission, this output SNR is notable for its mathematical simplicity, and for its consequent attractiveness in engineering terms.

A condition under which overall system design may properly be based on maximizing the processor output SNR at the receiver, is that there in a sense be a small channel SNR. This is a "low energy-coherence" (LEC) condition that by no means need imply small output SNR (poor system performance). Specifically, we have shown that the maximum output SNR R is the sum of the squared, noise-scaled eigenvalues (or coordinate signaling energies) $\{\beta_j\} = \{2\tilde{\lambda}_j/N_0\}$ (where N_0 is the noise spectral density), while $\beta_0 = \max_j \beta_j$ is an index of the maximum "design loss" sustained in employing the SNR criterion. ("Design loss" is defined as the reduction in amplitude of the transmission, keeping its waveshape unchanged, that lowers the best error performance attainable under given system constraints to that obtained using maximum-SNR transmission and receiver design under the same constraints.) Since we need only be sure that β_0 is small, however, all that is really required is a good upper bound to β_0 in terms of readily determined quantities. Such bounds are given in Refs. 2 and 4, among them being the LEC bound (2.5) and the spread-channel bound (2.7).

Section III establishes that the portion of the design loss that can be attributed to the use of a SNR-maximizing receiver is, for a given but arbitrary error performance, no greater than

$10 \log_{10} (1 + \beta_0)$ decibels, this being true for a wide variety of radar and binary communication systems. Section IV shows, with the aid of Pierce's work,⁵ that for the class of spread-channel, binary symmetric communication systems, extension of the output-SNR criterion to the design of the transmission as well as of the receiver, results in an overall design loss not exceeding $10 \log_{10} \{(1 + 3\beta_{0m}/2) \kappa[\sqrt{R_m}/4/(1 + 3\beta_{0m}/2)]\}$ decibels. Here $\kappa(G_1)$ is less than $2/\sqrt{\pi}$ and approaches unity as G_1 increases (error probability decreases), R_m is the maximum output SNR attainable under the system constraints, and β_{0m} is the attendant maximum-SNR value of β_0 .

These design-loss bounds are the main results, but there are a few incidental findings. Section II demonstrates, for example, that among a broad class of mixed linear-quadratic receiving processors, the processor that attains maximum output SNR corresponds to the asymptotic, $N_0 \rightarrow \infty$, solution of the integral equation (A-10) for optimum processing. Another result forms part of the development of the receiving-loss bound $10 \log_{10} (1 + \beta_0)$ decibels, where it has been necessary to prove (in Appendix B) the notion that with error-optimum reception, increasing or decreasing any or all of the noise-scaled coordinate signaling energies $\{\beta_j\}$ cannot, respectively, worsen or better the error performance. This notion is not true of the suboptimum, SNR-maximizing reception, however, as is shown by the performance of a particular communication system analyzed in Sec. III-B.

Under the more stringent conditions that β_0 be small compared to R^{-1} , the reciprocal of the maximized output SNR, while this SNR also is at least unity, it has been possible (in Sec. IV-C) to obtain error-probability bounds for binary symmetric communication that are tight to within ± 2.5 percent. Under general LEC conditions, however, the error-probability bounds will be loose in the normal sense although good in terms of bounding the design loss quite tightly.

In establishing the design-loss bounds we have confined our attention to radar, or equivalently to on-off binary communications, or to binary communication systems that use a pair of identical, statistically independent channels with the same waveform transmitted over either channel. Although the channels in this study have been assumed to be describable in terms of scattering functions [see the discussion associated with (1.1)], this is not an essential restriction. All that is actually required for the cited loss bounds is that the signal received in the absence of the white channel noise be narrow-band Gaussian (having the same statistics when received over either channel in the case of dual-channel binary communications) with correlation function $\varphi_z(t, \tau)$. The largest eigenvalue $\lambda_0 = N_0 \beta_0 / 2$ is then that of the integral equation (A-2), and (2.5) remains as a general upper bound to β_0 . It even seems likely that the narrow-band assumption is not necessary, although many of the equations would have to be reworked for single rather than paired eigenvalues.

The two particular communication systems examined in Sec. III-B show that the receiving-loss bound $10 \log_{10} (1 + \beta_0)$ may possibly be tightened[†] to $10 \log_{10} (1 + \beta_0^2)$ for $\beta_0 \leq 1$ and to $10 \log (1 + \sqrt{\beta_0})$ for $\beta_0 \geq 1$, but that if a bound of the form $10 \log_{10} (1 + k\beta_0)$ is retained, k must be at least 0.075. It would be rewarding to find minimum universal design-loss bounds as functions of β_0 and R , or failing that, at least to see how improvement may be made in the bounds presented here. Extension of loss-bound analysis from binary to M-ary communications would also be worth while.

