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UNDERWATER SOUND RADIATION FROM A FINITE CYLINDER: GENERAL ANALYSIS

P. W. Smith, Jr.
E. M. Kerwin, Jr.

Contract No. Nonr-4476(00)(FBM)

15 March 1965

Submitted to:

Dr. Nicholas Perrone, Code 439
Structural Mechanics Branch
Office of Naval Research
Department of the Navy
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ABSTRACT

The sound radiation from vibrations of a finite, air-filled, thin cylindrical shell is studied by a multi-modal approach. The modes are classified according to their resonance characteristics and their efficiencies of coupling to sound. The results are used to assess radiation from a shell 6.4 ft long, 2.1 ft in radius, in the frequency range of about 100 to 300 cps. A survey of the modal properties indicates that radiation is due principally to (i) modes of gross motion of shell and end caps, and (ii) a few highly resonant, very poorly coupled lobar modes. The importance of the latter is determined by internal damping. A detailed analysis is given for the coupling of the lobar modes.

UNDERWATER SOUND RADIATION FROM A FINITE CYLINDER: GENERAL ANALYSIS

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UNDERWATER SOUND RADIATION FROM A FINITE CYLINDER: GENERAL ANALYSIS

I. INTRODUCTION

We are concerned with predicting the sound radiated into an infinite body of water by a finite, air-filled cylindrical shell, when it is excited by a known force at one point. The frequency range of interest is such that the shell's length is less than the wavelength of sound in the water, but not less than about 1/6 of the wavelength. The shell's diameter is about 2/3 of its length.

This report outlines the general analytical techniques employed for the prediction. Useful formulas are developed. The characteristic features of sound radiation in different frequency ranges are discussed, without any attempt to make precise, detailed predictions for a particular shell. Occasional references will be made to the particular shells with which we are concerned - e.g., "the EB shell" - in order to illustrate typical values of parameters. Detailed calculations for particular shells will be given in subsequent reports.

The general approach is to describe the total response of the shell as a modal series. The modal characteristics differ, both in regard to resonance frequency and damping and also in regard to efficiency of coupling to sound. A survey of these modal characteristics makes it possible to focus attention on a restricted set of modes at any one time, the choice being determined by the characteristics of the force and of the structure.

The nature of subsequent analysis depends greatly on the population of the set of important modes. If there are few, detailed analysis of each may be indicated. If there are many, detailed individual analysis may be abandoned in favor of an analysis of the average characteristics of the set. Often it is found that several subsets must be distinguished and averages formed separately.

This latter approach, for the multi-mode problem, is a technique developed and tested for finite flat panels and cylinders radiating into air;[1-3] it has been called the "statistical energy method". The prediction of sound radiated by structures is essentially a statistical one. A single engineering drawing may be the "input" from which is derived not only a physical structure and experimental measurements but also a mathematical model and calculations. However, repetitions of this process, using the same drawing, will generate different measurements and calculations. (We assume that the analyst is not entirely deterministic and will try different approximations in constructing his models.) Many of the differences are uncontrollable and subject to random fluctuations.

Chapter II outlines the nomenclature and concepts pertinent to a modal description of the problem. There, also, the formal manipulations of the multi-modal approach are developed.

Chapters III and IV assess the major characteristics of the various modes, pointing out the existence of restricted sets of modes having similar characteristics. Resonance frequencies and virtual ("added") mass are evaluated in Chapter III. Radiation resistances are evaluated in Chapter IV.

This survey justifies a tremendous restriction of attention, to only a few modes, for the particular problem at hand. (It also suggests that many more modes and much more complicated response patterns would be involved at somewhat higher frequencies or for a much longer shell.)

Chapter V contains a detailed analysis of the radiation resistances of the few modes that resonate in the frequency range of interest.

II. GENERAL FORMULATION

A. INTRODUCTION

The total structural vibration is described by a series in the natural modes of the undamped structure in vacuo. To the mass and stiffness associated with those modes are added the effects of internal damping and radiation loading. It is postulated that none of these additional effects leads to significant coupling between modes. Thus, we assume that the structural dynamics is adequately described by the response of an uncoupled set of simple resonators, with one resonator for each mode. The typical k-th member of the set has the following constant parameters:

T_k: force transfer function, the ratio of modal generalized force to applied force;

M_k: modal generalized mass;

ω_k: modal resonance frequency;

R_k: total modal resistance;

$R_k^{rad} \equiv \mu_k R_k$: modal radiation resistance;

$\eta_k \equiv R_k/\omega_k M_k$: modal loss factor.

The characterization of each mode as a simple resonator is not strictly valid since the values of such parameters as radiation resistance and virtual mass will vary with frequency. However, the characterization is a reasonable approximation in two circumstances. First, when the structure is highly resonant, significant response in any one mode is limited to a narrow band of frequency within which the parameters can be approximated by constants. Second, when the force is a narrow-band noise or a pure tone, the values of the parameters that are pertinent to that frequency can be used.

We consider response to a general force f, characterized by its spectral density $S_f(\omega)$. The mean square applied force is

$$f^2 = \int_0^\infty S_f(\omega')d\omega' \quad . \tag{1}$$

Two special cases will interest us. The first is a band-limited, flat noise:

$$S_f(\omega') = \begin{cases} f^2/\delta\omega \, , & |\omega'-\omega_c| < \frac{1}{2}\delta\omega \\ 0 \, , & |\omega'-\omega_c| > \frac{1}{2}\delta\omega \end{cases} \quad . \tag{2}$$

The second is a pure tone with frequency ω:

$$S_f(\omega') = f^2 \, \delta(\omega'-\omega) \quad . \tag{3}$$

The mean-square response of a linear system to the band-limited noise is identical with the average response to pure tones having frequencies in the band. For convenience, we shall talk of response to noise, but the interpretation as an average over frequency is always implicit.

Consider the response of a single mode to the pure tone force,

$$\sqrt{2} \, f \, e^{i\omega t} \tag{4}$$

in the complex convention for time dependence. Then, according to familiar laws,[4] the instantaneous generalized velocity of modal response is

$$\sqrt{2}\, V_k\, e^{i\omega t} \quad ,$$

where

$$V_k = (T_k\, f)\, Y_k \quad . \tag{5}$$

The modal admittance Y_k is a complex function of frequency

$$Y_k^{-1} = R_k + i\omega M_k(1-\omega_k^2/\omega^2) \quad . \tag{6}$$

The parameters all admit of ready definition in terms of the energy functions, even when their values depend on frequency.