Finally, since in practical systems there will usually be some inaccuracy in realizing the SNR-maximizing receiver, useful studies might be made of the degradation in error performance caused by such receiver mismatch.

[†] Some recent and unpublished work of Dr. R. S. Kennedy also suggests that the receiving loss may not exceed $5 \log_{10} (1 + \beta_0)$ decibels.

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APPENDIX A

DERIVATION OF PROCESSOR OUTPUT FOR THE OPTIMUM RADAR RECEIVER

To form the ratio of the probability measure that the signal $w(t)$, received over one of the channels and observed in $(-T/2, T/2)$, is a result of transmission through that channel, to the probability measure that it arose from noise alone, we first need the zero-mean random-variable "observables" $\{w_j\}$:

$$w_j = \int_{-T/2}^{T/2} w(t) \psi_j(t) dt \quad ; \quad j = 0, 1, 2, \dots \quad (A-1)$$

Here the $\{\psi_j(t)\}$ are the eigenfunctions (excluding the trivial one that is identically zero) of the homogeneous linear integral equation involving the correlation function $\varphi_z(t, \tau)$ of the fluctuating-multipath perturbed transmission:

$$\int_{-T/2}^{T/2} \varphi_z(t, \tau) \psi_j(\tau) d\tau = \lambda_j \psi_j(t) \quad ; \quad -\frac{T}{2} < t < \frac{T}{2} \quad (A-2)$$

The $\{\lambda_j\}$ are the associated eigenvalues and form a bounded countable set. The eigenfunctions can be arranged to be orthonormal:

$$\int_{-T/2}^{T/2} \psi_j(t) \psi_k(t) dt = \delta_{jk} = \begin{cases} 0 & ; \quad j \neq k \\ 1 & ; \quad j = k \end{cases} \quad (A-3)$$

although they may not form a complete set. These properties of the $\{\lambda_j\}$ and $\{\psi_j(t)\}$ follow⁹ from the fact that

$$\iint_{-T/2}^{T/2} \varphi_z^2(t, \tau) dt d\tau \leq \iint_{-T/2}^{T/2} \varphi_z(t, t) \varphi_z(\tau, \tau) dt d\tau \leq (\overline{E_z})^2 < \infty \quad (A-4)$$

and the assumed finiteness of the total average energy $\overline{E_z}$ received in $(-\infty, \infty)$. The inequality in (A-4) is obtained from the condition that the square of the correlation coefficient, $\varphi_z^2(t, \tau)/[\varphi_z(t, t) \varphi_z(\tau, \tau)]$, existing between $z(t)$ and $z(\tau)$ cannot exceed unity. The eigenvalues are also non-negative since $\varphi_z(t, \tau)$ is non-negative definite.⁹

Because the $\{\psi_j(t)\}$ often do not form a complete set, especially one satisfactory for representing a process containing white noise, we have generally

$$\hat{w}(t) = \sum_{j=0}^{\infty} w_j \psi_j(t) \neq w(t) \quad (A-5)$$

in $(-T/2, T/2)$. Therefore, if we replace the received signal $w(t)$ by the set $\{w_j\}$ as the input to the receiver, we should also make available to the receiver the difference waveform $w(t) - \hat{w}(t)$; by the equality in (A-5), $w(t)$ can then in principle be recreated from the new receiver inputs and no information is lost. By the linearity of all the operations thus far involved and the Gaussian-ness of $w(t)$ under either hypothesis, the $\{w_j\}$ and the waveform $w(t) - \hat{w}(t)$ are jointly Gaussian. Letting $a = 0$ represent the hypothesis that the received signal is solely a sample of noise of

spectral density N_o , and letting $a = 1$ correspond to the other hypothesis, we find that the $\{w_j\}$ are mutually independent under either hypothesis. This is so because they are uncorrelated:

$$\begin{aligned} \overline{w_j w_k} &= \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \psi_j(t) \psi_k(\tau) \left[a \varphi_z(t, \tau) + \frac{N_o \delta(t - \tau)}{2} \right] d\tau dt \\ &= \left(a \lambda_n + \frac{N_o}{2} \right) \int_{-T/2}^{T/2} \psi_j(t) \psi_k(t) dt = \left(a \lambda_n + \frac{N_o}{2} \right) \delta_{jk} \quad ; \quad a = 0, 1 \quad . \quad (A-6) \end{aligned}$$

Here (A-3) and (A-4) have been used. Furthermore, all the $\{w_j\}$ are independent of $w(t) - \hat{w}(t)$:

$$\begin{aligned} \overline{w_j [w(t) - \hat{w}(t)]} &= \int_{-T/2}^{T/2} \psi_j(\tau) \left[\overline{w(\tau) w(t)} - \sum_{k=0}^{\infty} \psi_k(t) \int_{-T/2}^{T/2} \overline{w(\tau) w(\sigma)} \psi_k(\sigma) d\sigma \right] d\tau \\ &= \int_{-T/2}^{T/2} \psi_j(\tau) \left[a \varphi_z(t, \tau) + \frac{N_o \delta(t - \tau)}{2} - \sum_{k=0}^{\infty} \left(a \lambda_k + \frac{N_o}{2} \right) \psi_k(t) \psi_k(\tau) d\tau \right] \\ &= \left(a \lambda_j + \frac{N_o}{2} \right) \psi_j(t) - \left(a \lambda_j + \frac{N_o}{2} \right) \psi_j(t) = 0 \quad ; \\ & \quad j = 0, 1, 2, \dots \quad ; \quad a = 0, 1 \quad ; \quad \text{all } t \quad . \quad (A-7) \end{aligned}$$

In the absence of the additive white channel noise, $w(t)$ is equal,¹⁰ with probability one, to $\hat{w}(t)$ at any t , assuming that $\varphi_z(t, \tau)$ is continuous in t and τ . (This continuity assumption could be violated, for example, by a process formed by sharply gating a stationary process, although it will be satisfied by stationary processes themselves. As a reasonable engineering approximation we henceforth assume that continuity holds.) Thus under either hypothesis, $w(t) - \hat{w}(t)$ is a waveform generated solely by the noise and as such can contain no direct information about whether or not there was a transmission over the channel; neither can it convey information indirectly when taken in conjunction with the $\{w_j\}$ because of the statistical independence just shown.

We conclude by this plausibility argument[†] that the optimum decision-making receiver need only deal with the set of observables $\{w_j\}$, and furthermore that, by virtue of their statistical independence, the overall ratio of probability measures under the two hypotheses can be constructed by multiplying together the ratios for the individual w_j . The zero-mean Gaussian observable w_j having variance $(a \lambda_j + N_o/2)$ by (A-6), with $a = 0$ or 1 according to the hypothesis selected, we find for $\log \Lambda$, the logarithm of the likelihood ratio formed on the observation of one of the two received signals in the interval $(-T/2, T/2)$:

$$\log \Lambda = -\frac{1}{2} \sum_{j=0}^{\infty} \log \left(1 + \frac{2 \lambda_j}{N_o} \right) + \frac{2}{N_o} \sum_{j=0}^{\infty} \frac{\lambda_j w_j^2}{N_o + 2 \lambda_j} \quad (A-8)$$

[†] It would be presumptuous to call our argument even "formal," considering that the series for $\hat{w}(t)$ in (A-5) does not converge! However, it appears certain that (A-8) is the correct expression for the logarithm of the likelihood ratio in the wide variety of white-noise situations where both sums converge separately.¹⁹

From the discussion preceding (2.1) of the main text, we are allowed to add to (A-8) so that its first right-hand term is canceled, and are permitted as well to then multiply the result by $N_0^2/2$, receiver optimality being unaffected. Thus the modified but still-optimum radar processor reduces its input signal $w(t)$, received in the observation interval $(-T/2, T/2)$, to the quadratic form d^0 :

$$d^0 = \sum_{j=0}^{\infty} \frac{\lambda_j w_j^2}{1 + 2\lambda_j/N_0} = \iint_{-T/2}^{T/2} w(t) w(\tau) \left[\sum_{j=0}^{\infty} \frac{\lambda_j \psi_j(t) \psi_j(\tau)}{1 + 2\lambda_j/N_0} \right] dt d\tau \quad (A-9)$$

Here we have referred to (A-1). To obtain an explicit expression for the infinite-series processing kernel $F(t, \tau)$ that appears in the integrand of (A-9), one needs to solve the generally difficult inhomogeneous integral equation:

$$\int_{-T/2}^{T/2} \left[\frac{2\varphi_z(t, \sigma)}{N_0} + \delta(t - \sigma) \right] F(\sigma, \tau) d\sigma = \varphi_z(t, \tau) \quad ; \quad -\frac{T}{2} \leq t \leq \frac{T}{2} \quad (A-10)$$

As $N_0 \rightarrow \infty$ it can be seen from (A-10) that $F(t, \tau) \rightarrow \varphi_z(t, \tau)$.