The time average of total sound power radiated is

$$P_k^{rad} = V_k^2\, R_k^{rad} \tag{7}$$

where $R_k^{rad} \equiv \mu_k\, R_k$ is that part of the resistance associated with sound radiation.*

---

*We adopt the convention that A^2 is to be read as $|A|^2$ when A is complex. Thus V_k^2 indicates a real number which equals the mean square value of the real velocity:

$$V_k^2 = \langle [\mathrm{Re}(\sqrt{2}\, V_k\, e^{i\omega t})]^2 \rangle_t \ .$$

Far from the shell, at a range r in a direction specified by the unit vector $\underline{\Omega}$, this power is evidenced as sound intensity I and sound pressure p:

$$I = p^2/\rho_f c_f \tag{8}$$

where ρ_f and c_f are the density and sound speed of the fluid. The radiated power equals the surface integral of intensity over a sphere enclosing the source; thus

$$P_k^{rad} = 4\pi r^2 \langle I_k(r,\underline{\Omega})\rangle_{\underline{\Omega}} \quad , \tag{9}$$

where the angular braces indicate an average with respect to the subscript variable, $\underline{\Omega}$ or direction. The variation of intensity with direction is incorporated in the directivity function

$$D_k(\underline{\Omega}) = I_k(r,\underline{\Omega})/\langle I_k(r,\underline{\Omega})\rangle_{\underline{\Omega}} \quad . \tag{10}$$

The directivity is determined by the mode shape (eigenfunction, etc.) and is independent of V_k. However, the directivity will usually be found to change slowly with frequency.

With knowledge of these various modal functions, the computation of the total power radiated by the shell is, in principle, a straightforward task. Computation of total sound intensity at one point requires further consideration of directivity and of relative phase between modes.

B. RESONANT RESPONSE TO NOISE

When the force is a band-limited noise, we divide the modes into two subsets: those responding in a resonant manner, and all others. More precisely, by "resonant response" we indicate the response of modes having a resonance frequency in the noise band, ignoring all others. We compute the response of these resonant modes as though the bandwidth of the noise were "infinitely" broad. The validity of these approximations is based on assumptions that the noise bandwidth is significantly larger than the bandwidth of a single mode's resonance curve, and that the noise bandwidth contains an adequate number of resonance frequencies.*

From familiar formulas for the response to noise of a single resonator[4], we find that the modal contribution to power carried by radiated sound is

$$P_k^{rad} = \frac{\pi}{2} S_f \, T_k^2 \, \mu_k / M_k \quad , \quad \omega_k \text{ in band} \quad , \tag{11}$$

where S_f is given by Eq. 2. The total power radiated is the sum

$$P^{rad} = \Sigma_k \, P_k^{rad} \quad , \tag{12}$$

where the sum extends over those modes having ω_k in the band.

---

*More sophisticated justifications, interpreting the results as precise ensemble averages, have been constructed in some circumstances. See Ref. 5, Sect. II.9.

We shall express the sum as an average value times the number of modes with ω_k in the band. That number of modes is denoted by a product

$$n(\omega_c)\delta\omega = N\left(\omega_c - \tfrac{1}{2}\delta\omega < \omega_k < \omega_c + \tfrac{1}{2}\delta\omega\right) \quad . \tag{13}$$

The factor $n(\omega_c)$ is called the "modal density". Its value depends both on ω_c and bandwidth $\delta\omega$, but the percent fluctuations with variations in ω_c and $\delta\omega$ are small if $\delta\omega$ is broad enough to contain a fair number of ω_k.

The combination of Eqs. 2 and 11-13 yields the desired result for total sound power radiated per unit of mean-square force:

$$P^{rad}/f^2 = \frac{\pi}{2}\, n(\omega_c)\, \langle T_k^2\, \mu_k/M_k \rangle_k \quad , \tag{14}$$

where the average with respect to k is taken over all modes having ω_k in the band.

The replacement of the sum over modes by the product of number of modes and average value is a matter solely of convenience. When the modal density is large, it has been found that the average value is a slowly varying function of center frequency and is relatively insensitive to bandwidth.

C. RESONANT RESPONSE TO PURE TONES; FLUCTUATIONS WITH FREQUENCY

The power radiated by a single mode driven by a pure tone at its resonance frequency is

$$W_k^{rad}(\omega_k)/f^2 = T_k^2 \mu_k/R_k = T_k^2 \mu_k^2/R_k^{rad} \quad , \tag{15}$$

where resonance is defined by the vanishing of the imaginary part of the modal admittance or, equivalently, by the vanishing of the Lagrangian. (The symbol W is used for power when the force is a pure tone.) Note that this power is independent of the purely mechanical properties of the mode when radiation dominates the damping ($\mu_k \approx 1$).

If the modal loss factor η_k is small, the curve of pure-tone power as a function of frequency forms a narrow peak with a maximum at resonance. A measure of the breadth of that peak is the effective (or, noise) bandwidth[4]

$$\Delta_k = \frac{\pi}{2} \eta_k \omega_k \equiv \int_0^\infty W_k^{rad}(\omega)\, d\omega / W_k^{rad}(\omega_k) \; ; \tag{16}$$

it equals $\pi/2$ times the bandwidth at the half-power points. If the modal characteristics-resistance, mass, etc. - vary with frequency, the integral expression may become useless if interpreted strictly. The equations remain useful approximations if the integral is restricted to frequencies near resonance, in a band including ω_k and extending at least $\eta_k \omega_k$ on either side.

As was noted in the introductory discussion, the response to a band of noise equals the average in frequency of the response to a tone having the same mean square force as the noise:

$$P_k^{rad} = \frac{1}{\delta\omega}\int_{\omega_c \pm \frac{1}{2}\delta\omega} W_k^{rad}(\omega)d\omega = \langle W_k^{rad}(\omega)\rangle_{\omega \text{ in band}} \quad (17)$$

(The relation is equally valid, written without subscript k's, for the sum of power due to all modes.) Therefore, if $\delta\omega \gtrsim 2\Delta_k$ and the band includes ω_k, Eqs. 16 and 17 can be combined to yield

$$\Delta_k W_k^{rad}(\omega_k) = P_k^{rad}\,\delta\omega \quad (18)$$

where both powers are evaluated for the same mean square force.

Let us sum Eq. 18 over all modes having ω_k in the noise band, and replace the sum by the product of the average and the number of modes (cf. Eqs. 12, 13). The result can be expressed in terms of the total noise power (see Eqs. 5 and 6):

$$\langle \Delta_k W_k^{rad}(\omega_k)\rangle_k = P^{rad}/n(\omega_c) \quad . \quad (19a)$$

In many practical situations, especially where the losses are dominated by internal dissipation, the value of $\Delta_k \propto \eta_k \omega_k$ is more or less constant from mode to mode in any restricted range of frequency. Under this assumption, the left side of Eq. 19a can be factored with the result

$$\langle W_k^{rad}(\omega_k)\rangle_k = P^{rad}/\langle \Delta_k\rangle_k \, n(\omega_c) \quad . \quad (19b)$$

The denominator on the right hand side is the number of modal resonances in an average modal bandwidth.