Turning now to the consequences of the narrow-band character assumed for fluctuating-multipath perturbed transmission, we see from (1.1) that $\varphi_z(t, \tau)$ is locally sinusoidal in t , assuming that ω_0 is high relative to the bandwidths of the transmission and the channel fluctuations. Therefore by (A-2), the eigenfunctions $\{\psi_j(t)\}$ must likewise be locally sinusoidal with angular frequency ω_0 . Hence they are representable as $\psi_j(t) = \sqrt{2} \operatorname{Re} \{ \tilde{\psi}_j(t) \exp[i\omega_0 t + i\Theta] \}$, where Θ is an as yet unspecified real constant and $\tilde{\psi}_j(t)$ is a complex function whose variation is slow relative to that of $\exp[i\omega_0 t]$. Substituting (1.1) and this representation for $\psi_j(t)$ into (A-2), and remembering that $2 \operatorname{Re} \{A\} \operatorname{Re} \{B\} = \operatorname{Re} \{AB\} + \operatorname{Re} \{AB^*\}$, we have the condition

$$\begin{aligned} & \frac{1}{4} \operatorname{Re} \left\{ \exp[i\omega_0 t + i\Theta] \int_{-T/2}^{T/2} \left[\iint_{-\infty}^{\infty} \mathfrak{X}(t - \lambda) \mathfrak{X}^*(\tau - \lambda) \Phi_y(\omega, \lambda) \tilde{\psi}_j(\tau) \right. \right. \\ & \quad \times \exp[i\omega(t - \tau)] d\omega d\lambda \left. \right] d\tau \left. \right\} + \frac{1}{4} \operatorname{Re} \left\{ \exp[i\omega_0 t - i\Theta] \int_{-T/2}^{T/2} \left[\iint_{-\infty}^{\infty} \mathfrak{X}(t - \lambda) \right. \right. \\ & \quad \times \mathfrak{X}^*(\tau - \lambda) \Phi_y(\omega, \lambda) \tilde{\psi}_j^*(\tau) \exp[i\omega(t - \tau) - 2i\omega_0 \tau] d\omega d\lambda \left. \right] d\tau \left. \right\} \\ & = \lambda_j \operatorname{Re} \{ \tilde{\psi}_j(t) \exp[i\omega_0 t + i\Theta] \} \quad ; \quad -\frac{T}{2} \leq t \leq \frac{T}{2} \quad (A-11) \end{aligned}$$

Under the assumed narrow-band conditions, the integrand of the second integral in (A-11) oscillates so rapidly that the integral becomes negligible relative to the first integral. Thus a sufficient condition for $\sqrt{2} \operatorname{Re} \{ \tilde{\psi}_j(t) \exp[i\omega_0 t + i\Theta] \}$ to be an eigenfunction of (A-2) is that $\tilde{\psi}_j(t)$ be an eigenfunction of the integral equation:

$$\begin{aligned} & \int_{-T/2}^{T/2} \left[\iint_{-\infty}^{\infty} \mathfrak{X}(t - \lambda) \mathfrak{X}^*(\tau - \lambda) \Phi_y(\omega, \lambda) \exp[i\omega(t - \tau)] d\omega d\lambda \right] \tilde{\psi}_j(\tau) d\tau \\ & = 4\lambda_j \tilde{\psi}_j(t) \quad ; \quad -\frac{T}{2} \leq t \leq \frac{T}{2} \quad (A-12) \end{aligned}$$

That (A-12) is also a necessary condition on $\tilde{\psi}(\tau)$ can be established by considering (A-11) at two values of t spaced $\pi/(2\omega_0)$ apart, and noting that the functions multiplying $\exp[i\omega_0 t + i\theta]$ can change only negligibly in such a small time interval, again by the narrow-band assumption.

Thus far the value of θ in the representation of the narrow-band $\{\psi_j(t)\}$ has been left open; we now see that any value of θ will yield an eigenfunction $\psi_j(t)$ of (A-2) for a given solution $\tilde{\psi}_j(t)$ of (A-12). There are, however, just two linearly independent solutions of (A-2) for a given $\tilde{\psi}_j(t)$. This can be seen by forming the two solutions through choosing a pair of values of θ that are $\pi/2$ apart [(A-3) then being satisfied if, as usual, we neglect terms in $e^{2i\omega_0 t}$] and then finding it impossible to introduce a third solution at a value of θ that yields linear independence with respect to the other two. We conclude that for each eigenvalue of (A-12) [counting each by the number of linearly independent solutions $\tilde{\psi}_j(t)$ that are associated with it] there will be a pair of eigenvalues of (A-2), both equal to one-quarter the eigenvalue of (A-12). To avoid confusion in indexing, we shall now denote the eigenvalues of (A-12) by $\{4\tilde{\lambda}_j\}$ rather than $\{4\lambda_j\}$; those of (A-2) then become $\{\lambda_j\} = \{\tilde{\lambda}_0, \tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_2, \dots\}$.