Now, when the modal bandwidths Δ_k are significantly smaller than the frequency interval between resonances ($\approx 1/n(\omega)$), the total pure-tone response at each resonance frequency will be dominated by the response of the single resonant mode. The analytical expression of this statement is

$$W^{rad}(\omega_k) \approx W_k^{rad}(\omega_k), \text{ if } \Delta_k n(\omega) < 1 .$$

Then also, the total response will have a local maximum at that frequency, and Eqs. 19a and 19b indicate how much those maxima exceed the total power radiated when the same mean square force is redistributed as a noise with flat spectral density.

On the other hand, when $\Delta_k n(\omega) \gtrsim 1$, individual maxima of W^{rad} are no longer determined by individual modal responses but rather by fluctuations in the spacing of resonance frequencies. A much different analysis is then required.[6]

D. RESPONSE OF NONRESONANT MODES

The previous subsections considered the response only of such modes as are resonant at frequencies in the range of interest. The response of all other modes, for which the imaginary part of the impedance does not vanish, must also be considered. The task of numerical assessment is simplified because that response is a relatively slowly varying function of frequency.

In previous studies[1-3], it has been the general rule that the radiated sound can be ascribed entirely to the response of resonant modes, unless they are few in number and their vibration is poorly coupled to sound. Then, the response of nonresonant modes that are better coupled to sound may predominate in the sound power, if not in the structural vibration.

As a general procedure, the powers radiated by resonant and by nonresonant modes are evaluated separately.

A peculiar situation arises with structures radiating into liquids, for which the characteristic impedance $\rho_f c_f$ is so large. As frequency is increased, the radiation resistance of a mode also increases, eventually reaching an asymptotic value. It is often found by calculation that the radiation resistance at the resonance frequency is small, so that the mode is highly resonant (lightly damped) in that frequency range. However, at higher frequencies the radiation resistance becomes so large that it may not only dominate the total resistance (i.e. $\mu_k \approx 1$) but also exceed the mass reactance, viz. ωM_k. In such a situation, the response of the mode is governed by the equations for excitation at resonance (e.g. Eq. 15), over a broad range of frequencies. The mode might well be called quasi-resonant at these frequencies.

III. MODAL MASSES AND RESONANCE FREQUENCIES

For a general survey of the characteristics of the various modes of the finite shell, we idealize the ring-stiffened shell as an orthotropic, homogeneous thin cylinder which is simply supported at its ends by rigid and motionless end caps. Of course, the actual end caps are not constrained from motion. It is recognized that motion of the end caps will be important to low-frequency modes involving the gross motion of the shell (rigid body motion and the fundamental breathing and bending modes). Their analysis is deferred to later reports where the individual peculiarities of each shell will be considered more precisely.

For the idealized case described above, the normalized modes shapes are

$$\Psi_{nm} = \sqrt{2\epsilon_n} \cos n\phi \sin(m\pi x/\ell) \qquad (20)$$

where m and n are the axial and circumferential mode numbers ($\epsilon_o=1$, $\epsilon_{n>o}=2$). For these mode shapes, the modal mass in the absence of fluid load is just equal to the actual mass of the shell, without end caps. (The mode shapes are normalized, in the sense that their mean square values on the shell are all unity. Therefore, the kinetic energy in a vibration $V_{nm}(t)\Psi_{nm}$ is $\frac{1}{2} M_o V_{nm}^2$, where M_o is the shell's mass.)

When the shell vibrates in water the modal mass is increased by the virtual mass of fluid loading. This mass loading reduces the modal resonance frequencies from their values in vacuo, in proportion to the square root of total mass. We approximate the correction factor that must be applied to the in vacuo frequency by the expression

$$\left(1 + \frac{\rho_f A/M_o}{k_h}\right)^{-1/2} \tag{21}$$

where

ρ_f = fluid density

A = active surface area of the shell

$k_h = (k_a^2 + k_c^2)^{1/2}$ = helical wavenumber

$k_a = m\pi/\ell$ = axial wavenumber

$k_c = n/a$ = circumferential wavenumber

a = radius of cylinder

This correction is precisely correct for waves on a large flat plate. Numerical comparison with experimental data[7,8] has shown that it is approximately correct for the cylindrical shells of interest to us in this study. The helical, axial, and circumferential wavenumbers are introduced as convenient parameters for describing the mode shapes and their resonance frequencies.

Estimates for the resonance frequencies of shell modes, with fluid loading, will be based on an expression given by Heckl[9], with modifications to account for (i) virtual mass, (ii) kinetic energy of circumferential motion, and (iii) the orthotropic nature of the shell. The complete expression is

$$f_{nm} = f_r \left\{ \frac{k_a^4}{k_h^4} + \left[k_a^2 + \left(\frac{B_c}{B_a}\right)^{\frac{1}{2}} k_c^2 \right] \frac{h^3 a^2}{12 h_{eff}} \right\}^{\frac{1}{2}} \times \qquad (22)$$

$$\left(1 + \frac{1}{k_c^2 a^2}\right)^{-\frac{1}{2}} \times \left(1 + \frac{\rho_f A/M_o}{k_h}\right)^{-\frac{1}{2}}$$

where

f_r = $(E/\rho)^{\frac{1}{2}}/2\pi a$

ρ, a, h = shell density, radius, and plating thickness;

E = Young's modulus;

B_c/B_a = bending stiffness ratio (circumferential/axial), determined as for a flat orthotropic plate;

h_{eff} = M_o/ρ , an effective shell thickness based on total shell mass.

The last factor of Eq. (22) is the correction for virtual mass given in Eq. (21). The next to last factor is a correction for the kinetic energy of circumferential motion, which was neglected by Heckl. Numerical comparison with the tables of Baron and Bleich[10] show this factor to be correct within 1 or 2 percent for $n>1$ and $k_a a \leq \pi$. The first factor, in braces, is based on Heckl's Eq. (8) and includes two terms: the first arises from membrane effects; the second arises from shell bending effects and has been modified to reflect approximately the orthotropic nature of the shell.

The resonance frequency given by Eq. (22) is a function of k_a and k_c , and of various parameters determined from the shell configuration. Each mode has particular values of k_a and k_c, determined from the mode numbers m and n as in Eq. (21). It is convenient to plot these results in a graph with k_a and k_c as coordinate axes. Then there is a lattice of points corresponding to the modal values. Contours of constant resonance frequency can be plotted to represent Eq. (22). An example of such a plot is given in Fig. 1, using the dimensions of the EB 7-stiffener shell.[7]

It is evident from Fig. 1 that only a few shell modes are resonant in the low frequency range of interest, about 150-260 cps. Because of the symmetry of the excitation in the particular problems to be considered, response in modes with even mode numbers m can be neglected. Then, there remains to be considered only a few modes with m=1 and the gross motion of the shell and end caps.

IV. TYPICAL MODAL RADIATION RESISTANCES

The modal radiation resistance is a measure of the coupling between vibration and sound waves which is determined, in large part, by the scale of the vibrational pattern (its "wavelengths") in relation to the wavelength of free sound waves, i.e. by the ratios k_a/k_f and k_c/k_f where $k_f = \omega/c_f$. The dependence of R_{nm}^{rad} upon these parameters has been a matter for long and intensive study in the acoustical literature. Some recent work has revealed that modes of simple plates and shells can be organized into classes with similar radiation characteristics, for each of which simple estimates are available.[1-3]

We assume that the mode shapes have been normalized, so that the generalized modal velocity is the spatial rms surface velocity.

The first class consists of the surface modes, defined as those having a helical wavenumber $k_h < k_f$. (See Fig. 2A.) Such a mode's radiation resistance is nearly

$$\text{surface mode:} \qquad R^{rad} = A\,\rho_f\,c_f\;, \tag{23}$$

corresponding to the asymptotic law for high frequencies.[11] All the surface radiates power in proportion to the local mean square velocity.

The second radiation class consists of edge modes, defined as those having $k_a > k_f$ but $k_c < k_f$. (See Fig. 2B.)

Their axial wavelength is short and radiative contributions from the middle of the shell are mutually destructive. The radiation can be accounted for as arising from the vibration of narrow strips (rings) at the ends of the shell.[2,12] With simply supported ends, the radiation resistance of an edge mode is estimated as[3]

$$\text{edge mode:} \quad R^{rad} = \frac{4}{3}\,(k_f/k_a^2\ell)\; A\; \rho_f c_f \tag{24}$$

when both $k_f a > 1$ and $k_f \ell > 2$. When $k_f \ell < 2$, the rings accounting for the radiation are close together. In all directions from the shell, either constructive or destructive interference will take place, depending on the symmetry of the mode shape: constructive for mode number m odd, destructive for m even. Correspondingly, the estimate for radiation resistance must be multiplied by 2, or by 0.

An exceptional situation arises when $k_f a < 1$ for those modes with $k_a > k_f$ but with $k_c = 0$ $(n = 0)$; these might be called end modes. (See Fig. 2C.) The axial wavelength is short, as in the case of edge modes, but the shell diameter is small. The radiation can be accounted for by simple sources at the ends of the shell, the strength of each being the acoustic volume velocity associated with that part of the shell within one-quarter wavelength $(\pi/2k_a)$ of the end. The radiation resistance of such a mode is found to be

$$\text{end mode:} \quad R^{rad} = 2k_f a\;(k_f/k_a^2\ell)\; A\; \rho_f c_f \tag{25}$$

when the shell is long $(k_f \ell > 2)$. This is less than the value appropriate to the same mode at higher frequencies, given by the previous formula. The correction factors for interference between the two ends when $k_f \ell < 2$ are the same as for edge modes (2, m odd; 0, m even).

The remaining modes, for which $k_c > k_f$, are "non-radiating" modes, at least to a first approximation. (See Fig. 2 D.) Such a conclusion is justified by Junger's calculations,[13] which show that the radiation resistance falls off sharply when $k_f a < n$; the fall-off is sharper for larger values of n.

In the specific problems of present interest, the shell dimensions and frequency range are such that the only resonant modes are in this "non-radiating" class. Therefore, a more detailed investigation of the radiative properties of these modes is required.

V. MODAL RADIATION RESISTANCES; SPECIAL ANALYSIS

We consider sound radiation into an infinite fluid from a finite cylindrical shell with rigid, motionless, plane end caps. In cylindrical coordinates (x,R,ϕ) with the x axis aligned on the shell's center line, the shell surface is R=a between $x=\pm\ell/2$. We consider pure-tone vibration of frequency ω in a single m=1 mode: viz., a distribution of normal velocity on the cylinder which is

$$u(t) = e^{i\omega t} 2\sqrt{\epsilon_n}\, U \cos \pi x/\ell \cos n\phi \qquad (26)$$

where $\epsilon_0=1$, $\epsilon_{n\neq 0}=2$, and n is an integer. It is evident that U is the rms value of the real velocity on the surface, averaged both in position and in time.

A. SHORT SHELL

Consider a shell that is short compared with the acoustic wavelength, in the sense that

$$k_f^2 \ell^2 \ll 1 \quad .$$

In such a case, the actual distribution of vibratory velocity along any generator of the cylinder is, to a first approximation, immaterial to the sound radiated so long as the value of the integral

$$\int_{-\frac{1}{2}\ell}^{\frac{1}{2}\ell} u\, dx = e^{i\omega t} (4\sqrt{\epsilon_n}/\pi)\, U\ell \cos n\phi \qquad (27)$$

is conserved. (This is a one-dimensional analog of the familiar "volume velocity" approximation in acoustics.)

The ideal problem we shall analyze is the limiting case as $\ell \rightarrow 0$, holding constant the integral in Eq. (27). The plane x=0 is a plane of symmetry in the actual problem, on which the normal particle velocity vanishes. The radiation pattern for x>0 is identical to that for x<0 and the powers radiated into the two half-spaces are equal. Therefore, we consider the radiation into the half-space x>0 that results from a normal velocity boundary condition on x=0:

$$u_n = (2\sqrt{2\epsilon_n}/\pi)\, U\ell\, \delta(R-a) \cos n\phi \quad , \tag{28}$$

where the exponential time dependence has been suppressed. The directivity pattern will be identical with the pattern for radiation into full space. By virtue of the extra factor $\sqrt{2}$ in Eq. (28), the power radiated into the half-space will equal the total power radiated into full space.

The pressure generated at a vector position $\underline{r}$ (polar coordinates r, θ, α, where θ is the polar angle measured from the x axis) is given by the Helmholtz integral over the surface x=0. For far-field points ($r^2 \gg a^2$), subject to the Fraunhofer approximation, the integral becomes (Ref. 14, p. 326ff)

$$p(\underline{r}) = i e^{-ik_f r} \frac{\rho_f c_f k_f}{2\pi r} \int u_n\, e^{ik_f R \sin\theta \cos(\phi-\alpha)} R dR d\phi \ . \tag{29}$$

When the velocity of Eq. (28) is substituted, the integral in Eq. (29) becomes

$$(\sqrt{2\epsilon_n}/\pi^2)AU \int_0^{2\pi} e^{ik_f a \sin\theta \cos(\phi-\alpha)} \cos n\phi \, d\phi$$

where $A=2\pi a\ell$ is the surface area of the actual shell. This integral over ϕ is readily found from Bessel's integral definition of Bessel functions (Ref. 15, sect. 2.2). The result is an expression for the far-field pressure:

$$p(\underline{r}) = \left(i^{n+1} e^{-ik_f r}\right) \frac{\sqrt{2\epsilon_n}}{\pi^2} \frac{\rho_f c_f k_f AU}{r} \cos n\alpha \; J_n(k_f a \sin\theta) \quad . \qquad (30)$$