Finally, we note that the complex kernel $\tilde{\varphi}_z(t, \tau)$ that multiplies $\tilde{\psi}_j(\tau)$ in (A-12) is Hermitian symmetric: $\tilde{\varphi}_z(t, \tau) = \tilde{\varphi}_z^*(\tau, t)$. Since it is easily established that this kernel is also non-negative definite and (using the Schwarz inequality) square-integrable like $\varphi_z(t, \tau)$ in (A-4), its eigenvalues form a bounded, countable and non-negative set, just as in (A-2).

Under these conditions the eigenfunctions $\{\tilde{\psi}_j(t)\}$ can be arranged to be orthonormal⁹ [consistent with the orthonormality of those of (A-2)]:

$$\int_{-T/2}^{T/2} \tilde{\psi}_j(t) \tilde{\psi}_k^*(t) dt = \delta_{jk} \quad . \quad (A-13)$$

APPENDIX B
PROOF THAT FOR OPTIMUM RECEPTION THE ERROR PERFORMANCE
IS NEVER IMPROVED BY DECREASING THE NOISE-SCALED SYSTEM
EIGENVALUES β_j

In this Appendix we first establish that if, for a spread-channel, optimum-reception radar system, there is a decrease in any one of the $\{\beta_j\}$, with the remainder of the $\{\beta_j\}$ arbitrary but fixed, the false-alarm probability for a given but arbitrary miss probability will never decrease (actually, it will in general increase). We then consider a dual-spread-channel, optimum-reception binary communication system operating with a fixed but arbitrary probability of mistaking the single transmission to have been sent over one channel for its actually having been sent over the other. For this communication system it is shown that a decrease in any one of the β_j never produces any decrease in the probability for the opposite type of error. Because of differences in system structure, it is necessary to consider the radar and communication situations separately. The proof used for each system also demonstrates that error performance is never worsened by increasing any of the β_j .

I. THE RADAR SYSTEM

With the decision level set at $DN_0^2/4 > 0$, the miss and false-alarm probabilities for optimum reception are given by (2.25) and (2.27), respectively. In order to hold the miss probability P_M^O constant when there is a change in a particular β_j , say β_n , D must be adjusted. For an infinitesimal change $d\beta_n$ in β_n , the required change in D is dD , and the ratio $-dD/d\beta_n$ is found by taking the ratio of the partial derivative of (2.25) with respect to β_n to that with respect to D :

$$-\frac{dD}{d\beta_n} = \frac{\partial P_M^O / \partial \beta_n}{\partial P_M^O / \partial D} = - \frac{2 \int_{C_1} \left[\prod_{j=0}^{\infty} (1 - 2i\beta_j \mu) \right]^{-1} (1 - 2i\beta_n \mu)^{-1} e^{-i\mu D} d\mu}{\int_{C_1} \left[\prod_{j=0}^{\infty} (1 - 2i\beta_j \mu) \right]^{-1} e^{-i\mu D} d\mu} \quad (B-1)$$

On the other hand, the incremental change in the false-alarm probability P_F^O that occurs when β_n is changed by $d\beta_n$, but D is at the same time changed by dD to keep the miss probability P_M^O fixed, is given by

$$dP_F^O = \frac{\partial P_F^O}{\partial \beta_n} d\beta_n + \frac{\partial P_F^O}{\partial D} dD = \frac{\partial P_F^O}{\partial D} \left[\frac{\partial P_F^O / \partial \beta_n}{\partial P_F^O / \partial D} + \frac{dD}{d\beta_n} \right] d\beta_n \quad (B-2)$$

Since $\partial P_F^O / \partial D$ is clearly non-positive, the change in false-alarm probability will be non-negative for a non-positive $d\beta_n$ if we can show that the bracketed expression in (B-2) is non-negative for all values of β_n and D . From (2.27) we find

$$\frac{\partial P_F^O / \partial \beta_n}{\partial P_F^O / \partial D} = - \frac{2 \int_{C_1} \left[\prod_{j=0}^{\infty} (1 - 2i\beta_j \mu) \right]^{-1} (1 - 2i\beta_n \mu)^{-1} e^{i\mu D} d\mu}{(1 + \beta_n) \int_{C_1} \left[\prod_{j=0}^{\infty} (1 - 2i\beta_j \mu) \right]^{-1} e^{-i\mu D} d\mu} \quad (B-3)$$

which is non-positive, since the integrals in (B-3) represent certain probability density functions. Comparing (B-1) and (B-3), no β_n being negative, it is established that the bracketed expression in (B-2) is always non-negative, and hence that the error performance can never be improved by decreasing any (or all) β_j .