The time-averaged sound intensity $I(\underline{r})$ observed at a far-field point $\underline{r}$ is

$$I(\underline{r}) = \tfrac{1}{2}|p|^2/\rho_f c_f \quad ,$$

and the total power is the integral of intensity over the hemisphere of radius r:

$$P = \int I(\underline{r}) \, r^2 \sin\theta \, d\theta \, d\alpha \quad .$$

The directivity function describes the variations of intensity with direction $\underline{\Omega}$ to the observation point $\underline{r} = r\underline{\Omega}$:

$$D(\underline{\Omega}) = 2\pi r^2 \, I(r\underline{\Omega})/P \quad .$$

The modal radiation resistance is defined as

$$R^{rad} \equiv P/U^2 \quad ,$$

and the modal radiation efficiency is defined as the dimensionless ratio

$$\sigma^{rad} \equiv R^{rad}/A\ \rho_f c_f \quad .$$

Now, that part of the power integral involving α is readily performed. Thus, the radiation efficiency can be written

$$\sigma^{rad} = \frac{2}{\pi^3} k_f^2\ A\ G_n(k_f a) \quad , \tag{31}$$

where

$$G_n(k_f a) = \int_0^{\frac{1}{2}\pi} J_n^2(k_f a \sin\theta) \sin\theta \, d\theta \quad .$$

The directivity function is

$$D(\alpha,\theta) = \epsilon_n \cos^2 n\alpha \ J_n^2(k_f a \sin\theta)/G_n(k_f a) \quad . \tag{32}$$

The integral over θ, $G_n(k_f a)$, must be evaluated. For this mathematical discussion, we denote the argument by y; thus,

$$G_n(y) \equiv \int_0^{\frac{1}{2}\pi} J_n^2(y \sin\theta) \sin\theta \, d\theta \quad . \tag{33}$$

Term-by-term integration of the series for J_n^2 (Ref. 16, p. 97) yields a series useful for small y

$$G_n(y) = \frac{y^{2n}}{(2n+1)!}(1-\delta_n) \quad ,$$

$$\delta_n \approx y^2/(2n+3) \quad . \tag{34}$$

Thus, the leading term is seen to be a good approximation for all n, if $y^2 \ll 3$. The function $G_n(y)$ can also be shown to equal another integral

$$G_n(y) = (2y)^{-1} \int_0^{2y} J_{2n}(t)\, dt \quad , \tag{35}$$

by comparison of the series expansions [for example, the Neumann series given for Eq. (35) in Ref. 16, p. 159, and for Eq. (33) in Ref. 17, Eq. 7.14 (21).] As $y \longrightarrow \infty$, the integral in Eq. (35) approaches unity, so we have

$$G_n(y) \sim 1/2y \text{ as } y \longrightarrow \infty \quad .$$

1. Low Frequency Approximation

When $k_f a$ is small (in the sense that $k_f^2\, a^2 \ll 2n+3$) the leading term approximation of Eq. (34) can be used. Then the modal radiation efficiency [Eq. (31)] is given by

$$\sigma^{rad} \approx \frac{4}{\pi^2}\frac{\ell}{a}\frac{(k_f a)^{2n+2}}{(2n+1)!} \quad . \tag{36}$$

The low frequency approximation for directivity [Eq. (32)] depends on n. For n=0, the source is omnidirectional, and D=1. For n>0, the directivity at low frequencies is

$$D(\alpha,\theta) \approx D_{max} \cos^2 n\alpha \sin^{2n}\theta \tag{37a}$$

with maximum directivity occurring at broadside ($\theta=\pi/2$ and $\alpha=0$):

$$D_{max} \approx (2n+1)! \Big/ 2^{2n-1}(n!)^2 \quad . \tag{37b}$$

(The leading term in the series for J_n^2 has been used.) Values of the maximum directivity index are as follows:

n	$10 \log_{10} D_{max}$
1	4.8 dB
2	5.7
3	6.4
4	6.9
n>4	~ 3.5 + 5 log n .

The asymptotic law, gotten from Stirling's formula, is good within 1/3 dB for n>4.

The directivity functions for n=3 and n=4 are plotted in Figs. 3 and 4. The function, Eq. (37a), has been factored:

$$\begin{aligned} D(\alpha,\theta) &= D_1(\alpha)\, D_2(\theta) \\ D_1(\alpha) &= \epsilon_n \cos^2 n\alpha \quad , \end{aligned} \tag{37c}$$

and $D_1(\alpha)$ and $D_2(\theta)$ displayed separately.

B. LONG SHELL

We now consider a shell that is long, in the sense that

$$k_f^2 \ell^2 \gg 1 \quad ,$$

and that vibrates in one of the fundamental-axial modes [Eq. (26)]. We assume that the edge conditions become less important as the length increases, and adapt to this problem the results pertinent to the shell with rigid cylindrical extentions (baffles). Such a situation is easily analyzed by Fourier transforms in the x (axial) coordinate. (The transform variable conjugate to x will be denoted by k_x.)

The transform, $\int u(x) \exp(ik_x x)\, dx$, of the velocity given in Eq. (26) is

$$4\pi \ell U \sqrt{\epsilon_n} \cos n\phi \cos\left(\tfrac{1}{2}k_x \ell\right)\left[\pi^2 - (k_x \ell)^2\right]^{-1} , \qquad (38)$$

if the time dependence is suppressed. The largest values of this transform occur near k_x=0, i.e. for $k_x \ell \lesssim \pi$. Since, by assumption, $k_f \ell$ is much larger, these small values of the transform variable account for the sound radiation. Instead of formulating an integral for power in the manner of Laird and Cohen[15] we approximate the square of Eq. (38) by a Dirac δ-function in k_x, centered on k_x=0. The strength of the δ-function is easily gotten from Parseval's theorem. Then, we use the acoustic resistances that were previously investigated by Junger[13] to compute the power radiated, the radiation resistance, and the radiation efficiency.

In the present case, the radiation efficiency is thus found to be equivalent to Junger's specific acoustic impedance resistance ratios, θ_n:

$$\sigma^{rad} = \theta_n(k_f a) = \mathrm{Im}[H_n(k_f a)/H_n'(k_f a)] \tag{39}$$

where H_n is the Hankel function describing outgoing waves. The imaginary part of the ratio can be simplified by means of the expression for the Wronskian for J_n and N_n; the result is:

$$\sigma^{rad} = 2[\pi k_f a |H_n'(k_f a)|^2]^{-1} \quad . \tag{40}$$

When $k_f a/n$ is small, the asymptotic expression for the Hankel function can be used (Ref. 19, Chapter 11), with the result

$$\sigma^{rad} \approx \epsilon_n^2 \pi \left(\frac{1}{2} k_f a\right)^{2n+1} (n!)^{-2} \quad . \tag{41}$$

When $k_f a/n$ is large, asymptotic expressions lead to the expected result:

$$\sigma^{rad} \sim 1 \quad . \tag{42}$$

Junger's graphs can be used for intermediate values.