II. THE BINARY COMMUNICATION SYSTEM

Equation (2.29) gives the probability of wrongly judging that the communication transmission has been sent over the first of the dual channels, when in fact it has been sent over the second. Optimum detection is assumed, with the decision level set at $DN_o^2/4$. The symmetry argument preceding (2.24) shows that with this decision level the probability of making the opposite type of error is also given by (2.29), but with $-D$ substituted for D .

We study the behavior of $P_e^0(D)$ for $P_e^0(-D)$ held fixed as any selected β_n changes.[†] Parallelizing the radar proof, we find by taking partial derivatives of $P_e^0(D)$ and $P_e^0(-D)$ with respect to β_n and D , and consolidating, that for the infinitesimal change $d\beta_n$ in β_n ,

$$dP_e^0(D) = -d\beta_n \left[\frac{\partial P_e^0(D)}{\partial D} \right] \left[\frac{4\beta_n(2 + \beta_n)}{1 + \beta_n} \right] \\ \times \frac{\int_{-\infty}^{\infty} \left\{ \prod_{j=0}^{\infty} \left[(1 + \beta_j/2)^2 + (\beta_j\mu)^2 \right] \right\}^{-1} \left[(1 + \beta_n/2)^2 + (\beta_n\mu)^2 \right]^{-1} \exp \left[-\frac{i\mu D}{2} \right] d\mu}{\int_{-\infty}^{\infty} \left\{ \prod_{j=0}^{\infty} \left[(1 + \beta_j/2)^2 + (\beta_j\mu)^2 \right] \right\}^{-1} \exp \left[-\frac{i\mu D}{2} \right] d\mu} \quad (B-4)$$

when $P_e^0(-D)$ is held fixed. [In arriving at (B-4) we make the change $\mu' = -\mu$ in the integrals related to $P_e^0(-D)$ and then drop the prime.]

Since $\partial P_e^0(D)/\partial D$, β_n , and the integrals in (B-4) are all non-negative, the latter because they are proportional to certain probability densities, it is established that the error performance is never improved by decreasing any of the β_j .

[†] When $D = 0$ in binary symmetric signaling, the two error probabilities are equal and are given by (2.29). In this case separation of the integral in (2.29) into real and imaginary parts results in the latter vanishing; differentiation with respect to any β_n immediately establishes that the error probability is a nonincreasing function of the β_n (see Sec. IV-B).

APPENDIX C

A CANDIDATE BOUND FOR RADAR ERROR PROBABILITY

As discussed in Sec. IV-A, we wish to show that for the sum of the two kinds of radar error probability in optimum reception,

$$P_F^O + P_M^O \geq 1 - \operatorname{erf} \sqrt{\frac{R}{8}} \quad (C-1)$$

at any decision level, where the output SNR R is given by the sum of the squared, noise-scaled eigenvalues $\{\beta_j\}$ as in (4.7). By referring to (2.25) and (2.27), we find that $P_F^O + P_M^O$ attains its minimum value $P_{\min}$ when the decision level satisfies

$$D = 2 \log \left[\prod_{j=0}^{\infty} (1 + \beta_j) \right] \quad (C-2)$$

so that

$$P_F^O + P_M^O \geq P_{\min} = 1 - (4\pi)^{-1} \int_{C_1} \frac{\exp \left[-2i\mu \sum_{j=0}^{\infty} \log(1 + \beta_j) \right] d\mu}{(-i\mu) \left(-i\mu - \frac{1}{2}\right) \prod_{j=0}^{\infty} (1 - 2i\beta_j \mu)} \quad (C-3)$$

The derivative of $P_{\min}$ with respect to the value of a β_k that is repeated K times in (C-3) is

$$\frac{\partial P_{\min}}{\partial \beta_k} = -\frac{K\beta_k}{\pi(1 + \beta_k)} \int_{C_1} \frac{\exp \left[-2i\mu \sum_{j=0}^{\infty} \log(1 + \beta_j) \right] d\mu}{(1 - 2i\beta_k \mu) \prod_{j=0}^{\infty} (1 - 2i\beta_j \mu)} \quad (C-4)$$

Thus, if a K -fold eigenvalue β_k is incremented by $d\beta_k$ while a different eigenvalue β_ℓ (that is not considered multiple even if its value is duplicated) is simultaneously incremented by $d\beta_\ell = -K(\beta_k/\beta_\ell) d\beta_k$ so that R , the sum of the squared eigenvalues, remains fixed, the net change in $P_{\min}$ is given by

$$dP_{\min} = \frac{-\beta_k d\beta_k (\beta_\ell - \beta_k)}{\pi(1 + \beta_k)(1 + \beta_\ell)} \int_{C_1} \frac{[1 - 2i\mu(1 + \beta_k + \beta_\ell)] \exp \left[-2i\mu \sum_{j=0}^{\infty} \log(1 + \beta_j) \right] d\mu}{(1 - 2i\beta_k \mu)(1 - 2i\beta_\ell \mu) \prod_{j=0}^{\infty} (1 - 2i\beta_j \mu)} \quad (C-5)$$

As long as the (possibly multiple) eigenvalue β_k exceeds β_ℓ , and provided that the contour integral in (C-5) is always non-negative, a decrease in the eigenvalue β_k with a compensating, R -preserving increase in β_ℓ will result in no net increase for $P_{\min}$. Hence if the eigenvalues are ranked in a nonincreasing order, we may first uniformly trade as many of the largest as may exist against a single one of the next largest until equality is obtained within this set, then uniformly trade this new largest set against one of the new next-largest, and so on, while never increasing $P_{\min}$ nor altering R .

By following this procedure ad infinitum, we obtain (in a heuristic limit) an infinite number of vanishingly small but equal eigenvalues. The Central Limit Theorem then guarantees the Gaussianness of the optimum receiver output (2.21); it is also simple to show that by virtue of

the eigenvalue smallness, the presence of an echo has negligible effect on the output variance. In the limit, the minimum error probability is obtained with the decision level set halfway between the mean output for the echo present and that for the echo absent. From the definition of the output SNR R as the ratio of the squared difference in means to the common output variance, it then follows that the value of the minimum sum of error probabilities in the limiting situation is given by the right member of (C-1). This establishes the bound.

For our argument to be valid, however, it is necessary that the value of the contour integral in (C-5) be non-negative at every stage of the above procedure, and this is where the weakness lies. The sign of the contribution made by that part of the integrand attributable to "1" in $[1 - 2i\mu(1 + \beta_k + \beta_l)]$ is certainly never negative, for (as in Appendix B) this contribution is the value of a certain probability density taken at a particular level. Unfortunately, it appears difficult to prove that the contribution of the remainder, attributable to $-2i\mu(1 + \beta_k + \beta_l)$, is likewise always non-negative. In a large number of special cases, however, this has been found to be so without exception, provided that β_k is the largest (possibly multiple) eigenvalue, just as in the above argument.

It may be noted that the remainder contribution under question in (C-5) is proportional to the derivative, taken at the level $2 \sum_{j=0}^{\infty} \log(1 + \beta_j)$, of the probability density of the sum of a set of central chi-square variates. Each of these variates has two degrees of freedom and has a β_j for the common variance of its Gaussian components; in the case of β_k , there is one more variate than its multiplicity, and the same is true of β_l . Since this repetition, as observed in the denominator of the integrand in (C-5), does not occur in $\sum_{j=0}^{\infty} \log(1 + \beta_j)$, it would suffice to show that the maximum (unimodality can be assumed) in the aforementioned probability density (of "generalized chi-square" class) never occurs at a level less than twice the sum taken on the $\{\log(1 + \beta_j)\}$, after the largest of this set has been excluded (or one of the largest, if there is a multiplicity).

As a matter of fact, in all cases that we have examined, the peak has been observed to lie at or above the similarly censored-and-doubled sum of the component variances $\{\beta_j\}$ themselves, which implies a stronger lower bound than the logarithmic one, and even at or above the yet stronger candidate bound formed by subtracting from the uncensored variance sum (i.e., the output SNR R) the ratio of the uncensored sum of the $\{\beta_j^3\}$ to the uncensored sum of the $\{\beta_j^2\}$, and doubling the result. (Exact equality is met in these latter two bounds when there are a finite number of β_j , all of equal value.) There is an obvious circuit-theory parallel to these latter two bounds, conjectured but thus far unproven, which may be stated as follows:

Theorem.

The (single) peak in the impulse response of an RC or RL ladder network (i.e., all-real-pole) can occur no sooner than the sum of the modal time-constants less the ratio of the sum of their cubes to the sum of their squares, or (should this not prove to be true) minus any one time constant that is not exceeded by another.