Curves of the radiation efficiency as a function of frequency are given in Fig. 5 for various n. They have been calculated from Eqs. (36), (41), and (42) using a sound speed c_f = 5000 ft/sec and dimensions ℓ = 6.42 ft, a = 2.11 ft, which are the dimensions of the EB shell. Transition between Eqs. (36) and (41) has been set, somewhat arbitrarily, at $k_r \ell = 4$. The analysis with artificial baffles indicates that the tendency to formation of a directive beam, oriented to broadside, is well underway at that point. The onset of transition has been placed an octave on either side.

Detailed discussion of results will be made in later reports. However, note the extremely small values of radiation efficiency for n=3 and n=4 in the frequency range 150-250 cps. Calculations indicate that the loss factor (1/Q) of these modes should range near 10^{-6} if there were no dissipation in the shell. Obviously, internal dissipation will control the response amplitude of these modes in any actual shell.

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LIST OF FIGURES

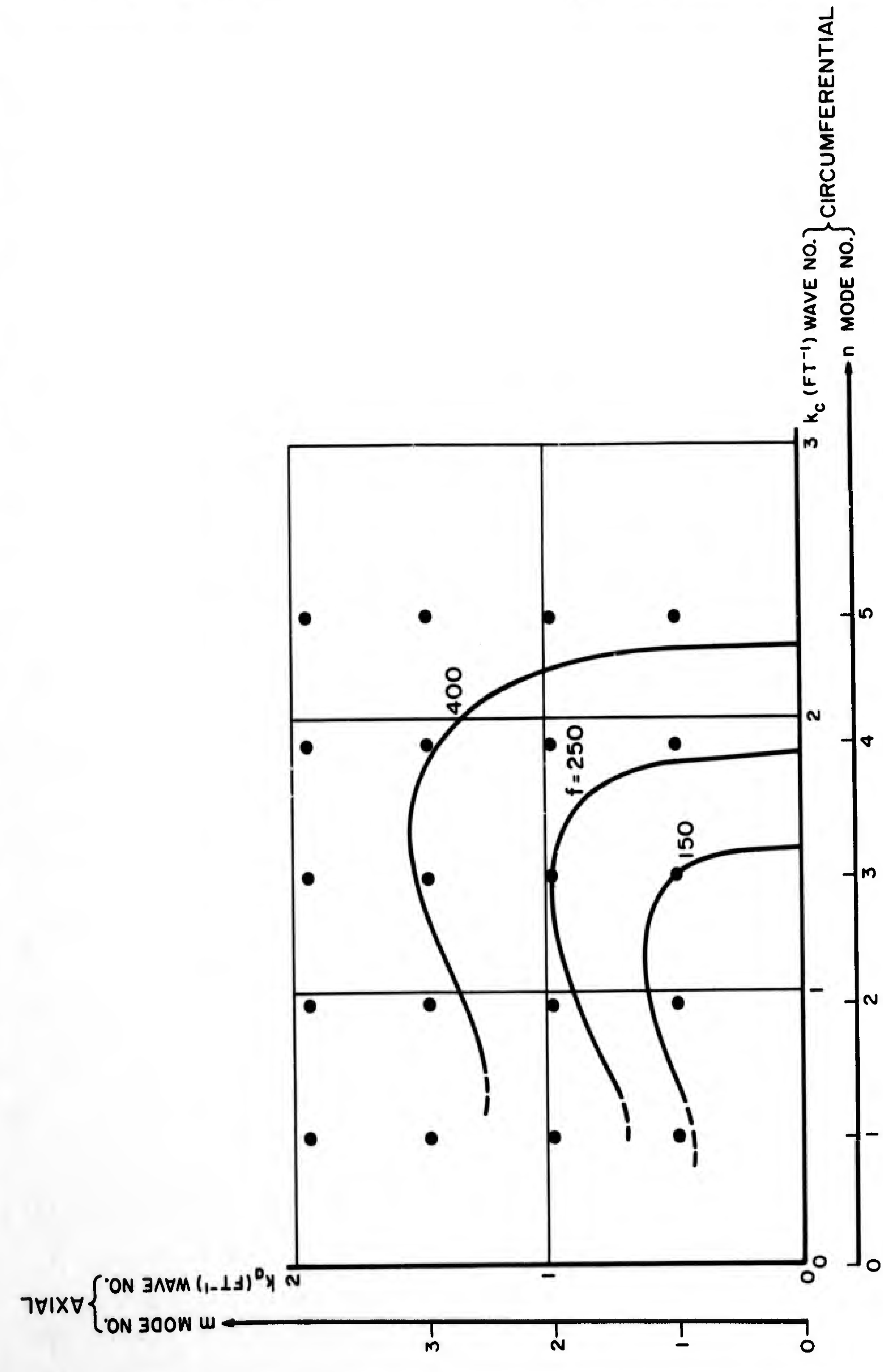