No counterexample to either part of this theorem has been found after examining several trial examples and making more than a dozen analog-computer tests (the latter with the kind assistance of Dr. Harold K. Knudsen). Furthermore, the error-probability bound (C-1) that it supports has been sustained in a number of radar cases that have been calculated exactly.¹⁶

REFERENCES

1. R. Price and P.E. Green, Jr., "Signal Processing in Radar Astronomy - Communication via Fluctuating Multipath Media," Technical Report 234, Lincoln Laboratory, M.I.T. (6 October 1960), DDC-246782, H-520.
2. R. Price, "Detectors for Radar Astronomy," in Radar Astronomy, J.V. Evans and T. Hagfors, editors (to be published).
3. E.J. Kelly, D.H. Lyons and W.L. Root, "The Sensitivity of Radiometric Measurements," J. Soc. Ind. Appl. Math. 11, 235 (June 1963).
4. P. Bello, "Some Results on the Problem of Discriminating between Two Gaussian Processes," IRE Trans. Inform. Theory IT-7, 224 (October 1961).
5. J.N. Pierce, "Approximate Error Probabilities for Optimal Diversity Combining," IEEE Trans. Comm. Systems CS-11, 352 (September 1963).
6. J.N. Pierce, "Error Probabilities for Certain Spread Channels," AFCRL Research Paper 72, AFCRL-64-978, United States Air Force, L.G. Hanscom Field, Bedford, Massachusetts (December 1964), Appendix A.
7. D. Middleton and D. Van Meter, "Detection and Extraction of Signals in Noise from the Point of View of Statistical Decision Theory," J. Soc. Ind. Appl. Math. 3, 192 (December 1955) and 4, 86 (June 1956).
8. W.W. Peterson, T.G. Birdsoll and W.C. Fox, "The Theory of Signal Detectability," IRE Trans. Inform. Theory, IT-4, 171 (September 1954).
9. W.B. Davenport, Jr., and W.L. Root, An Introduction to the Theory of Random Signals and Noise, Lincoln Laboratory Publications (McGraw-Hill, New York, 1958), Sec. A2-2.
10. Ibid., pp. 96-99.
11. R. Bellman, Introduction to Matrix Analysis (McGraw-Hill, New York, 1960), p. 115.
12. P. Bello, "Time-Frequency Duality," IEEE Trans. Inform. Theory IT-10, 18 (January 1964).
13. P. Rudnick, "A Signal-to-Noise Property of Binary Decisions," Nature 193, 604 (10 February 1962).
14. G.S. Sebestyen, "Classification Decisions in Pattern Recognition," Technical Report 381, Research Laboratory of Electronics, M.I.T. (25 April 1960), p. 51, Appendix D.
15. P. Whittle, Hypothesis Testing in Time Series Analysis (Almqvist and Wiksells AB, Uppsala, Sweden 1951).
16. R. Price, "Error Probabilities for the Ideal Detection of Signals Perturbed by Scatter and Noise," Group Report 34-40, Lincoln Laboratory, M.I.T. (3 October 1955) p. 6.
17. G.L. Turin, "On Optimal Diversity Reception," IRE Trans. Inform. Theory IT-7, 154 (July 1961).
18. _____ "On Optimal Diversity Reception, II," IRE Trans. Comm. Systems CS-10, 22 (March 1962).
19. J. Hájek, "On Linear Statistical Problems in Stochastic Processes," Czech. Math. J. 12, 404 (1962).
20. A.M. Yaglom, "On the Equivalence and Perpendicularity of Two Gaussian Probability Measures in Function Space," Chapter 22 of Time Series Analysis, M. Rosenblatt, editor (Wiley, New York, 1963).
21. D. Middleton, "On Singular and Nonsingular Optimum (Bayes) Tests for the Detection of Normal Stochastic Signals in Normal Noise," IRE Trans. Inform. Theory IT-7, 105 (April 1961). Revision: IRE Trans. Inform. Theory IT-8, 385 (October 1962).

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13. ABSTRACT In transmission and receiver design for radar or communication systems whose noisy channels contain Gaussianly fluctuating multipath, it is convenient to adopt a receiver output signal-to-noise ratio (SNR) criterion even though best error performance is actually sought. We investigate the loss (expressed as an equivalent transmitter output reduction) attending the use of this criterion. It is shown that when a Karhunen-Loève analysis of the signaling system yields a largest eigenvalue that is suitably small, this loss is minor or negligible at all levels of error probability. Furthermore, it is easily possible to have a channel-perturbed transmission that is sufficiently weak and incoherent for this eigenvalue to guarantee low loss, yet not so weak that high output SNR (good error performance) is precluded.		
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