FIG.1 MODAL LATTICE FOR EB 7-STIFFENER SHELL WITH COMPUTED CONSTANT-FREQUENCY CONTOURS

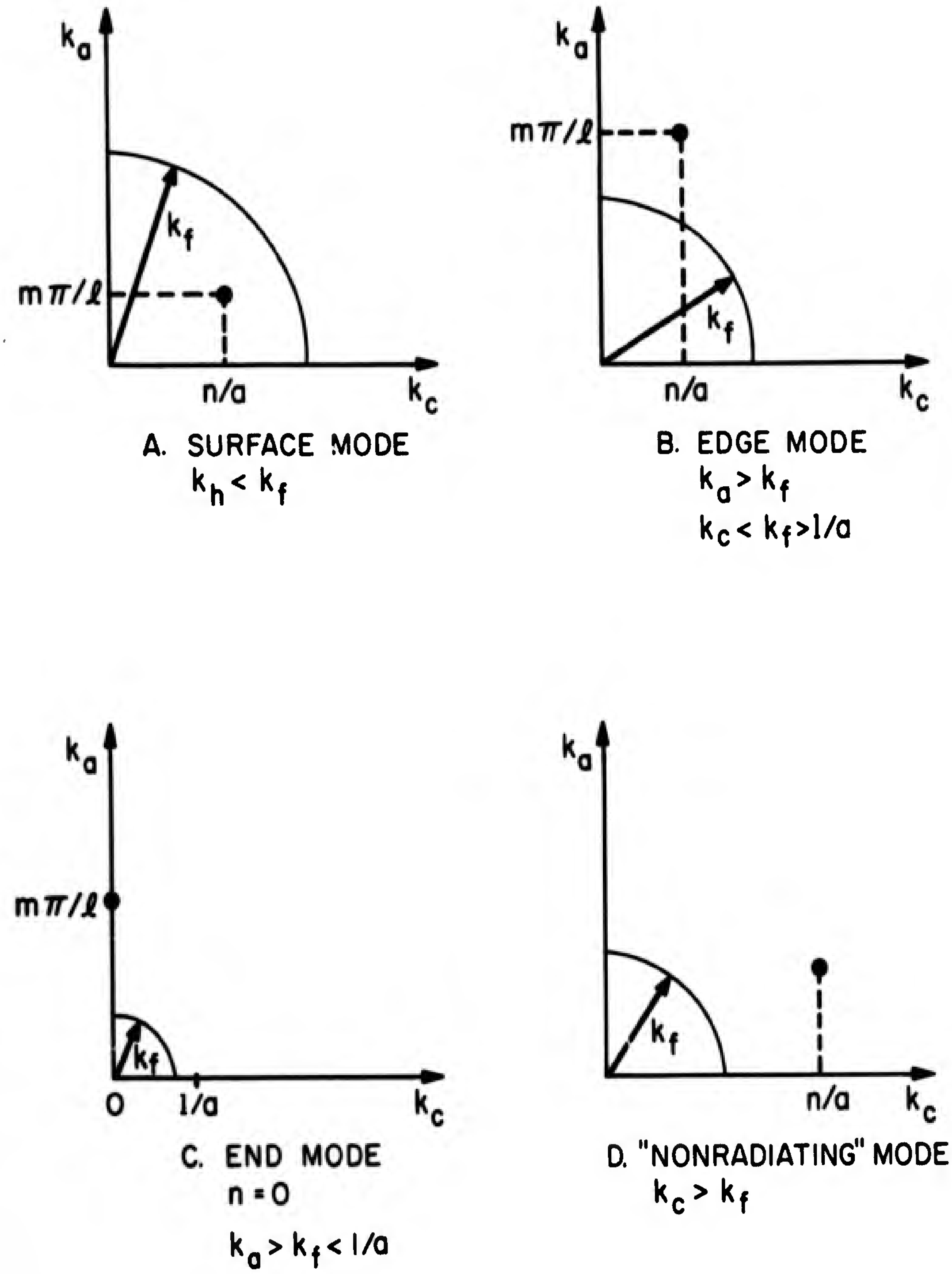

FIG.2 WAVENUMBER PLOTS FOR MODES OF VARIOUS RADIATION CLASSES

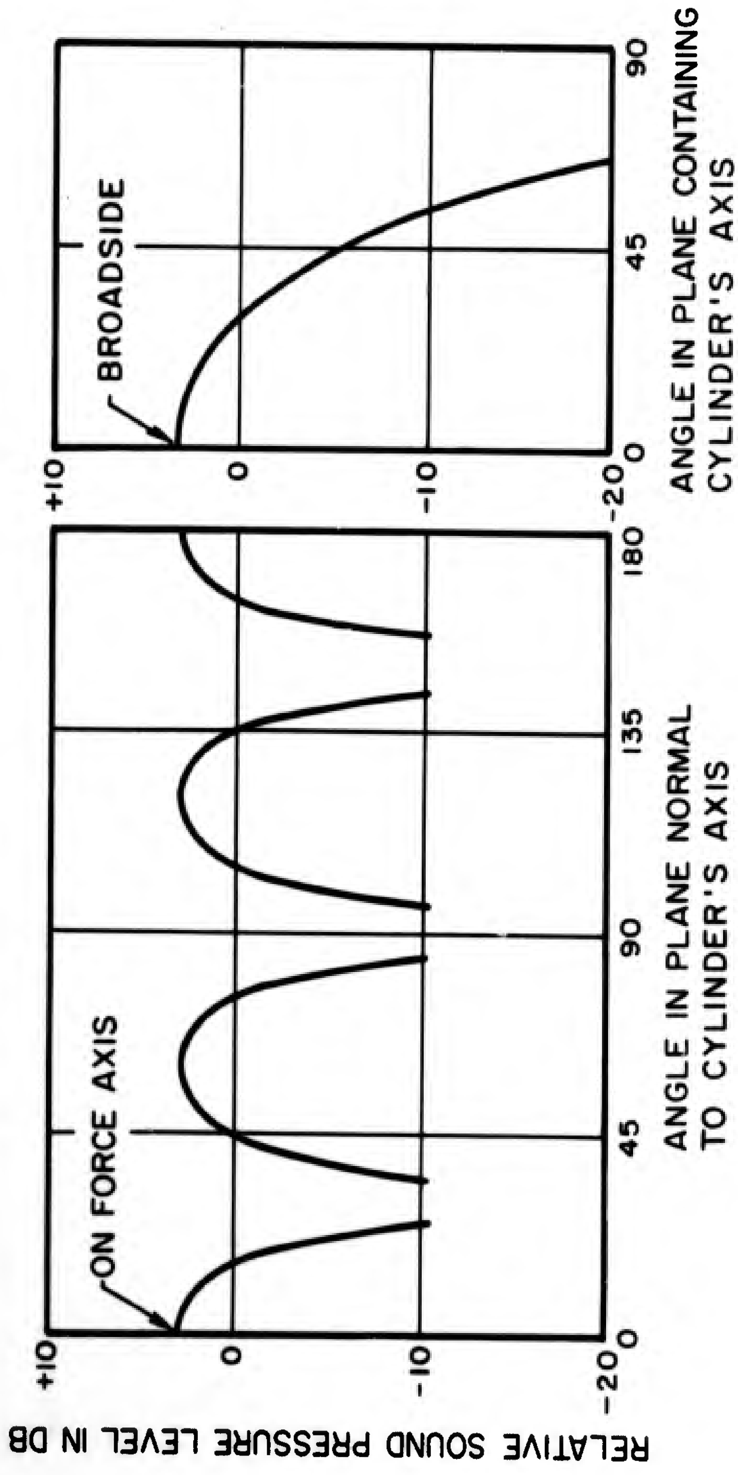

FIG. 3 DIRECTIVE PATTERN FOR (1, 3) MODE AT LOW FREQUENCY

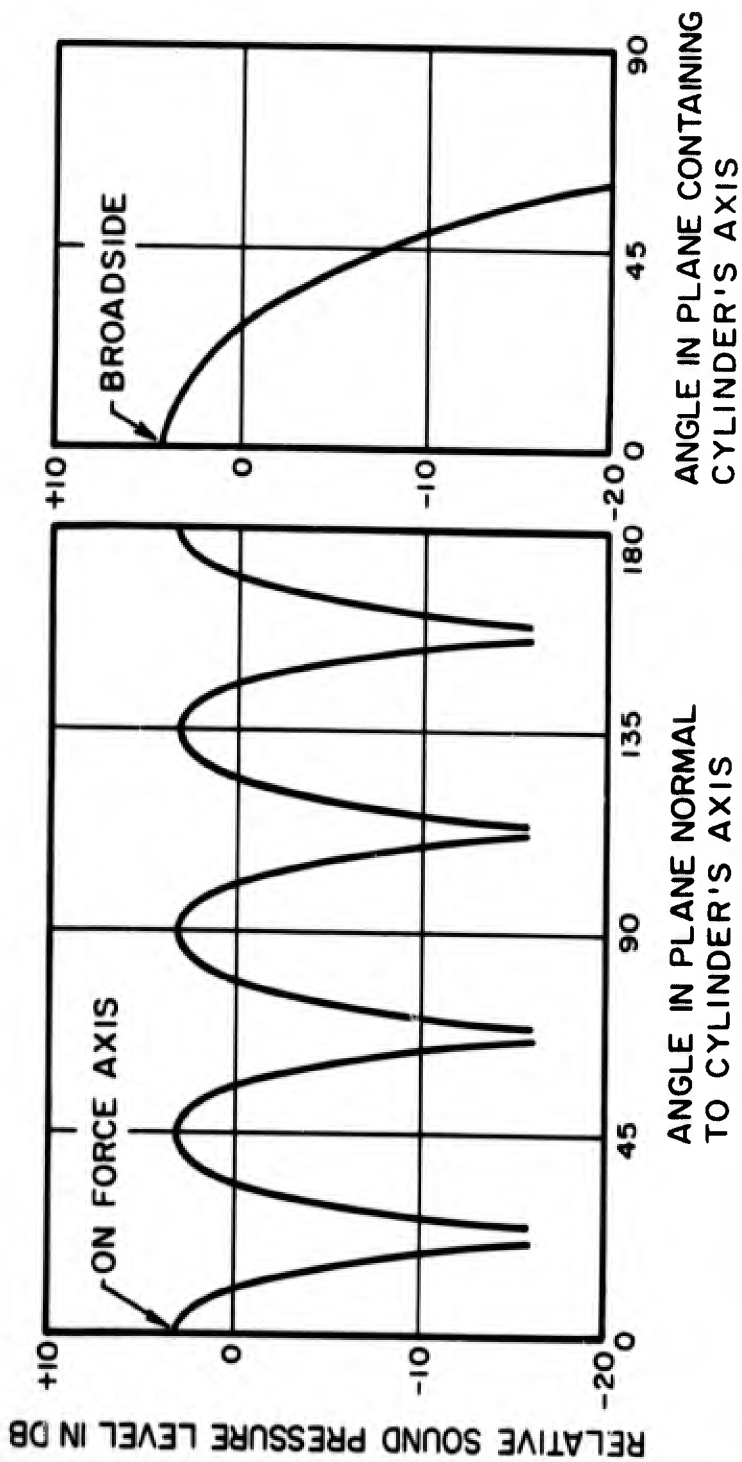

FIG. 4 DIRECTIVE PATTERN FOR (1, 4) MODE AT LOW FREQUENCY

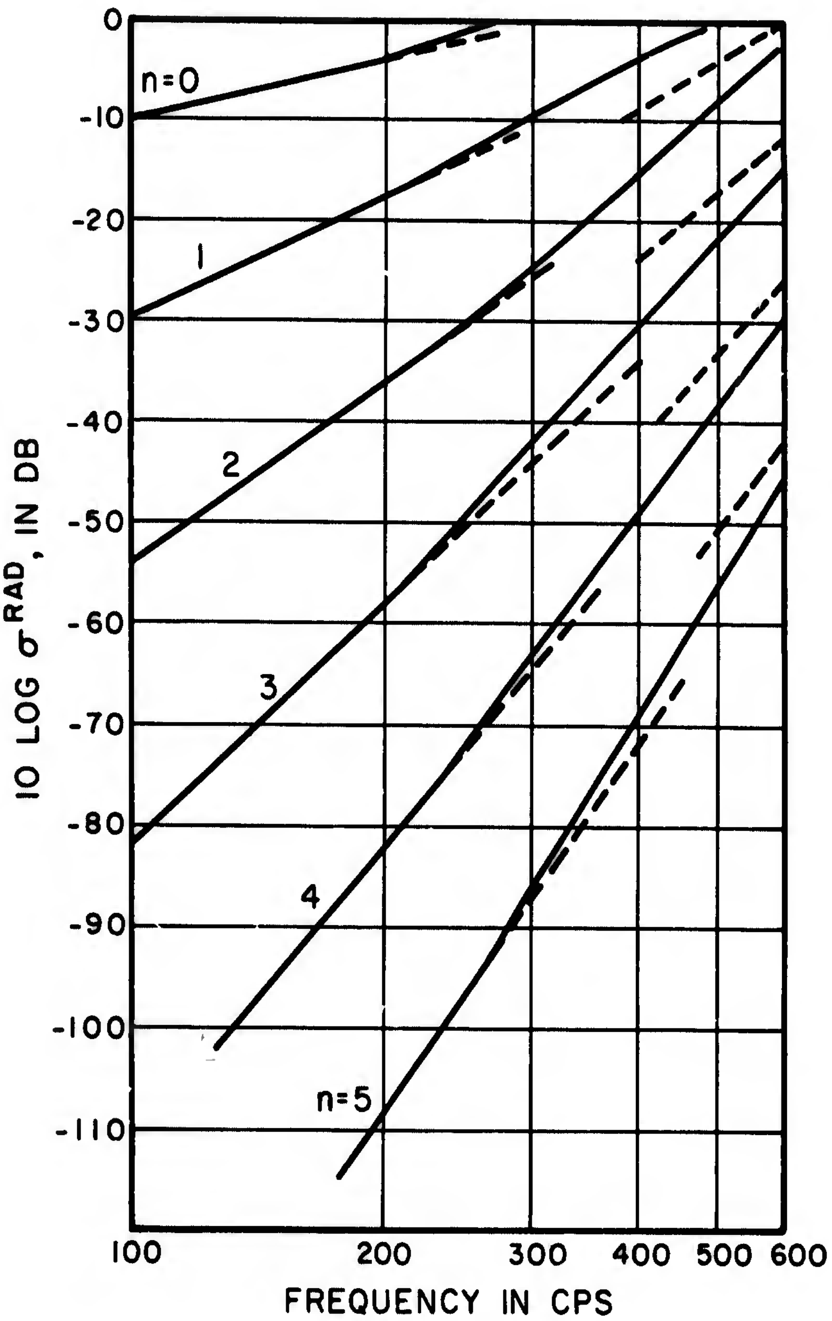

FIG. 5 RADIATION EFFICIENCIES FOR m=1 MODES (EB 7-STIFFENER SHELL)

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